JAYPEE INSTITUTE OF INFORMATION TECHNOLOGY
Electronics and Communication Engineering
Digital Signal Processing (15B11EC413)
Tutorial Sheet: 1_Solution

Q1. [CO1] Determine whether following sequence is periodic or aperiodic. If it is periodic, determine the fundamental period and frequency.

a) $x[n] = A\cos\left(\frac{3\pi}{7}n - \frac{\pi}{8}\right)$ b) $x[n] = \cos\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{3\pi}{5}n\right)$

c) $x[n] = \cos\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{2\pi}{7}n\right)$ d) $x[n] = e^{j\left(n/8 - \pi\right)}$

Sol. Q1. a)

x(n) is periodic if x(n) = x(n + N) for some integer value of N. For the sequence in (a),

$x(n + N) = A\cos\left(\frac{3\pi}{7}n + \frac{3\pi}{7}N - \frac{\pi}{8}\right)$

x(n + N) = x(n) if $\frac{3\pi}{7}$ N is an integer multiple of 2π. The smallest value of N for which this is true is N = 14. Therefore the sequence in (a) is periodic with period 14.

b) Following a)

$cos\left(\frac{2\pi}{5}n\right)$ is periodic with $N_1 = 5$, $cos\left(\frac{3\pi}{5}n\right)$ is periodic with $N_2 = 10$

The overall period N = LCM (N_1,N_2) = 10.

c) Following a)

$cos\left(\frac{2\pi}{5}n\right)$ is periodic with $N_1 = 5$, $cos\left(\frac{2\pi}{7}n\right)$ is periodic with $N_2 = 7$

The overall period N = LCM (N_1, N_2) = 35.

d)

$$x(n+N) = e^{j\left(\frac{n}{8}+\frac{N}{8}-\pi\right)}$$

$$= e^{j\left(\frac{n}{8}-\pi\right)} \; e^{j\frac{N}{8}} = x(n)\; e^{j\frac{N}{8}}$$

The factor $e^{j\frac{N}{8}}$ is unity for (N/8) an integer multiple of 2π. This requires that

$$\frac{N}{8} = 2\pi R$$

where N and R are both integers. This is not possible since π is an irrational number. Therefore this sequence is not periodic.

Q2. [CO1] Let $x_{ev}[n]$ and $x_{od}[n]$ be even and odd real sequences, respectively. Which one of the following sequences is an even sequence, and which one is an odd sequence?

a) $g[n] = x_{ev}[n]x_{ev}[n]$ b) $u[n] = x_{ev}[n]x_{od}[n]$ c) $v[n] = x_{od}[n]x_{od}[n]$

Sol. Q2.

(a) $g[n] = x_{ev}[n]x_{ev}[n]$. Thus, $g[-n] = x_{ev}[-n]x_{ev}[-n] = x_{ev}[n]x_{ev}[n] = g[n]$. Hence, $g[n]$ is an even sequence.

(b) $u[n] = x_{ev}[n]x_{od}[n]$. Thus, $u[-n] = x_{ev}[-n]x_{od}[-n] = x_{ev}[n](-x_{od}[n]) = -u[n]$. Hence, $u[n]$ is an odd sequence.

(c) $v[n] = x_{od}[n]x_{od}[n]$. Thus, $v[-n] = x_{od}[-n]x_{od}[-n] = (-x_{od}[n])(-x_{od}[n])$ $= x_{od}[n]x_{od}[n] = v[n]$. Hence, $v[n]$ is an even sequence.

Q3. [CO1] Which of the following sequences are bounded sequences?

a) $x_1[n] = A\alpha^n$, where A and α are complex numbers, and $|\alpha| < 1$.

b) $x_2[n] = A\alpha^n u[n]$, where A and α are complex numbers, and $|\alpha| < 1$.

c) $x_3[n] = C\beta^n u[n]$, where A and β are complex numbers, and $|\beta| > 1$.

d) $x_4[n] = 4\cos(\omega_0 n)$

e) $x_5[n] = \left(1 - \frac{1}{n^2}\right)u[n-1]$

Sol.

