MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet I: Hints and Solutions

1. Let F be a field. Show that

   (a) The additive identity in F is unique. **Hint:** Assume there are two additive identities 0 and $0'$. Write down $0+0'$ and use the defining property of an additive identity on each side to simplify. Ask yourself: which axiom lets you conclude $0 = 0'$.

   (b) The additive inverse of an element of F is unique. **Hint:** Suppose $a \in F$ has two inverses b and c. Use the definition of additive inverse and the fact that 0 is unique to write $a + b = 0 = b + a$ and $a + c = 0 = c + a$. Try to use the fact that adding the same element to equal elements preserves equality (why?), and use associativity/commutativity to rearrange terms to deduce $b = c$.

   (c) The multiplicative identity of F is unique. **Hint:** Assume there are two multiplicative identities 1 and $1'$. Consider $1.1'$ and apply the identity property to each factor. Deduce that $1 = 1'$.

   (d) The multiplicative inverse of a nonzero element of F is unique. **Hint:** Suppose $a \in F$ with $a \neq 0$ has two inverses b and c. By the definition of multiplicative inverse and teh fcat that 1 is unique, write $a.b = 1 = b.a$ and $a.c = 1 = c.a$. Use the multiplicative identity and associativity/commutativity to transform b to c.

2. Let F be a field. Show that $(-1) \cdot x = -x$ for every $x \in F$. We show that $(-1) \cdot x$ is an additive inverse of x, i.e.

   $$x + (-1) \cdot x = 0.$$

   Using the field axioms we proceed:

   $$\begin{aligned} x + (-1) \cdot x &= x + x \cdot (-1) && \text{(commutativity of multiplication)} \\ &= x \cdot 1 + x \cdot (-1) && \text{(multiplicative identity: } x = x \cdot 1) \\ &= x\big(1 + (-1)\big) && \text{(distributivity)} \\ &= x \cdot 0 && \text{(since } 1 + (-1) = 0) \\ &= 0 && \text{(see note below).} \end{aligned}$$

   Note (showing $x \cdot 0 = 0$). Using distributivity and existence of additive inverses,

   $$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0.$$

   Add the additive inverse $-(x \cdot 0)$ to both sides to obtain

   $$x \cdot 0 + (-(x \cdot 0)) = \big(x \cdot 0 + x \cdot 0\big) + (-(x \cdot 0)),$$

   and by associativity and the definition of additive inverse the left side is 0 and the right side simplifies to $x \cdot 0$. Hence $x \cdot 0 = 0$.

   Since $x + (-1) \cdot x = 0$, the element $(-1) \cdot x$ is an additive inverse of x. The additive inverse of x is unique, so $(-1) \cdot x = -x$.

   Axioms used.

   - Existence of multiplicative identity: $x \cdot 1 = x$.
   - Commutativity of multiplication (to swap factors when convenient). This could have been avoided.

- Distributive law: $a(b+c) = ab + ac$.
- Existence of additive inverse (for 1 there is -1 with $1 + (-1) = 0$).
- Additive identity 0.
- Associativity of addition and uniqueness of additive inverse (used in the final uniqueness step).

3. Let F be a field and let $a, b, c \in F$.

   (a) If $a + b = c + b$, then $a = c$. Suppose $a + b = c + b$. By the existence of additive inverses there is $-b \in F$ with $b + (-b) = 0$. Add $-b$ to both sides (Think: why $x = y \implies x + z = y + z$!) and use associativity of addition:

   $$(a + b) + (-b) = (c + b) + (-b).$$

   By associativity,

   $$a + (b + (-b)) = c + (b + (-b)).$$

   Since $b+(-b) = 0$, this becomes $a+0 = c+0$, and by the property of the additive identity $a = c$.

   (b) If $ab = cb$ and $b \neq 0$, then $a = c$. Suppose $ab = cb$ and $b \neq 0$. By the existence of multiplicative inverses (for nonzero elements) there exists $b^{-1} \in F$ with $bb^{-1} = 1$. Multiply the equality $ab = cb$ on the right by b^{-1} (Think: why $x = y \implies xz = yz$!) and use associativity of multiplication:

   $$(ab)b^{-1} = (cb)b^{-1}.$$

   By associativity,

   $$a(bb^{-1}) = c(bb^{-1}).$$

   Since $bb^{-1} = 1$, this gives $a \cdot 1 = c \cdot 1$, and then by the multiplicative identity property $a = c$.

   Axioms used.

   - Existence of additive identity 0 and additive inverses (to add $-b$).
   - Associativity of addition (to regroup when adding $-b$).
   - Existence of multiplicative identity 1 and multiplicative inverses for nonzero elements (to use b^{-1}).
   - Associativity of multiplication (to regroup when multiplying by b^{-1}).

