JAYPEE INSTITUTE OF INFORMATION TECHNOLOGY
Electronics and Communication Engineering
Digital Signal Processing (15B11EC413)
Tutorial Sheet: II

Q1. [CO1] Determine the z-transform of the following finite duration signals:

a) $x(n) = \left(\underset{\uparrow}{3,1,2,5,7,0,1}\right)$

b) $x(n) = (1,2,5,4,0,1)$

c) $x(n) = \delta(n)$

d) $x(n) = \delta(n-k)$ for $k > 0$

e) $x(n) = \delta(n+k)$ for $k > 0$

Sol. 1:

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Then,

(a)

$$X(z) = 3z^3 + z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

ROC of the signal is entire z plane except $z = 0$ and $z = \infty$ for which function is not defined

(b)

$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 4z^{-3} + z^{-5}$$

ROC of the signal is entire z plane except $z = 0$ for which function is not defined

(c)

$$x(n) = \delta(n) = 1 \; for \; n = 0 \; and \; 0 \; for \; all \; other \; values$$

i.e.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = 1. z^0|_{n=0} = 1$$

(d)

$x(n) = \delta(n-k)$ is the time shifted function of $\delta(n)$

$$x(n) = \delta(n-k) = 1 \; for \; n = k \; and \; 0 \; for \; all \; other \; values$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = 1. z^{-k}|_{n=k} = z^{-k}$$

ROC of the signal is entire z plane except $z = 0$ for which function is not defined

(e)

$x(n) = \delta(n+k)$ is the time shifted function of $\delta(n)$

$$x(n) = \delta(n+k) = 1 \; for \; n = -k \; and \; 0 \; for \; all \; other \; values$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = 1.z^k|_{n=-k} = z^k$$

ROC of the signal is entire z plane except $z = \infty$ for which function is not defined

Q2. [CO1] Determine the z-transform of the following signals:

a) $x(n) = a^n u(n)$

b) $x(n) = -a^n u(-n-1)$

c) $x(n) = cos(\omega_o n)u(n)$

d) $x(n) = \sin(\omega_o n)\, u(n)$

e) $x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & elsewhere \end{cases}$

Sol. 2:

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

Then,

(a)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n$$

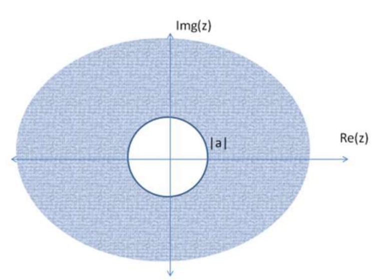
Above series a geometric series with common ratio of az^{-1}

Then for this infinite GP series sum will exist if and only if $|az^{-1}| < 1$

$$X(z) = \frac{(az^{-1})^0}{1 - az^{-1}} = \frac{1}{1 - az^{-1}}$$

The condition for the existence of this function is $|z| > |a|$

ROC of the signal is $|z| > |a|$ which function is defined



(b)

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

For $n = -l$

$$X(z) = -\sum_{l=1}^{\infty} (a^{-1}z)^l$$

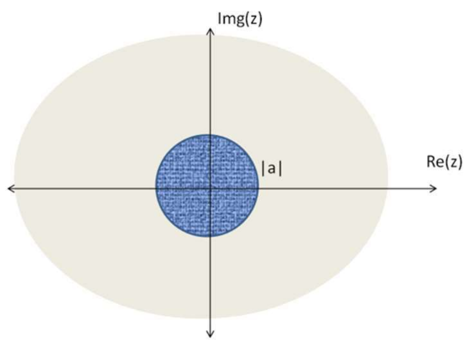
Above series a geometric series with common ratio of $a^{-1}z$

Then for this infinite GP series sum will exist if and only if $|a^{-1}z| < 1$

$$X(z) = -\frac{(a^{-1}z)^1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$

The condition for the existence of this function is $|z| < |a|$

ROC of the signal is $|z| < |a|$ which function is defined



(c)

$$x(n) = \cos(\omega_o n)\, u(n) = \frac{1}{2}\big(\exp(j\omega_o n)\, u(n) + \exp(-j\omega_o n)\, u(n)\big)$$

$$X(z) = Z\left(\frac{1}{2}\exp(j\omega_o n)\, u(n)\right) + Z\left(\frac{1}{2}\exp(-j\omega_o n)\, u(n)\right)$$

