Department of Mathematics
Indian Institute of Technology Delhi
MTL104 Linear Algebra and Applications
Midterm Examination
Answer Key and Scheme of Evaluation.

1. **Target:** Concept of vector space and subspace. Subspace verification. **Source:** Lecture + Practice Sheet.

   Suppose that S is a fixed, invertible $n \times n$ matrix. Let

   $$W = \{A \in M_{n\times n}(\mathbb{R}) : S^{-1}AS \text{ is diagonal}\}.$$

   Is W a vector space under the usual operations of matrix addition and scalar multiplication? Justify your answer.

   We want to check whether

   $$W = \{A \in M_{n\times n}(\mathbb{R}) : S^{-1}AS \text{ is diagonal}\}$$

   is a vector space.

   Since $W \subseteq M_{n\times n}(\mathbb{R})$, it is enough to prove that W is a **subspace** of the vector space $M_{n\times n}(\mathbb{R})$. **(1 Mark)**
   By the subspace criterion, we must check:

   (a) W is nonempty,
   (b) W is closed under addition,
   (c) W is closed under scalar multiplication.

   Step 1. Nonempty. The zero matrix belongs to W, since

   $$S^{-1}0S = 0,$$

   and 0 is diagonal. Thus $W \neq \varnothing$. **(0.5 Mark)**

   Step 2. Closed under addition. Take $A, B \in W$. Then

   $$S^{-1}AS = D_A, \qquad S^{-1}BS = D_B,$$

   where D_A, D_B are diagonal. Hence

   $$S^{-1}(A+B)S = S^{-1}AS + S^{-1}BS = D_A + D_B,$$

   and the sum of diagonal matrices is diagonal. Therefore $A + B \in W$. **(2 Marks)**

   Step 3. Closed under scalar multiplication. Let $\alpha \in \mathbb{R}$ and $A \in W$. With $D_A = S^{-1}AS$ diagonal, we have

   $$S^{-1}(\alpha A)S = \alpha S^{-1}AS = \alpha D_A,$$

   and a scalar multiple of a diagonal matrix is diagonal. Thus $\alpha A \in W$.

   Since all three conditions are satisfied, we conclude that W is a subspace of $M_{n\times n}(\mathbb{R})$.
   (1.5 Marks)

2. **Target:** Union of subspaces argument. **Source:** Practice sheet exercise.

Let V_1 and V_2 be two non-trivial subspaces of a vector space V over a field $\mathbb{F}$ (i.e. neither V_1 nor V_2 is $\{0\}$ or V). Show that there exists a vector $v \in V$ such that

$$v \notin V_1 \quad \text{and} \quad v \notin V_2.$$

Consider two cases.

Case 1: One subspace is contained in the other, say $V_1 \subseteq V_2$. Since V_2 is a proper subspace of V, there exists $v \in V \setminus V_2$. Then $v \notin V_2$ and, a fortiori, $v \notin V_1$. Thus v has the required property. **(1 Mark)**

Case 2: Neither subspace is contained in the other. Then there exist

$$x \in V_1 \setminus V_2 \qquad \text{and} \qquad y \in V_2 \setminus V_1.$$

Consider $x + y$. **(2 Marks)**
If $x + y \in V_1$ then

$$y = (x + y) - x \in V_1,$$

contradicting $y \notin V_1$. Hence $x + y \notin V_1$. Similarly, if $x + y \in V_2$ then

$$x = (x + y) - y \in V_2,$$

contradicting $x \notin V_2$. Thus $x + y \notin V_2$ as well. Therefore $v := x + y$ satisfies $v \notin V_1$ and $v \notin V_2$. **(2 Marks)**

In either case we have produced a vector $v \in V$ lying in none of V_1, V_2, as required. □

3. **Target:** Dimension formula. **Source:** Practice sheet.

Let U and W be subspaces of $\mathbb{R}^8$ with $\dim(U) = 3$, $\dim(W) = 5$, and $U + W = \mathbb{R}^8$. Show that

$$\mathbb{R}^8 = U \oplus W.$$

Step 1. By a result, $\mathbb{R}^8 = U \oplus W$ if and only if

$$U + W = \mathbb{R}^8 \quad \text{and} \quad U \cap W = \{0\}.$$

The first condition is given. It remains to check that $U \cap W = \{0\}$. **(1 Mark)**

Step 2. Dimension formula. For any subspaces U, W of a finite-dimensional vector space,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

(2 Marks)
Here, $\dim(U + W) = \dim(\mathbb{R}^8) = 8$, $\dim U = 3$, and $\dim W = 5$. Thus

$$8 = 3 + 5 - \dim(U \cap W).$$

Step 3. Solve. This gives

$$\dim(U \cap W) = 0,$$

hence $U \cap W = \{0\}$. **(2 Marks)**

Step 4. Conclusion. Since $U + W = \mathbb{R}^8$ and $U \cap W = \{0\}$, it follows that

$$\mathbb{R}^8 = U \oplus W.$$

4. **Target:** Independence test, parity of a natural number. **Source:** Lecture discussion.

