

MTL104 Linear Algebra and Applications
I Semester 2025-26
Practice Sheet III-A: Hints and Solutions

1. We are given a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

$$T(1, 0) = (1, 4), \quad T(1, 1) = (2, 5).$$

Step 1: Compute $T(0, 1)$. Since T is linear,

$$T(1, 1) = T((1, 0) + (0, 1)) = T(1, 0) + T(0, 1).$$

Substituting the known values,

$$(2, 5) = (1, 4) + T(0, 1).$$

Hence,

$$T(0, 1) = (2, 5) - (1, 4) = (1, 1).$$

Step 2: General formula for $T(x, y)$. Any $(x, y) \in \mathbb{R}^2$ can be written as

$$(x, y) = x(1, 0) + y(0, 1).$$

By linearity,

$$T(x, y) = xT(1, 0) + yT(0, 1).$$

Substitute the values of $T(1, 0)$ and $T(0, 1)$:

$$T(x, y) = x(1, 4) + y(1, 1) = (x + y, 4x + y).$$

Step 3: Evaluate at $(2, 3)$.

$$T(2, 3) = (2 + 3, 4 \cdot 2 + 3) = (5, 11).$$

2. Let $T : V \rightarrow W$ be a linear transformation between vector spaces over the same field. We show the three statements requested.

(1) Subspaces. If V_0 is a subspace of V , then the image

$$T(V_0) := \{Tx : x \in V_0\}$$

is a subspace of W .

Proof. To prove $T(V_0)$ is a subspace of W we check the three standard properties:

- i. $T(V_0) \neq \emptyset$. Since V_0 is a subspace it contains 0_V . Linearity gives

$$T(0_V) = 0_W,$$

so $0_W \in T(V_0)$.

- ii. Closed under addition. Let $y_1, y_2 \in T(V_0)$. Then there exist $x_1, x_2 \in V_0$ with $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Because V_0 is a subspace, $x_1 + x_2 \in V_0$. Linearity of T gives

$$y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2) \in T(V_0).$$

- iii. Closed under scalar multiplication. Let $y \in T(V_0)$ and α be any scalar. There exists $x \in V_0$ with $y = T(x)$. Since V_0 is a subspace, $\alpha x \in V_0$, and by linearity

$$\alpha y = \alpha T(x) = T(\alpha x) \in T(V_0).$$

Having verified these three properties, $T(V_0)$ is a subspace of W . \square

(2) Line segments and triangles.

First we prove the statement about line segments; the triangle result follows immediately.

Let $u, v \in V$ and let

$$S := \{(1 - \lambda)u + \lambda v : 0 \leq \lambda \leq 1\}$$

be the line segment joining u and v in V . If $T : V \rightarrow W$ is linear then

$$T(S) = \{(1 - \lambda)T(u) + \lambda T(v) : 0 \leq \lambda \leq 1\},$$

the line segment joining $T(u)$ and $T(v)$ in W .

Proof. Let $w \in T(S)$. Then $w = T(x)$ for some $x \in S$. By definition of S there exists $\lambda \in [0, 1]$ with $x = (1 - \lambda)u + \lambda v$. By linearity of T ,

$$w = T(x) = T((1 - \lambda)u + \lambda v) = (1 - \lambda)T(u) + \lambda T(v),$$

so w belongs to the right-hand set. This shows $T(S)$ is contained in $\{(1 - \lambda)T(u) + \lambda T(v) : 0 \leq \lambda \leq 1\}$.

Conversely, let $y = (1 - \lambda)T(u) + \lambda T(v)$ for some $\lambda \in [0, 1]$. Define $x := (1 - \lambda)u + \lambda v \in S$. By linearity,

$$T(x) = (1 - \lambda)T(u) + \lambda T(v) = y,$$

hence $y \in T(S)$. This proves equality of the two sets. Since the right-hand set is precisely the segment joining $T(u)$ and $T(v)$, the image $T(S)$ is that segment. \square