Lets

$$\alpha = \exp(\pm\, j\omega_o)$$

Then

$$|\alpha| = |\exp(\pm j\omega_o)| = 1$$

Hence using the z transform of $a^n u(n)$

$$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}}\ ROC\colon |z| > |a|$$

$$\exp(j\omega_o n)\, u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - \exp(j\omega_o)\, z^{-1}}\ ROC\colon |z| > 1$$

and

$$\exp(-j\omega_o n)u(n) \overset{z}{\leftrightarrow} \frac{1}{1-\exp(-j\omega_o)\,z^{-1}}\ ROC{:}\,|z|>1$$

then,

$$X(z) = \frac{1}{2}\left[\frac{1}{1-\exp(j\omega_o)\,z^{-1}} + \frac{1}{1-\exp(-j\omega_o)\,z^{-1}}\right]\ ROC{:}\,|z|>1$$

$$X(z) = \frac{1}{2}\left[\frac{2-z^{-1}(\exp(j\omega_o)+\exp(-j\omega_o))}{1-z^{-1}\big(\exp(j\omega_o)+exp(j\omega_o)\big)+\ z^{-2}}\right] ROC{:}\,|z|>1$$

$$X(z) = \frac{1-z^{-1}\cos(\omega_o)}{1-2z^{-1}\cos(\omega_o)+z^{-2}}\ ROC{:}\,|z|>1$$

(d)

$$x(n) = \sin(\omega_o n)\,u(n) = \frac{1}{2j}\big(\exp(j\omega_o n)\,u(n) - \exp(-j\omega_o n)\,u(n)\big)$$

$$X(z) = \ Z\left(\frac{1}{2j}\exp(j\omega_o n)\,u(n)\right) - \ Z\left(\frac{1}{2j}\exp(-j\omega_o n)\,u(n)\right)$$

Lets

$$\alpha = \exp(\pm\, j\omega_o)$$

Then

$$|\alpha| = |\exp(\pm j\omega_o)| = 1$$

Hence using the z transform of $a^n u(n)$

$$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1-az^{-1}}\ ROC{:}\,|z|>|a|$$

$$\exp(j\omega_o n)\,u(n) \overset{z}{\leftrightarrow} \frac{1}{1-\exp(j\omega_o)\,z^{-1}}\ ROC{:}\,|z|>1$$

and

$$\exp(-j\omega_o n)u(n) \overset{z}{\leftrightarrow} \frac{1}{1-\exp(-j\omega_o)\,z^{-1}}\ ROC{:}\,|z|>1$$

then,

$$X(z) = \frac{1}{2j}\left[\frac{1}{1-\exp(j\omega_o)\,z^{-1}} - \frac{1}{1-\exp(-j\omega_o)\,z^{-1}}\right]\ ROC{:}\,|z|>1$$

$$X(z) = \frac{1}{2j}\left[\frac{z^{-1}(\exp(j\omega_o) - \exp(-j\omega_o))}{1 - z^{-1}\left(\exp(j\omega_o) + exp(j\omega_o)\right) + \ z^{-2}}\right] \ ROC: |z| > 1$$

$$X(z) = \frac{z^{-1}\sin(\omega_o)}{1 - 2z^{-1}\cos(\omega_o) + z^{-2}} \ \ ROC: |z| > 1$$

(e)

$$x(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & elsewhere \end{cases}$$

$x(n)$ is a finite duration signal and can be represented as:

$$x(n) = u(n) - u(n-N)$$

We know

$$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}} \ ROC: |z| > |a|$$

So for $a = 1$

$$u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - z^{-1}} \ ROC: |z| > 1$$

And by applying time shifting property

$$x(n) \overset{z}{\leftrightarrow} X(z)$$

$$x(n-k) \overset{z}{\leftrightarrow} z^{-k}X(z)$$

Then,

$$u(n-N) \overset{z}{\leftrightarrow} \frac{z^{-N}}{1 - z^{-1}} \ ROC: |z| > 1$$

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{z^{-N}}{1 - z^{-1}} = \frac{1 - z^{-N}}{1 - z^{-1}} \qquad ROC: |z| > 1$$

