Discrete Fourier Transform

Frequency domain sampling of Discrete time signal

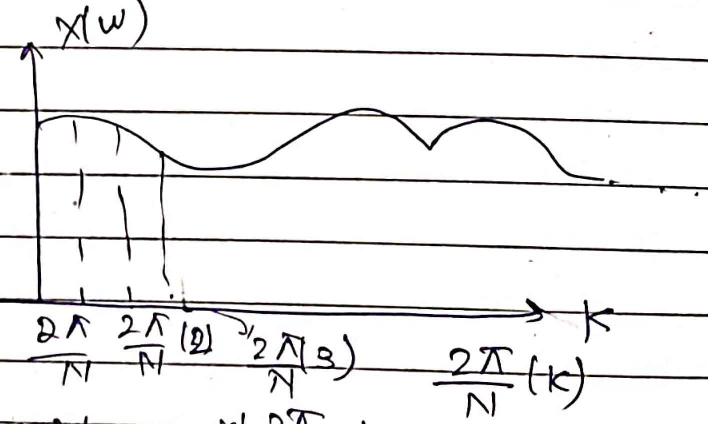
Fourier transform (DTFT) of any discrete time signal

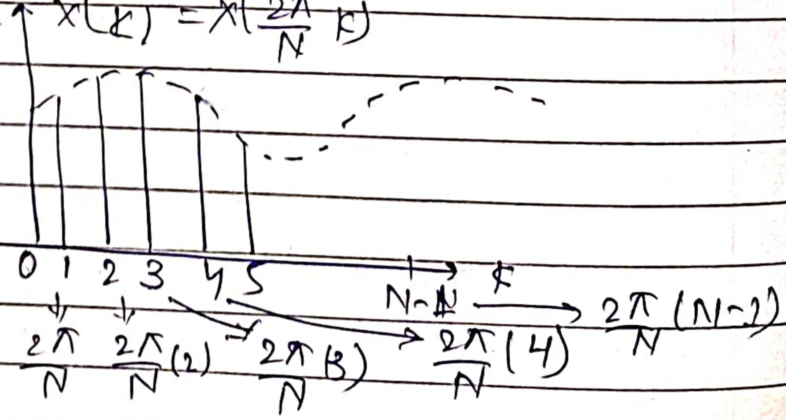
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

→ (i) $X(\omega)$ is continuous fn of ω

(ii) $X(\omega)$ is periodic with period 2π

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ betn two successful sample. Since $X(\omega)$ is periodic with period 2π. For convinience we take N equidistant samples in the interval $0 \le \omega \le 2\pi$, with spacing $\delta\omega = \frac{2\pi}{N}$ as shown in fig





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at $w = \frac{2\pi}{N}k$, $X\left(\frac{2\pi}{N}k\right) = X(k) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$

where $k = 0, 1, 2, \ldots N-1$

$$\sum_{n=-N}^{-1} x(n)e^{-jwn} + \sum_{n=0}^{N-1} x(n)e^{-jwn} + \sum_{n=N}^{2N-1} x(n)e^{-jwn} + \sum_{n=2N}^{3N-1} x(n)e^{-jwn} + \sum_{n=3N}^{4N-1} x(n)e^{-jwn}$$

$$\ldots \sum_{n=lN}^{lN+N-1} x(n)e^{-jwn}$$

$$X(k) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j\frac{2\pi kn}{N}}$$

Replacing the inner summation from n to $n-lN$ and interchange the order of summation

$$x(k) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)e^{-j\frac{2\pi k}{N}(n-lN)}$$

$$\sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)\, e^{-j\frac{2\pi kn}{N}} \, e^{j\frac{2\pi k}{N}(lN)} \to 1$$

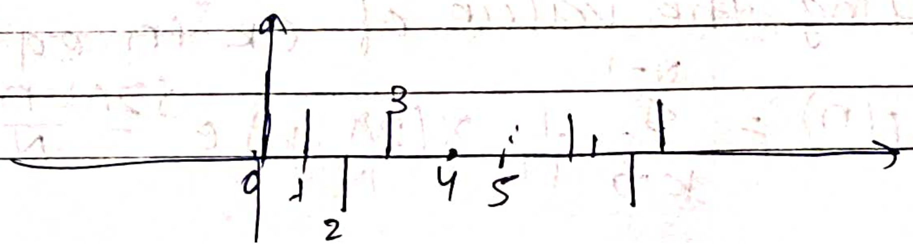
condition $N \geq L$

$$\sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)\, e^{-j\frac{2\pi kn}{N}}$$