Triangles. A (filled) triangle with vertices $a, b, c \in V$ is the set of all convex combinations

$$\{ \alpha a + \beta b + \gamma c : \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1 \},$$

i.e. the convex hull of $\{a, b, c\}$. The proof for segments generalizes: linear maps preserve affine (and hence convex) combinations, because for scalars α, β, γ with $\alpha + \beta + \gamma = 1$,

$$T(\alpha a + \beta b + \gamma c) = \alpha T(a) + \beta T(b) + \gamma T(c).$$

Therefore the image of the triangle (the convex hull of $\{a, b, c\}$) is the convex hull of $\{T(a), T(b), T(c)\}$, which is a (possibly degenerate) triangle in W . In particular, any linear transformation on \mathbb{R}^2 maps triangles to triangles; degeneracy can occur when the three image points are collinear or coincide (yielding a segment or a point).

(3) Circles are not a linear property.

Being a circle is not preserved in general by linear transformations: there exist linear maps that take a circle to a non-circle (for example to an ellipse, a line segment, or a single point).

Proof. It suffices to give concrete counterexamples.

(a) Non-circular image (ellipse). Consider $V = W = \mathbb{R}^2$ and the linear map

$$T(x, y) = (2x, y).$$

The unit circle $\mathcal{C} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is mapped to

$$T(\mathcal{C}) = \{(2x, y) : (x, y) \in \mathcal{C}\} = \{(u, v) : (u/2)^2 + v^2 = 1\},$$

which is the ellipse $\frac{u^2}{4} + v^2 = 1$. An ellipse that is not a circle shows that a linear map need not send a circle to a circle.

(b) Collapsed image (line segment). Consider the orthogonal projection onto the x -axis,

$$P(x, y) = (x, 0).$$

The same unit circle \mathcal{C} is sent to

$$P(\mathcal{C}) = \{(x, 0) : (x, y) \in \mathcal{C}\} = \{(x, 0) : -1 \leq x \leq 1\},$$

a line segment on the x -axis, which is certainly not a circle.

(c) Collapsed to a point. The zero map $Z(x, y) = (0, 0)$ sends every circle to the single point $(0, 0)$.

These examples demonstrate that circles are not preserved under arbitrary linear transformations; hence *being a circle* is not a linear property. \square

Remark 0.1. *The properties preserved by linear transformations include linear- and affine-structure related notions: subspaces, linear independence relations (up to collapse by non-injective maps), affine lines, line segments, convex combinations, parallelograms, and more. Geometric properties depending on distances or angles (e.g. being a circle, or preserving lengths and angles) require additional conditions such as orthogonality or isometries.*

3. Let V be an n -dimensional vector space over the field \mathbb{F} and let

$$\mathcal{B} = \{v_1, \dots, v_n\}$$

be an ordered basis of V . For each $v \in V$ there exist unique scalars $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1v_1 + \dots + a_nv_n.$$

Define

$$T : V \rightarrow \mathbb{F}^n, \quad T(v) = [v]_{\mathcal{B}} := (a_1, \dots, a_n).$$

We prove that T is a vector space isomorphism by showing it is linear and bijective.

The map T is linear.

Proof. Let $u, v \in V$ and $\alpha \in \mathbb{F}$. Write

$$u = b_1v_1 + \cdots + b_nv_n, \quad v = a_1v_1 + \cdots + a_nv_n.$$

Then

$$u + v = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n,$$

so

$$T(u + v) = (a_1 + b_1, \dots, a_n + b_n) = (a_1, \dots, a_n) + (b_1, \dots, b_n) = T(u) + T(v).$$

Also

$$\alpha v = (\alpha a_1)v_1 + \cdots + (\alpha a_n)v_n,$$

so

$$T(\alpha v) = (\alpha a_1, \dots, \alpha a_n) = \alpha(a_1, \dots, a_n) = \alpha T(v).$$

Hence T is linear. \square

The map T is bijective (and therefore an isomorphism).

Proof. Injectivity. Suppose $T(v) = \mathbf{0} \in \mathbb{F}^n$. If $v = a_1v_1 + \cdots + a_nv_n$ then

$$T(v) = (a_1, \dots, a_n) = \mathbf{0} \implies a_1 = \cdots = a_n = 0,$$

hence $v = 0$. Thus $\ker T = \{0\}$, so T is injective.

