MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet III-B

This Practice Sheet is based on the concepts: Duals, Annihilators, and Transpose.

1. Suppose V is finite-dimensional and U is a subspace of V. Show that:
   (a) $U^0 = \{0\} \iff U = V$;
   (b) $U^0 = V' \iff U = \{0\}$.

2. Suppose V and W are finite-dimensional vector spaces and $T \in L(V, W)$. Show that
$$T \text{ is injective } \iff T' \text{ is surjective.}$$

3. Explain why each linear functional is surjective or is the zero map.

4. Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Prove that there exists $\varphi \in V'$ such that
$$\varphi(u) = 0 \quad \text{for every } u \in U \quad \text{but} \quad \varphi \neq 0.$$

5. Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis of V'. Define
$$\Gamma : V \to \mathbb{F}^n, \quad \Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)),$$
$$\Lambda : \mathbb{F}^n \to V, \quad \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$
Explain why Γ and Λ are inverses of each other.

6. Suppose m is a positive integer. Show that the dual basis of the basis
$$1, x, \ldots, x^m \text{ of } \mathcal{P}_m(\mathbb{R})$$
is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where
$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$
Here, $p^{(k)}$ denotes the kth derivative of p, with the understanding that the 0th derivative of p is p itself.

7. Show that the dual map of the identity operator on V is the identity operator on V'.

8. Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map
$$\Gamma : V' \to \mathbb{F}^m, \quad \Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).$$
   (a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.
   (b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

9. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m \in V'$. Define a linear map
$$\Gamma : V \to \mathbb{F}^m, \quad \Gamma(v) = (\varphi_1(v), \ldots, \varphi_m(v)).$$
   (a) Prove that $\varphi_1, \ldots, \varphi_m$ spans V' if and only if Γ is injective.
   (b) Prove that $\varphi_1, \ldots, \varphi_m$ is linearly independent if and only if Γ is surjective.

10. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$.

11. Let f and g be linear functionals on a vector space V such that

$$f(v) = 0 \iff g(v) = 0 \quad \text{for every } v \in V.$$

Prove that $g = \alpha f$ for some scalar α.

12. Let $f : V \to \mathbb{F}$ be a linear functional and let $v \in V \setminus N(f)$. Show that

$$V = N(f) \oplus \{\alpha v : \alpha \in \mathbb{F}\}.$$

13. Let $t_1, \ldots, t_n \in \mathbb{R}$ be distinct. For any $p(t) \in \mathcal{P}_{n-1}(\mathbb{R})$, define

$$L_i(p) = p(t_i) \quad \text{for each } i \in \{1, \ldots, n\}.$$

Let

$$p_j(t) := \frac{(t - t_1) \cdots (t - t_{j-1})(t - t_{j+1}) \cdots (t - t_n)}{(t_j - t_1) \cdots (t_j - t_{j-1})(t_j - t_{j+1}) \cdots (t_j - t_n)}, \quad j \in \{1, \ldots, n\}.$$

Prove the following:

(a) $\{p_1, \ldots, p_n\}$ is a basis of $\mathcal{P}_{n-1}(\mathbb{R})$.

(b) $\{L_1, \ldots, L_n\}$ is a basis of the dual space of $\mathcal{P}_{n-1}(\mathbb{R})$.

(c) Given $a_1, \ldots, a_n \in \mathbb{R}$, there exists a unique polynomial $p(t) \in \mathcal{P}_{n-1}(\mathbb{R})$ such that

$$p(t_1) = a_1, \ldots, p(t_n) = a_n.$$

Note: The polynomials $p_j(t)$ are called the Lagrange polynomials. By doing this exercise, you have solved the interpolation problem, which asks for constructing a polynomial that takes prescribed values at prescribed points.