$L=4, N=6$

$$\sum_{l=-\infty}^{\infty} x(n-lN) = x(n) + x(n-N) + x(n-2N)$$

↳ It is periodic extension of $x(n)$ with period N



If $N \geq L$ $(N \geq 2, L = 4)$

$$\sum_{l=-\infty}^{\infty} x(n-lN) = \sum_{l=-\infty}^{\infty} x(n-2l)$$

If $N \geq L$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN), \quad 0 \leq n \leq N-1)$$

$$0, \quad \text{elsewhere}$$

Since $x_p(n)$ is periodic with period N. So, it can be expanded in Fourier series as

$$x_p(n) = \sum_{k=0}^{N-1} C_k e^{\frac{j2\pi kn}{N}}, \quad n = 0, 1, \ldots N-1 \quad (2)$$

where

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi kn}{N}}$$

$$C_k N = \sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi kn}{N}} \quad \text{—— } (3)$$

$$\sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi kn}{N}} \quad \text{—— } (i)$$

from (i) and (3)

$$C_k N = X\left(\frac{2\pi}{N} k\right)$$

$$C_k = \frac{1}{N} X\left(\frac{2\pi}{N} k\right)$$

putting the value of C_k in eq^n (2)

$$x_p(n) = \sum_{k=0}^{N-1} \frac{1}{N} X\left(\frac{2\pi}{N} k\right) e^{\frac{j2\pi kn}{N}}, \quad n = 0, 1, \ldots N-1$$

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$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) e^{\frac{j2\pi kn}{N}} , n = 0, 1 \ldots N-1 \quad \rightarrow 4$$

It is clear from eqn (4) $x_p(n)$ can be reconstructed or recovered from the sample of spectrum of $X(\omega)$ $\left[X\left(\frac{2\pi}{N} k\right)\right]$

But $x_p(n)$ is the periodic repeatable of $x(n)$ with period N samples.

So $x(n)$ can be recovered from $x_p(n)$ if there is no aliashing in time domain that is if $x(n)$ is time limited to less than the period N of $x_p(n)$ i.e. $N \geq L$

N-point DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn},\ k = 0,1,\ldots N$$

The DFT as linear transformation / matrix form representation

↳ Point DFT

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn},\ k = 0,1,2,3$$

$$X(0) = \sum_{n=0}^{3} x(n)\, W_4^{0} = x(0)W_4^0 + x(1)W_4^0 + x(2)$$
$$+ x(3)W_4^0$$

$$X(1) = \sum_{n=0}^{3} x(n)\, W_4^{n} = x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2$$
$$x(3)W_4^3$$

$$X(2) = \sum_{n=0}^{3} x(n)W_4^{2n} = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3$$

$$X(3) = \sum_{n=0}^{3} x(n)W_4^{3n} = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}_{4\times1} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}_{4\times4} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}_{4\times1}$$

$$[X(k)]_{4\times1} = [W_4]_{4\times4}\,[x(n)]_{4\times1}$$

$$\boxed{X(k) = W_4\, x(n)}$$

Matrix form representation

Generalised form

$$[X(k)]_{N\times 1} = [W_N]_{N\times N}\,[x(n)]_{N\times 1}$$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ \vdots\\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & W_N^{N-2} & W_N^{2(N-2)} & & W_N^{(N-2)(N-1)} \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ \vdots\\ x(N-2)\\ x(N-1) \end{bmatrix}$$

6-point DFT

$$\underset{6\times 1}{\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5) \end{bmatrix}} = \underset{6\times 6}{\begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix}} \underset{6\times 1}{\begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\\ x(4)\\ x(5) \end{bmatrix}}$$

$$x(n) = \{0, 1, 2, 3\}$$

$\boxed{W_N^0 = 1}$

$N = 4$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 0\\ 1\\ 2\\ 3 \end{bmatrix} = \begin{bmatrix} 6\\ 2j-2\\ -2\\ -2-2j \end{bmatrix}$$

$$X(k) = \{6,\ 2j-2,\ -2,\ -2-2j\}$$

$$\boxed{\begin{array}{l} W_N^{-k+N/2} = -W_N^{k} \rightarrow \text{symmetric property} \\[2ex] W_N^{k+N} = W_N^{k} \longrightarrow \text{periodic property} \end{array}}$$

$X(k) = W_N x(n)$

$W_N^{-1} \rightarrow$ inverse of W

$W_N^{-1} X(k) = (W_N^{-1} W_N) x(n) \quad = I x(n) = x(n) = \underset{\downarrow}{W_N^{-1} X_N}$

N-point IDFT

$x(n) = \frac{1}{N} W_N^{*}$

Properties of DFT

(1) Periodicity :

if $x(n)$ & $x(k)$ are N-point DFT pair

the

$$x(n+N) = x(n) \quad , \forall n$$
$$x(k+N) = X(k) \quad , \forall k$$

Proof: $X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}} , k = 0, 1 \ldots N-1$

$$x(k+N) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi n}{N}(k+N)}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi n k}{N}} = X(k)$$

$$\boxed{x(k+N) = X(k)}$$

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$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}, \quad n = 0,1 \cdots N-1$$

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi k}{N}(n+N)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi k}{N}(n+N)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} = x(n)$$

So, $x(n) = x(n+N), \; \forall n$

(2) Linearity Property

$x_1(n) \xleftrightarrow{DFT} X_1(k)$ & $x_2(n) \xleftrightarrow{DFT} X_2(k)$

then

$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(k) + a_2 X_2(k)$

(3) Circular symmetry of sequence

Circular shift of sequence by k unit:

A circular shift of N-point sequence is equivalent to linear shift of its periodic extension.

* Circular shift of sequence is represented as index modulo N.