Q3. [CO1] Determine the transform of the signal using the properties of the z-transform.

a) $x(n) = a^n(\cos(\omega_o n))u(n)$ using scaling in z-domain
b) $x(n) = \ u(-n)$ using time reversal property
c) $x(n) = \ n\left(\frac{1}{2}\right)^n u(n)$ using differentiation in z-domain

Sol. 3:

a). $x(n) = a^n(\cos(\omega_o n))u(n)$

We already know:

$$(\cos(\omega_o n))u(n) \overset{z}{\leftrightarrow} \frac{1 - z^{-1}\cos(\omega_o)}{1 - 2z^{-1}\cos(\omega_o) + z^{-1}} \ ROC: |z| > 1$$

And according to scaling property

$$x(n) \overset{z}{\leftrightarrow} X(z) \ ROC: r_1 < |z| < r_2$$

Then

$$a^n\big(x(n)\big) \overset{z}{\leftrightarrow} X(a^{-1}z) \ ROC: |a|r_1 < |z| < |a|r_2$$

Applying same we get

$$a^n(\cos(\omega_o n))u(n) \overset{z}{\leftrightarrow} \frac{1 - az^{-1}\cos(\omega_o)}{1 - 2az^{-1}\cos(\omega_o) + az^{-1}} \ ROC: |z| > a$$

b). $x(n) = u(-n)$

We already know:

$$u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - z^{-1}} \ ROC: |z| > 1$$

And according to time reversal property

$$x(n) \overset{z}{\leftrightarrow} X(z) \ ROC: r_1 < |z| < r_2$$

Then

$$x(-n) \overset{z}{\leftrightarrow} X(z^{-1}) \ ROC: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Applying same we get

$$u(-n) \overset{z}{\leftrightarrow} \frac{1}{1 - z} \ ROC: |z| < 1$$

c). $x(n) = n\left(\frac{1}{2}\right)^n u(n)$

We already know:

$$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}} \ ROC: |z| > a$$

And according to differentiation in z-domain property

$$x(n) \overset{z}{\leftrightarrow} X_1(z) \ ROC: r_1 < |z| < r_2$$

Then

$$nx(n) \overset{z}{\leftrightarrow} -z^{-1}\frac{dX_1(z)}{dz} ROC: r_1 < |z| < r_2$$

Applying same we get

$$na^n u(n) \overset{z}{\leftrightarrow} X(z) \quad ROC: |z| > a$$

And

$$X(z) = -z^{-1}\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2}\ ROC: |z| > |a|$$

Here, $a = \frac{1}{2}$

Then,

$$n\left(\frac{1}{2}\right)^n u(n) \overset{z}{\leftrightarrow} \frac{z^{-1}}{2-z^{-1}}\ \ ROC: |z| > \frac{1}{2}$$

Q4. [CO1] Find $Y(Z)$ for:

$$y(n) = x(n) * h(n)$$

Where, $x(n) = (0.5)^n u(n)$ and $h(n) = 3^n u(-n)$

Sol. 4:

We have

$$x(n) = (0.5)^n u(n)$$

We know

$$u(n) \overset{z}{\leftrightarrow} \frac{1}{1-z^{-1}}\ \ ROC: |z| > 1$$

Then by applying scaling property

$$(0.5)^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1-0.5z^{-1}}\ \ ROC: |z| > 0.5$$

And

$$h(n) = 3^n u(-n)$$

by applying time reversal property and scaling property

$$(3)^n u(-n) \overset{z}{\leftrightarrow} -\frac{3z^{-1}}{1-3z^{-1}}\ \ ROC: |z| < 3$$

As:

$$y(n) = x(n) * h(n)$$

By applying convolution property

$$Y(z) = -\left(\frac{1}{1-0.5z^{-1}}\right)\left(\frac{3z^{-1}}{1-3z^{-1}}\right)$$

The ROC is intersection of the regions $|z| > 0.5$ and $|z| < 3$, which is $0.5 < |z| < 3$.