(a) $x_1[n] = A\alpha^n$, where A and α are complex numbers with $|\alpha| < 1$. Here, $|\alpha|^n < 1$ only for n > 0 not for n< 0. Hence, $|x_1| < 1$ for only n > 0 implying $x_1[n] = A\alpha^n$ is not a bounded sequence.

(b) $y[n] = A\alpha^n \mu[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0, \end{cases}$ where A and α are complex numbers with $|\alpha| < 1$. Here, $|\alpha|^n \leq 1, n \geq 0$. Hence $|y[n]| \leq |A|$ for all values of n. Hence, $\{y[n]\}$ is a bounded sequence.

(c) $\{h[n]\} = C\beta^n \mu[n]$ where C and β are complex numbers with $|\beta| > 1$. Since for $n > 0$, $|\beta|^n$ can become arbitrarily large, $\{h[n]\}$ is not a bounded sequence.

(d) $\{g[n]\} = 4\cos(\omega_o n)$. Since $|g[n]| \leq 4$ for all values of n, $\{g[n]\}$ is a bounded sequence.

(e) $v[n] = \begin{cases} \left(1 - \frac{1}{n^2}\right), & n \geq 1, \\ 0, & n \leq 0. \end{cases}$ Since $\frac{1}{n^2} < 1$ for $n > 1$ and $\frac{1}{n^2} = 1$ for $n = 1$, $|v[n]| < 1$ for all values of n. Thus $\{v[n]\}$ is a bounded sequence.

Q4. [CO1] Compute the power of following sequences:

a) $x[n] = e^{j(n/8 - \pi)}$

b) $x[n] = 4\cos\left(\frac{2\pi}{5}n\right) + 3\cos\left(\frac{3\pi}{5}n\right)$

Sol. Q4. a) $P = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$, $P = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} 1 = 1$.

b) $\cos\left(\frac{2\pi}{5}n\right)$ is periodic with $N_1 = 5$, $\cos\left(\frac{3\pi}{5}n\right)$ is periodic with $N_2 = 10$

The overall period N = LCM (N_1, N_2) = 10.

$$P = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2\ ,\ P = \frac{1}{10}\sum_{n=0}^{9}\left|4\cos\left(\frac{2\pi}{5}n\right)+3\cos\left(\frac{3\pi}{5}n\right)\right|^2$$

$$= \frac{1}{10}\sum_{n=0}^{9}\left(16\cos^2\left(\frac{2\pi}{5}n\right)+9\cos^2\left(\frac{3\pi}{5}n\right)+24\left|\cos\left(\frac{2\pi}{5}n\right)\cos\left(\frac{3\pi}{5}n\right)\right|\right)$$

$$= \frac{1}{10}\sum_{n=0}^{9}\left(8\left(1+\cos\left(\frac{4\pi}{5}n\right)\right)+\frac{9}{2}\left(1+\cos\left(\frac{6\pi}{5}n\right)\right)+24\left|\cos\left(\frac{2\pi}{5}n\right)\cos\left(\frac{3\pi}{5}n\right)\right|\right)$$

$= 8 + 4.5 + 0 = 12.5.$

Q5. [CO1] Compute the energy of following sequences:

a) $x_1[n] = A\alpha^n u[n], |\alpha| < 1$

b) $x_2[n] = \left(\frac{1}{n^2}\right)u[n-1]$

Sol. Q5. a)

$$x[n] = A^{\alpha}\mu[n].\ \text{Then}\ \mathcal{E}_{x_a} = \sum_{n=-\infty}^{\infty}|x_a[n]|^2 = A^2\sum_{n=0}^{\infty}\alpha^{2n} = \frac{A^2}{1-\alpha^2}.$$

