MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet III-A

This Practice Sheet is based on Linear Transformations and related elementary results.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(1,0) = (1,4) \quad \text{and} \quad T(1,1) = (2,5).$$

   Find $T(2,3)$.

2. Properties preserved by linear transformations are called *linear properties.* Let $T : V \to W$ be a linear transformation. Show that the following are true:

   (a) **Subspaces:** If V_0 is a subspace of V, then

$$T(V_0) := \{Tx : x \in V_0\}$$

   is a subspace of W. That is, *being a subspace* is a linear property.

   (b) **Line segments and triangles:** The line segment joining u and v in a vector space V is the set

$$S := \{(1-\lambda)u + \lambda v : 0 \le \lambda \le 1\}.$$

   If $T : V \to W$ is linear, then

$$T(S) = \{(1-\lambda)T(u) + \lambda T(v) : 0 \le \lambda \le 1\},$$

   which is a line segment joining $T(u)$ and $T(v)$ in W.
   Hence, *being a triangle in the plane* is a linear property. This means that a triangle remains a triangle under any linear transformation on $\mathbb{R}^2$, including degenerate cases where it becomes a single point or a line segment.

   (c) **Circles:** Being a circle is *not* a linear property. Linear transformations in general map circles not necessarily to circles.

3. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis of a vector space V over a field $\mathbb{F}$. For each $v \in V$, there exists a unique n-tuple $(a_1, \ldots, a_n) \in \mathbb{F}^n$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

   Define a function

$$T : V \to \mathbb{F}^n \quad \text{by} \quad T(v) = [v]_{\mathcal{B}} := (a_1, \ldots, a_n),$$

   where $[v]_{\mathcal{B}} := a_1, \ldots, a_n)$ is called the coordinate vector of v with respect to the basis $\{v_1, \ldots, v_n\}$.

   Show that the function T defined this way encoding each vector $v \in V$ to its coordinate vector $[v]_{\mathcal{B}} \in \mathbb{F}^n$ is an *isomorphism.*

4. Let V be a one-dimensional vector space over a field $\mathbb{F}$, and let $T \in L(V)$ be a linear map. Show that there exists a scalar $\lambda \in \mathbb{F}$ such that

$$T(v) = \lambda v \quad \text{for all } v \in V.$$

   In particular, any linear transformation $T : \mathbb{R} \to \mathbb{R}$ is of the form $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.

5. Give an example to show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map. That is,

   (a) Give an example of a function
   $$\varphi : \mathbb{R}^2 \to \mathbb{R}$$
   such that
   $$\varphi(av) = a\,\varphi(v) \quad \text{for all } a \in \mathbb{R} \text{ and all } v \in \mathbb{R}^2,$$
   but φ is *not* linear.

   (b) Give an example of a function
   $$\varphi : \mathbb{C} \to \mathbb{C}$$
   such that
   $$\varphi(w+z) = \varphi(w) + \varphi(z) \quad \text{for all } w, z \in \mathbb{C},$$
   but φ is *not* linear.

6. Suppose V and W are finite-dimensional with $2 \le \dim V \le \dim W$. Show that
$$\{T \in L(V,W) \colon T \text{ is not injective}\}$$
is not a subspace of $L(V,W)$.

7. Suppose V and W are finite-dimensional with $\dim V \ge \dim W \ge 2$. Show that
$$\{T \in L(V,W) \colon T \text{ is not surjective}\}$$
is not a subspace of $L(V,W)$.

8. Suppose $T \in L(V,W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that
$$Tv_1, \ldots, Tv_n$$
is linearly independent in W.

9. Suppose $v_1, \ldots, v_n$ spans V and $T \in L(V,W)$. Show that
$$Tv_1, \ldots, Tv_n$$
spans range(T).

10. Let V be a vector space over a field $\mathbb{F}$ and let $\varphi \in L(V, \mathbb{F})$ be a nonzero linear functional. Suppose $u \in V$ is not in $\ker\varphi$. Then show that
$$V = \ker\varphi \oplus \{au : a \in \mathbb{F}\}.$$

11. Let $T : V \to W$ be a linear map and let $\{v_1, \ldots, v_n\}$ be a basis of V. If $\{Tv_1, \ldots, Tv_n\}$ is a basis of W, does it follow that T is an isomorphism? Justify.

12. Let $C^2(\mathbb{R})$ be the vector space of all functions defined on the real line $\mathbb{R}$ which have continuous second derivatives at each point of $\mathbb{R}$ and let $C(\mathbb{R})$ be the vector space of continuous functions on $\mathbb{R}$. Define the function
$$T : C^2(\mathbb{R}) \to C(\mathbb{R}), \qquad (Tf)(t) = f''(t) + f(t), \quad t \in \mathbb{R}.$$
(Notice that T is a linear map.) Assume that the kernel of T is two-dimensional. Then
$$\ker T = \operatorname{span}\{g, h\}$$
where $g(t) =$ __________ and $h(t) =$ __________ for all t. Thus the kernel of the linear map T is the solution space of the differential equation __________.