Now we have to find $y(n)$ using inverse z-transform

$$Y(z) = \frac{A}{1-0.5z^{-1}} + \frac{B}{1-3z^{-1}}$$

Where,

$$A = [(1-0.5z^{-1})Y(z)]_{z=0.5} = \frac{6}{5}$$

$$B = [(1-3z^{-1})Y(z)]_{z=3} = -\frac{6}{5}$$

Then,

$$y(n) = \left(\frac{6}{5}\right)\left(\frac{1}{2}\right)^n u(n) + \left(\frac{6}{5}\right) 3^n u(-n-1)$$

Q5. [CO1] Find the value of $x(0)$ for the sequence that has a z-transform.

$$X(Z) = \frac{z}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-2}\right)} \; for\; ROC\; |Z| > \frac{1}{2}$$

Sol. 5:

We can write

$$X(z) = \frac{z^4}{\left(z-\frac{1}{2}\right)(z^2-\frac{1}{3})} \; ROC: |z| > \frac{1}{2}$$

we see that $X(z) \to \infty$ as $|z| \to \infty$. Therefore, $x(n)$ is not causal. However, because $x(n)$ is right-sided, it may be delayed so that it is causal. Specifically, if we delay $x(n)$ by 1 to form the sequence $y(n) = x(n-1)$,

$$Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)\left(z^2-\frac{1}{3}\right)}$$

which approaches 1 as $|z| \to \infty$. Thus, $y(n)$ is causal, and we conclude that $y(0) = x(-1) = 1$. Because

$$X(z) = x(-1)z + \sum_{n=0}^{\infty} x(n)z^{-n}$$

$X(z) - x(-1)z$ is the z-transform of a causal sequence, and it follows from the initial value theorem that

$$x(0) = \lim_{|z|\to\infty} [X(z) - x(-1)z]$$

With

$$X(z) - x(-1)z = X(z) - z = \frac{z^4}{\left(z-\frac{1}{2}\right)\left(z^2-\frac{1}{3}\right)} - z$$

$$= \frac{z^4 - z\left(z^3 - \frac{1}{2}z^2 - \frac{1}{3}z + \frac{1}{6}\right)}{\left(z-\frac{1}{2}\right)\left(z^2-\frac{1}{3}\right)}$$

we have

$$x(0) = \lim_{|z|\to\infty} [X(z) - x(-1)z] = \frac{1}{2}$$

Q6. [CO1] The z-transform of a signal is given by $C(z) = \frac{\frac{1}{4}\left(z^{-1}(1-z^{-4})\right)}{(1-z^{-1})^2}$ find the final value of the signal.

Sol. 6:

According to final value theorem

$$\lim_{N\to\infty} x[n] = \lim_{z\to 1}(1-z^{-1})\, X^{+}(z)$$

Now,

$$\lim_{z\to 1}\frac{1}{4}\frac{(1-z^{-1})\left(z^{-1}(1-z^{-4})\right)}{(1-z^{-1})^2}$$
$$\lim_{z\to 1}\frac{1}{4}\frac{\left(z^{-1}(1-z^{-4})\right)}{1-z^{-1}}$$
$$\lim_{z\to 1}\frac{1}{4}\frac{1}{z}\frac{(z^4-1)/z^4}{(z-1)/z}$$
$$\lim_{z\to 1}\frac{1}{4}\frac{(z^2-1)(z^2+1)}{z^4(z-1)}$$
$$\lim_{z\to 1}\frac{1}{4}\,z^2\frac{(z+1)(z-1)(z^2+1)}{z-1}$$
$$\frac{1}{4}\times 4 = 1$$

Q7. [CO1] Determine the inverse z-transform

a) $X(Z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$ for ROC $|z| < 0.5$ using power series expansion

b) $X(Z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$ using partial fraction

Sol. 7:

a)

For the given $X(z)$ ROC is interior of a circle, i.e. $x(n)$ is non-causal.

