Solutions and marking scheme for TA's

Models (10) Give to all
(1.2) (2)
(1.3) $\frac{44}{4} = 11$
(1.5) $\frac{20}{4} = 5$
(1.6) $\frac{4}{2} = 2$
(1.7) $2 \times 5 = 10$
(40)

MLL100: Introduction to Materials Science and Engineering

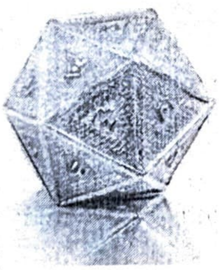
Experiment 1

Name | Entry Number

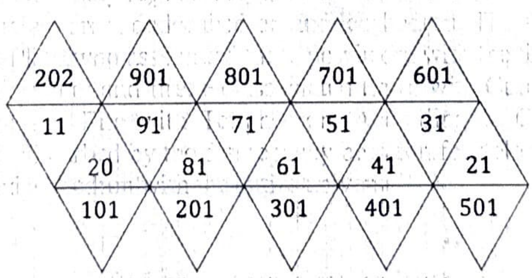
1.1 Introduction:

In the study of crystals, the geometry of polyhedra play an important role. A polyhedron is a three-dimensional solid with faces that are all polygons. Polygons are two-dimensional closed shapes with straight sides. A polygon is said to be regular if each face is made of sides that are of equal length and all angles are also equal. A polyhedron with congruent faces with the same number of faces meeting at each vertex is called a regular polyhedron or a Platonic solid.

Although there are infinitely many regular polygons in 2D we only have five regular polyhedra in 3D: tetrahedron, cube, octahedron, dodecahedron and icoshedron. The adjective Platonic comes from Greek philosopher Plato who associated these polyhedra with the four classical elements in his philosophy: earth, air, water and fire. He associated Earth with Cube, Air with Octahedron, Water with Icosahedron and Fire with Tetrahedron. According to Coxeter [*Introduction to Geometry*] Plato was not disturbed by the discrepancy between four elements and five solids, he simply associated the dodecahedron with the entire universe!



A well-crafted, small (handsize), hollow gold icosahedron with a number in Arabic on each face was found in the treasury of Tipu Sultan after he was overthrown by the British in 1799 at Seringapatam in southern India. The icosahedron can serve as a small container since five of its faces are formed together and hinged on one side to the rest of the body. This structure naturally gives a top (lid), middle, and bottom orientation to the icosahedron. The icosahedron remains a mystery till date as people struggle to understand the actual purpose and significance of the numbers inscribed on the faces of the box!



Numbers on Tipu's icosahedron.

Schläfli symbol: A Platonic solid is completely characterized if the number of edges p of the polygonal faces and the number of faces q meeting at each vertex is given. The pair of numbers $\{p, q\}$ is called the Schläfli symbol of the polyhedron. Thus, the symbol for cube is $\{4, 3\}$. Schläfli discovered regular polytopes, higher dimensional versions of regular polyhedra.

Archimedean Solid: A polyhedron with more than one type of regular polygons as faces with the same configuration around each vertex is called an Archimedean solid. We