4. Let F be a finite field of characteristic p. Then p is prime.

   By definition, the characteristic $\operatorname{char} F$ is the smallest positive integer m such that

   $$\underbrace{1 + \cdots + 1}_{m \text{ times}} = 0.$$

   If m were composite, say $m = ab$ with $1 < a, b < m$, then

   $$\big(\underbrace{1 + \cdots + 1}_{a \text{ times}}\big) \cdot \big(\underbrace{1 + \cdots + 1}_{b \text{ times}}\big) = \underbrace{1 + \cdots + 1}_{ab = m \text{ times}} = 0,$$

   so the product of two nonzero elements would be zero, contradicting that F is a field (How/why?). Hence $m = p$ must be prime.

5. (a) Let F be a subfield of $\mathbb{R}$. Since $1 \in F$, closure under addition implies that $n \cdot 1 \in F$ for all $n \in \mathbb{N}$, and existence of additive inverses gives $k \cdot 1 \in F$ for all $k \in \mathbb{Z}$. If $a/b \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$, then $a \cdot 1 \in F$ and $b \cdot 1 \in F$ with $b \cdot 1 \neq 0$. Since F is a field, $(b \cdot 1)^{-1} \in F$, and hence

$$\frac{a}{b} = (a \cdot 1)\,(b \cdot 1)^{-1} \in F.$$

Thus $\mathbb{Q} \subseteq F$. Consequently, $\mathbb{Q}$ is the smallest subfield of $\mathbb{R}$.

(b) Let d be a square-free integer. Show that

$$\mathbb{Q}(\sqrt{d}) := \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}$$

is a field with the usual addition and multiplication. In particular $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(i)$ (here $i^2 = -1$) are fields.

We verify the field axioms by elementary calculations.
(Closure and basic elements). Clearly $0 = 0 + 0\sqrt{d}$ and $1 = 1 + 0\sqrt{d}$ lie in $\mathbb{Q}(\sqrt{d})$. If

$$x = a + b\sqrt{d}, \qquad y = c + e\sqrt{d} \qquad (a, b, c, e \in \mathbb{Q}),$$

then

$$x + y = (a + c) + (b + e)\sqrt{d} \in \mathbb{Q}(\sqrt{d}),$$

so the set is closed under addition, and

$$x \cdot y = (a + b\sqrt{d})(c + e\sqrt{d}) = (ac + bed) + (ae + bc)\sqrt{d} \in \mathbb{Q}(\sqrt{d}),$$

so it is closed under multiplication. The additive inverse of x is $-x = (-a) + (-b)\sqrt{d}$, which belongs to the set.
(Multiplicative inverses). Let $x = a + b\sqrt{d}$ be nonzero. We show $x^{-1} \in \mathbb{Q}(\sqrt{d})$. Compute

$$x(a - b\sqrt{d}) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - b^2 d.$$

If $a^2 - b^2 d \neq 0$, then

$$x^{-1} = \frac{a - b\sqrt{d}}{a^2 - b^2 d} \in \mathbb{Q}(\sqrt{d}).$$

It remains to rule out the possibility $a^2 - b^2 d = 0$ when $x \neq 0$. Suppose $a^2 - b^2 d = 0$. If $b = 0$ then $a^2 = 0$ so $a = 0$ and $x = 0$, contradicting $x \neq 0$. Hence $b \neq 0$ and

$$d = \left(\frac{a}{b}\right)^2 \in \mathbb{Q},$$

so d would be a rational square. But d is an integer and square-free, so it cannot be a (nontrivial) rational square; this contradiction shows $a^2 - b^2 d \neq 0$. Therefore every nonzero element has an inverse in $\mathbb{Q}(\sqrt{d})$.
Thus $\mathbb{Q}(\sqrt{d})$ contains $0, 1$, is closed under $+$ and $\cdot$, every element has additive inverse, and every nonzero element has multiplicative inverse: it is a field.

Remarks.

- If $d > 0$ then $\mathbb{Q}(\sqrt{d}) \subset \mathbb{R}$; if $d < 0$ then $\mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$.
- The examples $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(i)$ follow immediately by taking $d = 2, 3, -1$.

6. Consider the following system of equations:

$$\begin{cases} x - y - 3z = 3, \\ 2x + z = 0, \\ 2y + 7z = c. \end{cases}$$

The system is consistent exactly when $c = -6$.