To obtain power series expansion:

$$\frac{1}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1} = 2z^2 + \frac{3z-2z^2}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1}$$

$$2z^2 + \frac{3z-2z^2}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1} = 2z^2 + 6z^3 + \frac{7z^2-6z^3}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1}$$

$$2z^2 + 6z^3 + \frac{7z^2-6z^3}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1} = 2z^2 + 6z^3 + 14z^4 + \frac{15z^3-14z^4}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1}$$

$$2z^2 + 6z^3 + 14z^4 + \frac{15z^3-14z^4}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + \frac{31z^4-30z^5}{\frac{1}{2}Z^{-2}-\frac{3}{2}z^{-1}+1}$$

And repeating the same step will give

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + --$$

So,

$$x(n) = \{----,62,30,14,6,2,0,\boxed{0}\}$$

b)

$$X(Z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

Re-writing the eqn.

$$\frac{X(z)}{z} = \frac{z + 1}{z^2 - z + 0.5}$$

Solving for the poles $z^2 - z + 0.5 = 0$ will give two complex poles

$$p_1 = \frac{1}{2} + j\frac{1}{2}$$
$$p_2 = \frac{1}{2} - j\frac{1}{2}$$

Then we can re-write the eqn. in form

$$\frac{X(z)}{z} = \frac{A}{z - p_1} + \frac{B}{z - p_2}$$

Where,

$$A = \left.\frac{(z - p_1)X(z)}{z}\right|_{z=p_1} = \frac{z + 1}{z - p_2} = \frac{1}{2} - j\frac{3}{2}$$
$$B = \left.\frac{(z - p_2)X(z)}{z}\right|_{z=p_2} = \frac{z + 1}{z - p_1} = \frac{1}{2} + j\frac{3}{2}$$

We know,

$$p_1 = p_2^*$$
$$A = B^*$$

Then,

$$A = \frac{\sqrt{10}}{2}\exp(-j71.565)$$
$$p_1 = \frac{1}{\sqrt{2}}\exp(j\pi/4)$$

Using the relation

$$Z^{-1}\left[\frac{A_k}{1 - p_k z^{-1}} + \frac{A_k^*}{1 - p_k^* z^{-1}}\right] = 2|A_k| r_k^n \cos(\beta_k n + \alpha_k)\, u(n)$$

Where

$$A_k = |A_k| \exp(j\alpha_k)$$
$$p_k = r_k \exp(j\beta_k)$$

Then,

$$x(n) = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi n}{4} - 71.656^o\right) u(n)$$

Q8. [CO1] Consider a system described by the difference equation

$$y(n) = y(n - 1) - y(n - 2) + 0.5x(n) + 0.5x(n - 1)$$

Find the response of this system to the input $x(n) = 0.5^n u(n)$. With initial conditions $y(-1) = 0.75$ and $y(-2) = 0.25$.

Sol. 8:

First, we take the one-sided z-transform of each term in the difference equation

$$Y(z) = z^{-1}Y(z) + y(-1) - [z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] + \tfrac{1}{2}X(z) + \tfrac{1}{2}z^{-1}X(z)$$

Substituting the given values for the initial conditions, we have

$$Y(z) = z^{-1}Y(z) + \tfrac{3}{4} - z^{-2}Y(z) - \tfrac{3}{4}z^{-1} - \tfrac{1}{4} + \tfrac{1}{2}X(z) + \tfrac{1}{2}z^{-1}X(z)$$

Collecting all of the terms that contain $Y(z)$ onto the left side of the equation gives

$$Y(z)[1 - z^{-1} + z^{-2}] = \tfrac{1}{2} - \tfrac{3}{4}z^{-1} + \tfrac{1}{2}X(z) + \tfrac{1}{2}z^{-1}X(z)$$

Because $x(n) = (\frac{1}{2})^n u(n)$,

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

which gives

$$Y(z) = \frac{\frac{1}{2} - \frac{3}{4}z^{-1}}{1 - z^{-1} + z^{-2}} + \frac{\frac{1}{2} + \frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1} + z^{-2})}$$

Expanding the second term using a partial fraction expansion, we have

$$Y(z) = \frac{\frac{1}{2} - \frac{3}{4}z^{-1}}{1 - z^{-1} + z^{-2}} + \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{z^{-1}}{1 - z^{-1} + z^{-2}}$$

