Minor Exam (20%)

Graded

Student

Kushagra Gupta

Total Points

41 / 50 pts

Question 1

Q1 (a)

5 / 5 pts

- ✔ **+ 1 pt** atn(p) = 1
- ✔ **+ 1 pt** atn($\bot$) = 0
- ✔ **+ 1 pt** atn($\neg\alpha$) = atn(α)
- ✔ **+ 1 pt** atn($\alpha \circ \beta$) = |atoms(α)$\cup$ atoms(β)| or atn(α) + atn(β) - |atoms(α)$\cap$ atoms(β)| $\circ \in \{\wedge, \vee, \supset\}$
- ✔ **+ 1 pt** Definition of atoms (inductive or sf(α) $\cap AP$)

+ 0 pts Incorrect/Not attempted

Question 2

Q1 (b)

5 / 5 pts

- ✔ **+ 0.5 pts** qd($t_1 \equiv t_2$) = 0
- ✔ **+ 0.5 pts** qd($P(t_1, \dots, t_n)$) = 0
- ✔ **+ 1 pt** qd($\neg\alpha$) = qd(α)
- ✔ **+ 1 pt** qd($\alpha \circ \beta$) = max(qd(α), qd(β)) $\circ \in \{\wedge, \vee, \supset\}$
- ✔ **+ 1 pt** qd($\exists x.[\alpha]$) = 1 + qd(α)
- ✔ **+ 1 pt** qd($\forall x.[\alpha]$) = 1 + qd(α)

+ 0 pts Incorrect/Not Attempt

Question 3

Q2 (a)

Resolved 3 / 4 pts

+ 4 pts Correct

+ 0 pts Incorrect

✔ **+ 1 pt** Disjunction to figure out the leftmost point j where the first flip from 0 to 1 occurs

✔ **+ 1 pt** Conjunction to say that the c and d values are the same from 1 through $j-1$

✔ **+ 1 pt** Conjunction to say that after j, if there are any positions, c might have been 1, but got flipped to 0 in d

+ 1 pt Something of the form $\bigvee_{1\leq j\leq n}[(\neg c_j \wedge d_j) \wedge \bigwedge_{1\leq i\leq j-1}((c_i \wedge d_i) \vee (\neg c_i \wedge \neg d_i)) \wedge \bigwedge_{j+1\leq i\leq n}((c_i \wedge \neg d_i))]$

Unclear answer

Regrade Request **Submitted on: Oct 01**

Dear sir/ma'am,
Could you please review my answer and consider the following request? I mentioned the exact same expression as the rubric -
$\bigvee_{n\geq j\geq 1}\left[(\neg c_j \wedge d_j) \wedge \bigwedge_{1\leq i\leq j-1}((c_i \wedge d_i) \vee (\neg c_i \wedge \neg d_i)) \wedge \bigwedge_{j+1\leq i\leq n}(c_i \wedge \neg d_i)\right]$ as a disjunction of formulae starting from $j = n$ to $j = 1$
(the disjunction is for j where first flip occurs starting from n to 1, in my answer - $d_n \wedge \neg c_n$ in first and $d_{n-1} \wedge \neg c_{n-1}$ in second line).

I also correctly wrote the conjunction (XNOR) to say c and d values are same from j-1 to 1, $\bigwedge_{n-1\geq i\geq 1}((c_i \wedge d_i) \vee (\neg c_i \wedge \neg d_i))$ in the first line and $\bigwedge_{n-2\leq i\leq 1}((c_i \wedge d_i) \vee (\neg c_i \wedge \neg d_i))$ in the second line. (in expanded form)

I also wrote the conjunction to represent the flipped bits, $\neg d_n \wedge c_n$ in the second line and similarly for the last one.
(However, due to time constraints in the last line, I couldn't mention $d_1 \vee \neg c_1$ at the end but it is clear from above - we shift $d_i \wedge \neg c_i$ towards end, I sincerely apologise.)

I feel I have mentioned everything in accordance with the rubric (the three points - Disj, Conj_same, Conj_flip as well as the expanded formula is same with indices from n to 1), I humbly request you to regrade my answer as I should get more marks.