b)

$$x_b[n] = \frac{1}{n^2}\mu[n-1].\ \text{Then}\ \mathcal{E}_{x_b}\ \sum_{n=-\infty}^{\infty}|x_b[n]|^2 = \sum_{n=1}^{\infty}\left|\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Q6. [CO1] Express signal x[n] (Fig. 1) as linear combination of

a) unit impulses

b) unit step sequences

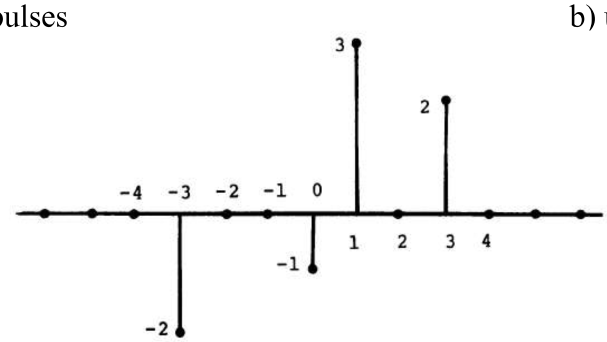


Fig. 1

Sol. Q6.

a) $x[n]=-2\delta[n+3]-\delta[n]+3\delta[n-1]+2\delta[n-3]$

where $\delta[n]$ is unit impulse sequence.

b) $x[n]=-2u[n+3]+2u[n+2]-u[n]+4u[n-1]-3u[n-2]+2u[n-3]-2u[n-4]$

where $u[n]$ is unit step sequence.

Q7. [CO1] Consider the following systems, x[n] and y[n] are input and output of the systems, respectively. Determine whether following discrete-time systems are linear, shift-invariant, and causal. Justify your answer.

a) $y[n]=2x[n]+3$

b) $y[n]=\sin\left(\frac{2\pi}{7}n+\frac{\pi}{6}\right)x[n]$

c) $y[n]=(x[n])^2$

d) $y[n]=\sum_{m=-\infty}^{n}x[m]$

e) $y[n]=x^2[n]-x[n-1]x[n+1]$

Sol. Q7.

a) Since output depends only on present values of input, system is causal.

(a), $T[x_1(n)] = 2x_1(n) + 3$

$T[x_2(n)] = 2x_2(n) + 3$

Since $T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + 3$

and $aT[x_1(n)] + bT[x_2(n)] = 2ax_1(n) + 2bx_2(n) + 3(a + b)$

The system is not linear. The system is, however, shift-invariant since $T[x(n-n_o)] = 2x(n-n_o) + 3 = y(n-n_o)$.

b) Since output depends only on present values of input, system is causal.

$$T[x_1[n]] = sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)x_1[n]$$

$$T[x_2[n]] = sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)x_2[n]$$

Since, $T[ax_1[n] + bx_2[n]] = sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)(ax_1[n] + bx_2[n])$

and

$$aT[x_1[n]] + bT[x_2[n]] = a\,sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)x_1[n] + b\,sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)x_2[n])$$
$$= sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)(ax_1[n] + bx_2[n])$$

The system is linear.

$$T[x[n-n_0]] = sin\left(\frac{2\pi}{7}n + \frac{\pi}{6}\right)x\ [n-n_0]$$

$$y[n-n_0]] = sin\left(\frac{2\pi}{7}(n-n_0) + \frac{\pi}{6}\right)x[n-n_0] \neq T[x[n-n_0]]$$

The system is not shift-invariant.

c) Since output depends only on present values of input, system is causal.

$$T[x_1[n]] = (x_1[n])^2$$
$$T[x_2[n]] = (x_2[n])^2$$

Since, $T[ax_1[n] + bx_2[n]] = (ax_1[n] + bx_2[n])^2 = a^2\ (x_1[n])^2 + b^2(x_2[n])^2 + 2abx_1[n]x_2[n]$

and

$$aT[x_1[n]] + bT[x_2[n]] = a(x_1[n])^2 + b(x_2[n]))^2$$

The system is not linear.

