MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet II: Hints and Solutions

1. (a) Since $S = \{v_1, v_2, v_3, v_4\}$ spans V, every vector in V is a linear combination of vectors from S. Therefore, to prove that

$$S' = \{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4\}$$

also spans V, it suffices to show that each vector in S can be expressed as a linear combination of the vectors in S' (why/how?). Indeed,

$$\begin{aligned}
v_4 &= v_4, \\
v_3 &= (v_3 - v_4) + v_4, \\
v_2 &= (v_2 - v_3) + v_3 = (v_2 - v_3) + (v_3 - v_4) + v_4, \\
v_1 &= (v_1 - v_2) + v_2 \\
&= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4.
\end{aligned}$$

Hence, each $v_i \in S$ is a linear combination of vectors in S', and thus S' spans V as well.

(b) Consider

$$a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4 = 0,$$

where $a, b, c, d \in \mathbb{R}$.
Expanding,

$$av_1 - av_2 + bv_2 - bv_3 + cv_3 - cv_4 + dv_4 = av_1 + (-a+b)v_2 + (-b+c)v_3 + (-c+d)v_4.$$

Since $\{v_1, v_2, v_3, v_4\}$ is linearly independent, each coefficient must vanish:

$$a = 0, \quad -a + b = 0, \quad -b + c = 0, \quad -c + d = 0.$$

From $a = 0$, it follows that $b = 0$, then $c = 0$, and finally $d = 0$. Hence the only solution is $a = b = c = d = 0$.
Therefore, the set

$$\{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4\}$$

is linearly independent.

2. Using trigonometric identities, we expand f_2 and f_3:

$$\begin{aligned}
f_2(x) &= \cos\big(x + \tfrac{\pi}{6}\big) = \cos x \cos \tfrac{\pi}{6} - \sin x \sin \tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} \cos x - \tfrac{1}{2} \sin x, \\
f_3(x) &= \sin\big(x - \tfrac{\pi}{4}\big) = \sin x \cos \tfrac{\pi}{4} - \cos x \sin \tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2} \sin x - \tfrac{\sqrt{2}}{2} \cos x.
\end{aligned}$$

Now consider

$$\alpha f_1(x) - 2f_2(x) - \beta f_3(x).$$

Substituting the expressions:

$$\begin{aligned}
&\alpha \sin x - 2\Big(\tfrac{\sqrt{3}}{2} \cos x - \tfrac{1}{2} \sin x\Big) - \beta\Big(\tfrac{\sqrt{2}}{2} \sin x - \tfrac{\sqrt{2}}{2} \cos x\Big) \\
&= \big(\alpha + 1 - \tfrac{\beta\sqrt{2}}{2}\big) \sin x + \big(-\sqrt{3} + \tfrac{\beta\sqrt{2}}{2}\big) \cos x.
\end{aligned}$$

For this expression to vanish identically on $[0, 2\pi]$, the coefficients of $\sin x$ and $\cos x$ must both be zero (why/how?):

$$\begin{cases} \alpha + 1 - \dfrac{\beta\sqrt{2}}{2} = 0, \\ -\sqrt{3} + \dfrac{\beta\sqrt{2}}{2} = 0. \end{cases}$$

From the second equation we obtain

$$\frac{\beta\sqrt{2}}{2} = \sqrt{3} \quad \Longrightarrow \quad \beta = \sqrt{6}.$$

Substituting into the first equation gives

$$\alpha = \frac{\beta\sqrt{2}}{2} - 1 = \sqrt{3} - 1.$$

Thus,

$$(\sqrt{3} - 1) f_1(x) - 2 f_2(x) - \sqrt{6}\, f_3(x) \equiv 0, \quad 0 \le x \le 2\pi.$$

3. The functions $f(x) = x,\ g(x) = e^x,\ h(x) = e^{-x}$ on $[0, 1]$ are linearly independent in $C[0, 1]$. Suppose constants $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy

$$\alpha f(x) + \beta g(x) + \gamma h(x) \equiv 0 \qquad (0 \le x \le 1),$$

i.e.

$$\alpha x + \beta e^x + \gamma e^{-x} = 0 \qquad \text{for all } x \in [0, 1].$$

Evaluate at three distinct points $x = 0,\ x = \frac{1}{2},\ x = 1$. We obtain

$$\beta + \gamma = 0, \tag{1}$$
$$\tfrac{1}{2}\alpha + \beta e^{1/2} + \gamma e^{-1/2} = 0, \tag{2}$$
$$\alpha + \beta e + \gamma e^{-1} = 0. \tag{3}$$

From (1) we get $\gamma = -\beta$. Substitute into (2) and (3):

$$\tfrac{1}{2}\alpha + \beta\big(e^{1/2} - e^{-1/2}\big) = 0, \tag{2'}$$
$$\alpha + \beta\big(e - e^{-1}\big) = 0. \tag{3'}$$

From (2') we express α in terms of β:

$$\alpha = -2\beta\big(e^{1/2} - e^{-1/2}\big).$$

Substitute this into (3'):

$$-2\beta\big(e^{1/2} - e^{-1/1/2}\big) + \beta\big(e - e^{-1}\big) = 0,$$

so

$$\beta\Big(e - e^{-1} - 2\big(e^{1/2} - e^{-1/2}\big)\Big) = 0.$$

The above product forces $\beta = 0$.