Thanks and best regards,

Fixed. Your answer was unclear and I couldn't understand the meaning of the expression that was written

Reviewed on: Oct 02

Question 4

Q2 (b) (i)

2 / 2 pts

- ✔ **+ 0.5 pts** $\mathscr{C} = \{0\}$
- ✔ **+ 1 pt** $\mathscr{F} = \{*/2\}$ (it is ok if it is sq/1 where sq(x) = x * x)
- ✔ **+ 0.5 pts** $\mathscr{P} = \{< /2\}$ or any other relation that can be used instead of $<$

+ 0 pts Incorrect

Question 5

Q2 (b) (ii)

4 / 4 pts

+ 0 pts Incorrect

$\alpha = \forall x.\forall y.\,[x < y \supset \exists z.[x < z \wedge z < y]]$

- ✔ **+ 0.5 pts** $\forall x \forall y$
- ✔ **+ 1 pt** $x < y$
- ✔ **+ 0.5 pts** $\exists z.[x < z \wedge z < y]$

+ 0 pts Incorrect

$\beta = \forall x.\big[0 < x \supset \exists y.[\exists z.[x = y * y \wedge x = z * z \wedge 0 < y \wedge z < 0]\big]$

- ✔ **+ 0.5 pts** Three variables, $\forall x, \exists y, \exists z$
- ✔ **+ 0.5 pts** $0 < x$ (use the same symbol as in the relation set)
- ✔ **+ 0.5 pts** $x = y * y \wedge x = z * z$
- ✔ **+ 0.5 pts** $0 < y$ and $z < 0$

+ 0 pts Incorrect

Question 6

Q3 (a)

4 / 4 pts

- ✔ **+ 0.5 pts** $(p \wedge s) \supset (r \vee s) \Leftrightarrow \neg(p \wedge s) \vee (r \vee s)$ and $(r \vee s) \supset (p \wedge s) \Leftrightarrow \neg(r \vee s) \vee (p \wedge s)$
- ✔ **+ 0.5 pts** $\neg(p \wedge s) \vee (r \vee s) \Leftrightarrow (\neg p \vee \neg s) \vee (r \vee s)$ and $\neg(r \vee s) \vee (p \wedge s) \Leftrightarrow (\neg r \wedge \neg s) \vee (p \wedge s)$
- ✔ **+ 0.5 pts** $(\neg p \vee \neg s) \vee (r \vee s) \Leftrightarrow (\neg p \vee r \vee T)$
- ✔ **+ 0.5 pts** $(\neg p \vee r \vee T) \Leftrightarrow T$
- ✔ **+ 0.5 pts** $(\neg r \wedge \neg s) \vee (p \wedge s) \Leftrightarrow (\neg r \vee (p \wedge s)) \wedge (\neg s \vee (p \wedge s))$
- ✔ **+ 0.5 pts** $(\neg r \vee (p \wedge s)) \wedge (\neg s \vee (p \wedge s)) \Leftrightarrow (\neg r \vee p) \wedge (\neg r \vee s)) \wedge (\neg s \vee p) \wedge (\neg s \vee s))$
- ✔ **+ 0.5 pts** $(\neg r \vee p) \wedge (\neg r \vee s)) \wedge (\neg s \vee p) \wedge (\neg s \vee s)) \Leftrightarrow (\neg r \vee p) \wedge (\neg r \vee s)) \wedge (\neg s \vee p) \wedge T$
- ✔ **+ 0.5 pts** $T \wedge ((\neg r \vee p) \wedge (\neg r \vee s)) \wedge (\neg s \vee p) \wedge T) \Leftrightarrow (\neg r \vee p) \wedge (\neg r \vee s)) \wedge (\neg s \vee p)$
- **+ 0 pts** Incorrect/Not attempted

Did not clean the expressions containing s and ~s

Question 7

Q3 (b)

6 / 6 pts

- ✔ **+ 1 pt** CNF expression to consider:
  $\varphi = \{\{\neg r, p\}, \{\neg r, s\}, \{\neg s, p\}, \{r \vee s\}, \{\neg s\}\}$
- ✔ **+ 1 pt** Clean this expression since $\{\neg s\} \subseteq \{\neg s, p\}$. Get
  $\varphi^* = \{\{\neg r, p\}, \{\neg r, s\}, \{r \vee s\}, \{\neg s\}\}$
- ✔ **+ 1.5 pts** Resolve using r. ($\{s\} \cup \{s\} = \{s\}$, and $\neg s$ stays untouched.
  resolve$(\varphi^*, r) = \{\{p, s\}, \{s\}, \{\neg s\}\}$
- ✔ **+ 1 pt** Clean this expression since $\{s\} \subseteq \{p, s\}$. Get
  $\varphi_1^* = \{\{s\}, \{\neg s\}\}$
- ✔ **+ 1.5 pts** Resolve using s.
  resolve$(\varphi_1^*, s) = \{\emptyset\}$ so the logical consequence holds
- **+ 0 pts** Incorrect/Not attempted

Question 9

Q4 (b)

Resolved **2 / 4 pts**

+ 0 pts Incorrect

$\Rightarrow$ direction: Suppose ϕ_{sat} is valid.