For $c = -6$ put $z = t$. Then

$$x = -\tfrac{1}{2}t, \qquad y = -3 - \tfrac{7}{2}t, \qquad z = t.$$

Hence the solution set is the line

$$(x, y, z) = (0, -3, 0) + s(-1, -7, 2), \qquad s \in \mathbb{R},$$

obtained by setting $t = 2s$.

For part (c), the first two planes intersect in the above line. The normal vector of the third plane $2y + 7z = 4$ is $(0, 2, 7)$, which satisfies

$$(0, 2, 7) \cdot (-1, -7, 2) = 0,$$

so the third plane is parallel to the line of intersection of the first two planes. Since the required right-hand side for the third plane to meet that line is -6 (not 4), the plane $2y+7z = 4$ is a parallel translate that does not meet the line; therefore the three planes do not have a common point.

7. Let $V = \{x \in \mathbb{R} : x > 0\}$ with operations

$$x \oplus y := xy, \qquad \alpha \odot x := x^{\alpha} \quad (\alpha \in \mathbb{R}).$$

Then $(V, \oplus, \odot)$ is a real vector space.

All operations are well-defined since $xy > 0$ and $x^{\alpha} > 0$ for $x > 0$ and $\alpha \in \mathbb{R}$.

Additive structure ("$\oplus$").

- Commutativity: $x \oplus y = xy = yx = y \oplus x$.
- Associativity: $(x \oplus y) \oplus z = (xy)z = x(yz) = x \oplus (y \oplus z)$.
- Additive identity (zero vector): 1, since $x \oplus 1 = 1 \oplus x = x$.
- Additive inverse: for each x, $x^{-1} \in V$ and $x \oplus x^{-1} = 1$.

Scalar multiplication ("$\odot$").

- Unital: $1 \odot x = x^1 = x$.
- Compatibility: $\alpha \odot (\beta \odot x) = (x^{\beta})^{\alpha} = x^{\alpha\beta} = (\alpha\beta) \odot x$.
- Distributivity over vector addition:

$$\alpha \odot (x \oplus y) = (xy)^{\alpha} = x^{\alpha} y^{\alpha} = (\alpha \odot x) \oplus (\alpha \odot y).$$

- Distributivity over scalar addition:

$$(\alpha + \beta) \odot x = x^{\alpha+\beta} = x^{\alpha} x^{\beta} = (\alpha \odot x) \oplus (\beta \odot x).$$

- Zero scalar: $0 \odot x = x^0 = 1$, which is the additive identity, as required.

Thus all vector space axioms hold; hence V is a real vector space.

Remark (isomorphism with $\mathbb{R}$). The map $\phi : V \to \mathbb{R}$, $\phi(x) = \ln x$, is a vector space isomorphism (this concept will be revisited later) because

$$\phi(x \oplus y) = \ln(xy) = \ln x + \ln y = \phi(x) + \phi(y), \qquad \phi(\alpha \odot x) = \ln(x^\alpha) = \alpha \ln x = \alpha\phi(x).$$

So $(V, \oplus, \odot) \cong (\mathbb{R}, +, \cdot)$.

8. Let V be the set of all $n \times n$ matrices of real numbers. Define an operation of "addition" by

$$A \diamond B = \tfrac{1}{2}(AB + BA)$$

for all $A, B \in V$. Define an operation of "scalar multiplication" by

$$\alpha \star A = 0$$

for all $\alpha \in \mathbb{R}$ and $A \in V$. Under the operations $\diamond$ and $\star$ the set V is not a vector space. Identify all the vector space axioms which fail to hold.

We verify the standard vector-space axioms one by one.

- *Closure of addition.* For $A, B \in V$, $\frac{1}{2}(AB + BA)$ is an $n \times n$ real matrix, so $A \diamond B \in V$. **Holds.**
- *Commutativity of addition.* $A \diamond B = \frac{1}{2}(AB + BA) = \frac{1}{2}(BA + AB) = B \diamond A$. **Holds.**
- *Associativity of addition.* We would need

$$(A \diamond B) \diamond C \stackrel{?}{=} A \diamond (B \diamond C).$$

In general these are not equal because

$$(A \diamond B) \diamond C = \tfrac{1}{4}(ABC + BAC + CAB + CBA),$$

while

$$A \diamond (B \diamond C) = \tfrac{1}{4}(ABC + ACB + BCA + CBA),$$

and the middle two terms differ in general.