With $\beta = 0$ we get $\gamma = -\beta = 0$, and then from (2') (or (3')) $\alpha = 0$.

Hence the only solution of the identity is $\alpha = \beta = \gamma = 0$. Therefore f, g, h are linearly independent in $C[0, 1]$.

Warning. To show that functions f_1, f_2, f_3 are linearly dependent, we must find constants α, β, γ (not all zero) such that

$$\alpha f_1(x) + \beta f_2(x) + \gamma f_3(x) \equiv 0 \quad \text{for all } x.$$

If we merely substitute three distinct points x_1, x_2, x_3, then we obtain a system of equations that is necessary but not sufficient. A nontrivial solution of this finite system only guarantees that the linear combination vanishes at those three points, not everywhere.

Example: a harmless but misleading first test, corrected by a second test. Let

$$f_1(x) = \sin x, \qquad f_2(x) = \cos x, \qquad f_3(x) = \sin 2x,$$

on the interval $[0, 2\pi]$. We illustrate the danger of testing at only one triple of points.

First triple (misleading). Consider

$$\alpha f_1(x) + \beta f_2(x) + \gamma f_3(x) \equiv 0 \quad \text{for all } x \in [0, 2\pi].$$

Test at

$$x_1 = 0, \qquad x_2 = \pi, \qquad x_3 = 2\pi.$$

We obtain

$$\beta = 0; \quad \alpha, \gamma \quad \text{arbitrary}.$$

But any such nontrivial vector $(\alpha, 0, \gamma)$ merely gives a linear combination that vanishes at the three chosen points; it does *not* give a relation that vanishes for all x. Thus the first test is inconclusive.

Second triple (decisive). Now test at

$$x_1 = \tfrac{\pi}{6}, \qquad x_2 = \tfrac{\pi}{4}, \qquad x_3 = \tfrac{\pi}{3}.$$

he only solution of

$$\alpha f_1(x_i) + \beta f_2(x_i) + \gamma f_3(x_i) = 0 \quad (i = 1, 2, 3)$$

is the trivial one, $\alpha = \beta = \gamma = 0$.

We can conclude the functions are linearly independent.

Conclusion.

- A nontrivial solution obtained from a single 3-point evaluation shows only that the combination vanishes at those points — it *does not* imply the functions are dependent everywhere.

Practical rule for students. If a 3-point substitution yields a nontrivial solution, do *not* stop: either (i) try another set of points and check, or (ii) use a structural test (Wronskian, or a direct algebraic argument) to decide dependence or independence.

Takeaway. Checking finitely many points can at best rule out independence, but it can never establish dependence unless one verifies the identity on the entire domain. For differentiable functions, the Wronskian provides a systematic and reliable tool: if

$$W(f_1, f_2, f_3)(x) \not\equiv 0,$$

then the functions are linearly independent.

Theorem. Let $f_1, \dots, f_n \in C^{n-1}(I)$ on an interval I. If $f_1, \dots, f_n$ are linearly dependent on I (that is, there exist constants $c_1, \dots, c_n$, not all zero, with $\sum_{i=1}^{n} c_i f_i(x) \equiv 0$ for all $x \in I$), then the Wronskian

$$W(f_1, \dots, f_n)(x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

vanishes identically on I.

Proof. By linear dependence there are constants $c_1, \dots, c_n$, not all zero, such that

$$\sum_{i=1}^{n} c_i f_i(x) = 0 \qquad \text{for all } x \in I.$$

Since each $f_i \in C^{n-1}(I)$, we may differentiate this identity k times for $k = 0, 1, \dots, n-1$. For each k we obtain

$$\sum_{i=1}^{n} c_i f_i^{(k)}(x) = 0 \qquad \text{for all } x \in I.$$

Fix any $x \in I$. Writing the n identities above (for $k = 0, 1, \dots, n-1$) in matrix form gives

$$\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The above system has nontrivial solution implies that the detwrminant of the coefficient matrix is zero. The choice of $x \in I$ was arbitrary, so the determinant vanishes for every $x \in I$. Therefore $W(f_1, \dots, f_n)(x) \equiv 0$ on I. $\square$

Remark. This proves the one-way implication: linear dependence $\Rightarrow$ Wronskian identically zero or equivalently, Wronskian is not identically zero $\Rightarrow$ Linear Independence. The converse (zero Wronskian $\Rightarrow$ dependence) is not true in full generality without extra hypotheses.

