Major (40%)

Graded

Student
Kushagra Gupta

Total Points
43 / 60 pts

Question 1

Q1 — 3 / 3 pts

- ✔ **+ 0.5 pts** Need to show cons(cons(Γ)) = Γ
- ✔ **+ 0.5 pts** Let $S = \mathrm{cons}(\Gamma) = \{\alpha \in PL \mid \Gamma \models \alpha\}$. Then, $\mathrm{cons}(S) = \{\beta \in PL \mid S \models \beta\}$.
- ✔ **+ 1 pt** Consider any $\beta \in \mathrm{cons}(S)$. $S \models \beta$, so there is a model M s.t. $M \models S$ and $M \models \beta$.
- ✔ **+ 1 pt** But if $M \models S$, then $M \models \Gamma$ (since $S = \mathrm{cons}(\Gamma)$ and Γ logically entails every $\alpha \in S$). So $\Gamma \models \beta$.
- **+ 0 pts** Incorrect/Not attempted

Question 2

Q2 — 10 / 10 pts

- ✔ **+ 2 pts** Not sound, but complete

Not sound: counterexample

- ✔ **+ 1 pt** Not sound means a satisfiable CNF formula returns the empty clause
- ✔ **+ 2 pts** Resolving $\{p, \neg q\}, \{\neg p, q\}$ on p gives $\{q\}, \{\neg q\}$, resolving which gives $\{\emptyset\}$, but the original formula is satisfiable

Complete

- ✔ **+ 1 pt** Complete means an unsatisfiable formula returns the empty clause (as expected)
- ✔ **+ 1 pt** Each clause in $\mathrm{resolve}(\varphi^*, p)$ is obtained by removing p or $\neg p$ from an existing clause in φ^*
- ✔ **+ 2 pts** Easy to see that if there is some valuation $v \models \mathrm{resolve}(\varphi^*, p)$, then $v \models \varphi^*$. If φ^* is unsat, then so is $\mathrm{resolve}(\varphi^*, p)$ (which is the same but does not mention the literal p at all)
- ✔ **+ 1 pt** Can keep repeating this till we have no literals left, so if φ^* could yield $\{\emptyset\}$ then so can $\mathrm{resolve}(\varphi^*, p)$ (in one fewer step)

- **+ 0 pts** Incorrect

Question 3

Q3 — 2 / 10 pts

- ✔ **+ 1 pt** The class of connected graphs is NOT Δ-elementary
- ✔ **+ 1 pt** Towards a contradiction, let X be a set of formulas that characterizes the class of connected graphs.
- **+ 2 pts** Set $\phi_n := \neg(x \equiv y) \wedge \neg\exists x_1 ... \exists x_n (x \equiv x_1) \wedge (x_n \equiv y) \wedge E(x_1, x_2) \wedge E(x_2, x_3) \wedge \cdots \wedge E(x_{n-1}, x_n)$
- **+ 2 pts** Set $Y := X \cup \{\phi_n \mid n \geq 2\}$

Every $Y_0 \subseteq_{\text{fin}} Y$ is satisfiable

- **+ 1 pt** Suppose $Y_0 \subseteq X \cup \{\phi_n \mid 2 \leq n \leq n_0\}$
- **+ 1 pt** A model of Y_0 is the regular polygon with vertices 0 through $2n_0$, where x is interpreted as 0 and y is interpreted as n_0

- **+ 1 pt** By compactness, since every finite subset of Y is satisfiable, Y is satisfiable too
- **+ 1 pt** Any model of Y is also a model of X. Consider any $\mathscr{I} \models Y$. Then, there is no path connecting $\mathscr{I}(x)$ and $\mathscr{I}(y)$. This is a contradiction.
- **+ 0 pts** Incorrect

Question 4

Q4 — 4.5 / 10 pts

4.1 4(a) — 1 / 3 pts

- **+ 1 pt** Basic idea: there is a function from the domain to itself which is one-one but not onto (or equivalent, recall Quiz 1)
- **+ 2 pts** $\forall x.[\forall y.[(f(x) \equiv f(y)) \supset (x \equiv y)]] \wedge \exists y.[\forall x.[\neg(f(x) \equiv y)]]$
- **+ 0 pts** Incorrect
- **+ 1 pt** Point adjustment