Counterexample (2×2)**.** Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$(A \diamond B) \diamond C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A \diamond (B \diamond C) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$$

so associativity fails. **Fails.**
- *Existence of additive identity.* We seek $E \in V$ with $A \diamond E = A$ for all A. For $E = I$,

$$A \diamond I = \tfrac{1}{2}(AI + IA) = \tfrac{1}{2}(A + A) = A.$$

Hence the additive identity exists and is I. **Holds.**
- *Existence of additive inverses.* We would need for each A some B with $A \diamond B = I$. But for the usual zero matrix 0,

$$0 \diamond B = \tfrac{1}{2}(0 \cdot B + B \cdot 0) = 0 \quad \text{for every } B,$$

and $0 \neq I$. Thus there is no B with $0 \diamond B = I$, so additive inverses do not exist for all elements. **Fails.**

- *Closure under scalar multiplication.* For any $\alpha \in \mathbb{R}$ and $A \in V$, $\alpha \star A = 0 \in V$. **Holds.**
- *Identity of scalars:* $1 \star A \stackrel{?}{=} A$. But $1 \star A = 0 \neq A$ (unless $A = 0$), so this axiom **fails**.
- *Compatibility of scalar multiplication:* $(\alpha\beta) \star A = \alpha \star (\beta \star A)$. Both sides equal 0, so this axiom **holds**.
- *Distributivity of scalar addition over vector:* $(\alpha + \beta) \star A \stackrel{?}{=} \alpha \star A \diamond \beta \star A$. The left-hand side is 0. The right-hand side is $0 \diamond 0 = \frac{1}{2}(0 \cdot 0 + 0 \cdot 0) = 0$. So equality holds; this axiom **holds**.
- *Distributivity of scalar multiplication over vector addition:* $\alpha \star (A \diamond B) \stackrel{?}{=} \alpha \star A \diamond \alpha \star B$. Both sides are 0, so this axiom **holds**.

9. **Result 1.** For every $x \in V$, one has $0 \cdot x = 0$.

   Proof. If $x \in V$, then

   $$0 \cdot x = (0 + 0) \cdot x \qquad \text{(reason: } 0 = 0 + 0 \text{ in the field)}$$

   $$= 0 \cdot x + 0 \cdot x \qquad \text{(reason: distributive property of scalar multiplication over field addition)}$$

   Now add the additive inverse $-(0 \cdot x)$ to both sides to obtain

   $$0 = 0 \cdot x \qquad \text{(reason: adding the additive inverse yields the zero vector).}$$

   This proves $0 \cdot x = 0$.

   Result 2. For every $x \in V$, one has $(-1) \cdot x = -x$.

   Proof. Recall $1 \cdot x = x$. Then

   $$x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x \qquad \text{(reason: distributive property } a \cdot x + b \cdot x = (a+b) \cdot x)$$

   $$= 0 \cdot x \qquad \text{(reason: } 1 + (-1) = 0 \text{ in the field)}$$

   $$= 0 \qquad \text{(reason: Result 1: } 0 \cdot x = 0).$$

   Thus $x + (-1) \cdot x = 0$, so $(-1) \cdot x$ is the additive inverse of x. By uniqueness of additive inverses we conclude

   $$(-1) \cdot x = -x.$$

10. Yes, a nonzero vector space can have finite cardinality if it is defined over a finite field. For instance, take a finite field F and consider $V = F$ as a vector space over F.
    In fact, we have the following result, which can be proved after recalling the ideas of basis and dimension.
    Claim 1. Let $\mathbb{F}$ be a finite field with q elements. A nonzero vector space V over $\mathbb{F}$ is finite *if and only if* $\dim V$ is finite. In that case, the cardinality of V denoted by $|V|$ is given by

    $$|V| = q^{\dim V}.$$

Proof. If $\dim V = n < \infty$, choose a basis $\{e_1, \dots, e_n\}$. Every vector in V has a unique coordinate n-tuple over $\mathbb{F}$ and conversely every n-tuple over F determines a vector in V, so

$$|V| = q^n,$$

which is finite. Conversely, suppose V is finite. If $\dim V$ were infinite then V would contain arbitrarily large finite linearly independent subsets. Let $S = \{v_1, \dots, v_n\}$ be any finite linearly independent subset; then $\operatorname{span}(S)$ has exactly q^n elements. As n grows the spans give arbitrarily large finite subsets of V, hence V would be infinite — contradiction. Therefore $\dim V$ must be finite, and the previous paragraph gives $|V| = q^{\dim V}$.

Claim 2. If F is an infinite field and V is a nonzero vector space over $\mathbb{F}$, then V is infinite.

Proof. Pick a nonzero vector $v \in V$. The map

$$F \to V, \qquad a \mapsto av,$$

is injective because $av = a'v$ implies $(a - a')v = 0$ and $v \neq 0$ forces $a = a'$. Thus $|F| \leq |V|$. Since F is infinite, V is infinite.