Counterexample (Wronskian zero but functions independent). Let

$$f_1(x) = x^2, \qquad f_2(x) = x|x| \qquad (x \in \mathbb{R}).$$

1. The functions are linearly independent. Assume there exist constants $a, b \in \mathbb{R}$ with

$$a f_1(x) + b f_2(x) \equiv 0 \qquad \text{for all } x \in \mathbb{R},$$

i.e.

$$ax^2 + bx|x| = 0 \qquad \text{for all } x.$$

Take $x > 0$. Then $x|x| = x^2$, so the identity becomes

$$(a+b)x^2 = 0 \quad \text{for all } x > 0,$$

hence $a + b = 0$. Take $x < 0$. Then $x|x| = -x^2$, so the identity becomes

$$(a-b)x^2 = 0 \quad \text{for all } x < 0,$$

hence $a - b = 0$. Solving $a + b = 0$ and $a - b = 0$ gives $a = b = 0$. Thus the only coefficients giving the identically zero combination are trivial, so f_1, f_2 are linearly independent on $\mathbb{R}$.

2. The Wronskian vanishes identically. Both f_1 and f_2 are differentiable on $\mathbb{R}$ with

$$f_1'(x) = 2x, \qquad f_2'(x) = 2|x| \quad (x \in \mathbb{R}).$$

Compute the Wronskian:

$$W(f_1, f_2)(x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = x^2 \cdot 2|x| - x|x| \cdot 2x = 2x^2|x| - 2x^2|x| = 0$$

for every $x \in \mathbb{R}$. Thus $W(f_1, f_2) \equiv 0$ while f_1, f_2 are independent.

4. **(a)** Suppose, for contradiction, that a linearly independent set S contains the zero vector 0. Then $1 \cdot 0 = 0$ is a nontrivial linear relation (the coefficient 1 is nonzero), so S would be linearly dependent. This contradiction shows $0 \notin S$ for any linearly independent set S.

   (b) Let D be a linearly dependent set and let T be any set with $D \subseteq T$. By definition of dependence there exist $v_1, \ldots, v_m \in D$ and scalars $\alpha_1, \ldots, \alpha_m$, not all zero, with

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0.$$

   Since each $v_i \in D \subseteq T$, the same nontrivial relation shows T is linearly dependent. Hence every superset of a dependent set is dependent.

   (c) Let B be a linearly independent set and let $S \subseteq B$. If S were linearly dependent, then there would exist $v_1, \ldots, v_m \in S$ and scalars $\alpha_1, \ldots, \alpha_m$, not all zero, with

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0.$$

   But these v_i also lie in B, so this would be a nontrivial linear relation among elements of B, contradicting the independence of B. Therefore S must be linearly independent.

5. Let E_1 and E_2 be linearly independent subsets of a vector space V with

$$\operatorname{span}(E_1) \cap \operatorname{span}(E_2) = \{0\}.$$

   We must show that $E_1 \cup E_2$ is linearly independent.

   Take a finite linear relation among elements of $E_1 \cup E_2$:

$$\sum_{i=1}^{m} \alpha_i e_i + \sum_{j=1}^{n} \beta_j f_j = 0,$$

where each $e_i \in E_1$, each $f_j \in E_2$, and $\alpha_i, \beta_j \in \mathbb{F}$ (the field of scalars). Rearranging,

$$\sum_{i=1}^{m} \alpha_i e_i \;=\; -\sum_{j=1}^{n} \beta_j f_j.$$

The left-hand side belongs to $\operatorname{span}(E_1)$ and the right-hand side belongs to $\operatorname{span}(E_2)$; hence their common value lies in $\operatorname{span}(E_1) \cap \operatorname{span}(E_2)$. By hypothesis this intersection is $\{0\}$. Therefore

$$\sum_{i=1}^{m} \alpha_i e_i = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \beta_j f_j = 0.$$

Since E_1 and E_2 are linearly independent, all coefficients α_i and all β_j must be zero. Thus the only linear relation among elements of $E_1 \cup E_2$ is the trivial one, so $E_1 \cup E_2$ is linearly independent.

6. (a) **Claim.** A vector space V is infinite dimensional if and only if there exists a sequence $\{v_1, v_2, v_3, \dots\}$ of vectors in V such that for every $m \in \mathbb{N}$ the set $\{v_1, \dots, v_m\}$ is linearly independent.

   Proof. ($\Rightarrow$) Suppose V is infinite dimensional. Construct inductively: choose $v_1 \neq 0$. Having chosen $v_1, \dots, v_m$, note that $\operatorname{span}\{v_1, \dots, v_m\} \neq V$, so we can pick $v_{m+1} \in V \setminus \operatorname{span}\{v_1, \dots, v_m\}$. Thus for each m the set $\{v_1, \dots, v_m\}$ is linearly independent.