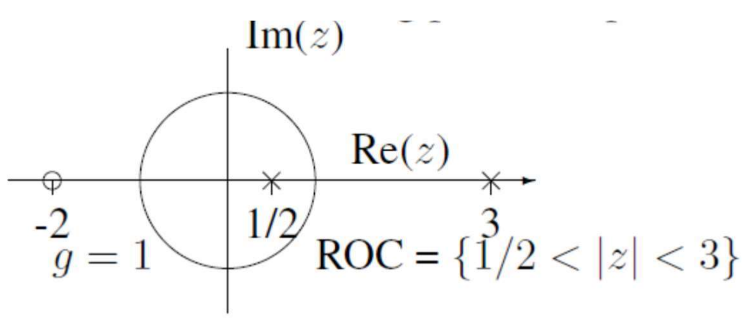
or

$$Y(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{2} + \frac{1}{4}z^{-1}}{1 - z^{-1} + z^{-2}}$$

Therefore, the solution is

$$y(n) = \left(\tfrac{1}{2}\right)^{n+1}u(n) + \left[\tfrac{\sqrt{3}}{6}\sin\left(\frac{n\pi}{3}\right) + \tfrac{\sqrt{3}}{3}\sin(n-1)\frac{\pi}{3}\right]u(n)$$

Q9. [CO1] Find the signal $x(n)$ whose z-transform has the following pole-zero plot.



Sol. 9:

Finding $X(z)$ from the plot

$$X(z) = \frac{z + 2}{\left(z - \frac{1}{2}\right)(z - 3)}$$

Representing in normal form

$$X(z) = \frac{(z^{-1} + 2z^{-2})}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 3z^{-1})} = \frac{z^{-1} + 2z^{-2}}{1 - \frac{7}{2}z^{-1} + \frac{3}{2}z^{-2}}$$

Writing in proper form using power series expansion in reverse order

$$\sqrt[\frac{3}{2}z^{-2} - \frac{7}{2}z^{-1} + 1]{2z^{-2} + z^{-1}} = \frac{4}{3} + \frac{-\frac{4}{3} + \frac{17}{3}z^{-1}}{1 - \frac{7}{2}z^{-1} + \frac{3}{2}z^{-2}}$$

Can be written as:

$$X(z) = \frac{4}{3} + \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - 3z^{-1}} = \frac{4}{3} + P(z)$$

Then,

$$\frac{P(z)}{z} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - 3} = \frac{-\frac{4}{3}z + \frac{17}{3}}{\left(z - \frac{1}{2}\right)(z - 3)}$$

Then, multiplying $P(z)$ with $\left(z - \frac{1}{2}\right)$ then substituting $z = \frac{1}{2}$

$$A = \left.\frac{\left(z - \frac{1}{2}\right)P(z)}{z}\right|_{z=\frac{1}{2}} = \left.\frac{\left(-\frac{4}{3}z + \frac{17}{3}\right)}{z - 3}\right|_{z=\frac{1}{2}} = -2$$

Then, multiplying $P(z)$ with $(z - 3)$ then substituting $z = 3$

$$B = \left.\frac{(z - 3)P(z)}{z}\right|_{z=3} = \left.\frac{\left(-\frac{4}{3}z + \frac{17}{3}\right)}{z - \frac{1}{2}}\right|_{z=3} = \frac{2}{3}$$

Then,

$$X(z) = \frac{4}{3} - \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{2}{3}\frac{1}{1 - 3z^{-1}}$$

Finding $x(n)$

$$x(n) = \frac{4}{3}\delta(n) - (2)\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}(3)^n u(-n - 1)$$

Q10. [CO1] Find the impulse response of the system described by the following equation:

$$y(n) = \frac{4}{3}y(n-1) - \frac{7}{12}y(n-2) + \frac{1}{12}y(n-3) + x(n) - x(n-3)$$

Sol. 10:

First finding the system function in z-domain using shifting and linearity property

$$Y(z) = \frac{4}{3}z^{-1}Y(z) - \frac{7}{12}z^{-2}Y(z) + \frac{1}{12}z^{-3}Y(z) + X(z) - z^{-3}X(z)$$

Rearranging the terms

$$Y(z)\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right] = [1 - z^{-3}]X(z)$$

Then using convolution property in z-domain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}$$

Now, performing power series expansion in reverse order to obtain $H(z)$ in proper format.

$$-\tfrac{1}{12}z^{-3}+\tfrac{7}{12}z^{-2}-\tfrac{4}{3}z^{-1}+1\sqrt{-z^{-3} + 1} = 12 + \frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}$$

Finding the roots of the $1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3} = \left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right)$