Examples. Over the finite field $\mathbb{Z}_2$ the space $\mathbb{Z}_2^3$ is nonzero and has $2^3 = 8$ elements. Over the infinite field $\mathbb{R}$ the 1-dimensional space $\mathbb{R}$ (viewed as a vector space over itself) is infinite.

11. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers are as usual, and for $t \in \mathbb{R}$ define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$t + \infty = \infty + t = \infty + \infty = \infty, \qquad t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$
$$\infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

In any vector space the cancellation law must hold:

$$x + y = x + z \implies y = z.$$

We show cancellation fails in this V. Take

$$x = \infty, \qquad y = 1, \qquad z = 2.$$

Using the addition rules we have

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty,$$

hence

$$\infty + 1 = \infty + 2.$$

But $1 \neq 2$. Thus cancellation does not hold in V, so V cannot be a vector space over $\mathbb{R}$.

12. Let V be the set of all complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$f(-t) = \overline{f(t)} \qquad \text{for all } t \in \mathbb{R},$$

where the bar denotes complex conjugation.

Define the operations:

$$(f+g)(t) = f(t) + g(t), \qquad (cf)(t) = cf(t) \quad (c \in \mathbb{R}).$$

(a) Show that V, with these operations, is a vector space over the field of real numbers $\mathbb{R}$.

(b) Give an example of a function in V which is not real-valued.

(a) To check that V is a real vector space, we verify closure properties.

If $f, g \in V$, then

$$(f+g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{f(t) + g(t)} = \overline{(f+g)(t)}.$$

Hence $f + g \in V$.

If $c \in \mathbb{R}$ and $f \in V$, then

$$(cf)(-t) = cf(-t) = c\,\overline{f(t)} = \overline{cf(t)}.$$

Thus $cf \in V$.

The other vector space axioms (associativity, distributivity, existence of additive identity and inverses) hold automatically since they are inherited from the structure of functions and the field $\mathbb{C}$. Therefore, V is a vector space over $\mathbb{R}$.

Alternatively, one may view V as a subset of the vector space $\mathbb{C}^{\mathbb{R}}$ equipped with the standard operations. Since V is nonempty and closed under addition and scalar multiplication, it follows that V is a subspace of $\mathbb{C}^{\mathbb{R}}$. Hence V itself is a vector space.

(b) An example of a function in V which is not real-valued is

$$f(t) = it.$$

Indeed,

$$f(-t) = -it, \qquad \overline{f(t)} = \overline{it} = -it.$$

So $f(-t) = \overline{f(t)}$ and hence $f \in V$. But $f(t)$ is purely imaginary for $t \neq 0$, so f is not real-valued.

13. Let V be the set of all fifth-degree polynomials with standard operations. Then V is not a vector space. The reason is that V is not closed under addition. For example, consider

$$f(x) = x^5 + x - 1 \quad \text{and} \quad g(x) = -x^5.$$

Both $f, g \in V$ since they are fifth-degree polynomials. However,

$$f(x) + g(x) = (x^5 + x - 1) + (-x^5) = x - 1,$$

which is not a fifth-degree polynomial, and hence $f + g \notin V$.

Therefore, V is not a vector space.

14. We that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that

$$f'(-1) = 3f(2)$$

is a subspace of $V = \mathbb{R}^{(-4,4)}$.

Let

$$W = \{f : (-4, 4) \to \mathbb{R} \mid f \text{ is differentiable and } f'(-1) = 3f(2)\}.$$

We check the subspace criteria:

- The zero function $f(x) = 0$ belongs to W it is differentiable, $f'(-1) = 0$ and $3f(2) = 0$. Hence $0 \in W$.
- If $f, g \in W$, then $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. For any scalars $\alpha, \beta \in \mathbb{R}$, we have

$$(\alpha f + \beta g)'(-1) = \alpha f'(-1) + \beta g'(-1) = \alpha \cdot 3f(2) + \beta \cdot 3g(2) = 3(\alpha f(2) + \beta g(2)),$$

so $\alpha f + \beta g \in W$.

Thus W is a subspace of the vector space $V = \mathbb{R}^{(-4,4)}$.

15. Suppose $b \in \mathbb{R}$. Show that the set

$$W_b = \{\, f \in C([0, 1]) : \int_0^1 f(x)\, dx = b \,\}$$

of continuous real-valued functions on $[0, 1]$ whose integral equals b is a subspace of $C([0, 1])$ if and only if $b = 0$.

First note that continuity guarantees the Riemann integral on $[0, 1]$ is well defined for each $f \in C([0, 1])$.