Surjectivity. Let $(c_1, \dots, c_n) \in \mathbb{F}^n$ be arbitrary. Set

$$v = c_1v_1 + \cdots + c_nv_n \in V.$$

By definition $T(v) = (c_1, \dots, c_n)$. Hence every element of \mathbb{F}^n has a preimage in V , so T is surjective.

Since T is linear and bijective, it is a linear isomorphism between V and \mathbb{F}^n . \square

Alternatively, one may exhibit the inverse explicitly: define

$$S : \mathbb{F}^n \rightarrow V, \quad S((c_1, \dots, c_n)) := c_1v_1 + \cdots + c_nv_n.$$

Then S is linear and $S \circ T = \text{id}_V$, $T \circ S = \text{id}_{\mathbb{F}^n}$, so $S = T^{-1}$ and T is an isomorphism. \square

4. Let V be a one-dimensional vector space over a field \mathbb{F} , and let $T \in L(V)$ be a linear map. Show that there exists a scalar $\lambda \in \mathbb{F}$ such that

$$T(v) = \lambda v \quad \text{for all } v \in V.$$

Proof. Since $\dim V = 1$, there exists a basis consisting of a single nonzero vector, say $\{v_0\}$. Every $v \in V$ can be uniquely written as

$$v = av_0, \quad \text{for some } a \in \mathbb{F}.$$

Now consider $T(v_0) \in V$. Since V is one-dimensional, $T(v_0)$ must be a scalar multiple of v_0 . That is, there exists $\lambda \in \mathbb{F}$ such that

$$T(v_0) = \lambda v_0.$$

Now for an arbitrary vector $v = av_0 \in V$, we compute:

$$T(v) = T(av_0) = aT(v_0) \quad (\text{by linearity of } T).$$

Substituting $T(v_0) = \lambda v_0$, we get

$$T(v) = a(\lambda v_0) = \lambda(av_0) = \lambda v.$$

Thus, T acts as scalar multiplication by λ on all of V .

Special case: $V = \mathbb{R}$. Here V is the one-dimensional vector space \mathbb{R} over itself. Take the standard basis $\{1\}$. Then

$$T(1) = \lambda \cdot 1 = \lambda$$

for some $\lambda \in \mathbb{R}$. For any $x \in \mathbb{R}$,

$$T(x) = T(x \cdot 1) = xT(1) = x\lambda = \lambda x.$$

Hence any linear map $T : \mathbb{R} \rightarrow \mathbb{R}$ has the form $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$. \square

5. Neither homogeneity nor additivity alone implies linearity.

(a) **Homogeneous but not additive.**

Define

$$\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \varphi(x, y) = (x^3 + y^3)^{1/3}.$$

Check homogeneity: For any scalar $\alpha \in \mathbb{R}$,

$$\varphi(\alpha x, \alpha y) = ((\alpha x)^3 + (\alpha y)^3)^{1/3} = (\alpha^3(x^3 + y^3))^{1/3} = \alpha(x^3 + y^3)^{1/3} = \alpha \varphi(x, y),$$

so φ is homogeneous of degree 1.

Check additivity: Consider $(1, 0)$ and $(0, 1)$:

$$\varphi((1, 0) + (0, 1)) = \varphi(1, 1) = (1^3 + 1^3)^{1/3} = 2^{1/3} \approx 1.26,$$

but

$$\varphi(1, 0) + \varphi(0, 1) = 1 + 1 = 2.$$

Since $2^{1/3} \neq 2$, φ is not additive.

Hence φ is homogeneous but not additive.