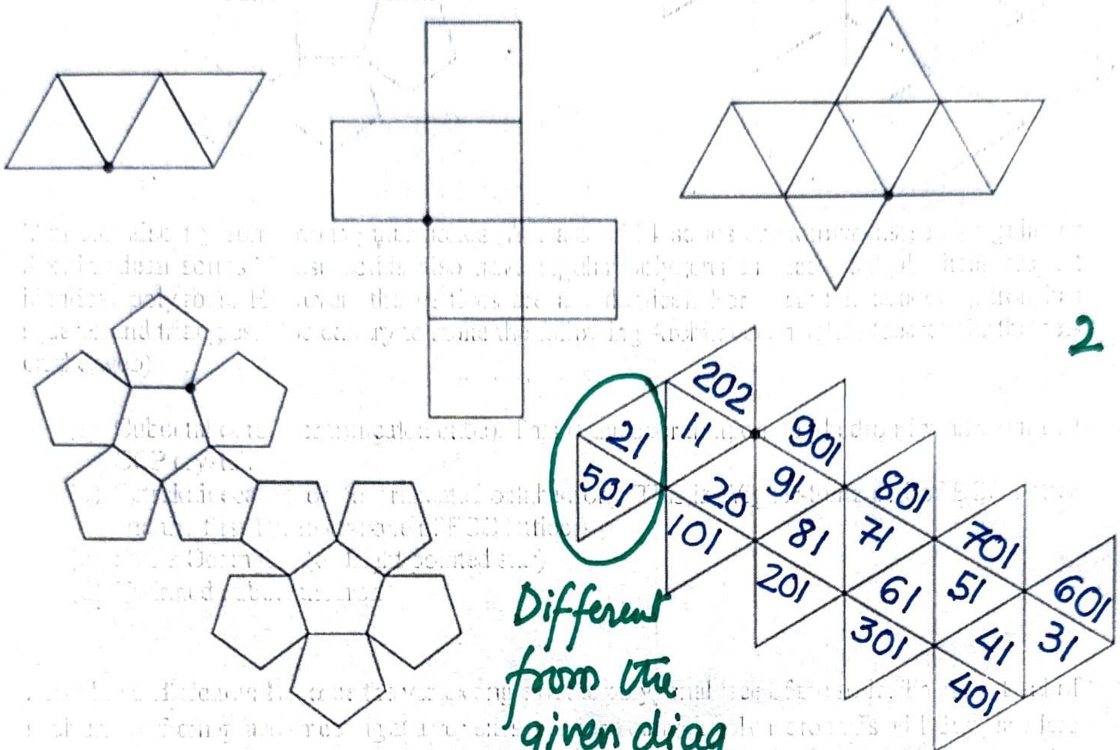
1.2 Construction of Paper Models

Construct paper models of the five Platonic solids. The nets to make these models are given. You

10 Marks for paper models

willfind that folding along edges becomes easier if you score the edges with a sharp point (eg. Point of a compass). In the figures below, label which net corresponds to which Platonic solid. For icosahedron, just for fun, you may like to add Tipu's number on the faces.

2 marks.



Different from the given diag.

You can also try some nonregular solids. A class of 11 solids are known as semi-regular or Archimedean solids. These solids also have regular polygons as faces, but all faces are not identical polygons. However, the vertices are all identical. For example, cuboctahedron has squares and triangles. You can try to make the following Archimedean solids (search for the nets on the web)

(a) Cuboctahedron (or truncated cube). This is the coordination polyhedron for an atom in a CCP crystal.
(b) Tetrakaidecahedron (or truncated octahedron) [This is Wigner-Seitz cell of BCC lattice and the first Brillouin zone of FCC lattice.}
(c) Stella Octangula (or Eight pointed star)
(d) Twinned cuboctahedron

Angular Deficiency: Let α be the vertex angle of the polygonal face of the solid. Then the total of such angles from q faces meeting at a vertex is $q\alpha$. For a convex polyhedron this will alwaysbe less than 360°. The angular deficiency θ is defined as $\theta = 360° - q\alpha$. The sum of angular deficiencies at all the vertices of the solid is the total angular deficiency of the solid.

1.3 Properties of the Platonic Solids

After you have made the models of Platonic solids you can fill the table II. The column for tetrahedron has been filled out for you as an example.

$\frac{44}{4} = 11$ Marks.

Table I: Properties of Platonic Solids

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Polygonal face	Equilateral Triangle	square	Equilateral △	Regular Pentagon	Equilateral △
Number of edges in polygon face, p	3	4	3	5	3
Number of faces meeting at a vertex, q	3	3	4	3	5
Schläfli Symbol $\{p,q\}$	{3,3}	{4, 3}	{3, 4}	{5, 3}	{3, 5}
No. of faces, F	4	6	8	12	20
No. of edges, E	6	12	12	30	30
No. of vertices, V	4	8	6	20	12
V-E+F (Euler's Polyhedron Formula)	2	2	2	2	2
Interior angle of polygon face, α	60°	90°	60°	108°	60°
Angular deficiency at a vertex $\theta = 360° - q\alpha$	180°	90°	120°	36°	60°
Total angular deficiency $V\theta$	720° (4π)	720°	720°	720°	720°

1.4 Some interesting results for polyhedra

You must have discovered these while filling Table I.

Euler's Polyhedron Formula: You will find that $V - E + F = 2$ to be true for all Platonic solids. This is known as Euler's polyhedron formula. It is frequently rates as one of the most beautiful results in all of mathematics (Google it, but outside class). It is true for irregular solids as well.