(Necessity.) Suppose W_b is a subspace of $C([0, 1])$. Then the zero vector (the zero function) must lie in W_b. The zero function 0 satisfies

$$\int_0^1 0\, dx = 0,$$

so $0 \in W_b$ implies $b = 0$.

(Sufficiency.) Conversely, assume $b = 0$. We show W_0 is a subspace.

- The zero function 0 belongs to W_0 since $\int_0^1 0\, dx = 0$.
- Closure under addition: if $f, g \in W_0$, then by linearity of the integral

$$\int_0^1 (f + g)(x)\, dx = \int_0^1 f(x)\, dx + \int_0^1 g(x)\, dx = 0 + 0 = 0,$$

so $f + g \in W_0$.

- Closure under scalar multiplication: if $f \in W_0$ and $\alpha \in \mathbb{R}$, then

$$\int_0^1 (\alpha f)(x)\, dx = \alpha \int_0^1 f(x)\, dx = \alpha \cdot 0 = 0,$$

so $\alpha f \in W_0$.

Thus W_0 contains 0 and is closed under addition and scalar multiplication, hence is a subspace of $C([0,1])$.

Combining the two directions, W_b is a subspace of $C([0,1])$ iff $b = 0$.

16. If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses, then U is a subspace of $\mathbb{R}^2$,

    Counterexample. Let
    $$U = \{(m, 0) : m \in \mathbb{Z}\} \subset \mathbb{R}^2.$$
    Then $U \neq \varnothing$. If $(m, 0), (n, 0) \in U$, then
    $$(m, 0) + (n, 0) = (m + n, 0) \in U,$$
    so U is closed under addition. Also for every $(m, 0) \in U$ its additive inverse is
    $$-(m, 0) = (-m, 0) \in U,$$
    so U is closed under additive inverses.

    However U is *not* a subspace of $\mathbb{R}^2$ because it is not closed under scalar multiplication by real numbers. For example,
    $$\tfrac{1}{2} \cdot (1, 0) = (1/2, 0) \notin U,$$
    since $1/2 \notin \mathbb{Z}$. Therefore U is not a vector subspace.

    Remark. The two properties given (nonempty, closed under addition and additive inverses) say precisely that U is an additive subgroup of $(\mathbb{R}^2, +)$. Additive subgroups of $\mathbb{R}^2$ need not be linear subspaces over $\mathbb{R}$; to be a subspace one must additionally require closure under multiplication by *all* real scalars.

17. Let $\{W_i : i \in I\}$ be any collection of subspaces of a vector space V, where I is an indexing set. Define
    $$W = \bigcap_{i \in I} W_i = \{v \in V : v \in W_i \text{ for every } i \in I\}.$$

    - Since each W_i is a subspace, $0 \in W_i$ for all $i \in I$. Hence $0 \in W$.
    - If $u, v \in W$, then $u, v \in W_i$ for all $i \in I$. As each W_i is a subspace, $u + v \in W_i$ for all i, so $u + v \in W$.
    - If $u \in W$ and α is a scalar, then $u \in W_i$ for all i, hence $\alpha u \in W_i$ for all i, so $\alpha u \in W$.

    Thus W is a subspace of V.

    Remark on empty intersections. If $I = \varnothing$, then by convention
    $$\bigcap_{i \in \varnothing} W_i = V,$$
    since every element of V vacuously belongs to all sets in the empty collection. Hence the intersection of an empty family of subspaces is V itself, which is a subspace.

Union of subspaces. In general, the union of subspaces need not be a subspace. For example, in $\mathbb{R}^2$ let

$$W_1 = \{(x, 0) : x \in \mathbb{R}\}, \quad W_2 = \{(0, y) : y \in \mathbb{R}\}.$$

Both are subspaces, but

$$W_1 \cup W_2$$

is not closed under addition: $(1, 0) \in W_1$, $(0, 1) \in W_2$, but $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

Remark on empty unions. If $I = \varnothing$, then by convention

$$\bigcup_{i \in \varnothing} W_i = \varnothing,$$

which is not a subspace (since it does not contain 0).

18. Let W_1 and W_2 be subspaces of a vector space V. Then we prove that

$$W_1 \cup W_2$$

is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. (If.) If $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$, which is a subspace. Similarly if $W_2 \subseteq W_1$.

(Only if.) Suppose $W_1 \cup W_2$ is a subspace. We show one of W_1, W_2 is contained in the other. Assume for contradiction that neither is contained in the other. Then there exist

$$u \in W_1 \setminus W_2 \qquad \text{and} \qquad v \in W_2 \setminus W_1.$$

Both u and v lie in the subspace $W_1 \cup W_2$, so their sum $u + v$ must also lie in $W_1 \cup W_2$ (closure under addition). Two cases:

- If $u+v \in W_1$, then since $u \in W_1$ and W_1 is a subspace, subtracting u (equivalently adding $-u$) gives

$$v = (u + v) - u \in W_1,$$

  contradicting $v \notin W_1$.
- If $u + v \in W_2$, then since $v \in W_2$ and W_2 is a subspace, subtracting v gives

$$u = (u + v) - v \in W_2,$$

  contradicting $u \notin W_2$.