Linear shift of sequence by k unit

$x'(n) = x(n-k)$

If $x(n) = \{1, 2, 3, 4\}$

then $x(n-1) = \{$ shifted value of $x(n)$ in right direction

then $x(n+1) = \{$ shifted value in left.

circular shift

thus we can write $x'(n) = x(n-k)$, modulo N

$$= x((n-k)_N) = x(n-k+pN) \text{ where } p \in I$$

~~circular shift of sequence by k unit~~

e.g. $x(n) = \{1,2,3,4\}$

$$x((n-1))_4 = x'(n) = x(n-1+N)$$
$$= x(n-1+4) = x(n+3)$$

$x'(-1)$

$x'(0) = x(3) = 4$

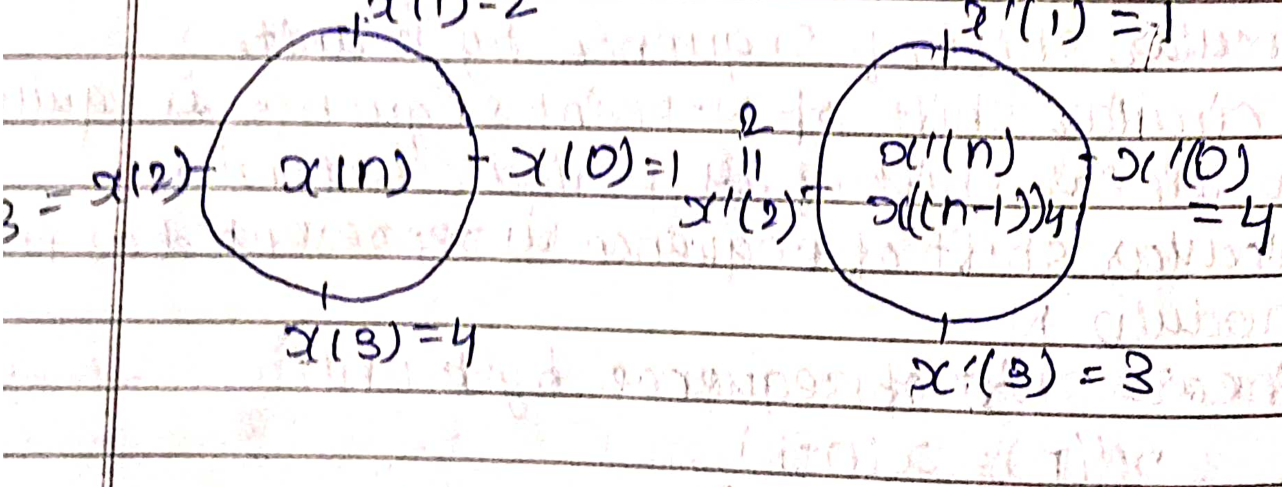
$x'(1) = x(4) = x(0+4) = x(0) = 1$

$x'(2) = x(2+3) = x(1+4) = x(1) = 2$

$x'(3) = x(6) = x(2+4) = x(2) = 3$

$x'(4) = x(7) = x(3+4) = x(3) = 4$

$$x'(n) = \{x'(0); x'(1), x'(2), x'(3)\}$$
$$= \{4,1,2,3\}$$



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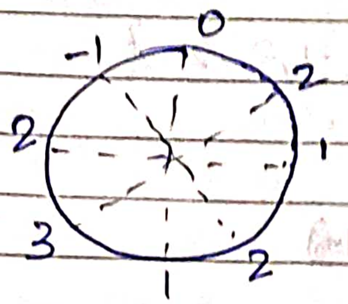
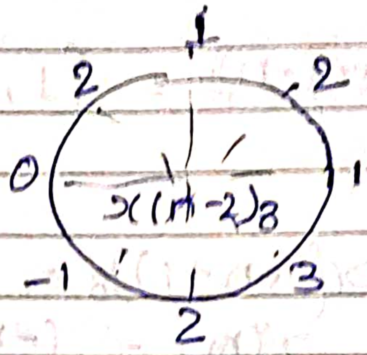
$x(n) = \{1,2,0,-1,2,3,1,2\}$





$x(n) = \{1,2,0,-1,2,3,1,2\}$

$x((n-1))_8 = \{2,1,2,0,-1,2,3,1\}$

$x(n-2)_8 = \{1,2,1,2,0,-1,2,3\}$

2 units shift ke liye (1,2) dono ko le jao

$x(n-2)$.

Q. $x(n) = \{1,2,3,4,-1,-2,0,2\}$

$x(n-1)_8 = \{2,1,2,3,4,-1,-2,0\}$

$x(n-2)_8 = \{0,2,1,2,3,4,-1,-2\}$

$x'(n) = x((-n))_8 = x(-n+8)$

$x(-n)_8 = \{1,2,3,4,-1,-2,0,2\}$

$= \{1,2,0,-2,-1,4,3,2\}$

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$x(n) = \{1, 2, 3, -1, 2, 5, 6\}$

$x((-n))_7 = \{1, 6, 5, 2, -1, 3, 2\}$

$x((-n+1))_8 = x(-n+1+8)$

$x'(n) = (-n+9)$

$x'(0) = x(9) = x(1+8) = x(1) = 2$
$x'(1) = x(8) = x(0) = x(0) = 1$
$x'(2) = x(7) = 2$
$x'(3) = x(6) = 0$
$x'(4) = x(5) = -2$
$x'(5) = x(4) - 1$
$x'(6) = x(3) = 4$
$x'(7) = x(2) = 3$
$x'(8) = x(1) =$