Descartes Formula: The sum of angular deficiencies for all vertices of a convex polyhedron is 720° (4π).

1.5 Symmetry of Platonic Solids:

The Platonic solids are highly symmetric objects. By symmetry we mean that the object, as a whole, should come into self-coincidence after some geometric operation such as rotation or reflection. A rotation axis of symmetry is characterized by its 'fold': an axis of rotational symmetry is called an n-fold axis of rotation if the minimum angle of rotation to bring the object into self-coincidence 360°. This can be expressed alternatively by stating that the object comes into self-coincidence n times in one complete rotation about an n-fold axis.

Observe the symmetry elements in each of the Platonic solids and fill the table below. The rotation axes and mirror planes for a tetrahedron are shown in the figure. Observe any relationship between symmetries of different solids.

$\frac{20}{4} = 5$ marks.

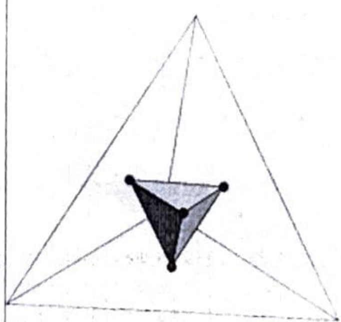
	No. of Rotational Symmetry Axes				No. of Mirror Planes
	2-fold	3-fold	4-fold	5-fold	
Tetrahedron	3	4	0	0	6
Octahedron	6	4	3	0	9
Cube	6	4	3	0	9
Dodecahedron	15	10	0	6	15
Icosahedron	15	10	0	6	15

Duals have the same symmetry

Two opposite edges define a mirror. $\therefore \frac{30}{2} = 15$.

1.6 Dual or reciprocal solid:

Mark the centres of the faces of all the Platonic solid with a marker/pen. Consider creating a solid by joining all the marked points, as shown in the image in the table for the tetrahedron. This new solid is called the reciprocal or dual of the given solid. Find the dual of all the five platonic solids

	Platonic solid	Dual
1	Tetrahedron	 Tetrahedron (Self-dual)
2	Cube	Octahedron
3	Octahedron	Cube
4	Dodecahedron	Icosahedron
5	Icosahedron	Dodecahedron

$\cancel{\frac{4}{4}}$ $\frac{4}{2} = 2$ marks

Actually must be less than 360°.

1.7 Questions:

Q1 Explain why there should be only 5 Platonic solids. (Hint: Note that at a vertex at least three faces should meet and the sum of the angles at a corner cannot be greater than 360°)?

With Δ faces $3\Delta \Rightarrow$ Tetrahedron

5 marks.

$4\Delta \Rightarrow$ Octahedron

$5\Delta \Rightarrow$ Icosahedron

$6\Delta \Rightarrow 360° \Rightarrow$ Not possible. Higher numbers also not possible.

With squares 3 squares $\Rightarrow$ Cube.

4 squares $\Rightarrow 360°$ Not possible.

With Pentagons, 3 Pentagons $\Rightarrow$ Dodecahedron

4 Pentagons $\Rightarrow > 360° \Rightarrow$ Not possible.

Q2 The Schläfli symbol $\{p, q\}$ completely defines a regular solid. Express the number of vertices (V), edges (E) and faces (F) of a regular polyhedron in terms of p and q.

$\because$ q-faces meet at each vertex $\therefore$ q edges meet at each vertex And since each edge is shared by two vertices we have total no. of edges $E = \frac{qV}{2}$ ①

① Mark

Again since q faces meet at a vertex and a face is shared by p vertices we have the number of faces as $F = \frac{qV}{p}$ ②

① Marks

By Euler's relation

$$V - E + F = 2$$

Substituting ① and ②

$$\Rightarrow V - \frac{qV}{2} + \frac{qV}{p} = 2 \Rightarrow V\left(1 - \frac{q}{2} + \frac{q}{p}\right) = 2$$

$$\Rightarrow \boxed{V = \frac{4p}{2p + 2q - pq}} \quad ④$$

Substituting ④ in ① and ② we get

$$\boxed{E = \frac{2pq}{2p + 2q - pq}} \quad \& \quad \boxed{F = \frac{4q}{2p + 2q - pq}}$$

(3×1) Marks