+ 0.5 pts For any τ, $\tau \models \bigvee\{\phi'_\tau \mid \tau' \text{ a valuation over } \mathsf{atoms}(\phi)\}$

+ 1 pt $\tau \models \phi'_\tau$ for some τ' (by the satisfaction of disjunction)

+ 0.5 pts There is some τ_0 such that $\tau_0 \models \phi$ (by the induction statement for (a))

$\Leftarrow$ direction: Suppose ϕ is satisfiable.

✔ **+ 1 pt** There is some τ such that $\tau \models \phi$, iff $\models \phi_\tau$ (by (a))

✔ **+ 1 pt** If $\models \phi_\tau$ then $\models \phi_\tau \vee \psi$ for any ψ, and $\phi_{\text{sat}} = \phi_\tau \vee \phi'$ (where $\phi' = \bigvee_{\tau' \neq \tau}\{\phi'_\tau \mid \tau' \text{ a valuation over } \mathsf{atoms}(\phi)\}$

Regrade Request **Submitted on: Oct 01**

Dear sir/ma'am,

For the => direction,
I mentioned that if ϕ_{sat} is valid, then there would exist one τ' such that $\phi_{\tau'}$ is valid by part a. (this is true as every $\phi_{\tau'}$ for some τ' is either a validity or unsat based on our construction and we take the disjunction.)

Then I concluded by the part a statement, as $\phi_{\tau'}$ is valid, $\tau' \models \phi$ and hence ϕ is satisfiable. So, I humbly request you to please regrade as I feel I it is correct and I deserve more marks for the => direction as per the three/last 2 rubric points.

(I apologise for the mistake I did in part a - my construction replaced the positive literals in τ by validity $\alpha \supset \phi$ but I forgot to mention that we replace the negative literals by negation of the validity - however the proof for => direction in b still holds.)

Thanks and best regards,

Just because a disjunction is valid does not imply that a single disjunct is valid, merely that each valuation satisfies some disjunct. No points.

Reviewed on: Oct 02

Question 10

Q5

10 / 10 pts

- ✔ **+ 0.5 pts** Apply DT to $(p \supset q) \supset (p \supset r), \neg q \supset \neg p \vdash \neg r \supset \neg p$ to get $(p \supset q) \supset (p \supset r) \vdash (\neg q \supset \neg p) \supset (\neg r \supset \neg p)$

- ✔ **+ 0.5 pts** Get $\neg r \supset \neg p$ by applying MP to $(p \supset r) \supset (\neg r \supset \neg p)$ (still to be proved) and $p \supset r$ (also to be proved)

Show that $\vdash (p \supset r) \supset (\neg r \supset \neg p)$

- ✔ **+ 1 pt** Use MP on $(\neg\neg p \supset r) \supset \neg p$ and $\neg\neg p \supset r$ (still needs a proof) and DT to get the result

- ✔ **+ 1.5 pts** Use MP on $(\neg\neg p \supset \neg r) \supset (\neg\neg p \supset r) \supset \neg p$ (obtained by H3) and $(\neg\neg p \supset \neg r)$ (obtained by Ax + DT) to get $(\neg\neg p \supset r) \supset \neg p$

Show that $p \supset r \vdash \neg\neg p \supset r$

- ✔ **+ 0.5 pts** Apply DT to $p \supset r, \neg\neg p \vdash r$ to get the result

  Get r by MP on $p \supset r$ (obtained by Ax) and p

- ✔ **+ 1 pt** Get p by MP on $(\neg p \supset \neg p) \supset p$ and $\neg p \supset \neg p$ (proved by Ax + DT)

- ✔ **+ 1.5 pts** Get $(\neg p \supset \neg p) \supset p$ by MP on $(\neg p \supset p) \supset (\neg p \supset \neg p) \supset p$ (obtained by H3) and $\neg p \supset \neg\neg p$ (obtained by Ax + DT)