   ($\Leftarrow$) Conversely, suppose such a sequence exists. If V were finite dimensional, say $\dim V = N$, then any $N+1$ vectors would be linearly dependent. But $\{v_1, \dots, v_{N+1}\}$ is independent, contradiction. Hence V is infinite dimensional. $\square$

   (b) Take the monomials $1, x, x^2, \dots$. For each m, the set $\{1, x, \dots, x^{m-1}\}$ is linearly independent since a polynomial identity

$$a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} \equiv 0$$

   implies $a_0 = \cdots = a_{m-1} = 0$. By part (1), $\mathcal{P}(F)$ is infinite dimensional.

   (c) **Function spaces.**

   - In $\mathbb{R}^{[a,b]}$, the functions t^n $(n = 0, 1, 2, \dots)$ are linearly independent.
   - In $C[a, b]$, the same monomials t^n are continuous and linearly independent.
   - In $C'[a, b]$, the monomials t^n are continuously differentiable and linearly independent.

   Thus each space is infinite dimensional.

   (d) Let $[a, b] \subset \mathbb{R}$ and define for each $\lambda \in [a, b]$ the function

$$u_\lambda(t) = e^{\lambda t}, \qquad t \in [a, b].$$

   The parameter set $[a, b]$ is uncountable. Moreover, the map

$$\lambda \mapsto u_\lambda$$

   is injective: if $u_\lambda = u_\mu$ as functions, then $e^{\lambda t} = e^{\mu t}$ for all $t \in [a, b]$. Taking $t \neq 0$ gives $e^{(\lambda-\mu)t} = 1$, which forces $\lambda = \mu$. Therefore, the set

$$\{\, u_\lambda : \lambda \in [a, b] \,\} \subset C[a, b]$$

   is uncountable.

Let $u_\lambda(t) = e^{\lambda t}$ for $\lambda \in [a,b]$. Take distinct parameters $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and suppose

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t} \equiv 0 \qquad (t \in [a,b]).$$

Divide by $e^{\lambda_1 t}$ to obtain

$$c_1 + \sum_{j=2}^{n} c_j e^{\mu_j t} \equiv 0, \qquad \mu_j := \lambda_j - \lambda_1 > 0.$$

Differentiate this identity k times and evaluate at $t = 0$ for $k = 1, 2, \ldots, n-1$. Since the derivative of the constant c_1 vanishes for $k \geq 1$, we get the homogeneous system

$$\sum_{j=2}^{n} c_j \mu_j^k = 0, \qquad k = 1, 2, \ldots, n-1.$$

Writing this in matrix form gives an $(n-1) \times (n-1)$ system

$$\begin{pmatrix} \mu_2 & \mu_3 & \cdots & \mu_n \\ \mu_2^2 & \mu_3^2 & \cdots & \mu_n^2 \\ \vdots & \vdots & & \vdots \\ \mu_2^{n-1} & \mu_3^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

This matrix equals $\operatorname{diag}(\mu_2, \ldots, \mu_n)$ times the usual Vandermonde matrix built from $\mu_2, \ldots, \mu_n$ (with powers $0, \ldots, n-2$). Hence its determinant is

$$\det = \Big(\prod_{j=2}^{n} \mu_j\Big) \prod_{2 \leq i < j \leq n} (\mu_j - \mu_i),$$

which is nonzero because the μ_j are distinct and nonzero. Therefore the matrix is invertible, so $c_2 = \cdots = c_n = 0$. Substituting back into the original relation yields $c_1 = 0$ as well. Thus any finite set $\{u_{\lambda_1}, \ldots, u_{\lambda_n}\}$ of distinct exponentials is linearly independent.

7. **Over $\mathbb{R}$.** Suppose $a, b \in \mathbb{R}$ and

$$a(1+i) + b(1-i) = 0 + i0.$$

Expanding gives

$$(a+b) + i(a-b) = 0 + i0.$$

Equating real and imaginary parts:

$$a + b = 0, \qquad a - b = 0.$$

Adding yields $2a = 0 \implies a = 0$, and then $b = 0$. Hence the only real solution is $a = b = 0$. Therefore, $\{1+i,\ 1-i\}$ is linearly independent over $\mathbb{R}$.

Over $\mathbb{C}$. We want to find complex scalars α, β, not both zero, such that

$$\alpha(1+i) + \beta(1-i) = 0.$$

Take $\alpha = i$ and $\beta = 1$. Then

$$i(1+i) + 1(1-i) = i + i^2 + 1 - i = i - 1 + 1 - i = 0.$$

Since $(\alpha, \beta) = (i, 1)$ is a nontrivial solution, the set $\{1+i,\ 1-i\}$ is linearly dependent over $\mathbb{C}$.