   Let $n \in \mathbb{N}$ and $v_1, v_2, \ldots, v_n$ be n linearly independent vectors in a vector space V. Suppose that the set
   $$\{\, v_1 + v_2,\ v_2 + v_3,\ \ldots,\ v_{n-1} + v_n,\ v_n + v_1 \,\}$$
   is also linearly independent. What can you conclude about the parity of n?

   Let $\{v_1, \ldots, v_n\}$ be linearly independent and suppose
   $$a_1(v_1 + v_2) + a_2(v_2 + v_3) + \cdots + a_{n-1}(v_{n-1} + v_n) + a_n(v_n + v_1) = 0.$$
   Expanding and collecting coefficients of the v_i gives the system
   $$\begin{aligned} a_1 + a_n &= 0, \\ a_1 + a_2 &= 0, \\ a_2 + a_3 &= 0, \\ &\vdots \\ a_{n-2} + a_{n-1} &= 0, \\ a_{n-1} + a_n &= 0. \end{aligned}$$
   (2 Marks) From $a_1 + a_2 = 0$ we get $a_2 = -a_1$; from $a_2 + a_3 = 0$ we get $a_3 = -a_2 = a_1$; and so on, hence
   $$a_k = (-1)^{k-1} a_1 \qquad (k = 1, 2, \ldots, n).$$
   Using $a_1 + a_n = 0$ and $a_n = (-1)^{n-1} a_1$ gives
   $$a_1\big(1 + (-1)^{n-1}\big) = 0.$$
   If n is even then $1 + (-1)^{n-1} = 0$, so we get nonzero a_1. **(2 Marks)** Therefore, for the set $\{v_1 + v_2, \ldots, v_n + v_1\}$ to be linearly independent we must have $1 + (-1)^{n-1} \neq 0$, i.e. n is odd. **(1 Mark)**

5. **Target:** Basis in function spaces. **Source:** Practice sheet.

   Let $\mathbb{R}^{\{a,b,c\}}$ be the vector space of all functions $f : \{a, b, c\} \to \mathbb{R}$. Define the functions $e_a, e_b, e_c : \{a, b, c\} \to \mathbb{R}$ by
   $$e_a(k) = \begin{cases} 1, & k = a, \\ 0, & k = b \text{ or } c, \end{cases} \qquad e_b(k) = \begin{cases} 1, & k = b, \\ 0, & k = a \text{ or } c, \end{cases} \qquad e_c(k) = \begin{cases} 1, & k = c, \\ 0, & k = a \text{ or } b. \end{cases}$$
   Is the set $\{e_a, e_b, e_c\}$ a basis for $\mathbb{R}^{\{a,b,c\}}$? Justify.

   By definition, a subset of a vector space is a basis if it is linearly independent and spans the space.

   Linear independence. Suppose there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that
   $$\alpha e_a + \beta e_b + \gamma e_c = 0,$$
   where 0 denotes the zero function (the function taking the value 0 at each of a, b, c). Evaluate this equality at the point a:
   $$(\alpha e_a + \beta e_b + \gamma e_c)(a) = \alpha \cdot e_a(a) + \beta \cdot e_b(a) + \gamma \cdot e_c(a) = \alpha \cdot 1 + \beta \cdot 0 + \gamma \cdot 0 = \alpha.$$
   Because the right-hand side is the zero function, its value at a is 0, so $\alpha = 0$. Evaluating at b gives $\beta = 0$, and evaluating at c gives $\gamma = 0$. Thus the only linear relation is the trivial one, so $\{e_a, e_b, e_c\}$ is linearly independent. **(2 Marks)**

Spanning. Let $f \in \mathbb{R}^{\{a,b,c\}}$ be an arbitrary function. Define the function

$$g := f(a)\, e_a + f(b)\, e_b + f(c)\, e_c.$$

For each point of the domain we have

$$g(a) = f(a) \cdot e_a(a) + f(b) \cdot e_b(a) + f(c) \cdot e_c(a) = f(a),$$

and similarly $g(b) = f(b)$ and $g(c) = f(c)$. Hence $g = f$. Therefore every f is a linear combination of e_a, e_b, e_c, so these functions span $\mathbb{R}^{\{a,b,c\}}$.