$$T[x[n-n_0]] = (x\ [n-n_0])^2$$

$$y[n-n_0]] = (x\ [n-n_0])^2 = T[x[n-n_0]]$$

The system is shift-invariant.

d) Since output depends on present and past values of input, system is causal.

$$T[x_1[n]] = \sum_{m=-\infty}^{n} (x_1[m])$$

$$T[x_2[n]] = \sum_{m=-\infty}^{n} (x_2[m])$$

Since, $T[ax_1[n]+bx_2[n]] = \sum_{m=-\infty}^{n} (ax_1[m]+bx_2[m]) = a\sum_{m=-\infty}^{n} x_1[m] + b\sum_{m=-\infty}^{n} x_2[m]$

and

$$aT[x_1[n]] + bT[x_2[n]] = a\sum_{m=-\infty}^{n} x_1[m] + b\sum_{m=-\infty}^{n} x_2[m]$$

The system is linear.

$$T[x[n-n_0]] = \sum_{m=-\infty}^{n} (x\ [m-n_0]) = \sum_{k=-\infty}^{n-n_0} (x\ [k])$$

$$y[n-n_0]] = \sum_{m=-\infty}^{n-n_0} (x\ [m]) = T[x[n-n_0]]$$

The system is shift-invariant.

e) Since output depends on future values of input, system is not causal.

$$T[x_1[n]] = (x_1[n])^2 - (x_1[n-1])(x_1[n+1])$$

$$T[x_2[n]] = (x_2[n])^2 - (x_2[n-1])(x_2[n+1])$$

Since, $T[ax_1[n]+bx_2[n]] = (ax_1[n]+bx_2[n])^2 - (ax_1[n-1]+bx_2[n-1])(ax_1[n+1]+bx_2[n-1])$

and

$aT[x_1[n]] + bT[x_2[n]] = a((x_1[n])^2 - (x_1[n-1])(x_1[n+1])) + b((x_1[n])^2 - (x_1[n-1])(x_1[n+1]))$The system is not linear.

$$T[x[n-n_0]] = (x\ [n-n_0])^2 - (x\ [n-1-n_0])(x_1[n+1-n_0])$$

$$y[n-n_0]] = \left(x\ [n-n_0]\right)^2 - \left(x\ [n-n_0-1]\right)\left(x_1[n-n_0+1]\right) = T[x[n-n_0]]$$

The system is shift-invariant.

Q8. [CO1] For each of the following pairs of sequence x[n] represents the input to an LTI system with impulse response h[n] (Fig. 2). Determine each output y[n]. Sketch your results.

a)

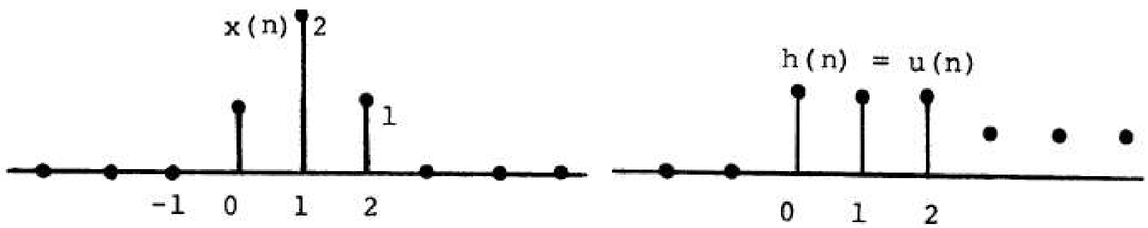


Fig. 2(a)

b)