$x''(n) = \{2, 1, 2, 0, -2, 1, 4, 3\}$

$x((-n+1))_8 =$
$x(-n) = \{1, 2, 0, -2, -1, 4, 3, 2\}$

$x(-n+1)_8 = \{2, 1, 2, 0, -2, -1, 4, 3\}$
circular shift of $x(-n)_8$ by one unit

Note: (i) An N point sequence is called circularly even if it is symmetric about the point zero on the circle.

i.e. $\boxed{x(N-n) = x(n) \quad x((-n))_N}$, $0 \le n \le N-1$

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(ii) An N-point sequence is called circularly odd if it is ~~sy~~ antisymmetric about the point zero on the circle

$$x((-n))_N = -x(n)$$

(iii) If $x(n)$ is real valued sequence then

$$X(N-k) = X((-k))_N = X^*(k) = X(-k)$$

Proof: $X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$

$$X^*(k) = \left[\sum_{n=0}^{N-1} (x(n) e^{\frac{-j2\pi kn}{N}}\right]^* = \sum_{n=0}^{N-1} x(n) e^{\frac{j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n(-k)}{N}} = X(-k)$$

$$X((-k))_N = X(N-k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n}{N}(N-k)}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{j2\pi nk}{N}} = X^*(k)$$

E.g. $x(n) = \{0, 1, 2, 3\}$

$X(k) = \{6, (-2+2j), -2, (-2-2j)\}$

~~Find DFT $[x((-n))_4]$~~
N-K

$x(4-k) = X((-k))_4$

$X((-k))_4 = \{6, (-2-2j), -2, -2+2j\}$

(4) circular time shift property

If $x(n) \xleftrightarrow[N]{DFT} X(K)$
then
$$x((n-l))_N \xleftrightarrow[N]{DFT} X(K)e^{-j2\pi kl/N}$$

$L=2$

Ex: $x(n) = \{0,1,2,3\}$
Find $x(K)$, $DFT[(x(n-3))_4]$, $DFT[x(n)e^{\frac{j2\pi(2n)}{4}}]$

$X(K) = \{6, -2+2j, -2, -2-2j\}$

$DFT[x((n-3)_4] = X(K)e^{\frac{-j2\pi k(3)}{4}} = X'(k)$

$X'(K) = \{6, (-2+2j)e^{\frac{-j2\pi(3)}{4}}, -2e^{-j3}, (-2-2j)e^{\frac{-j2\pi 9}{4}}$

$X((K-2))_N = \{-2, -2-2j, 6, -2+2j\}$

Proof: $IDFT\left[x(k)e^{\frac{-j2\pi kl}{N}}\right]$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{\frac{-j2\pi kl}{N}} \cdot e^{\frac{j2\pi kn}{N}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{\frac{j2\pi K}{N}(n-l)}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{\frac{j2\pi K}{N}(n-l+N)}$$

$= x(n-l+N)$

$= x((n-l))_N$

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(5) circularly frequency shift property

if $x(0) \xleftrightarrow[N]{DFT} X(k)$

$x(n)e^{\frac{j2\pi nl}{N}} \xleftrightarrow[N]{DFT} X((k-l))_N \rightarrow x(k-l+N)$

Proof:

$$DFT\left[x(n)e^{\frac{j2\pi kl}{N}}\right] = \sum_{n=0}^{N-1} x(n) e^{\frac{j2\pi kl}{N}} e^{\frac{-j2\pi kn}{N}}, k=0 \text{ -- } N-1$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n}{N}(-l+k)}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n}{N}(k-l+N)}$$

$$= X(k-l+N) = X((k-l))_N$$

(6) Complex Conjugate Property

$\boxed{x(n) = x_R(n) + jx_I(n)}$

if $x(n) \xleftrightarrow[N]{DFT} X(K)$

then

(a) $x^*(n) \xleftrightarrow[N]{DFT} X^*((-K))_N = X^*(N-k)$

(b) $x^*((-n))_N \xleftrightarrow[N]{DFT} X^*(K)$

Eg: $x(n) = \{1, (1+2j), -3, (1-2j)\}$

$\{X^*(0), X^*(3), X^*(2), X^*(1)\}$

$X(K) = \{X(0), X(1), X(2), X(3)\}$

$DFT[x^*(n)] = \{X(0), X(3), X(2), X(1)\}^*$

Q. $x(n) = \{x(0), x(1), x(2), x(3)\}$
$h(n) = \{h(0), h(1), h(2), h(3)\}$

$DFT[x(n)] = X(k) = \{1, j, -1, -j\}$

Find (a) DFT of $\{x(3), x(0), x(1), x(2)\}$

$\{x(3), x(0), x(1), x(2)\} = x((n-1))_4$

$X'(k) = DFT[x((n-1))_4] = X(k)e^{\frac{-j2\pi k}{4}}$

$= \{X'(0), X'(1), X'(2), X'(3)\}$

$= \{1, (j)(-j), (-1)(-1), (-j)(j)\}$

(b) DFT of $\{h(0), h(1), h(2), h(3)\}$

$x(n)e^{\frac{j2\pi nl}{N}} \xleftrightarrow{DFT} X((k-l))_N$

DFT of $h'(n) = \{h(0), h(1), h(2), h(3)\}$

$= h(n)e^{j\frac{2\pi n}{N}(2)}$

$= H((k-2))_4$

$= \{1, (1-j), 0, (1+j)\}$

$$DF[x^*(n)]$$

$$\sum_{n=0}^{N-1} \cancel{DFT} [x^*(n) e^{-\frac{j2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n}{N}(-k+N)}\right] = [X(N-k)]^* = X^*(N-k)$$