Since $\{e_a, e_b, e_c\}$ is both linearly independent and spanning, it is a basis of $\mathbb{R}^{\{a,b,c\}}$. $\square$

(3 Marks)

6. **Target:** Concept of Basis, new basis from old. **Source:** Practice Sheet.

Case 1: $\dim V = 0$. Then $V = \{0\}$ and the only basis is the empty set. Hence V has exactly one basis. **(1 Mark)**

Case 2: $\dim V = 1$. In this case V is isomorphic to $\mathbb{F}$. Every basis consists of a single nonzero vector. Thus the number of bases is equal to the number of nonzero vectors in V, which is $|\mathbb{F}| - 1$. Hence V has exactly one basis if and only if $|\mathbb{F}| - 1 = 1$, i.e. $|\mathbb{F}| = 2$. Therefore, the 1-dimensional vector spaces over a field with 2 elements have exactly one basis.**(2 Marks)**

Case 3: $\dim V \geq 2$ *(finite or infinite)*. Let B be a basis of V. Since $|B| \geq 2$, choose distinct $u, v \in B$. Consider

$$B' := (B \setminus \{u\}) \cup \{u + v\}.$$

It is easy to verify that B' is again a basis of V and $B' \neq B$. Thus V has at least two distinct bases. **(2 Marks)**

Conclusion: The only vector spaces over $\mathbb{F}$ that have exactly one basis are

$$V = \{0\} \quad \text{(over any field)}, \qquad \text{and} \qquad \dim V = 1 \text{ over } \mathbb{F},$$

where $\mathbb{F}$ is a field with two elements.

7. (a) **Target:** Understanding range space. **Source:** Lecture.
Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a mapping (not identically zero) whose range is contained in the paraboloid

$$z = x^2 + y^2.$$

State in a short sentence or two how you know that T is not a linear transformation.

Let $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and $\text{range}(T) \subseteq P$. If $\text{range}(T) \neq \{0\}$ pick a nonzero

$$v = (x, y, x^2 + y^2) \in \text{range}(T) \subset P.$$

For $\alpha \in \mathbb{R}$ the vector αv in general does not lie in P. Thus P is not closed under arbitrary scalar multiplication, and the only linear subspace of $\mathbb{R}^3$ contained in P is $\{0\}$. This contradicts $\text{range}(T) \neq \{0\}$. Therefore T cannot be linear.
(2 Marks)

(b) **Target:** Exploring beyond standard result. **Source:** Lecture.
Prove or disprove: If V is a vector space and $U, V, W \subseteq V$, are subspaces of V, then

$$\begin{aligned}\dim(U + V + W) = {} & \dim(U) + \dim(V) + \dim(W) - \dim(U \cap V) - \dim(V \cap W) - \dim(W \cap U) \\ & + \dim(U \cap V \cap W).\end{aligned}$$

There is a minor notational issue in the question. As announced in the exam hall, let us take the vector space as V' and subspaces as U, V, W.

The statement is *false* in general.

Counterexample. Consider $V = \mathbb{R}^2$ and define

$$U = \operatorname{span}\{(1,1)\}, \quad V = \operatorname{span}\{(1,0)\}, \quad W = \operatorname{span}\{(0,1)\}.$$

(1 Mark)
Then

$$\dim U = \dim V = \dim W = 1.$$

Moreover, all pairwise and triple intersections are trivial:

$$U \cap V = \{0\}, \quad U \cap W = \{0\}, \quad V \cap W = \{0\}, \quad U \cap V \cap W = \{0\}.$$

Hence the right–hand side of the proposed formula equals

$$1 + 1 + 1 - 0 - 0 - 0 + 0 = 3.$$

On the other hand,

$$U + V + W = \mathbb{R}^2,$$

so

$$\dim(U + V + W) = \dim \mathbb{R}^2 = 2.$$

Thus the proposed equality would assert $2 = 3$, which is a contradiction.
(2 Marks)

8. Let $A, B, C, D \in M_{n\times n}(\mathbb{R})$ be square matrices. Define

$$T : M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad T(X) = AXB + CX + XD.$$

(a) **Target:** Verify additivity and homogeneity. **Source:** Practice Sheet.
Prove that T is a linear operator on $M_{n\times n}(\mathbb{R})$.