8. (a) $W_1 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_3 - x_4 = 0\}$.
The defining equation is $x_1 = x_3 + x_4$. Take free parameters

$$x_2 = s, \quad x_3 = t, \quad x_4 = u, \quad x_5 = v \qquad (s, t, u, v \in \mathbb{R}).$$

Then

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) =&(t + u,\ s,\ t,\ u,\ v)\\ =&t(1, 0, 1, 0, 0) + u(1, 0, 0, 1, 0) + s(0, 1, 0, 0, 0) + v(0, 0, 0, 0, 1).\end{aligned}$$

Hence

$$\mathcal{B}_1 = \{(1, 0, 1, 0, 0),\ (1, 0, 0, 1, 0),\ (0, 1, 0, 0, 0),\ (0, 0, 0, 0, 1)\},$$

spans W_1. Note that $\mathcal{B}_1$ is linearly independent as well. Consequently, $\dim(W_1) = 4$.

(b) $W_2 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4,\ x_1 + x_5 = 0\}$.
Set $x_2 = x_3 = x_4 = r$ and $x_1 = a,\ x_5 = -a$ with parameters $a, r \in \mathbb{R}$. Thus

$$(x_1, x_2, x_3, x_4, x_5) = (a, r, r, r, -a) = a(1, 0, 0, 0, -1) + r(0, 1, 1, 1, 0).$$

So

$$\mathcal{B}_2 = \{(1, 0, 0, 0, -1),\ (0, 1, 1, 1, 0)\},$$

spans W_2. Further $\mathcal{B}_2$ is linearly independent , and hence a basis. As a consequence, $\dim W_2 = 2$.

(c) $W_3 = \operatorname{span}\{v_1, \ldots, v_6\}$, where

$$\begin{aligned}v_1 &= (1, -1, 0, 2, 1),\\ v_2 &= (2, 1, -2, 0, 0),\\ v_3 &= (0, -3, 2, 4, 2),\\ v_4 &= (3, 3, -4, -2, -1),\\ v_5 &= (2, 4, 1, 0, 1),\\ v_6 &= (5, 7, -3, -2, 0).\end{aligned}$$

Form the 6×5 matrix A with these vectors as rows. Then the row space of A coincides with W_3. Therefore the question boils down to finding a basis and dimension for the row space of A. Find the row-reduce echelon form A_R corresponding to A. The non zero rows of A_R form a basis for the row space of A and hence for W_3.

9. Recall $\mathcal{P}_n(F)$ denotes the space of polynomials of degree $\leq n$, which has dimension $n + 1$. The set $\mathcal{S}$ has $n + 1$ elements, so it suffices to show linear independence.

Suppose
$$a_0(1 + t^n) + a_1(t + t^n) + \cdots + a_{n-1}(t^{n-1} + t^n) + a_n t^n \equiv 0$$
as the zero polynomial, where $a_0, \ldots, a_n \in F$. Collect coefficients of each power of t.

- Constant term: $a_0 = 0$. - For $1 \leq k \leq n - 1$, the coefficient of t^k is $a_k = 0$. - The coefficient of t^n is $a_0 + a_1 + \cdots + a_{n-1} + a_n$. Since $a_0 = \cdots = a_{n-1} = 0$, this gives $a_n = 0$.

Thus all $a_j = 0$; $\mathcal{S}$ is linearly independent. Because $|\mathcal{S}| = n + 1 = \dim \mathcal{P}_n(F)$, $\mathcal{S}$ is a basis.

The statement is false; give a counterexample.

Counterexample from part (1). By part (1) the set

$$\{\, 1+t^3,\ t+t^3,\ t^2+t^3,\ t^3 \,\}$$

is a basis of $\mathcal{P}_3(F)$. Each member has degree 3 (in particular, none has degree 2), so this is a basis of $\mathcal{P}_3(F)$ consisting entirely of polynomials whose degree is *not* 2. Hence the statement in (2) is false.

10. A general 2×2 real matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The trace-zero condition $a + d = 0$ gives $d = -a$. Hence every $A \in V$ has the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad a, b, c \in \mathbb{R}.$$

Therefore the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

spans V. These three matrices are easily seen to be linearly independent, so $\mathcal{B}$ is a basis of V.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \qquad \dim V = 3$$

11. (a) Parametrize U by free parameters s, t, u via

$$x_2 = s, \quad x_4 = t, \quad x_5 = u,$$

so

$$(x_1, x_2, x_3, x_4, x_5) = (3s, s, 7t, t, u) = s(3,1,0,0,0) + t(0,0,7,1,0) + u(0,0,0,0,1).$$

Hence

$$\mathcal{B}_U = \{(3,1,0,0,0),\ (0,0,7,1,0),\ (0,0,0,0,1)\},$$

spans U. In fact, it is a basis and $\dim U = 3$.

(b) The subspace U is cut out by the relations $x_1 = 3x_2$ and $x_3 = 7x_4$. To extend a basis of U to a basis of $\mathbb{R}^5$ we need two additional vectors that are *not* in U. Concretely, choose vectors that violate each relation separately so they cannot lie in U. Thus pick

$$e_1 = (1,0,0,0,0) \quad (\text{violates } x_1 = 3x_2), \qquad e_3 = (0,0,1,0,0) \quad (\text{violates } x_3 = 7x_4).$$

Because each breaks one defining relation of U, neither lies in U.