1. as m is given value by the domain of discourse if the domain is finite m is bounded

4.2 4(b) — Resolved 0 / 2 pts

- **+ 0.5 pts** Basic idea: x < y if there is some non-zero natural number z such that adding x and z gives y
- **+ 1.5 pts** $\mathrm{lt}(x, y) := \exists z.[\exists w.[z \equiv succ(w)] \wedge (x + z \equiv y)]$
- ✔ **+ 0 pts** Incorrect

2. Succ is a function
3. What does this mean?

Regrade Request — Submitted on: Nov 29

> Dear sir/ma'am,
>
> In Q4(b), I mentioned the idea for $<$ that we could define it (recursively) as the $\vee$ of succ(y) ≡x, succ(succ(y)) ≡x . . . [which I wrote hurriedly succ(y,x), succ(succ(y,x)) - so what I wrote meant that y is some kth successor of x if x+k=y].
>
> So, could you please review my answer for the alternate idea rubric and confirm if such a formulation could be expressed in first order logic.
>
> Thanks and best regards,

No, we can only evaluate your written answer. Also this would define an infinite-length formula

Reviewed on: Nov 29

4.3 4(c) — 1 / 2 pts

- **+ 0.5 pts** Basic idea: y is a successor of x if x < y and there is no other element in between
- **+ 1.5 pts** $\mathrm{succR}(x, y) := (x < y) \wedge \neg\exists z.[x < z \wedge z < y]$
- **+ 0 pts** Incorrect
- **+ 1 pt** Point adjustment

4. This will not hold for z = 0 and w = 0

4.4 4(d) — 2.5 / 3 pts

- **+ 0 pts** Incorrect
- ✔ **+ 0.5 pts** Basic idea: 1 is defined to be the element multiplying any number by which yields the same number. Can combine this with + to get successor.
- ✔ **+ 2.5 pts** $\mathrm{succR}(x, y) := \exists z.[\forall w.[w \times z \equiv w] \wedge x + z \equiv y]$
- **− 0.5 pts** Point adjustment

6. Constants are not given to you. You need to define a zero

Question 5

Q5 — 14 / 15 pts

5.1 5(a) — 6 / 7 pts

- ✔ **+ 1 pt** $\Sigma = (\emptyset, \emptyset, \{< /2, \mathrm{cfd}/1, \mathrm{cft}/2\})$

  where $<$ stands for "earlier",
  cfd stands for "collects fees from drivers", and
  cft stands for "collects fees from all earlier toll booths which collect fees from drivers"
- ✔ **+ 1 pt** Each toll booth collects fees from drivers or from all earlier toll booths which collect fees from drivers:

  $\forall x.[\mathrm{cfd}(x) \vee \forall y.[(y < x) \wedge \mathrm{cfd}(y) \supset \mathrm{cft}(x, y)]]$
- ✔ **+ 2 pts** Graph arrangement:

  Edges: TC < TB, TS < TB, TB < TF, TF < TW
  Paths: Either write a formula for transitivity, or the following:
  TC < TF, TC < TW, TS < TF, TS < TW, TB < TW
- ✔ **+ 1 pt** TW does not collect fees from TF or from drivers:

  $\neg\mathrm{cft}(TW, TF) \wedge \neg\mathrm{cfd}(TW)$
- ✔ **+ 1 pt** TB collects fees from TC but not from TS:

  $\mathrm{cft}(TB, TC) \wedge \neg\mathrm{cft}(TB, TS)$
- **+ 1 pt** TC and TS collect fees from drivers:

  $\mathrm{cfd}(TC)$
  $\mathrm{cfd}(TS)$
- **+ 0 pts** Incorrect/ Not Attempted

5.2 5(b) — 8 / 8 pts

- **+ 0 pts** Incorrect / Not attempted
- ✔ **+ 1 pt** Statement to be proved:
  $\exists x.[\exists y.[\mathrm{cft}(x, y) \wedge \mathrm{cfd}(y)]]$ (or equivalent, based on model in prior part of the question)
- ✔ **+ 1 pt** Proof strategy: Either Resolution (full marks) or
  Proof tree (0.5 for proof strategy, 0.5 for appealing to soundness)
- ✔ **+ 6 pts** Resolution procedure or proof tree, as applicable