(b) **Additive but not homogeneous.**

Define

$$\varphi : \mathbb{C} \rightarrow \mathbb{C}, \quad \varphi(z) = \bar{z}.$$

Check additivity: For all $w, z \in \mathbb{C}$,

$$\varphi(w + z) = \overline{w + z} = \overline{w} + \overline{z} = \varphi(w) + \varphi(z).$$

Failure of homogeneity (over \mathbb{C}): If $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}$, then

$$\varphi(\alpha z) = \overline{\alpha z} = \overline{\alpha} \overline{z}.$$

But

$$\alpha \varphi(z) = \alpha \bar{z}.$$

These are equal for all α only when $\alpha \in \mathbb{R}$. In general, if α is not real,

$$\overline{\alpha} \overline{z} \neq \alpha \bar{z}.$$

For instance, with $\alpha = i$ and $z = 1$,

$$\varphi(i \cdot 1) = \overline{i} = -i \neq i = \alpha \varphi(1).$$

Thus φ is additive but not linear as a transformation over \mathbb{C} .

6. Let $n = \dim V$. Choose a basis $\{v_1, \dots, v_n\}$ of V and linearly independent vectors $\{w_1, \dots, w_n\}$ in W (possible since $n \leq \dim W$). Define linear maps $T_1, T_2 \in L(V, W)$ by

$$\begin{aligned} T_1(v_1) &= 0, & T_1(v_i) &= w_i \quad (i = 2, \dots, n), \\ T_2(v_1) &= w_1, & T_2(v_i) &= 0 \quad (i = 2, \dots, n). \end{aligned}$$

Each of T_1 and T_2 is not injective: indeed, $T_1(v_1) = 0$ and $\ker T_2$ contains $\text{span}\{v_2, \dots, v_n\}$. However, for their sum we have

$$(T_1 + T_2)(v_i) = w_i \quad \text{for all } i = 1, \dots, n.$$

Thus $T_1 + T_2$ maps the basis $\{v_i\}$ to the linearly independent set $\{w_i\}$, so $T_1 + T_2$ is injective.

Therefore, the set

$$\{T \in L(V, W) : T \text{ is not injective}\}$$

is not closed under addition, and hence is not a subspace of $L(V, W)$.

7. Let $n = \dim V$ and $m = \dim W$; by hypothesis $n \geq m \geq 2$. Choose a basis $\{v_1, \dots, v_n\}$ of V and a basis $\{w_1, \dots, w_m\}$ of W . Define linear maps $T_1, T_2 \in L(V, W)$ by their action on the basis of V :

$$\begin{aligned} T_1(v_1) &= w_1, & T_1(v_i) &= 0 \quad (i = 2, \dots, n), \\ T_2(v_1) &= 0, & T_2(v_i) &= w_i \quad (i = 2, \dots, m), \\ T_2(v_i) &= 0 & & (i = m + 1, \dots, n). \end{aligned}$$

Each T_i is not surjective: $\text{im } T_1 = \text{span}\{w_1\}$ and $\text{im } T_2 = \text{span}\{w_2, \dots, w_m\}$, both proper subspaces of W (here we use $m \geq 2$). However, for the sum we have

$$(T_1 + T_2)(v_i) = w_i \quad \text{for } i = 1, \dots, m,$$

so $T_1 + T_2$ maps $\{v_1, \dots, v_m\}$ onto the basis $\{w_1, \dots, w_m\}$ of W , hence $T_1 + T_2$ is surjective.

Thus two non-surjective maps can sum to a surjective map, so the set

$$\{T \in L(V, W) : T \text{ is not surjective}\}$$

is not closed under addition and therefore is not a subspace of $L(V, W)$.

8. Suppose $T \in L(V, W)$ is injective and v_1, \dots, v_n is linearly independent in V . We want to show that

$$Tv_1, \dots, Tv_n$$

is linearly independent in W .

Assume there exist scalars a_1, \dots, a_n such that

$$a_1Tv_1 + \dots + a_nTv_n = 0.$$

By linearity of T , this can be written as

$$T(a_1v_1 + \dots + a_nv_n) = 0.$$

Since T is injective, $\ker(T) = \{0\}$, so we must have

$$a_1v_1 + \dots + a_nv_n = 0.$$

But v_1, \dots, v_n are linearly independent, so $a_1 = \dots = a_n = 0$. Hence, Tv_1, \dots, Tv_n are linearly independent.

9. Suppose v_1, \dots, v_n spans V and $T \in L(V, W)$. We want to show that

$$Tv_1, \dots, Tv_n$$

spans $\text{range}(T)$.