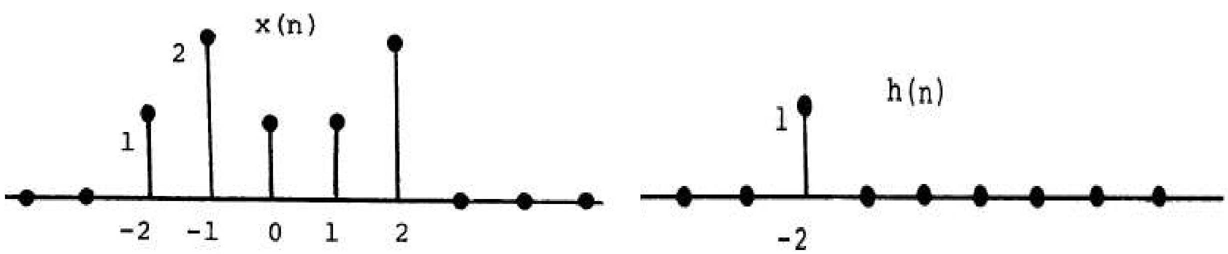
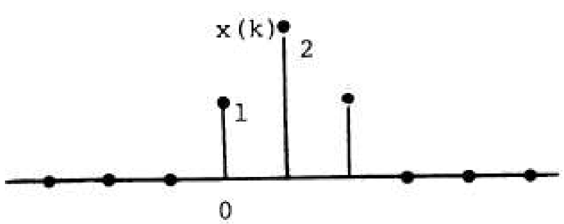
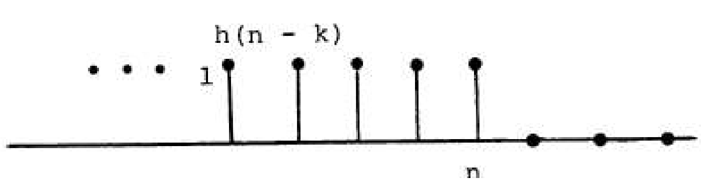


Fig. 2(b)

Sol. Q8.

a)

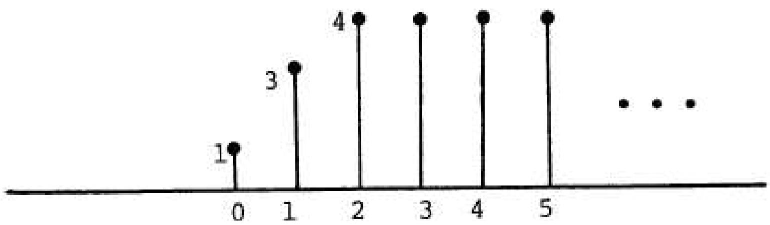




Since $h(n - k)$ is zero for $k > n$, and is unity for $k < n$,

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{n} x(k)$$

as sketched below:



Part **(b)** can likewise be done graphically. Alternatively since

$$h(n) = \delta(n + 2) \quad ,$$

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{+\infty} \delta(k+2)\, x(n-k)$$

Since $\delta(k + 2) = 0$ except for $k = -2$, and is unity for $k = -2$
$y(n) = x(n + 2)$.