Question 6

Q6

Resolved 9.5 / 12 pts

+ 0 pts Incorrect / Not Attempted

- ✔ **+ 1 pt** Let $\Gamma = \{\forall x.[P(x) \supset Q(x)] \vee \forall y.[Q(y) \supset P(y)]\}$
  $\Gamma \vdash \forall x.[\forall y.[(P(x) \wedge Q(y)) \supset (Q(x) \vee P(y))]]$ using $\forall I$
- ✔ **+ 1 pt** $\Gamma \vdash \forall y.[(P(w) \wedge Q(y)) \supset (Q(w) \vee P(y))]$ using $\forall I$
- ✔ **+ 1 pt** $\Gamma \vdash [(P(w) \wedge Q(z)) \supset (Q(w) \vee P(z))]$ using $\supset I$ (subproof in item 5)
- ✔ **+ 1 pt** $\Gamma, P(w) \wedge Q(z) \vdash Q(w) \vee P(z)$ using $\vee E$ (three subproofs in items 6,7,8)
- ✔ **+ 1 pt** Subproof 1 for or-elim: $\Gamma, P(w) \wedge Q(z) \vdash \forall x.[P(x) \supset Q(x)] \vee \forall y.[Q(y) \supset P(y)]$ using ax

Subproof 2 for or-elim: $\Gamma' = \Gamma \cup \{P(w) \wedge Q(z)\}$

- ✔ **+ 0.5 pts** $\Gamma', \forall x.[P(x) \supset Q(x)] \vdash Q(w) \vee P(z)$ using $\vee I$ (subproof detailed in next items)
- ✔ **+ 1 pt** $\Gamma', \forall x.[P(x) \supset Q(x)] \vdash Q(w)$ using $\supset E$
- ✔ **+ 0.5 pts** $\Gamma', \forall x.[P(x) \supset Q(x)] \vdash P(w) \supset Q(w)$ using $\forall E$
- ✔ **+ 0.5 pts** $\Gamma', \forall x.[P(x) \supset Q(x)] \vdash \forall x.[P(x) \supset Q(x)]$ using ax
- ✔ **+ 0.5 pts** $\Gamma', \forall x.[P(x) \supset Q(x)] \vdash P(w)$ using $\wedge E$
- ✔ **+ 0.5 pts** $\Gamma', \forall x.[P(x) \supset Q(x)] \vdash P(w) \wedge Q(z)$ using ax

Subproof 3 for or-elim: $\Gamma' = \Gamma \cup \{P(w) \wedge Q(z)\}$

- **+ 0.5 pts** $\Gamma', \forall y.[Q(y) \supset P(y)] \vdash Q(w) \vee P(z)$ using $\vee I$ (subproof detailed in next items)
- **+ 1 pt** $\Gamma', \forall y.[Q(y) \supset P(y)] \vdash P(z)$ using $\supset E$
- **+ 0.5 pts** $\Gamma', \forall y.[Q(y) \supset P(y)] \vdash Q(z) \supset P(z)$ using $\forall E$
- **+ 0.5 pts** $\Gamma', \forall y.[Q(y) \supset P(y)] \vdash \forall y.[Q(y) \supset P(y)]$ using ax
- **+ 0.5 pts** $\Gamma', \forall y.[Q(y) \supset P(y)] \vdash Q(z)$ using $\wedge E$
- **+ 0.5 pts** $\Gamma', \forall y.[Q(y) \supset P(y)] \vdash P(w) \wedge Q(z)$ using ax

+ 12 pts Correct

+ 1 pt Point adjustment

5 Incomplete proof

Regrade Request

Submitted on: Nov 28

Dear sir/ma'am,

In Q6, I applied the OR elimination rule and explicitly mentioned that the same proof subtree will be used for the 2 rightmost parts of the tree in the Note.