Let $w \in \text{range}(T)$. By definition, there exists $v \in V$ such that $T(v) = w$. Since v_1, \dots, v_n spans V , we can write

$$v = a_1v_1 + \dots + a_nv_n$$

for some scalars a_1, \dots, a_n . By linearity of T , we have

$$w = T(v) = a_1Tv_1 + \dots + a_nTv_n.$$

Thus every element of $\text{range}(T)$ is a linear combination of Tv_1, \dots, Tv_n , showing that they span $\text{range}(T)$.

10. Let V be a vector space over a field \mathbb{F} and let $\varphi \in L(V, \mathbb{F})$ be a nonzero linear functional. Suppose $u \in V$ is not in $\ker \varphi$. We want to show that

$$V = \ker \varphi \oplus \{au : a \in \mathbb{F}\}.$$

Step 1: Show that $V = \ker \varphi + \{au : a \in \mathbb{F}\}$.

Let $v \in V$ be arbitrary. Since $\varphi(u) \neq 0$, define

$$a := \frac{\varphi(v)}{\varphi(u)} \in \mathbb{F}.$$

Then consider

$$v - au.$$

By linearity of φ , we have

$$\varphi(v - au) = \varphi(v) - a\varphi(u) = \varphi(v) - \frac{\varphi(v)}{\varphi(u)}\varphi(u) = 0.$$

Thus $v - au \in \ker \varphi$, and we can write

$$v = (v - au) + au \in \ker \varphi + \{au : a \in \mathbb{F}\}.$$

Hence every $v \in V$ can be expressed as a sum of an element of $\ker \varphi$ and a scalar multiple of u .

Step 2: Show that the sum is direct.

Suppose

$$v \in \ker \varphi \cap \{au : a \in \mathbb{F}\}.$$

Then $v = au$ for some $a \in \mathbb{F}$ and $\varphi(v) = 0$ (since $v \in \ker \varphi$). Hence

$$0 = \varphi(v) = \varphi(au) = a\varphi(u).$$

Since $\varphi(u) \neq 0$, it follows that $a = 0$, so $v = 0$.

Thus $\ker \varphi \cap \{au : a \in \mathbb{F}\} = \{0\}$, so the sum is direct.

Combining Steps 1 and 2, we conclude that

$$V = \ker \varphi \oplus \{au : a \in \mathbb{F}\}.$$

11. Let $T : V \rightarrow W$ be a linear map and let $\{v_1, \dots, v_n\}$ be a basis of V . If $\{Tv_1, \dots, Tv_n\}$ is a basis of W , then T is an isomorphism.

Injectivity: Suppose

$$T\left(\sum_{i=1}^n a_i v_i\right) = 0.$$

By linearity of T , we have

$$\sum_{i=1}^n a_i T v_i = 0.$$

Since $\{Tv_1, \dots, Tv_n\}$ is linearly independent, it follows that $a_1 = \dots = a_n = 0$. Hence T is injective.

Surjectivity: Since $\{Tv_1, \dots, Tv_n\}$ spans W , for any $w \in W$ there exist scalars a_1, \dots, a_n such that

$$w = \sum_{i=1}^n a_i T v_i = T\left(\sum_{i=1}^n a_i v_i\right).$$

Thus T is surjective.

Being both injective and surjective, T is a linear isomorphism.

12. Let $T : C^2(\mathbb{R}) \rightarrow C(\mathbb{R})$ be defined by $(Tf)(t) = f''(t) + f(t)$. Given that $\ker T$ is two-dimensional.

Since T is linear, $\ker T$ is a subspace of $C^2(\mathbb{R})$. Because $\dim \ker T = 2$, there exist two linearly independent functions $g, h \in \ker T$ such that

$$\ker T = \text{span}\{g, h\}.$$

By definition of the kernel, every element of $\ker T$ satisfies

$$Tf = f'' + f = 0.$$

Thus the kernel of T is the solution space of the differential equation

$$f'' + f = 0.$$

It can be easily seen that $\sin t$ and $\cos t$ are solutions of the above differential equation, and also linearly independent. Hence one can choose $g(t) = \sin t$ and $h(t) = \cos t$.