Now show the five vectors

$$(3,1,0,0,0),\ (0,0,7,1,0),\ (0,0,0,0,1),\ (1,0,0,0,0),\ (0,0,1,0,0)$$

are linearly independent. Suppose

$$\alpha(3,1,0,0,0) + \beta(0,0,7,1,0) + \gamma(0,0,0,0,1) + \delta(1,0,0,0,0) + \varepsilon(0,0,1,0,0) = 0.$$

Comparing coordinates gives

$$(3\alpha+\delta,\ \alpha,\ 7\beta+\varepsilon,\ \beta,\ \gamma)=(0,0,0,0,0).$$

Thus $\alpha=0,\ \beta=0,\ \gamma=0$, and then $\delta=0,\ \varepsilon=0$. So all coefficients vanish and the set is independent. As it has 5 vectors in $\mathbb{R}^5$, it is a basis. Therefore an extension is

$$\mathcal{B}=\mathcal{B}_U\cup\{(1,0,0,0,0),(0,0,1,0,0)\}.$$

(c) A complementary subspace is the span of the two added vectors:

$$W=\operatorname{span}\{(1,0,0,0,0),\ (0,0,1,0,0)\}.$$

Indeed $\dim W=2$, $\dim U+\dim W=3+2=5$, and $U\cap W=\{0\}$ (a vector in W has form $(a,0,b,0,0)$; if it belongs to U then $a=3\cdot 0=0$ and $b=7\cdot 0=0$). Hence $\mathbb{R}^5=U\oplus W$.

12. The only vector spaces that have exactly one basis are

$$\{0\}\quad\text{and}\quad\text{any 1-dimensional vector space over a field of two elements}$$

Proof.

(1) **Zero space.** The zero vector space $V=\{0\}$ over any field F has dimension 0. By definition the empty set $\varnothing$ is a basis (it is linearly independent and spans $\{0\}$), and no nonempty set can be a basis. Hence $\varnothing$ is the unique basis of $\{0\}$.

(2) **One-dimensional spaces.** Let V be 1-dimensional over a field F. A basis of V is any single nonzero vector. Hence the number of different bases equals the number of nonzero vectors in V, which is $|V|-1$. Since V is 1-dimensional over F, V and F are isomorphic. Hence, $|V|=|F|$, and the number of bases is $|F|-1$. Thus a 1-dimensional space has exactly one basis if and only if $|F|-1=1$, i.e. $|F|=2$.

(3) **All other cases fail.**

(i) If $\dim V\geq 1$ and $|F|\geq 3$: Let $\{v_1,\ldots,v_n\}$ be a basis ($n\geq 1$). Pick $c\in F$ with $c\neq 0,1$ (possible since $|F|\geq 3$). Then $\{cv_1,v_2,\ldots,v_n\}$ is another basis, different from the original because $cv_1\neq v_1$.

(ii) If $\dim V\geq 2$ and $F=\mathbb{F}_2$: Let $\{v_1,v_2,\ldots,v_n\}$ be a basis with $n\geq 2$. Over $\mathbb{F}_2$ the only nonzero scalar is 1, so the trick in (i) does not work. However, we can replace v_1 by v_1+v_2. The set

$$\{v_1+v_2,\ v_2,\ v_3,\ \ldots,v_n\}$$

is again a basis (why/how?). This basis is different from the original.

Combining the cases shows the only possibilities for exactly one basis are the two listed at the start. $\square$

13. It is easy to see that each of the listed sets is a subspace of $\mathbb{R}^3$. What is nontrivial is to show that these are the *only* subspaces.

Let V be a subspace of $\mathbb{R}^3$. Since V is itself a vector space, its dimension must satisfy

$$0\leq\dim V\leq 3.$$

We consider cases according to the possible values of $\dim V$.

- If $\dim V = 0$, then $V = \{0\}$.
- If $\dim V = 1$, then $V = \operatorname{span}\{v\}$ for some nonzero v, i.e., a line through the origin.
- If $\dim V = 2$, then $V = \operatorname{span}\{u, v\}$ for some linearly independent u, v, i.e., a plane through the origin.
- If $\dim V = 3$, then $V = \mathbb{R}^3$.

These four cases exhaust all possibilities, and hence the classification is complete.

14. Let
$$U = \{\, p \in \mathcal{P}_4(F) : p(6) = 0 \,\}.$$

(a) A basis of U.
Every $p \in U$ is divisible by $x - 6$, so $p(x) = (x-6)q(x)$ with $\deg q \leq 3$.
A convenient basis of $\mathcal{P}_3(F)$ is $\{1, x, x^2, x^3\}$, and multiplying by $(x-6)$ yields
$$\mathcal{B}_U = \{\, (x-6),\ x(x-6),\ x^2(x-6),\ x^3(x-6) \,\} \subset U.$$
Clearly $\mathcal{B}_U$ spans U. These four polynomials are linearly independent as well. Thus $\mathcal{B}_U$ is a basis of U and $\dim U = 4$.