The reason being that replacing P with Q and x with y in
$\Gamma, \forall x\ [(P(x) \supset Q(x))] \vdash \forall x\ [\forall y\ ((P(x) \wedge Q(y)) \supset (Q(x) \vee P(y)))]$
yields

$\Gamma, \forall y\ [(Q(y) \supset P(y))] \vdash \forall y\ [\forall x\ ((Q(y) \wedge P(x)) \supset (P(y) \vee Q(x)))]$

which is the same tree used in the right part, as I explained in the note.
(Please note that what $\Gamma, \_ \vdash \_$ means is explained in note 2, it is just used for modus ponens premise in the $e_{\supset}$ rule in both parts)

Now the proof of both these subtrees π_2 is shown above, and I have correctly mentioned all the steps completely till the leaves (including $e_{\vee}$ and replacement of variable with terms).
(one step amongst two involving two substitutions of variables $w/x, z/y$ is left for brevity)

Since I have mentioned everything correctly in accordance with the rubric (and no part of the proof is incomplete), I humbly request you to regrade my answer as I should get more marks.

Thanks and best regards,

Both subtrees had marks and you have lost marks for the subtree you have not written.

Reviewed on: Nov 29

Name: KUSHAGRA GUPTA Entry Number: 2021CS50592 1

Indian Institute of Technology Delhi

COL703: Logic for Computer Science **END-TERM EXAM**

DATE: Tuesday the 19th of November 2024 **DURATION:** 2 hours **MAXIMUM MARKS:** 60

Q1 (3)	Q2 (10)	Q3 (10)	Q4 (10)	Q5 (15)	Q6 (12)	Total (60)

Instructions: Write your name and entry number at the top of each sheet. Answer all questions only in the boxes provided. Solutions outside the assigned box will not be graded. Make reasonable assumptions and state them wherever necessary. You may use the last page for rough work; nothing written on that page will be graded.

Attestation: I agree to abide by the Honour Code of IIT Delhi. **Signature:** Kushagra

Q1. (3 marks) Let $\Gamma \subseteq \mathsf{PL}$. Let $\mathsf{cons}(\Gamma) = \{\alpha \in \mathsf{PL} \mid \Gamma \vDash \alpha\}$. Show that cons is idempotent.

To show cons is idempotent we show cons (Γ) = Cons (cons(Γ)).
(i) Consider any arbitrary element of cons (Γ), c $\therefore c \in PL \mid \Gamma \vDash c$
Now as $c \in$ cons(Γ), cons$(\Gamma) \vDash c$ $\therefore c \in PL \mid$ cons$(\Gamma) \vDash c$
or $c \in$ cons(cons(Γ)) $\Rightarrow$ cons$(\Gamma) \subseteq$ cons(cons(Γ))
(ii) Consider any arbitrary element of cons(cons(Γ)), a $\therefore a \in PL \mid$ cons$(\Gamma) \vDash a$
Now if $a \notin$ cons(Γ), $\Gamma \nvDash a$ $\therefore$ cons$(\Gamma) \nvDash a$ (as it can only derive what Γ can derive)
$\therefore a \in$ cons(Γ) or cons$(\Gamma) \supseteq$ cons(cons(Γ)) $\therefore$ cons(Γ) = cons(cons(Γ))

Q2. (10 marks) Suppose we start with a clean φ^* and use the following definition of resolve in our PL resolution procedure.

$$\mathsf{resolve}(\varphi^*, p) \triangleq \left(\varphi^* \setminus (\Delta_p \cup \overline{\Delta}_p)\right) \cup \{\delta \setminus \{p\} \mid \delta \in \Delta_p\} \cup \{\delta' \setminus \{\neg p\} \mid \delta' \in \overline{\Delta}_p\}$$

where $\Delta_p = \{\delta \in \varphi^* \mid p \in \delta\}$, and $\overline{\Delta}_p = \{\delta' \in \varphi^* \mid \neg p \in \delta'\}$. The resolution procedure using this new resolve is (tick one)

☐ Sound ☑ Complete ☐ Both ☐ Neither Justify your answer below.

