Indian Institute of Technology Delhi

MTL104 Linear Algebra and Its Applications

Quiz III: Answer Key and Scheme of Evaluation

1. Let V be a 10-dimensional vector space and let $T : V \to V$ be a linear operator. If $T^{2025} = 0$, then $T^{10} = 0$.

Answer: True.
Since $T^{2025} = 0$, $f(x) = x^{2025}$ is an annihilating polynomial for T. Hence, the minimal polynomial $p(x)$ divides x^{2025}. So, $p(x) = x^r$ for some $r \leq 2025$. (**1.5 Marks**)
But $\deg p(x) \leq \dim V = 10$, hence $r \leq 10$ and therefore $T^{10} = 0$.

(1 Mark)
(Alternatively, all eigenvalues are 0, so by Cayley–Hamilton, $T^{10} = 0$.)

2. Similar matrices have the same minimal polynomial.

Answer: True.
If $B = P^{-1}AP$, then $p(B) = P^{-1}p(A)P$.(**1.5 Marks**)
Thus any polynomial annihilating A also annihilates B, and vice versa. Hence both have the same monic minimal polynomial. **(1 Mark)**

3. Every matrix A satisfying $A^2 = A$ is diagonalizable.

Answer: True.
From the given condition, it follows that $f(x) = x^2 - x = x(x-1)$ is an annihilating polynomial for A. **(1 Mark)**
Since the minimal polynomial $p(x)$ of A divides $f(x)$, the possibilities are $p(x) = x$, $p(x) = x - 1$ and $p(x) = x(x-1)$. Hence A is diagonalizable.
(**1.5 Marks**)

4. Let $T : V \to V$ be a linear operator, and let c be an eigenvalue of T. Let

$$W = \{v \in V : T(v) = cv\}$$

be the corresponding space of eigenvectors. Then the restriction

$$T|_W : W \to W$$

is always a scalar multiple of the identity operator.

Answer: True.
For every $w \in W$, $T(w) = cw$. (**1 Mark**)

Hence the restriction $T|_W$ acts as scalar multiplication by c, i.e.

$$T|_W = cI_W.$$

(1.5 Marks)