- ✔ **+ 0.5 pts** Use MP with $(p \supset q) \supset (p \supset r)$ and $p \supset q$ to get $p \supset r$ (Still have to produce a proof of $p \supset q$)

- ✔ **+ 0.5 pts** Get $(p \supset q) \supset (p \supset r), \neg q \supset \neg p, \neg r \vdash \neg r$ by Ax

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Show that $\neg q \supset \neg p \vdash p \supset q$

- ✔ **+ 1 pt** Get $\neg q \supset \neg p \vdash p \supset q$ by DT on $\neg q \supset \neg p, p \vdash q$

  Obtain $\neg q \supset \neg p, p \vdash q$ by MP on $(\neg q \supset p) \supset q$ and $\neg q \supset p$ (by Ax + DT)

- ✔ **+ 1.5 pts** Obtain $(\neg q \supset \neg p) \supset q$ by MP on $(\neg q \supset \neg p) \supset (\neg q \supset p) \supset q$ (obtained by H3) and $\neg q \supset \neg p$ (obtained by Ax)

+ 0 pts Incorrect

Indian Institute of Technology Delhi
COL703: Logic for Computer Science **MID-TERM EXAM**

DATE: Friday the 13th of September 2024 **DURATION:** 2 hours **MAXIMUM MARKS:** 50

Q1 (10)	Q2 (10)	Q3 (10)	Q4 (10)	Q5 (10)	Total (50)

Instructions: Write your name and entry number at the top of each sheet. Answer all questions only in the boxes provided. Solutions outside the assigned box will not be graded. Make reasonable assumptions and state them wherever necessary.

Attestation: I agree to abide by the Honour Code of IIT Delhi.

Signature: Kushagra

Q1. (10 marks) The length of an expression is often not a very good measure of how complex it is. To this end, we define new notions for propositional and first-order expressions.

(a) (5 marks) For propositional expressions, use the number of distinct propositional atoms, denoted by atn. For example,

$\mathsf{atn}(p) = 1$ $\mathsf{atn}(p \wedge (q \vee p)) = 2$ $\mathsf{atn}(p \vee (\neg p \wedge q)) = 2$ $\mathsf{atn}(p \wedge (\neg(p \supset r))) =$ X 2.

Define atn inductively for expressions generated by the following grammar.

$$\alpha, \beta := p \mid \bot \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \supset \beta \quad \text{where } p \in AP$$

~~$atn(p) = 1$ $atn(\bot) = 0$ $atn(\neg\alpha) = atn(\alpha)$~~

We also define ats inductively in order to define atn :

$ats(P) = \{p\}$, $ats(\bot) = \phi$, $ats(\neg\alpha) = ats(\alpha)$

$ats(\alpha \wedge \beta) = ats(\alpha) \cup ats(\beta)$ $ats(\alpha \vee \beta) = ats(\alpha) \cup ats(\beta)$

$ats(\alpha \supset \beta) = ats(\alpha) \cup ats(\beta)$

Now, $atn(P) = |ats(P)|$, $atn(\bot) = |ats(\bot)|$, $atn(\neg\alpha) = |ats(\neg\alpha)|$

$atn(\alpha \wedge \beta) = |ats(\alpha \wedge \beta)|$, $atn(\alpha \vee \beta) = |ats(\alpha \vee \beta)|$, $atn(\alpha \supset \beta) = |ats(\alpha \supset \beta)|$

(b) (5 marks) For first-order expressions, we use the maximum nesting depth of quantifiers, denoted by qd. For example,

$\mathsf{qd}(t_1 \equiv t_2) = 0$ $\mathsf{qd}(\forall x.\ [x \equiv c]) = 1$ $\mathsf{qd}(\exists x.\ [\exists y.\ [x \equiv y]] \wedge y \equiv c) = 2$ $\mathsf{qd}(\exists x.\ [P(t_1) \wedge \forall y.\ [y \equiv c]]) = 2$

Write an inductive definition for qd for first-order expressions generated by the following grammar.

$$\alpha, \beta := t_1 \equiv t_2 \mid P(t_1, \ldots, t_n) \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \supset \beta \mid \exists x.\ [\alpha] \mid \forall x.\ [\alpha] \quad \text{where } P \text{ is an } n\text{-ary predicate symbol.}$$

$qd(t_1 \equiv t_2) = 0$ $qd(P(t_1 -- t_n)) = 0$

$qd(\neg\alpha) = qd(\alpha)$ $qd(\alpha \wedge \beta) = max(qd(\alpha), qd(\beta))$

$qd(\alpha \vee \beta) = max(qd(\alpha), qd(\beta))$

$qd(\alpha \supset \beta) = max(qd(\alpha), qd(\beta))$

$qd(\exists x [\alpha]) = qd(\alpha) + 1$, $qd(\forall x [\alpha]) = qd(\alpha) + 1$.