(b) Extend to a basis of $\mathcal{P}_4(F)$.
Since $\dim \mathcal{P}_4(F) = 5$, we need one more polynomial outside U. The constant polynomial 1 satisfies $1(6) = 1 \neq 0$, so $1 \notin U$. Hence
$$\mathcal{B}_{\mathcal{P}_4} = \{\, 1,\ (x-6),\ x(x-6),\ x^2(x-6),\ x^3(x-6) \,\}$$
is a basis of $\mathcal{P}_4(F)$. (Every $p \in \mathcal{P}_4(F)$ decomposes as $p(6) \cdot 1 + (p(x) - p(6))$, with $p(x) - p(6) \in U$.)

(c) Complement W with $\mathcal{P}_4(F) = U \oplus W$.
Take $W = \operatorname{span}\{1\}$. Then $U \cap W = \{0\}$ because the only constant polynomial vanishing at $x = 6$ is the zero polynomial. Every $p \in \mathcal{P}_4(F)$ has the unique decomposition
$$p(x) = p(6) \cdot 1 + \big(p(x) - p(6)\big),$$
with $p(6) \cdot 1 \in W$ and $p(x) - p(6) \in U$. Therefore $\mathcal{P}_4(F) = U \oplus W$.

15. Recall the dimension formula for subspaces U, W of a finite-dimensional vector space:
$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$
Here $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$, so
$$8 = \dim(\mathbb{R}^8) = \dim(U + W) = 3 + 5 - \dim(U \cap W).$$
Hence $\dim(U \cap W) = 0$, which implies $U \cap W = \{0\}$. Therefore, by a theorem, every vector in $\mathbb{R}^8 = U + W$ can be written uniquely as a sum of a vector from U and a vector from W, so
$$\mathbb{R}^8 = U \oplus W.$$

16. Let V be finite-dimensional with $\dim V = n \geq 1$. Choose a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of V. For each $i = 1, \ldots, n$ set
$$V_i = \operatorname{span}\{v_i\},$$
which is a one-dimensional subspace of V.

We claim that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

First, every vector $v \in V$ can be written as a linear combination of the basis vectors, so $v = \sum_{i=1}^{n} a_i v_i$ for some scalars a_i. Hence

$$v = \sum_{i=1}^{n} a_i v_i \in V_1 + \cdots + V_n,$$

so $V = V_1 + \cdots + V_n$.

Second, the sum is direct. Suppose

$$u_1 + \cdots + u_n = 0$$

with $u_i \in V_i$. Writing $u_i = \alpha_i v_i$ for scalars α_i, we have

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

Since $\{v_1, \ldots, v_n\}$ is a basis (hence linearly independent), all $\alpha_i = 0$. Thus each $u_i = 0$, showing the representation is unique. Therefore the sum is direct.

Combining the two parts, $V = V_1 \oplus \cdots \oplus V_n$, as required.

17. Suppose, for contradiction, that $\mathbb{R}$ is finite dimensional over $\mathbb{Q}$. Suppose the dimension is n, for some $n \in \mathbb{N}$. Choose an ordered basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$. Every real number r can then be written uniquely as

$$r = q_1 v_1 + \cdots + q_n v_n$$

with $q_i \in \mathbb{Q}$. The correspondence

$$r \longmapsto (q_1, \ldots, q_n)$$

is therefore a bijection between $\mathbb{R}$ and $\mathbb{Q}^n$.

But $\mathbb{Q}$ is countable, and a finite Cartesian product of countable sets is countable; hence $\mathbb{Q}^n$ is countable. Therefore $\mathbb{R}$ would be countable. This contradicts the well-known fact that $\mathbb{R}$ is uncountable (has the cardinality of the continuum).

Thus $\mathbb{R}$ cannot be finite-dimensional over $\mathbb{Q}$.

18. **Idea.** We want a basis $\{A_1, A_2, A_3, A_4\}$ of $M_{2\times 2}(\mathbb{R})$ consisting of idempotent matrices (that is, $A_i^2 = A_i$ for all i). Recall that the standard basis consists of the four matrix units

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Two of these, E_{11} and E_{22}, are already idempotent. So the problem reduces to finding idempotent matrices that "generate" E_{12} and E_{21} as linear combinations.

Notice that if we add E_{11} to E_{12} we get

$$E_{11} + E_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

which squares to itself. Similarly, $E_{22} + E_{21}$ is also idempotent. Thus we can take these two combinations, together with E_{11} and E_{22}, to form our desired basis.