if $x(n) \xleftrightarrow{DFT} X(k)$

$$x(n) = x_R(n) + jx_I(n)$$

Find $DFT[x_R(n)]$, $DFT[x_I(n)]$

$$x(n) = x_R(n) + jx_I(n)$$
$$x^*(n) = x_R(n) - jx_I(n)$$

$$x_R(n) = \frac{x(n) + x^*(n)}{2} \Rightarrow DFT[x_R(n)] = \frac{1}{2}(X(k) + X^*((-k))_N$$

$$x_I(n) = \frac{x(n) - x^*(n)}{2j} \Rightarrow DFT(x_I(n)] = \frac{1}{2j}[X(k) - X^*((-k))_N$$

3. $x(n) = \{0, (1+j), 1, (1-j)\}$

$DFT[x(n)] = X(k)$

Define the DFT of following

(i) $y(n) = e^{j(\pi/2)n} x(n)$

(ii) $y(n) = \cos(\pi/2 n) \cdot x(n)$

(iii) $y(n) = x((n-1))_4$

(iv) $y(n) = \{0, 0, 1, 0\} \circledast x(n)$ → circular convolution

- Linear convolution of two signals $x_1(n)$ & $x_2(n)$

$$= x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

- Circular convolution of two signals $x_1(n)$ & $x_2(n)$

or $\circledast$

$$= x_1(n) \text{ (N) } x_2(n) = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N$$

↓

circular convolution

Ex: Find the circular convolution of following signal

$x_1(n) = \{2, 1, 2, 1\}$ $\quad$ $x_2(n) = \{1, 2, 3, 4\}$

Four point circular conv. ($N=4$)

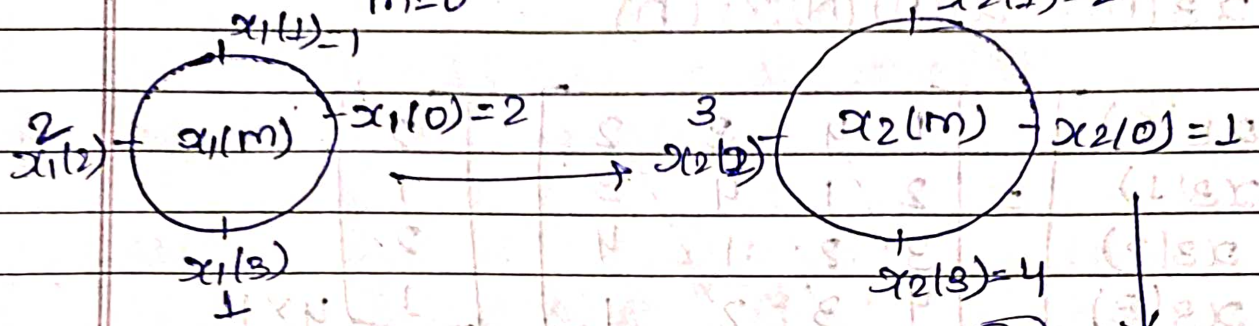
$$x_3(n) = x_1(n) \text{ (N) } x_2(n) = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N$$