We clearly see that it is unsound
Counter Example: $\varphi^* = \{\{p, q\}, \{\neg p, \neg q\}\}$
Resolving once on p: $\{\{q\}, \{\neg q\}\}$, again on q: $\{\phi, \phi\}$
Algorithm returns ϕ but φ^* is not UNSAT.
We now prove that it is complete
The standard resolution procedure adds $\{(\delta \cup \delta') \setminus \{p, \neg p\} \mid \delta \in \Delta p, \delta' \in \overline{\Delta} p\}$
★ We show completeness in similar manner by Second theorem in resolve and that a clean set of a set is equivalent to it logically
[Here, if $p \in (\delta_1 \setminus \delta_2)$, $\neg p \in (\delta_2 \setminus \delta_1)$ for some $p \in AP$ &
$\delta_1 \setminus \{p\}$ and $\delta_2 \setminus \{\neg p\}$ are SAT then δ_1 & δ_2 are SAT]
It is easy to see why this theorem holds true because irrespective of $\neg p, p$ the clauses δ_1 and δ_2 are SAT.
Hence our completeness proof of completeness
(i) Termination is guaranteed (similar to standard resolve)
(ii) For each step if Δ' is the clean set obtained after one iteration on Δ, if Δ is UNSAT then Δ' is also UNSAT shown in ★ (contrapositive)
$\therefore$ resolve (φ^*, P) is complete.

Q3. (10 marks) A graph $G = (V, E)$ is said to be connected if, for arbitrary $a, b \in V$ with $a \neq b$, there is an $n \geqslant 2$, and some $a_1, \dots, a_n \in V$ with $a_1 = a$, $a_n = b$, and $(a_i, a_{i+1}) \in E$ for each $1 \leqslant i \leqslant n-1$. Is the class of connected graphs Δ-elementary? Prove your claim.

No we show by compactness theorem that it is not elementary }

Q4. (10 marks)

(a) (3 marks) Consider $\Sigma = (\emptyset, \{f/1\}, \emptyset)$. Write a sentence in FO_Σ whose every model is infinite.

X is set of sentence with arbitrary finite'

$\{\varphi_{\geq m} \cup X\} \mid m \geq 2$ $\varphi_{\geq n} = \exists x_1 [\exists x_2 \dots x_n [\bigwedge_{1 \leq i \leq j \leq n} \neg(x_i \equiv x_j)]]]$

(b) (2 marks) Show that < is elementary definable in the FO structure (ℕ, +, succ), where succ is the successor function.

Recursive

$\varphi(x, y) := succ(y, x) \lor succ(succ(y, x)) \lor \dots$ succ(succ(... forall n

(c) (2 marks) Show that the successor function succ is elementary definable in the FO structure (ℕ, +, <).

$\varphi(x, y) := x < y \land [\exists z [\forall w [z < w + w]] \land (y < (x+z)+z))]$

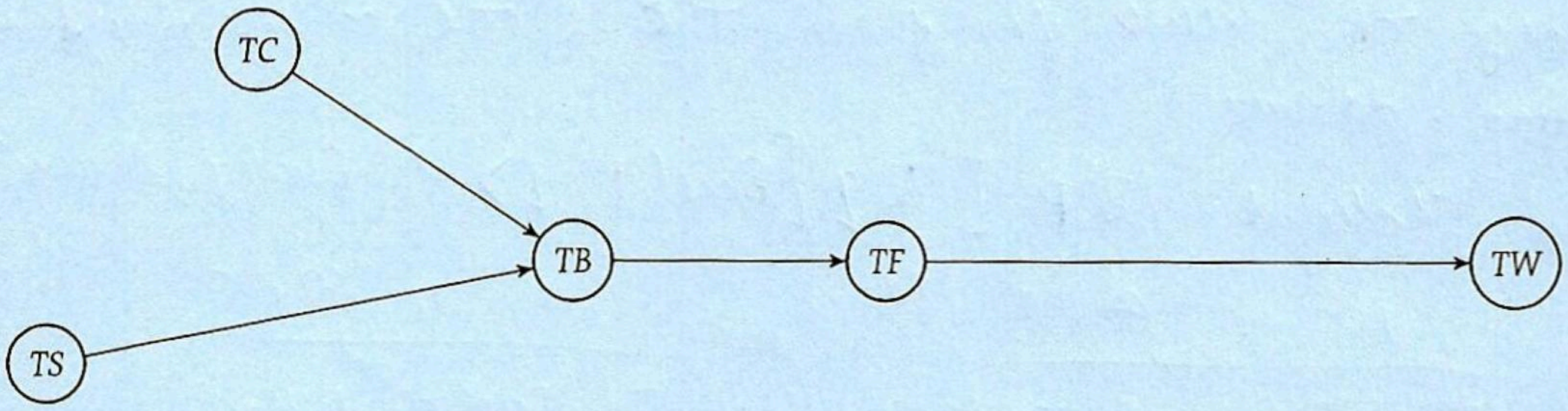
$\varphi(x, y) := x < y \land [\exists z [\forall y [z < w+w]] \land y < (x+z)+z)]$