Q2. (10 marks) Modelling in logics.

(a) (4 marks) Consider two n-bit binary numbers $c_1c_2 \dots c_n$ and $d_1d_2 \dots d_n$. Let $AP = \{c_1, \dots, c_n, d_1, \dots, d_n\}$ be the set of propositional atoms. Any valuation τ sets c_i to T iff $c_i = 1$, and similarly for d_i. Write a propositional expression for $d = c + 1$.

$(d_n \wedge \neg c_n \wedge ((d_{n-1} \wedge c_{n-1}) \vee (\neg d_{n-1} \wedge \neg c_{n-1})) \wedge \dots ((d_1 \wedge c_1) \vee (\neg d_1 \wedge \neg c_1)))$

$\vee\ (\neg d_n \wedge c_n \wedge (d_{n-1} \wedge \neg c_{n-1}) \wedge ((d_{n-2} \wedge c_{n-2}) \vee (\neg d_{n-2} \wedge \neg c_{n-2})) \dots)$

$\vee\ (\neg d_n \wedge c_n \wedge \neg d_{n-1} \wedge c_{n-1} \wedge \neg d$

(b) i. (2 marks) Define a signature Σ required to express the following statements in first-order logic.

$\mathcal{C} = \{0\}$, $\mathcal{F} = \{sq/1\}$

$\mathcal{P} = \{L/2, LE/2\}$

ii. (4 marks) Express the following in FO_Σ.

- [α]: Between any two real numbers, there is another real number.

$\forall p [\forall q [\exists r [(LE(p,r) \wedge LE(r,q)) \vee (LE(q,r) \wedge LE(r,p))]]]$

- [β]: Every positive real number has two real square roots, one positive and one negative.

$(\forall p [L(0,p) \supset (\exists r [L(0,r) \wedge (sq(r) \equiv p)]]) \wedge (\forall p [L(0,p) \supset (\exists r [L(r,0) \wedge (sq(r) \equiv p)]])])$

Q3. (10 marks) CNF and Resolution.

(a) (4 marks) Convert $((p \wedge s) \supset (r \vee s)) \wedge ((r \vee s) \supset (p \wedge s))$ to an equivalent CNF expression. Justify each step clearly.

$(p \wedge s) \supset (r \vee s) \equiv \neg(p \wedge s) \vee (r \vee s) \equiv (\neg p \vee \neg s \vee r \vee s)$. ①

$(r \vee s) \supset (p \wedge s) \equiv \neg(r \vee s) \vee (p \wedge s) \equiv (\neg r \wedge \neg s) \vee (p \wedge s)$ ②

By distributivity of $\wedge$ over $\vee$. $\equiv \cancel{(\neg r \vee p) \wedge (\neg r} (\neg r \vee (p \wedge s)) \wedge (\neg s \vee (p \wedge s))$

$\equiv ((\neg r \vee p) \wedge (\neg r \vee s)) \wedge ((\neg s \vee p) \wedge (\neg s \vee s))$

$\equiv ((\neg r \vee p) \wedge (\neg r \vee s) \wedge (\neg s \vee p) \wedge (\neg s \vee s))$.

① & ② followed by identity $a \supset b \equiv \neg a \vee b$ & $\neg(a \wedge b) \equiv \neg a \vee \neg b$
$\neg(b \vee a) \equiv \neg a \wedge \neg b$

So $((p \wedge s) \supset (r \vee s)) \wedge ((r \vee s) \supset (p \wedge s))$

$\equiv ((\neg p \vee \neg s \vee r \vee s) \wedge (\neg r \vee p) \wedge (\neg r \vee s) \wedge (\neg s \vee p) \wedge (\neg s \vee s))$

In CNF expression $\{\{\neg p, \neg s, r, s\}, \{\neg r, p\}, \{\neg r, s\}, \{\neg s, p\}, \{\neg s, s\}\}$

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(b) (6 marks) Use resolution to show that $\{((p \wedge s) \supset (r \vee s)) \wedge ((r \vee s) \supset (p \wedge s)), r \vee s\} \models s$.