Construction. Define

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Each is idempotent:

$$A_1^2 = A_1, \quad A_2^2 = A_2, \quad A_3^2 = A_3, \quad A_4^2 = A_4.$$

Verification that they form a basis. The standard matrix units are obtained as linear combinations:

$$E_{11} = A_1, \quad E_{22} = A_2, \quad E_{12} = A_3 - A_1, \quad E_{21} = A_4 - A_2.$$

Hence $\{A_1, A_2, A_3, A_4\}$ spans $M_{2\times 2}(\mathbb{R})$. Since there are 4 matrices in a 4-dimensional space, they form a basis.

Therefore $\{A_1, A_2, A_3, A_4\}$ is a basis of $M_{2\times 2}(\mathbb{R})$ with $A_i^2 = A_i$ for each i.

19. Let

$$U = \left\{ \begin{bmatrix} u & -u \\ -x & x \end{bmatrix} : u, x \in \mathbb{R} \right\} \quad \text{and} \quad V = \left\{ \begin{bmatrix} v & 0 \\ w & -v \end{bmatrix} : v, w \in \mathbb{R} \right\}.$$

(a) **Basis for U.** Any element of U can be written as

$$\begin{bmatrix} u & -u \\ -x & x \end{bmatrix} = u \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Hence

$$B_U = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \right\}$$

spans U. Also B_U is LI and hence a basis for U.

(b) **Basis for V.** Any element of V can be written as

$$\begin{bmatrix} v & 0 \\ w & -v \end{bmatrix} = v \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + w \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Hence

$$B_V = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

spans V. In fact, one can see that B_V is a basis for V.

(c) **Basis for $U + V$.** The sum $U + V$ is spanned by the four matrices from $B_U \cup B_V$, which are linearly independent. Therefore $U + V = M_{2\times 2}(\mathbb{R})$. A convenient basis is the standard one:

$$B_{U+V} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(d) **Basis for $U \cap V$.** Suppose

$$\begin{bmatrix} u & -u \\ -x & x \end{bmatrix} = \begin{bmatrix} v & 0 \\ w & -v \end{bmatrix}.$$

Comparing entries gives $u = 0$, $v = 0$, $x = 0$, $w = 0$. Thus $U \cap V = \{0\}$, whose basis is the empty set.

Conclusion. We have

$$M_{2\times 2}(\mathbb{R}) = U \oplus V.$$

20. **Claim.** The set

$$V = \left\{ x \in C^k([a,b];\mathbb{R}) : a_0 x^{(k)} + a_1 x^{(k-1)} + \cdots + a_{k-1}x' + a_k x = 0 \text{ on } [a,b] \right\}$$

is a real vector space and $\dim V = k$ (provided the leading coefficient $a_0 \neq 0$; if the highest nonzero coefficient is a_m with $m \leq k$, then $\dim V = m$).

Proof. (Vector space.) Let $x, y \in V$ and $\lambda \in \mathbb{R}$. Since derivatives are linear,

$$a_0(x+y)^{(k)} + \cdots + a_k(x+y) = \big(a_0 x^{(k)} + \cdots + a_k x\big) + \big(a_0 y^{(k)} + \cdots + a_k y\big) = 0 + 0 = 0,$$

and similarly

$$a_0(\lambda x)^{(k)} + \cdots + a_k(\lambda x) = \lambda\big(a_0 x^{(k)} + \cdots + a_k x\big) = \lambda \cdot 0 = 0.$$

Thus $x+y \in V$ and $\lambda x \in V$. The zero function belongs to V as it satisfies the equation. Hence V is a subspace of $C^k([a,b];\mathbb{R})$.

(Dimension when $a_0 \neq 0$.) Assume $a_0 \neq 0$ so the differential equation is truly of order k. Consider the linear map

$$\Phi : V \longrightarrow \mathbb{R}^k, \qquad \Phi(x) = \big(x(a),\, x'(a),\, \ldots,\, x^{(k-1)}(a)\big).$$

The standard existence-and-uniqueness theorem for linear ordinary differential equations with continuous coefficients (here the coefficients are the constants a_i) guarantees that for any prescribed initial data

$$\big(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\big) \in \mathbb{R}^k$$

there exists a unique function $x \in C^k([a,b];\mathbb{R})$ satisfying the differential equation and

$$x(a) = \alpha_0,\ x'(a) = \alpha_1,\ \ldots,\ x^{(k-1)}(a) = \alpha_{k-1}.$$

Thus Φ is surjective, and uniqueness implies Φ is injective. Therefore Φ is a linear isomorphism $V \cong \mathbb{R}^k$, so $\dim V = k$.

(Remark — vanishing leading coefficient.) If $a_0 = 0$ but some $a_m \neq 0$ is the coefficient of the highest derivative occurring in the equation (so the equation reduces to an mth-order linear equation), then the same argument shows $\dim V = m$. In the extreme case all $a_i = 0$ the equation is $0 = 0$ and $V = C^k([a,b];\mathbb{R})$ which is infinite dimensional.

This completes the proof.