In either case we obtain a contradiction. Hence our assumption was false, so one of the subspaces must be contained in the other. This completes the proof. □

19. Statement: If V_1, V_2, U are subspaces of V with $V_1 + U = V_2 + U$, then $V_1 = V_2$.

Answer: False.

Counterexample. Let $V = \mathbb{R}^2$, let

$$U = \{(x, 0) : x \in \mathbb{R}\} \quad \text{(the } x\text{-axis)},$$

and let

$$V_1 = \{(0, y) : y \in \mathbb{R}\} \quad \text{(the } y\text{-axis)},$$

$$V_2 = \{(t,t) : t \in \mathbb{R}\} \quad \text{(the line } y = x\text{)}.$$

Then V_1 and V_2 are distinct subspaces of $\mathbb{R}^2$. For every $(a,b) \in \mathbb{R}^2$ we have

$$(a,b) = (a,0) + (0,b) \in U + V_1,$$

and also

$$(a,b) = (a-b,0) + (b,b) \in U + V_2.$$

Hence $U + V_1 = \mathbb{R}^2 = U + V_2$, so $V_1 + U = V_2 + U$ but $V_1 \neq V_2$. Thus the statement is false.

Statement. If V_1, V_2, U are subspaces of V with $V = V_1 \oplus U$ and $V = V_2 \oplus U$, then $V_1 = V_2$.

Answer: False.

Counterexample. Use the same V, U, V_1, V_2 as above. Note that

$$\mathbb{R}^2 = V_1 \oplus U \qquad \text{and} \qquad \mathbb{R}^2 = V_2 \oplus U,$$

because in each case the sum is direct (each complementary subspace meets U only in 0) and equals all of $\mathbb{R}^2$. Yet $V_1 \neq V_2$. Thus even when both decompositions are direct sums with the same complement U, the complementary subspace need not be unique.

Remark. In general, for a fixed subspace U there are many different complements W with $V = W \oplus U$. The complement is unique only if one imposes an extra structure (for example a specific projection operator or an inner product and one demands the orthogonal complement), but not in the purely algebraic setting.

20. **There is a minor notational issue in the question.** Note that both $F^{n\times 1}$ and $M_{n\times 1}(F)$ is used to denote the vector space of all $n \times 1$ (column) matrices with entries from F.

    Let A be a fixed $m \times n$ matrix over a field F. The set

    $$\ker A := \{X \in F^{n\times 1} : AX = 0\}$$

    is a subspace of the vector space $F^{n\times 1}$.

    We must check that $\ker A$ is nonempty and closed under vector addition and scalar multiplication.

    - *Nonempty:* The zero vector $0 \in F^{n\times 1}$ satisfies $A0 = 0$, so $0 \in \ker A$.
    - *Closed under linear combinations:* Let $X, Y \in \ker A$ and let $c \in F$. Then $AX = 0$ and $AY = 0$. Using the distributive and compatibility properties of matrix multiplication with scalar multiplication,

      $$A(cX + Y) = c(AX) + AY = c \cdot 0 + 0 = 0.$$

      Hence $cX + Y \in \ker A$.

    Therefore $\ker A$ is a subspace of $F^{n\times 1}$.

    Remark. This is the general fact that the kernel (null space) of any linear transformation (here $X \mapsto AX$) is a subspace of the domain. This will be revisited later in the course.

21. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Let

$$V_e = \{f \in V : \ f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$$

be the subset of even functions, and

$$V_o = \{f \in V : \ f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$$

the subset of odd functions.

(a) Prove that V_e and V_o are subspaces of V.
(b) Prove that $V_e + V_o = V$.
(c) Prove that $V_e \cap V_o = \{0\}$.

(a) V_e and V_o are subspaces.

We show the argument for V_e; the proof for V_o is analogous.

Nonempty. The zero function $0(x) \equiv 0$ satisfies $0(-x) = 0 = 0(x)$, so $0 \in V_e$.

Closed under addition. If $f, g \in V_e$ then for all x,

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

so $f + g \in V_e$.

Closed under scalar multiplication. If $f \in V_e$ and $\alpha \in \mathbb{R}$ then

$$(\alpha f)(-x) = \alpha f(-x) = \alpha f(x) = (\alpha f)(x),$$

so $\alpha f \in V_e$.