Then,

$$H(z) = 12 + \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 (1 - \frac{1}{3}z^{-1})}$$

We can see there is one repeated root so $H(z)$ can be written as

$$H(z) = 12 + P(z)$$

$$P(z) = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 (1 - \frac{1}{3}z^{-1})} = z.\frac{-11z^2 + 16z - 7}{\left(z - \frac{1}{2}\right)^2 \left(z - \frac{1}{3}\right)}$$

$$\frac{P(z)}{z} = \frac{A}{z - \frac{1}{2}} + \frac{B}{\left(z - \frac{1}{2}\right)^2} + \frac{C}{z - \frac{1}{3}}$$

Multiplying complete equation with $z - \frac{1}{3}$ and substituting $z = \frac{1}{3}$, then

$$C = \left.\frac{\left(z-\frac{1}{3}\right)P(z)}{z}\right|_{z=\frac{1}{3}} = \left.\frac{-11z^2+16z-7}{\left(z-\frac{1}{2}\right)^2}\right|_{z=\frac{1}{3}} = -104$$

Multiplying complete equation with $\left(z-\frac{1}{2}\right)^2$ and substituting $z=\frac{1}{2}$, then

$$B = \left.\frac{\left(z-\frac{1}{2}\right)^2 P(z)}{z}\right|_{z=\frac{1}{2}} = \left.\frac{-11z^2+16z-7}{\left(z-\frac{1}{3}\right)}\right|_{z=\frac{1}{2}} = -10.5$$

Multiplying complete equation with $\left(z-\frac{1}{2}\right)^2$ and then differentiating the equation with $\frac{d}{dz}$

$$\frac{P(z)}{z}\left(z-\frac{1}{2}\right)^2 = A\left(z-\frac{1}{2}\right) + B + \frac{C\left(z-\frac{1}{2}\right)^2}{z-\frac{1}{3}}$$

$$\frac{d}{dz}\left[\frac{P(z)}{z}\left(z-\frac{1}{2}\right)^2\right] = A + C\left(2\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)-\left(z-\frac{1}{2}\right)^2\right)/\left(z-\frac{1}{3}\right)^2$$

For $z=\frac{1}{2}$

$$\frac{d}{dz}\left[\frac{\left(z-\frac{1}{2}\right)^2 P(z)}{z}\right]_{z=\frac{1}{2}} = A$$

Then,

$$A = \frac{d}{dz}\left[\frac{\left(z-\frac{1}{2}\right)^2 P(z)}{z}\right]_{z=\frac{1}{2}} = \frac{d}{dz}\left[\frac{-11z^2+16z-7}{\left(z-\frac{1}{3}\right)}\right]_{z=\frac{1}{2}} = \left.\frac{(-22z+16)}{z-\frac{1}{3}} - \frac{-11z^2+16z-7}{\left(z-\frac{1}{3}\right)^2}\right|_{z=\frac{1}{2}}$$

$$\left.\frac{(-22z+16)\left(z-\frac{1}{3}\right)+11z^2-16z+7}{\left(z-\frac{1}{3}\right)^2}\right|_{z=\frac{1}{2}} = \frac{\left(\frac{5}{6}+\frac{11}{4}-1\right)}{\left(\frac{1}{6}\right)^2} = 93$$

Then,

$$P(z) = \frac{Az}{z-\frac{1}{2}} + \frac{Bz}{\left(z-\frac{1}{2}\right)^2} + \frac{Cz}{z-\frac{1}{3}}$$

$$P(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{Bz^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{C}{1-\frac{1}{3}z^{-1}} = \frac{93}{1-\frac{1}{2}z^{-1}} - \frac{(21)\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} - \frac{104}{1-\frac{1}{3}z^{-1}}$$

Now, we can write

$$p(n) = (93)\left(\frac{1}{2}\right)^n u(n) - (21)\left(\frac{1}{2}\right)^n nu(n) - (104)\left(\frac{1}{3}\right)^n u(n)$$

Then,

$$h(n) = 12\,\delta(n) + (93)\left(\frac{1}{2}\right)^n u(n) - (21)\left(\frac{1}{2}\right)^n nu(n) - (104)\left(\frac{1}{3}\right)^n u(n)$$