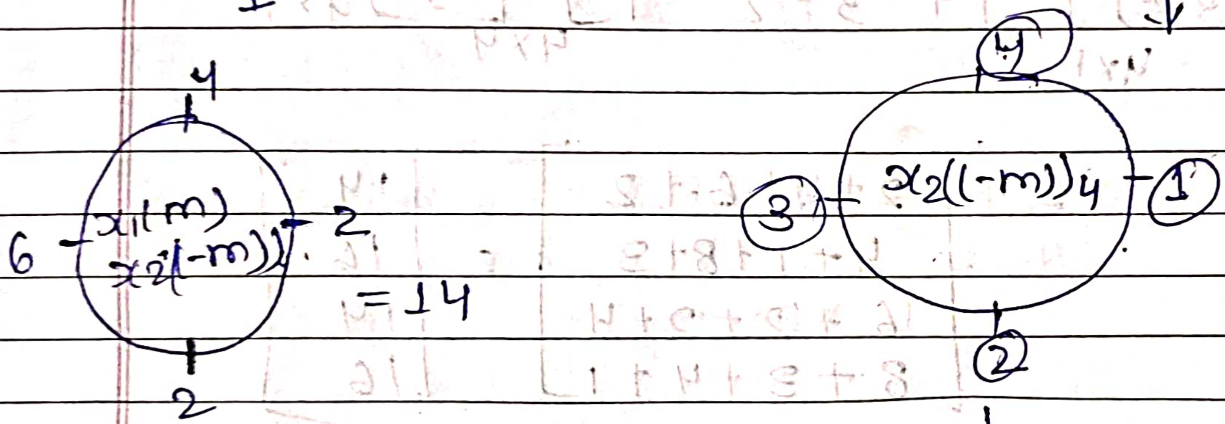
Graphical method

$$x_3(n) = \sum_{m=0}^{3} x_1(m)\, x_2((n-m))_4$$

$$= \{x_3(0),\ x_3(1),\ x_3(2),\ x_3(3)\}$$

$$\rightarrow x_3(0) = \sum_{m=0}^{3} x_1(m)\, x_2((-m))_4 = 14$$





$$x_3(1) = \sum_{m=0}^{3} x_1(m)\, x_2((1-m))_4$$

$x_1(m) = \{2, 1, 2, 1\}$
$x_2(-m) = \{1, 4, 3, 2\}$
$x_2(1-m) = \{2, 1, 4, 3\}$

$x_3(1) = 4 + 1 + 8 + 3 = 16$

$x_3(3) =$
$x_1(m) = \{2, 1, 2, 1\}$
$x_2(3-m) = \{4, 3, 2, 1\}$

$8 + 3 + 4 + 1$
$= 16$

$x_3(n) = \{14, 16, 14, 16\}$

$$x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_2((2-m))_4$$

$x_1(m) = \{2, 1, 2, 1\}$
$x_2(2-m) = \{3, 2, 1, 4\}$

$= 6 + 2 + 2 + 4$
$= 14$

matrix form method

8 point circular convolution

$x_1(n) = \{2, 1, 2, 1, 0, 0, 0, 0\}$

$x_2(n) = \{1, 2, 3, 4, 0, 0, 0, 0\}$

$x_3(n) = x_1(n) \circledN x_2(n)$

$$\underset{4\times1}{\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}} = \underset{4\times4}{\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 2+4+6+2 \\ 4+1+8+3 \\ 6+2+2+4 \\ 8+3+4+1 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

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Tut - 3

1. (a) $x[n] = [1, 0, -1, 0]$

$$X(k) = \sum_{n=0}^{3} x(n) e^{\left(-\frac{j2\pi}{N} kn\right)}, \quad k = 0, 1 \cdots N-1$$

$k = 0$ $$= \sum_{n=0}^{3} x(n) e$$

$$= x(0) + x(1) + x(2) + x(3)$$
$$= 1 + 0 - 1 + 0 = 0$$

$k = 1$ $$\sum_{n=0}^{3} x(n) e^{\frac{-j2\pi n}{\cancel{3}2}} = -\frac{j\pi n}{2}$$

$$= x(0) e^{0} + x(1) e^{-\frac{j\pi}{2}} + x(2) e^{-j\pi} + x(3) e^{-\frac{j\pi 3}{2}}$$

$$= 1 + 0 + -1 \times -1$$

$$= 2$$

$k = 2$ $$\sum_{n=0}^{3} x(n) e^{\frac{-j2\pi n}{2}} \quad -j\pi n$$

$$e^{-\frac{j2\pi \times 3n}{4/2}}$$

$$= e^{-\frac{j\pi 3 \times 2}{2}}$$

$$x(0) + \cancel{x(1)} + x(2) e^{-j\pi 2}$$

$$\{0, 2, 0, 2\}$$

$$1 - 1 \times 1 = 0$$

$k = 3$

$$\sum_{n=0} x(0) + x(2) e^{-\frac{j2\pi}{4/2} \times 3 \cancel{2}}$$

$$1 + (-1)(-1) = 2$$

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(c) $x(n) = \{1,1,1,1,1,1,1,1\}$

$$\frac{\overset{45}{\cancel{3\times 180}}}{\cancel{4}}$$

$$X(k) = \sum_{n=0}^{7} x(n) e^{\frac{-j2\pi kn}{8}}$$

135

sin 45

$\frac{1}{\sqrt{2}}$

k = 0

$$= \sum_{n=0}^{7} x(n) = 8$$

k = 1

$$= \sum_{n=0}^{7} x(n) e^{-\frac{j2\pi n}{\cancel{8}4}}$$

$$\frac{3\ \overset{45}{\cancel{\times 180}}}{\cancel{4}}$$

135

$$\frac{\overset{30}{\cancel{3\times 180}}}{\cancel{4}}$$

$$= \sum_{n=0}^{7} x(n) e^{\frac{-j\pi n}{4}}$$

$e^{-\frac{j3\pi}{4}}$

$\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$

$$= 1 + 1\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) - j +$$

$$= 1 + e^{\frac{-j2\pi k}{N}} + e^{\frac{-j2\pi k(2)}{N}} + e^{\frac{-j2\pi k(3)}{N}} + \cdots e^{\frac{-j2\pi k(N-1)}{N}}$$

$$= \frac{1-r^N}{1-r} = \frac{1-\left(e^{\left(-\frac{j2\pi k}{N}\right)}\right)^N}{1-e^{\frac{-j2\pi k}{N}}}, \quad k \neq 0$$