$\Gamma = \{((p \wedge s) \supset (r \vee s)) \wedge ((r \vee s) \supset (p \wedge s)), \; r \vee s\}$

First we convert each expression in Γ into CNF form

$(\neg p \vee \neg s \vee r \vee s) \wedge ((\neg r \wedge \neg s) \vee (p \wedge s))$ as $a \supset b \equiv \neg a \vee b$

$(\neg p \vee \neg s \vee r \vee s) \wedge (\neg r \vee p) \wedge (\neg r \vee s) \wedge (\neg s \vee p) \wedge (\neg s \vee s)$.

So $\Delta = \{\{\neg p, \neg s, r, s\}, \{\neg r, p\}, \{\neg r, s\}, \{\neg s, p\}, \{\neg s, s\}, \{r, s\}, \{\neg s\}\}$

(added neg. of consequence s for resol^n refutation)

First we clean Δ

$\Delta' = \{\{\neg r, p\}, \{\neg r, s\}, \cancel{\{\neg s, p\}}, \{r, s\}, \{\neg s\}\}$ (as $\neg s \subseteq \{\neg s, p\}$)

Now resolve with "p" $= r$, taking $\delta_1 = \{r, s\}$ & $\delta_2 = \{\neg r, s\}, \{\neg r, p\}$.

$\Delta'' = \{\{p, s\}, \{s\}, \{\neg s\}\}$

Now, resolve with "p" $= s$, taking $\delta_1 = \{s\}, \{p, s\}$ & $\delta_2 = \{\neg s\}$.

$\Delta''' = \{\{p\}, \phi\}$

Since Δ''' has ϕ in set of clauses, it is UNSAT.

$\therefore$ $\Gamma \wedge \neg s$ is UNSAT or $\neg(\Gamma \supset s)$ is UNSAT

$\therefore$ $\Gamma \models s$ by resolution refutation

Q4. (10 marks) Encoding meta-logical concepts as PL expressions.

(a) (6 marks) For $\varphi, \psi \in$ PL, we say that ψ is an instance of φ, if one can simultaneously replace each atom p occurring in φ by some expression α_p, and obtain ψ. For any φ, and a valuation τ over atoms(φ), define an instance φ_τ of φ such that φ_τ is valid iff $\tau \models \varphi$.

Given valuation τ, construct $\alpha = \bigwedge_{p_i \in \tau} p_i \wedge \bigwedge_{p_j \notin \tau} \neg p_j$ (if $\tau: p=0, q=1$ $p \notin \tau, q \in \tau$)

Consider $\varphi \wedge \alpha$.

Replace each atom in φ by $\varphi \Leftrightarrow \alpha$ to get φ_τ ($\varphi \Leftrightarrow \alpha$ is $\varphi \supset \alpha \wedge \alpha \supset \varphi$)

It is easy to see why it is valid iff $\tau \models \varphi$.

(b) (4 marks) Let $\varphi_{sat} = \bigvee\{\varphi_\tau \mid \tau$ a valuation over atoms(φ)$\}$. Show that φ_{sat} is valid iff φ is satisfiable.

$\Leftarrow$ If φ is satisfiable $\exists$ a valuation τ' s.t $\tau' \models \varphi$

$\Rightarrow$ $\varphi_{\tau'}$ is valid (by part a)

Since φ_{sat} is disjunction of φ_τ $\Rightarrow$ φ_{sat} is valid.

$\Rightarrow$ If φ_{sat} is valid, $\exists$ some τ' such that $\varphi_{\tau'}$ is valid.