(d) (3 marks) Show that successor succ defined as $succ(x) := x + 1$ is elementary definable in the FO structure (ℝ, +, ×)

$\varphi_1(x, y) := \exists z [\neg(z \equiv 0) \land \exists w [z \equiv w \times w] \land x + z \equiv y]$

$\Rightarrow \varphi(x, y) := \varphi_1(x, y) \land [\exists z [\forall y [\varphi_1(z, w+w) \land \varphi_1(y, (x+z)+z))]$

Q5. (15 marks) Each toll booth collects fees from drivers or from all earlier toll booths which collect fees from drivers. *TW*, *TF*, *TB*, *TC*, and *TS* are toll booths. They are arranged as follows, where *P* is earlier than *Q* iff there is a (directed) path from *P* to *Q*, for any *P*, *Q*.



TW does not collect fees from *TF* or from drivers. *TB* collects fees from *TC* but not from *TS*. *TC* and *TS* collect fees from drivers.

(a) (7 marks) Model the above situation in an appropriate logic. Specify all elements of your syntax precisely.

$C = \{ Drivers, TC, TB, TF, TW, TS \}$ F.O.L.

$P \cancel{\phi} = \{ Fees/2, E/2 \}$ $F = \phi$ $M = (C, P, F)$

Where $Fees(x, y)$ is true if x collects fees from y
$E/2(x; y)$ is true if there is an edge from x to y in the directed graph.

Propositions are as follows:

1. $\forall y \left[\forall x \left[Fees(x, Drivers) \wedge E(x, y) \supset Fees(y, x) \right] \vee Fees(y, Drivers) \right]$

Every toll both y collects from drivers or ....

2. $\neg Fees(TW, TF)$
3. $\neg Fees(TW, drivers)$
4. $Fees(TB, TC)$
5. $\neg Fees(TB, TS)$
6. $Fees(TC, drivers)$
7. $Fees(TS, drivers)$

Together 1, 2 ... 7 are Γ.

(b) (8 marks) Prove that there is a toll booth P that collects fees from a toll booth Q, where Q collects fees from drivers. Which toll booths are these?

Clearly TB collects fees from TC & TC collects fees from ~~drivers~~ . drivers.

Given Statement: $\exists p \, [\exists q \, [Fees(p,q) \wedge Fees(q, drivers)]]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Ax as } \oplus \in \Gamma}{\Gamma \vdash Fees(TB, TC)} \qquad \dfrac{\text{Ax as } \circledast \in \Gamma}{\Gamma \vdash Fees(TC, drivers)}}{\Gamma \vdash Fees(TB, TC) \wedge Fees(TC, drivers)} \wedge i}{\Gamma \vdash \exists q \, [Fees(TB, q) \wedge Fees(q, drivers)]} \exists i \; (TC/q)}{\Gamma \vdash \exists p \, [\exists q \, [Fees(p,q) \wedge Fees(q, drivers)]]} \exists i \; (TB/p)$$

Hence Proved.

Now I list out all such P (& respective Q)

~~Poss~~ (TB, TC)

(

[~~TF may collect only from drivers~~]

TB collects from drivers as TB dosnt ~~Since~~ collect from TS.

Since TW does not collect from drivers, it collects from ~~previous~~ earlier booths who collect from drivers.