Solution. Recall that $M_{n\times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$ with matrix addition and scalar multiplication defined entrywise. To show that T is linear we must check

$$T(X + Y) = T(X) + T(Y) \qquad \text{and} \qquad T(\lambda X) = \lambda T(X)$$

for all $X, Y \in M_{n\times n}(\mathbb{R})$ and all scalars $\lambda \in \mathbb{R}$.
Let $X, Y \in M_{n\times n}(\mathbb{R})$. Using distributivity and associativity of matrix multiplication,

$$\begin{aligned} T(X + Y) &= A(X + Y)B + C(X + Y) + (X + Y)D \\ &= (AX + AY)B + CX + CY + XD + YD \\ &= AXB + AYB + CX + CY + XD + YD \\ &= \big(AXB + CX + XD\big) + \big(AYB + CY + YD\big) \\ &= T(X) + T(Y). \end{aligned}$$

(1 Mark)
Next let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} T(\lambda X) &= A(\lambda X)B + C(\lambda X) + (\lambda X)D \\ &= \lambda(AXB) + \lambda(CX) + \lambda(XD) \\ &= \lambda\big(AXB + CX + XD\big) \\ &= \lambda T(X), \end{aligned}$$

where we used that scalar multiplication distributes over matrix multiplication. **(1 Mark)** Since both additivity and homogeneity hold, T is a linear operator on $M_{n\times n}(\mathbb{R})$.

(b) **Target:** To assess the ability to formulate intuitive conjectures and subsequently provide a rigorous proof.

Source: Lecture, Practice Sheet.

Let $C = D = 0$. Investigate a sufficient condition in terms of matrices A, B that makes the transformation T invertible.

Assume $C = D = 0$. Then T is the linear map

$$T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}), \qquad T(X) = AXB.$$

Claim. T is invertible if and only if both A and B are invertible. In particular, a convenient sufficient condition is that A and B are invertible (and this condition is also necessary). **(1 Mark)**

First suppose A and B are invertible. Define

$$S : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}), \qquad S(Y) = A^{-1}YB^{-1}.$$

Then for every X,

$$S(T(X)) = A^{-1}(AXB)B^{-1} = (A^{-1}A)X(BB^{-1}) = X,$$

and similarly

$$T(S(Y)) = A(A^{-1}YB^{-1})B = Y.$$

Thus $S = T^{-1}$ and T is invertible.

(2 Marks)

9. **Target:** Verification of theorem. **Source:** Lecture, Practice Sheet.

Let $P_n(\mathbb{R})$ be the vector space of real polynomials of degree at most n. Define the linear transformation $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ by

$$T(p)(x) = xp'(x) - p(x).$$

Verify the fundamental theorem for linear transformation (rank–nullity theorem) for T.

Write a general polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

with real coefficients $a_0, \ldots, a_n$. Then

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1},$$

so

$$xp'(x) = a_1x + 2a_2x^2 + 3a_3x^3 + \cdots + na_nx^n.$$

Therefore

$$T(p)(x) = xp'(x) - p(x) = (-a_0) + 0 \cdot x + (2-1)a_2x^2 + (3-1)a_3x^3 + \cdots + (n-1)a_nx^n.$$

Equating coefficients, the image polynomial $T(p)$ has coefficients

$$b_k = (k-1)a_k \qquad \text{for } k = 0, 1, 2, \ldots, n,$$

(where by convention b_k is the coefficient of x^k in $T(p)$). Concretely:

$$b_0 = -a_0, \quad b_1 = 0, \quad b_k = (k-1)a_k \ (k \geq 2).$$

Kernel. Solve $T(p) = 0$. The coefficient relations give $(k-1)a_k = 0$ for each k. Thus

$$a_k = 0 \quad \text{for all } k \neq 1,$$

while a_1 is arbitrary. Hence

$$\ker T = \operatorname{span}\{x\},$$

so $\dim \ker T = 1$ when $n \geq 1$. (If $n = 0$ then trivially $\ker T = \{0\}$ and $\dim \ker T = 0$.) **(1 Mark)**

Image. From the coefficient relations we see that the coefficient of x in any $T(p)$ is zero, while any choice of coefficients for the monomials $1, x^2, \ldots, x^n$ occurs: indeed, for $k \in \{0\} \cup \{2, \ldots, n\}$ the map $a_k \mapsto b_k = (k-1)a_k$ is a bijection $\mathbb{R} \to \mathbb{R}$ (since $k - 1 \neq 0$). Hence

$$\operatorname{Im} T = \operatorname{span}\{1, x^2, x^3, \ldots, x^n\},$$

and therefore

$$\dim \operatorname{Im} T = \begin{cases} n, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

