MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet II

This Practice Sheet is based on Span, Linear Independence and Dependence, Basis, Dimension and Basic results on these notions.

1. Prove or give a counterexample:

   (a) If v_1, v_2, v_3, v_4 spans V, then the set

   $$\{v_1 - v_2, \quad v_2 - v_3, \quad v_3 - v_4, \quad v_4\}$$

   also spans V.

   (b) Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the set

   $$\{v_1 - v_2, \quad v_2 - v_3, \quad v_3 - v_4, \quad v_4\}$$

   is also linearly independent.

2. By exhibiting constants α and β with

   $$\alpha f_1 - 2f_2 - \beta f_3 \equiv 0.$$

   show that the set $\{f_1, f_2, f_3\}$ is linearly dependent in $C[0, 2\pi]$, the space of all real-valued continuous functions on $[0, 2\pi]$, where

   $$f_1(x) = \sin x, \qquad f_2(x) = \cos\big(x + \tfrac{\pi}{6}\big), \qquad f_3(x) = \sin\big(x - \tfrac{\pi}{4}\big), \qquad 0 \le x \le 2\pi.$$

3. Show that the functions

   $$f(x) = x, \qquad g(x) = e^x, \qquad h(x) = e^{-x} \qquad (0 \le x \le 1)$$

   in $C[0, 1]$ are linearly independent.

4. Establish the following claims:

   (a) 0 does not belong to any linearly independent set.
   (b) Every superset of a linearly dependent set is linearly dependent.
   (c) Every subset of a linearly independent set is linearly independent.

5. Show that if E_1 and E_2 are linearly independent subsets of a vector space V such that

   $$\operatorname{span}(E_1) \cap \operatorname{span}(E_2) = \{0\},$$

   then $E_1 \cup E_2$ is linearly independent.

6. (a) Prove that a vector space V is infinite-dimensional if and only if there exists a sequence $\{v_1, v_2, v_3, \dots\}$ of vectors in V such that for every positive integer m, the vectors $\{v_1, v_2, \dots, v_m\}$ is linearly independent.
   (b) By using the above result, show that $\mathcal{P}(F)$ is infinite dimensional.
   (c) Show that $\mathbb{R}^{[a,b]}$, the space $C[a, b]$ of all real-valued continuous functions on $[a, b]$, the space $C'[a, b]$ of all continuously differentiable functions are all infinite dimensional.

(d) Let $[a, b] \subset \mathbb{R}$ and for each $\lambda \in [a, b]$ define

$$u_\lambda : [a, b] \to \mathbb{R}, \qquad u_\lambda(t) = e^{\lambda t} \quad (t \in [a, b]).$$

Show that the set

$$\{ u_\lambda : \lambda \in [a, b] \} \subset C[a, b]$$

is uncountable and is a linearly independent subset of $C[a, b]$.

7. (a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the set $\{1 + i, \quad 1 - i\}$ is linearly independent.

   (b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the set $\{1 + i, \quad 1 - i\}$ is linearly dependent.

8. Find bases and dimensions of the following subspaces of $\mathbb{R}^5$:

   (a) $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_3 - x_4 = 0\}$.

   (b) $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4, \; x_1 + x_5 = 0\}$.

   (c) $\operatorname{span}\{(1, -1, 0, 2, 1), (2, 1, -2, 0, 0), (0, -3, 2, 4, 2),$
   $(3, 3, -4, -2, -1), (2, 4, 1, 0, 1), (5, 7, -3, -2, 0)\}$.

9. (a) Is the set

$$\{ 1 + t^n, \; t + t^n, \; \ldots, \; t^{n-1} + t^n, \; t^n \}$$

   a basis for $\mathcal{P}_n(F)$?

   (b) Prove or give a counterexample: If $\{p_0, p_1, p_2, p_3\}$ is a set in $\mathcal{P}_3(F)$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then $\{p_0, p_1, p_2, p_3\}$ is not a basis of $\mathcal{P}_3(F)$.

10. Exhibit a basis for V, the vector space of all 2×2 matrices of real numbers with zero trace.

11. (a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

   Find a basis of U.

   (b) Extend the basis in (a) to a basis of $\mathbb{R}^5$.

   (c) Find a subspace W of $\mathbb{R}^5$ such that

$$\mathbb{R}^5 = U \oplus W.$$

12. Find all vector spaces that have exactly one basis.

13. The subspaces of $\mathbb{R}^3$ are exactly the following:

$$\{0\}, \quad \text{all lines in } \mathbb{R}^3 \text{ containing the origin}, \quad \text{all planes in } \mathbb{R}^3 \text{ containing the origin}, \quad \mathbb{R}^3.$$

14. Let

$$U = \{ p \in \mathcal{P}_4(F) : p(6) = 0 \}.$$

   (a) Find a basis of U.

   (b) Extend the basis in (a) to a basis of $\mathcal{P}_4(F)$.

(c) Find a subspace W of $\mathcal{P}_4(F)$ such that

$$\mathcal{P}_4(F) = U \oplus W.$$

15. Let U and W be subspaces of $\mathbb{R}^8$ with $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Show that

$$\mathbb{R}^8 = U \oplus W.$$

16. Suppose V is finite-dimensional with $\dim V = n \geq 1$. Then show that there exist one-dimensional subspaces $V_1, \dots, V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

17. Show that $\mathbb{R}$ is not finite-dimensional vector space over $\mathbb{Q}$.

18. Let V be a vector space of all 2×2 real matrices. Find a basis $\{A_1, A_2, A_3, A_4\}$ such that $A_i^2 = A_i$ for each i.

19. Let

$$U = \left\{ \begin{bmatrix} u & -u \\ -x & x \end{bmatrix} : u, x \in \mathbb{R} \right\} \quad \text{and} \quad V = \left\{ \begin{bmatrix} v & 0 \\ w & -v \end{bmatrix} : v, w \in \mathbb{R} \right\}.$$

Exhibit a basis for U, for V, for $U + V$, and for $U \cap V$.

20. Let $a_0, a_1, \dots, a_k \in \mathbb{R}$ and consider the linear homogeneous differential equation

$$a_0\, x^{(k)} + a_1\, x^{(k-1)} + \cdots + a_{k-1}\, x' + a_k\, x = 0 \tag{*}$$

on an interval $[a, b]$. Let

$$V = \left\{ x \in C^k([a, b]; \mathbb{R}) : x \text{ satisfies } (*) \text{ on } [a, b] \right\}.$$

Show that V is a vector space over $\mathbb{R}$. What is $\dim V$? Here $C^k([a, b]; \mathbb{R})$ is the space of all real valued functions that are k-times continuously differentiable, that is, all the derivatives up to k-th order exist and are continuous.