14. (a) Let U and W be subspaces of a finite-dimensional vector space V. Prove that

$$(U \cap W)^0 = U^0 + W^0 \quad \text{and} \quad (U + W)^0 = U^0 \cap W^0.$$

(b) Let U and W be subspaces of a finite-dimensional vector space V. Prove that if $V = U \oplus W$, then

$$U' \simeq W^0, \quad W' \simeq U^0, \quad \text{and} \quad V' = U^0 \oplus W^0.$$

(c) Show that the restriction of the natural isomorphism $T : V \to V''$, defined by $(Tv)(g) = g(v)$, to U is an isomorphism from U to U^{00}.

(d) If U and W are subspaces of a finite-dimensional vector space V, prove that

$$U = W \iff U^0 = W^0.$$

15. Let V be a finite dimensional real vector space and $T \in L(V)$. If W is a subspace of V that is invariant under T, is it true that W^0 is invariant under the adjoint operator T^t?

16. Read the following example carefully. In contrast to a finite dimensional vector space, what important aspect about the structure or properties of a general vector space and its duals does this example illuminate?

    Let V be a vector space over the field $\mathbb{Z}_2$ with a basis $\{e_k\}_{k\in\mathbb{N}}$, where e_k is an infinite binary sequence with a 1 at the k-th place and $0'$s elsewhere. Thus, V consists of all infinite binary sequences with only a finite number of 1's. Define the order $o(v)$ of any $v \in V$ to be the largest coordinate of v that is equal to 1. Consider the dual functionals $\{f_i\}_{i\in\mathbb{N}}$ in V' defined by $f_i(e_k) = \delta_{ik}$. For any $v \in V$, the evaluation functional $F_v \in V''$ satisfies
$$F_v(f_k) = f_k(v) = 0 \quad \forall\ k > o(v).$$
    Since the dual functionals f_i, $i \in \mathbb{N}$ are linearly independent, there exists $G \in V''$ such that $G(f_i) = 1$ for all $i \geq 1$. Clearly this $G \neq F_v$ for any $v \in V$.

17. A *hyperspace* in a vector space V is a maximal proper subspace of V. Prove the following:

    (a) If f is a nonzero linear functional on V, then $N(f)$ is a hyperspace in V, where $N(f) = \{v \in V : f(v) = 0\}$ is the null space of f.

    (b) Each hyperspace in V is the null space of some linear functional on V.

    (c) Such a linear functional in part (b) need not be unique.

18. (For those who know cardinality concept) For a finite dimensional vector space V it is known that $\dim V = \dim V'$ and consequently, $V \cong V'$. Now consider an infinite dimensional vector space V over $\mathbb{F}$. One of your friends has tried to prove that $\dim(V') > \dim(V)$ using the following approach. Can you assist in completing the proof by providing all the necessary details?

    Proof. (outline): Let B be a basis of V over $\mathbb{F}$. Let us denote the cardinality of a set A by $|A|$. We have
$$|V| = |B||\mathbb{F}| = \max\{|B|, |\mathbb{F}|\}\ldots\ldots(*)$$
    Next we claim that $\dim(V') = |V'|$. In view of $(*)$ applied to V', to establish this claim, it is enough to prove that $\dim(V') \geq |\mathbb{F}|$. Assume $\mathbb{F}$ is infinite. Let $E := \{e_i : i \in \mathbb{N}\} \subseteq B$ be countably infinite. For $a \in \mathbb{F}$, define $\phi_a : V \to \mathbb{F}$ by
$$\phi_a(e_i) = a^i \text{ for } i \in \mathbb{N} \quad\text{and}\quad \phi_a(b) = 0 \quad\text{for}\quad b \in B \setminus E.$$
    Note that $\{\phi_a : a \in \mathbb{F} \setminus \{0\}\}$ is linearly independent. Since V' contains a linearly independent set of size $|\mathbb{F}|$, it follows that $\dim(V') \geq |\mathbb{F}|$. We have $V' \cong \mathbb{F}^B$, since every linear functional is uniquely determined by its values on B. Hence,
$$\dim(V') = |\mathbb{F}|^{|B|} > |B| = \dim(V),$$
    which completes the proof. $\square$