$\therefore$ by part a, $\tau' \models \varphi$

So φ is satisfiable

Hence Both dir^n proved

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Q5. (10 marks) Prove that $(p \supset q) \supset (p \supset r) \vdash_{\mathcal{H}} (\neg q \supset \neg p) \supset (\neg r \supset \neg p)$.

$$\dfrac{\dfrac{\dfrac{H3}{\Gamma\vdash(\neg a\supset\neg\neg a)\supset((\neg a\supset\neg a)\supset a)} \quad \dfrac{\dfrac{H1}{\Gamma\vdash\neg\neg a\supset(\neg a\supset\neg\neg a)}\quad\dfrac{Ax}{\Gamma\vdash\neg\neg a}}{\Gamma\vdash\neg a\supset\neg\neg a}MP}{\Gamma\vdash(\neg a\supset\neg a)\supset a}MP \quad \dfrac{\dfrac{Ax}{\Gamma,\neg a\vdash\neg a}}{\Gamma\vdash\neg a\supset\neg a}DT}{\neg\neg a\vdash a}MP$$

π_3 $\neg\neg a \vdash a$ Proof tree $\Gamma = \neg\neg a$

$$\dfrac{\dfrac{\dfrac{H3}{\Gamma\vdash(\neg\neg\neg a\supset\neg a)\supset((\neg\neg\neg a\supset a)\supset(\neg\neg a))} \quad \dfrac{\dfrac{\pi_3 \text{ with } a=\neg a}{\Gamma,\neg\neg\neg a\vdash\neg a}}{\Gamma\vdash\neg\neg\neg a\supset\neg a}DT}{\Gamma\vdash(\neg\neg\neg a\supset a)\supset(\neg\neg a)}MP \quad \dfrac{\dfrac{Ax}{\Gamma,\neg\neg\neg a\vdash a}}{\Gamma\vdash\neg\neg\neg a\supset a}DT}{a\vdash\neg\neg a}MP$$

π_4 $a \vdash \neg\neg a$ Proof tree $\Gamma = a$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{H3}{\Gamma\vdash(\neg\neg a\supset\neg b)\supset((\neg\neg a\supset b)\supset\neg a)} \quad \dfrac{\dfrac{Ax}{\Gamma,\neg\neg a\vdash\neg b}}{\Gamma\vdash(\neg\neg a\supset\neg b)}DT}{\Gamma\vdash(\neg\neg a\supset b)\supset\neg a} \quad \dfrac{\dfrac{\dfrac{Ax}{\Gamma,\neg\neg a\vdash a\supset b}\quad\dfrac{\pi_3}{\Gamma,\neg\neg a\vdash a}}{\Gamma,\neg\neg a\vdash b}MP}{\Gamma\vdash\neg\neg a\supset b}DT}{a\supset b,\neg b\vdash\neg a}}{a\supset b\vdash\neg b\supset\neg a}DT}{\vdash(a\supset b)\supset(\neg b\supset\neg a)}DT$$

π_1 where $\Gamma = a\supset b, \neg b$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Ax}{\Gamma'\vdash\neg\neg p\supset\neg\neg q}\quad\dfrac{\pi_2}{p\vdash\neg\neg p}}{\neg\neg p\supset\neg\neg q, p\vdash\neg\neg q}MP\quad\dfrac{\pi_3}{\vdash\neg\neg q\supset q}}{\cancel{\Gamma\vdash\neg\neg p\supset\neg\neg q \ldots q}}MP}{\Gamma\vdash(\neg\neg p\supset\neg\neg q)\supset(p\supset q)}DT^2 \quad \dfrac{\dfrac{\pi_1}{\vdash\neg q\supset\neg p\supset(\neg\neg p\supset\neg\neg q)}}{\Gamma\vdash\neg\neg p\supset\neg\neg q}DT}{\neg q\supset\neg p\vdash p\supset q}MP}{\vdash(\neg q\supset\neg p)\supset(p\supset q)}$$

π_2

Here $\Gamma' = (\neg\neg p\supset\neg\neg q, p)$.

& DT^2 is DT applied twice

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma\vdash(p\supset r)\supset(\neg r\supset\neg p)}\quad\dfrac{\dfrac{Ax}{\Gamma\vdash(p\supset q)\supset(p\supset r)}\quad\dfrac{\dfrac{\pi_2}{\Gamma\vdash(\neg q\supset\neg p)\supset(p\supset q)}\quad\dfrac{Ax}{\Gamma\vdash(\neg q\supset\neg p)}}{\Gamma\vdash(p\supset q)}}{\Gamma\vdash(p\supset r)}}{(p\supset q)\supset(p\supset r), \neg q\supset\neg p\vdash(\neg r\supset\neg p)}MP}{(p\supset q)\supset(p\supset r)\vdash_{\mathcal{H}}(\neg q\supset\neg p)\supset(\neg r\supset\neg p)}DT$$

$\Gamma\vdash(\neg r\supset\neg p)$