TW doesn't collect from TF $\Rightarrow$ TF doesnt collect from drivers

$\therefore$ TF collects from TB ✓, TC ✓ & TS ✓

TW collects from TB ✓, TC ✓, TS ✓

all such toll booths are

(TB, TC) , (TF, TB) , (TF, TC) & (TF, TS)

as T(W) doesn't collect from drivers

Name: KUSHAGRA GUPTA Entry Number: 2021CS50592

Q6. (12 marks) Show that $\forall x.\ [P(x) \supset Q(x)] \vee \forall y.\ [Q(y) \supset P(y)] \vdash_{\mathcal{G}} \forall x.\ [\forall y.\ [(P(x) \wedge Q(y)) \supset (Q(x) \vee P(y))]]$
(Rotate the sheet and draw **a single** proof tree in landscape mode if it does not fit clearly vertically)

Here Π_2 is the proof tree for proving Π_1 (used twice)
Π_1 is main proof tree

replaced x with w/x for some term w

Π_2:-

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Ax}{\Gamma, \forall x[P(x) \supset Q(x)] \vdash \forall x[P(x) \supset Q(x)]}}{\Gamma, \forall x[P(x) \supset Q(x)] \vdash P(w/x) \supset Q(w/x)}\ \forall e \qquad \dfrac{\dfrac{Ax}{\Gamma, \forall x[\ ], P(w/x) \wedge Q(z/y) \vdash P(w/x) \wedge Q(z/y)}}{\Gamma, \forall x[P(x) \supset Q(x)], P(w/x) \wedge Q(z/y) \vdash P(w/x)}\ \wedge e_0}{\Gamma, \forall x[P(x) \supset Q(x)], P(w/x) \wedge Q(z/y) \vdash Q(w/x)}\ \supset e}{\Gamma, \forall x[P(x) \supset Q(x)], P(w/x) \wedge Q(z/y) \vdash Q(w/x) \vee P(z/y)}\ \vee i_0}{\Gamma, \forall x[P(x) \supset Q(x)] \vdash (P(w/x) \wedge Q(z/y)) \supset (Q(w/x) \vee P(z/y))}\ \supset i}{\Gamma, \forall x[P(x) \supset Q(x)] \vdash}\ \forall i (z\ \text{fresh})}{\Gamma, \forall x[P(x) \supset Q(x)] \vdash \forall x[\forall y[(P(x) \wedge Q(y)) \supset Q(x) \vee P(y))]]}\ \forall i (w\ \text{fresh})}{\Gamma \vdash \forall x[P(x) \supset Q(x)] \supset \forall x[\forall y[(P(x) \wedge Q(y)) \supset Q(x) \vee P(y))]]}\ \supset i$$

Note: Π_2 can also be written by replacing x with y & Q with P then it can be used exactly as it as for ②

Note② $\Gamma, - \vdash -$
here $-$ is $\forall x\ [P(x) \supset Q(x)]$

Here $-$ is $\forall y\ [Q(y) \supset P(y)]$

Π_1:

$$\dfrac{\dfrac{Ax}{\Gamma \vdash \forall x[P(x) \supset Q(x)] \vee \forall y[Q(y) \supset P(y)]} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma \vdash \forall x[P(x) \supset Q(x)] \supset \psi} \quad \dfrac{Ax}{\Gamma, - \vdash -}}{\Gamma, \forall x[P(x) \supset Q(x)] \vdash \psi}\ \supset e \qquad \dfrac{\dfrac{\Pi_2 ②}{\Gamma \vdash \forall y[Q(y) \supset P(y)]} \quad \dfrac{Ax}{\Gamma, - \vdash -}}{\Gamma, \forall y[Q(y) \supset P(y)] \vdash \psi}\ \supset e}{\Gamma \vdash_{G} \forall x\ [\forall y\ [(P(x) \wedge Q(y)) \supset (Q(x) \vee P(y))]] = \psi}\ \vee e$$

$\Gamma = \forall x\ [P(x) \supset Q(x)] \vee \forall y\ [Q(y) \supset P(y)]$
$\psi = \forall x\ [\forall y\ [(P(x) \wedge Q(y)) \supset (Q(x) \vee P(y)))]]$

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