$$= \frac{1-1}{} = 0, \quad k \neq 0$$

$(k = 1,2,3 \ldots 7)$

$$X(k) = \{8,0,0,0,0,0,0,0\}$$

* $\cos n\pi = (-1)^n$
$\cos 2\pi n = 1$

(d) $x(n) = \cos(0.25\pi n)$, $n = 0, 1, \ldots 7$

$$= \cos\left(\frac{\pi}{4} n\right) \quad , n = 0, 1, \ldots 7$$

$$= \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}$$

$$X(k) = \frac{1}{2}\sum_{n=0}^{N-1}\left(e^{j\frac{\pi}{4}n} e^{-\frac{j2\pi kn}{8}} + e^{-j\frac{\pi}{4}n} e^{-\frac{j2\pi kn}{8}}\right)$$

$$= \frac{1}{2}\Bigg\{\sum_{n=0}^{7} e^{j\frac{\pi}{4}(1-k)} + e^{j\frac{\pi}{4}(2)(1-k)} + \ldots e^{j\frac{\pi}{4}(7)(1-k)}$$

$$= \quad + 1 + e^{-j\frac{\pi}{4}(1+k)} + e^{j\frac{\pi}{4}(2)(1+k)} \ldots\ldots e^{-j\frac{\pi}{4}(7)(1+k)}\Bigg\}$$

$$= \frac{1}{2}\left[\frac{1-\left(e^{j\frac{\pi}{4}(1-k)}\right)^8}{1-e^{j\pi/4(1-k)}} + \frac{1-\left(e^{-j\frac{\pi}{4}(1+k)}\right)^8}{1-e^{-j\frac{\pi}{4}(1+k)}}\right], \quad k \neq 1, k \neq 7$$

$$= \begin{cases} 0, & k \neq 1 \\ \frac{1}{2}(8+0) = 4 & k = 1 \\ \frac{1}{2}(0+8) = 4 & k = 7 \end{cases}$$

$$1 + e^{-j\frac{\pi}{4}(1+1)} + e^{-j\frac{\pi}{4}(2)(2)} + e^{-j\frac{\pi}{4}(3)(2)}$$

$$= \frac{1-\left(e^{-j\frac{\pi}{4}(2)}\right)^8}{1-e^{-j\frac{\pi}{4}(2)}} = 1 - 1 = 0$$

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(e) $x(n) = 0.9^n,\ n = [0, 1, \ldots, 7]$

$$X(k) = \sum_{n=0}^{N-1} 0.9^n e^{-j2\pi kn/N}$$

$$= \frac{1 - \left(0.9e^{-j2\pi k/N}\right)^N}{1 - 0.9e^{-j2\pi k/N}}$$

$$= (0.1)8 = 0.8$$

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Circular convolution

$x_3(n) = x_1(n) \, (N) \, x_2(n)$ $\qquad x_1(k)\, x_2(n-k)$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

method to calculate circular conv (periodic conv)

(1) Graphical method (Based on circle)
(2) matrix multiplication method
(3) By the means of DFT & IDFT

Properties:

(1) if $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$ & $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then $x_1(n) \, (N) \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\, X_2(k)$

$IDFT[X_1(k) \cdot X_2(K)] = x_1(n) \, (N) \, x_2(n)$

Q. By means DFT & IDFT determine the circular convolution of the sequence $x_1(n)$ & $x_2(n)$ where $x_1(n) = \{2,1,2,1\}$ & $x_2(n) = \{1,2,3,4\}$
$N = 4$

$$X_1(K) = \sum_{n=0}^{3} x_1(n)\, e^{-\frac{j2\pi kn}{4}} \quad , K = 0,1,2,3$$

$$x_1(0) + x_1(1)\, e^{-\frac{j2\pi k}{4}} + x_1(2)\, e^{-\frac{j4\pi k}{4}} + x_1(3)\, e^{-\frac{j6\pi k}{4}}$$

$X_1(0) = 6 \qquad X_1(1) = 0 \qquad X_1(2) = 2 \qquad X_1(3) = 0$

$X_1(K) = \{6, 0, 2, 0\}$

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$$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{\frac{-j2\pi kn}{4}} \quad , k = 0,1,2,3$$

$k=0$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) = 10$$

$k=1$

$$X_2(1) = \sum_{n=0}^{3} x_2(n) e^{\frac{-j\pi n}{2}}$$

$$= x_2(0) + x_2(1) e^{\frac{-j\pi}{2}} + x_2(2) e^{-j\pi} + x_2(3) e^{\frac{-j3\pi}{2}}$$

$$= 1 + 2j - 3$$

$$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{\frac{-j2\pi kn}{4}} \quad , k = 0,1,2,3$$

$$= 1 + 2e^{\frac{-j2\pi k}{4}} + 3e^{\frac{-j2\pi k(2)}{4}} + 4e^{\frac{-j2\pi k(3)}{4}}$$