Q9. [CO1] Find the overall impulse response h[n] for system shown in Fig. 3**.**

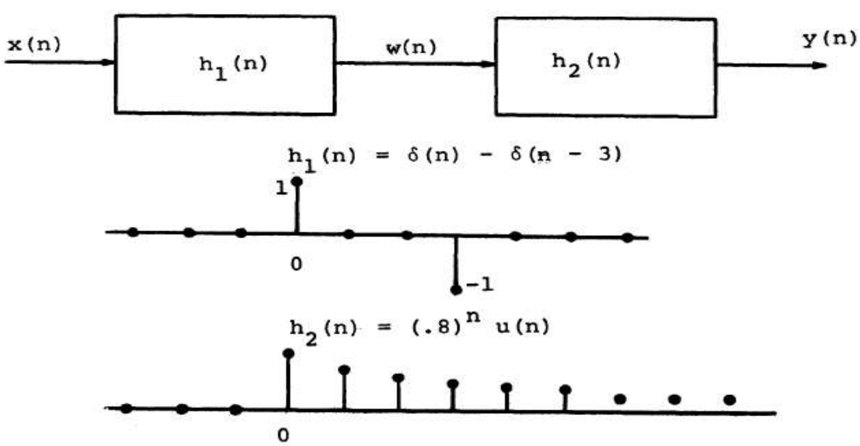
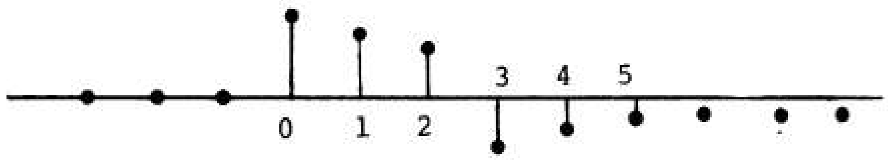


Fig. 3

Sol.

$$h(n) = h_1(n) * h_2(n) = (.8)^n u(n) - (.8)^{(n-3)} u(n-3)$$



Q10. [CO1] Consider a causal system for which the input x[n] and output y[n] are related by the linear constant coefficient difference equation $y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{2}x[n-1]$

a) Determine the impulse response of the system.

b) Determine the response of the system using result obtained in part a). (Use $x[n]=e^{j\omega n}$)

c) Determine the frequency response of the system. (Use $x[n]=e^{j\omega n}$)

d) Determine the response of the system to input $x[n]=\cos\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)$.

Sol. Q10.

(a) Rewriting the difference equation, with $x(n) = \delta(n)$ and $h(n)$ denoting the unit-sample response

$$h(n) = \frac{1}{2} h(n-1) + \delta(n) + \frac{1}{2}\delta(n-1) \quad .$$

Since the system is causal, $h(n)$ is zero for $n < 0$. For $n \geq 0$,

$$h(0) = \frac{1}{2} h(-1) + \delta(0) + \frac{1}{2}\delta(-1) = 1$$

$$h(1) = \frac{1}{2} h(0) + \delta(1) + \frac{1}{2}\delta(0) = 1$$

$$h(2) = \frac{1}{2} h(1) + \delta(2) + \frac{1}{2}\delta(1) = \frac{1}{2}$$

$$h(n) = 2\left(\frac{1}{2}\right)^n \quad n \geq 1$$

$h(n)$ can also be expressed as

$$h(n) = \left(\frac{1}{2}\right)^n [u(n) + u(n-1)] \quad .$$

(b) Substituting x(n) and h(n) into the convolution sum we obtain

$$y(n) = \sum_{k=-\infty}^{\infty} (\tfrac{1}{2})^k \left[u(k) + u(k-1)\right] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \left[\sum_{k=0}^{\infty} (\tfrac{1}{2})^k \; e^{-j\omega k} + \sum_{k=1}^{\infty} (\tfrac{1}{2})^k \; e^{-j\omega k}\right]$$

$$= e^{j\omega n} \left[\frac{1 + \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}\right]$$

(c) The frequency response of the system is the complex amplitude of the response with an excitation $e^{j\omega n}$. Thus, from part (a)

$$H(e^{j\omega}) = \frac{1 + \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

(d) With $H(e^{j\omega})$ expressed in polar form as $|H(e^{j\omega})| \; e^{j\Theta(\omega)}$, the response to the specified input is

$$y(n) = |H(e^{j\frac{\pi}{2}})| \; \cos(\tfrac{\pi n}{2} + \tfrac{\pi}{4} + \Theta(\tfrac{\pi}{2}))$$

From part (c),

$$|H(e^{j\frac{\pi}{2}})| = 1$$

$$\Theta(\tfrac{\pi}{2}) = -2 \tan^{-1}(\tfrac{1}{2})$$