(1 Mark) Rank -Nullity Theorem **(2 marks)**
Rank–nullity. For $n \geq 1$,

$$\dim \ker T + \dim \operatorname{Im} T = 1 + n = n + 1 = \dim P_n(\mathbb{R}),$$

and for $n = 0$,

$$\dim \ker T + \dim \operatorname{Im} T = 0 + 1 = 1 = \dim P_0(\mathbb{R}).$$

Thus the rank–nullity theorem holds for T in all cases. □

(1 Mark)

10. **Target:** Ability to identify errors and produce a correct proof. **Source:** Lecture.

Result. Suppose V is a finite-dimensional vector space and U is a subspace of V. Then there exists a subspace W of V such that

$$V = U \oplus W.$$

Task. The "proof" given by one of your classmates is reprinted in the Appendix below. Read the proof carefully. If the proof is valid, fill in any missing details. If the proof is invalid, point out the errors and help your classmate write a correct proof. You will be rewarded for your assistance!

Appendix (classmate's proof).

Proof. Since V is finite-dimensional, choose a basis $\{v_1, v_2, \ldots, v_n\}$ of V. Because U is a subspace of V, some of the vectors $v_1, \ldots, v_n$ lie in U. Without loss of generality, assume $v_1, \ldots, v_m \in U$ and $v_{m+1}, \ldots, v_n \notin U$. Then $v_1, \ldots, v_m$ span U, so they form a basis of U. Let

$$W = \operatorname{span}\{v_{m+1}, \ldots, v_n\}.$$

Every $v \in V$ can be written as

$$v = a_1v_1 + \cdots + a_mv_m + a_{m+1}v_{m+1} + \cdots + a_nv_n = u + w,$$

where $u = a_1v_1 + \cdots + a_mv_m \in U$ and $w = a_{m+1}v_{m+1} + \cdots + a_nv_n \in W$. Thus $V = U + W$.

To see that $U \cap W = \{0\}$, suppose $x \in U \cap W$. Then

$$x = b_1v_1 + \cdots + b_mv_m = c_{m+1}v_{m+1} + \cdots + c_nv_n.$$

Hence

$$b_1v_1 + \cdots + b_mv_m - c_{m+1}v_{m+1} - \cdots - c_nv_n = 0.$$

Since $v_1, \ldots, v_n$ is a basis of V and therefore linearly independent, all coefficients must be zero, so $x = 0$. Therefore $U \cap W = \{0\}$.

Consequently $V = U \oplus W$, as desired. □

Errors in the classmate's proof

The proof in the Appendix makes the following unjustified assumptions:

(a) It assumes that, given an arbitrary basis $\{v_1, \dots, v_n\}$ of V, some of these vectors lie in U. This is not always true. *Counterexample:* In $V = \mathbb{R}^2$ with basis $\{(1,0),(0,1)\}$, take $U = \operatorname{span}\{(1,1)\}$. Neither basis vector belongs to U.

(b) It further assumes that the basis vectors lying in U automatically span U. This also need not hold: even if some basis vectors happen to lie in U, they might not span the entire subspace.

Because of these issues, the proof as written is invalid. **(2 Marks)**

Correct Proof

We correct the argument using the basis-extension theorem.

Proof. Let $\{u_1, \dots, u_m\}$ be a basis of U. By the basis-extension theorem, we can find vectors $w_1, \dots, w_k \in V$ such that

$$\mathcal{B} = \{u_1, \dots, u_m, w_1, \dots, w_k\}$$

is a basis of V. Define

$$W = \operatorname{span}\{w_1, \dots, w_k\}.$$

Step 1: $V = U + W$. Every $v \in V$ can be expressed uniquely as

$$v = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_kw_k.$$

The first sum belongs to U and the second to W. Hence $v \in U + W$, so $V = U + W$.

Step 2: $U \cap W = \{0\}$. Suppose $x \in U \cap W$. Then

$$x = \sum_{i=1}^{m} \alpha_i u_i = \sum_{j=1}^{k} \beta_j w_j.$$

Subtracting gives

$$\sum_{i=1}^{m} \alpha_i u_i - \sum_{j=1}^{k} \beta_j w_j = 0.$$

Since $\mathcal{B}$ is linearly independent, all coefficients are zero, so $x = 0$. Therefore $U \cap W = \{0\}$.

Combining Step 1 and Step 2, we conclude

$$V = U \oplus W.$$

(3 Marks)