Hence V_e is a subspace. The same three checks with $f(-x) = -f(x)$ show V_o is a subspace.

(b) $V_e + V_o = V$.

Let $f \in V$. Define functions f_e and f_o by

$$f_e(x) = \tfrac{1}{2}\big(f(x) + f(-x)\big), \qquad f_o(x) = \tfrac{1}{2}\big(f(x) - f(-x)\big).$$

Then for all x,

$$f_e(-x) = \tfrac{1}{2}\big(f(-x) + f(x)\big) = f_e(x),$$

so $f_e \in V_e$, and

$$f_o(-x) = \tfrac{1}{2}\big(f(-x) - f(x)\big) = -\tfrac{1}{2}\big(f(x) - f(-x)\big) = -f_o(x),$$

so $f_o \in V_o$. Moreover

$$f_e(x) + f_o(x) = \tfrac{1}{2}\big(f(x) + f(-x)\big) + \tfrac{1}{2}\big(f(x) - f(-x)\big) = f(x),$$

hence $f = f_e + f_o \in V_e + V_o$. Since f was arbitrary, $V_e + V_o = V$.

(c) $V_e \cap V_o = \{0\}$.

Let $h \in V_e \cap V_o$. Then for all x we have both $h(-x) = h(x)$ and $h(-x) = -h(x)$. Combining these equalities gives

$$h(x) = -h(x) \quad \text{for all } x,$$

so $2h(x) = 0$ and hence $h(x) = 0$ for every x. Thus h is the zero function, and $V_e \cap V_o = \{0\}$. **(d)** From the last two items it follows that, V is the direct sum

$$V = V_e \oplus V_e.$$

That is, every function $f : \mathbb{R} \to \mathbb{R}$ can be decomposed as a sum of an even function $g : \mathbb{R} \to \mathbb{R}$ and an odd function $h : \mathbb{R} \to \mathbb{R}$ in a unique way, $f = g + h$.

22. Let $\mathbb{R}^\infty$ denote the vector space of all real sequences (addition and scalar multiplication defined coordinatewise). For each subset below write **Yes** if it is a subspace of $\mathbb{R}^\infty$ and **No** if it is not; give a short reason.

    (a) Sequences that have infinitely many zeros (for example, $(1, 1, 0, 1, 1, 0, 1, 1, 0, \dots)$).
    Answer: No. *Reason:* Sum of two such sequences need not have infinitely many zeros. Example: $a = (1, 0, 1, 0, \dots)$ and $b = (0, 1, 0, 1, \dots)$ each have infinitely many zeros but $a + b = (1, 1, 1, 1, \dots)$ has no zero.

    (b) Sequences which are eventually zero (there exists N with $x_n = 0$ for all $n \geq N$).
    Answer: Yes. *Reason:* The sum and scalar multiples of eventually-zero sequences are eventually zero (take the maximum tail index).

    (c) Absolutely summable sequences ($\sum_{k=1}^{\infty} |x_k| < \infty$).
    Answer: Yes. *Reason:* The set $\ell^1 = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k| < \infty\}$ is a linear space: absolute summability is preserved under addition and scalar multiplication (triangle inequality and homogeneity).

    (d) Bounded sequences (there exists M with $|x_k| \leq M$ for all k).
    Answer: Yes. *Reason:* Sum and scalar multiples of bounded sequences are bounded (use triangle inequality and scalar factor).

    (e) Decreasing sequences ($x_{n+1} \leq x_n$ for every n).
    Answer: No. *Reason:* Not closed under scalar multiplication by negative scalars: if x is decreasing and $\alpha < 0$, then αx is nondecreasing, so may fail to be decreasing.

    (f) Convergent sequences.
    Answer: Yes. *Reason:* Limits are linear: sum and scalar multiple of convergent sequences converge to the corresponding sums/products of limits.

    (g) Arithmetic progressions (sequences of the form $a, a+k, a+2k, \dots$ for some $a, k \in \mathbb{R}$).
    Answer: Yes. *Reason:* If $x_n = a + (n-1)k$ and $y_n = b + (n-1)\ell$ then $\alpha x + \beta y$ has the form $(\alpha a + \beta b) + (n-1)(\alpha k + \beta \ell)$, so closed under linear combinations; the zero sequence corresponds to $a = k = 0$.

    (h) Geometric progressions (sequences of the form $a, ak, ak^2, ak^3, \dots$ for some $a, k \in \mathbb{R}$).
    Answer: No. *Reason:* Sum of two geometric sequences with different ratios is typically not geometric (so not closed under addition).