$X_2(0) = 10$ $\quad\quad X_2(2) = -2$

$X_2(1) = -2+2j$ $\quad\quad X_2(3) = -2-j2$

$$X_2(k) = \{10, (-2+2j), -2, (-2-j2)\}$$

$$X_1(k) X_2(k) = \{60, 0, -4, 0\}$$

$$x_1(n) \, N \, x_2(n) = \text{IDFT}[X_1(k) X_2(k)]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} (X_1(k) X_2(k)) e^{\frac{j2\pi kn}{N}}$$

$$= \frac{1}{4} \sum_{k=0}^{3} X_1(k) X_2(k) e^{\frac{j2\pi kn}{4}}$$

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$$= \frac{1}{4}\left[60 - 4e^{\frac{j2\pi(2)n}{4}}\right] \quad n = 0,1,2,3$$

$$= \frac{1}{4}\left[60 - 4e^{j\pi n}\right]$$

$$x_3(n) = x_1(n) \; (N) \; x_2(n) = \{14, 16, 14, 16\}$$

Proof:

$$x_3(n) = IDFT\left[X_1(k)X_2(k)\right] = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)e^{\frac{j2\pi kn}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m)e^{\frac{-j2\pi km}{N}}\right]\left[\sum_{l=0}^{N-1} x_2(l)e^{\frac{-j2\pi kl}{N}}\right] e^{\frac{j2\pi kn}{N}}$$

Interchanging the order of summation

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m)\sum_{l=0}^{N-1} x_2(l)\left(\sum_{k=0}^{N-1} e^{\frac{-j2\pi k}{N}(m+l-n)}\right)$$

$$= \sum_{k=0}^{N-1} e^{\frac{-j2\pi k}{N}(m+l-n)}, \quad \text{if } m+l-n \to PN, \text{ where } P \in I$$

$$l = PN - m + n$$

$$= 1 + 1 \cdots = N$$

$$= \frac{1 - \left(e^{\frac{-j2\pi}{N}(m+l-n)}\right)^N}{1 - e^{\frac{-j2\pi}{N}(m+l-n)}} \quad \text{if } l \neq PN - m + N$$

$$= 0$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \left(N x_2(pN+n-m) + 0 + 0 \ldots \right)$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) x_2(n-m+pN)$$

$$= \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N = x_1(n) \, (N) \, x_2(n)$$

circular conv

Property-2:

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ & $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

$$x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} \left[X_1(k) \, (N) \, X_2(k) \right]$$

Proof:

$$DFT[x_1(n) \cdot x_2(n)] = \sum_{n=0}^{N-1} x_1(n) x_2(n) e^{-\frac{j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1}$$

Linear filtering approach to determine the Response of the System

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

- if system is finite length ~~sequence~~ sequence (Tabular Method)

$y(n) = x(n) * h(n)$

$x_1(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$ M $\qquad x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ L

$x(n) = x_1(n) * x_2(n)$

$x_1(n)$ \ $x_2(n)$	1	2	3	4
1	1	2	3	4
1	1	2	3	4
2	2	4	6	8
2	2	4	6	8

$\boxed{\text{* Add the diagonals}}$

$x(n) = \{1, 3, 7, 13, 14, 14, 8\} = x_1(n) * x_2(n)$

Note: Linear convolution can also be calculated by the circular convolution if $N \geq M+L-1$ (for response)

$x_1(n) \, (N) \, x_2(n)$

$N \geq M+L-1$
$N \geq 7$

$\rightarrow$ matrix form
$x_1(n) \, (N) \, x_2(n)$

$x_1(n) = \{1,1,2,2\} = \{1,1,2,2,0,0,0\}$
$x_2(n) = \{1,2,3,4\} = \{1,2,3,4,0,0,0\}$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 7 \\ 13 \\ 14 \\ 14 \\ 8 \end{bmatrix}$$

3. (i) $x_1(n) = \delta(n) + \delta(n-1) - \delta(n-2) - \delta(n-3)$
$x_2(n) = \delta(n) - \delta(n-2) + \delta(n-4)$

$x_1(n) = \{1, 1, -1, -1\}$
$x_2(n) = \{1, 0, -1, 0, 1\}$

Note: if we have to calculate linear convolution with the help of circular the $N \geq M+L-1$

if we have to calculate circular convolution only then $M = L$

5. $x(n) = \begin{cases} 2, & n \to \text{even} \\ -3, & n \to \text{odd} \end{cases}$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n \to \text{even}} x(n) e^{\frac{-j2\pi kn}{N}} + \sum_{n \to \text{odd}} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$= 2\left(1 + e^{\frac{-j2\pi k(2)}{N}} + e^{\frac{-j2\pi k(4)}{N}} - - \right) - 3\left(e^{\frac{-j2\pi(1)}{N}} + e^{\frac{-j2\pi k(3)}{N}} + e^{\frac{-j2\pi k(5)}{N}} + - - \right)$$

Perseval's Theorem

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

$$\boxed{\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(K)\, X_2^*(K)}$$

→ if $x_1(n) = x_2(n)$

$$\sum_{n=0}^{N-1} x_1(n)\, x_1^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(K)\, X_1^*(K)$$

$$\Rightarrow \sum_{n=0}^{N-1} |x_1(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X_1(K)|^2$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

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$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{K=0}^{N-1} |X(K)|^2$$

Proof:

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x(n)\, x^*(n)$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi kn}{N}}\right) x^*(n)$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \left(\sum_{n=0}^{N-1} x^*(n)\, e^{\frac{j2\pi kn}{N}}\right)$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \left[\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}}\right]^*$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \left[X(K)\right]^*$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, X^*(K) = \frac{1}{N}\sum_{K=0}^{N-1} |X(K)|^2$$