MTL104 Linear Algebra and Applications
I Semester 2025-26
Practice Sheet III-B: Hints and Solutions

4. Let $\dim V = n$ and $\dim U = k$. Since $U \neq V$, we have $k < n$. Choose a basis $\{u_1, \ldots, u_k\}$ of U and extend it to a basis of V:

$$\{u_1, \ldots, u_k, v_{k+1}, \ldots, v_n\}.$$

Define a linear functional $\varphi : V \to \mathbb{F}$ (where $\mathbb{F}$ is the underlying field) by setting

$$\varphi(u_i) = 0 \quad (1 \leq i \leq k), \qquad \varphi(v_{k+1}) = 1, \qquad \varphi(v_j) = 0 \quad (k+2 \leq j \leq n),$$

and extending φ linearly to all of V.

By construction, $\varphi(u) = 0$ for every $u \in U$, and $\varphi(v_{k+1}) = 1 \neq 0$. Hence, φ is a nonzero linear functional satisfying

$$\varphi(u) = 0 \quad \text{for all } u \in U.$$

That is, $0 \neq \varphi \in U^\circ$. Alternatively, note that the *annihilator* of U is defined as

$$U^\circ = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

Since $\dim U^\circ = \dim V - \dim U = n - k > 0$, the subspace U° contains nonzero functionals, so such a φ exists.

5. Let $v \in V$ and $(a_1, \ldots, a_n) \in \mathbb{F}^n$. Using the defining property of the dual basis, $\varphi_i(v_j) = \delta_{ij}$, we compute

$$\Gamma\big(\Lambda(a_1, \ldots, a_n)\big) = \Gamma\Big(\sum_{j=1}^{n} a_j v_j\Big) = \big(\varphi_1(\sum_j a_j v_j), \ldots, \varphi_n(\sum_j a_j v_j)\big).$$

Since each φ_i is linear,

$$\varphi_i\Big(\sum_j a_j v_j\Big) = \sum_j a_j \varphi_i(v_j) = \sum_j a_j \delta_{ij} = a_i,$$

so $\Gamma(\Lambda(a_1, \ldots, a_n)) = (a_1, \ldots, a_n)$. Thus $\Gamma \circ \Lambda = \mathrm{Id}_{\mathbb{F}^n}$.

Conversely, for $v \in V$ write $v = \sum_{j=1}^{n} b_j v_j$ (unique coordinates b_j). Then

$$\Lambda\big(\Gamma(v)\big) = \Lambda\big(\varphi_1(v), \ldots, \varphi_n(v)\big) = \sum_{i=1}^{n} \varphi_i(v)\, v_i.$$

But $\varphi_i(v) = \varphi_i\big(\sum_j b_j v_j\big) = \sum_j b_j \varphi_i(v_j) = b_i$, so

$$\Lambda(\Gamma(v)) = \sum_{i=1}^{n} b_i v_i = v.$$

Hence $\Lambda \circ \Gamma = \mathrm{Id}_V$. Therefore Γ and Λ are mutual inverses.

6. Let $p_j(x) = x^j$ for $j = 0, \ldots, m$. For fixed k with $0 \leq k \leq m$ compute the k-th derivative of p_j:

$$p_j^{(k)}(x) = \begin{cases} \dfrac{j!}{(j-k)!}\, x^{j-k}, & j \geq k, \end{cases}$$

Evaluating at $x = 0$ gives

$$p_j^{(k)}(0) = \begin{cases} 0, & j > k, \\ k!, & j = k, \\ 0, & j < k, \end{cases}$$

so

$$\varphi_k(p_j) = \frac{p_j^{(k)}(0)}{k!} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Thus $\varphi_k(p_j) = \delta_{kj}$. Since each φ_k is linear and $\{\varphi_0, \ldots, \varphi_m\}$ satisfies $\varphi_k(x^j) = \delta_{kj}$ for all $0 \leq k, j \leq m$, it follows by the defining property of the dual basis that $\{\varphi_0, \ldots, \varphi_m\}$ is the dual basis of $\{1, x, \ldots, x^m\}$.

7. Let $I : V \to V$ be the identity operator on V, denoted by Id_V. Let $I' : V' \to V'$ denote its dual map. By definition, for each $\varphi \in V'$ we have

$$I'(\varphi) = \varphi \circ I.$$

Since $I(v) = v$ for all $v \in V$, it follows that $\varphi \circ I = \varphi$. Therefore $I'(\varphi) = \varphi$ for every $\varphi \in V'$, i.e. $I' = \mathrm{Id}_{V'}$.

8. Let V be finite-dimensional and fix $v_1, \ldots, v_m \in V$. Define

$$\Gamma : V' \to \mathbb{F}^m, \qquad \Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).$$

(a) $v_1, \ldots, v_m$ *spans* V *iff* Γ *is injective.*

(⇒) Suppose $\operatorname{span}\{v_1, \ldots, v_m\} = V$. If $\varphi \in \ker \Gamma$ then $\varphi(v_i) = 0$ for all i, hence φ vanishes on $\operatorname{span}\{v_1, \ldots, v_m\} = V$. Thus $\varphi \equiv 0$, so $\ker \Gamma = \{0\}$ and Γ is injective.

(⇐) Conversely, if $\operatorname{span}\{v_1, \ldots, v_m\} \neq V$ set $S = \operatorname{span}\{v_1, \ldots, v_m\}$. Choose $w \in V \setminus S$ and extend a basis of S by adding w (and further vectors if necessary) to obtain a basis of V. Define a linear functional $\varphi \in V'$ by prescribing φ to be 0 on all vectors of the basis coming from S and $\varphi(w) = 1$, then extend linearly. This φ is nonzero but $\varphi(v_i) = 0$ for all i, so $\varphi \in \ker \Gamma \setminus \{0\}$. Hence Γ is not injective. This proves the equivalence.

(b) $v_1, \ldots, v_m$ *is linearly independent iff* Γ *is surjective.*

(⇒) Assume $v_1, \ldots, v_m$ are linearly independent. Extend $\{v_1, \ldots, v_m\}$ to a basis $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ of V. Given any $(a_1, \ldots, a_m) \in \mathbb{F}^m$ define $\varphi \in V'$ by

$$\varphi(v_i) = a_i \ (1 \leq i \leq m), \qquad \varphi(v_j) = 0 \ (m+1 \leq j \leq n),$$

and extend linearly. Then $\Gamma(\varphi) = (a_1, \ldots, a_m)$, so every vector of $\mathbb{F}^m$ is attained and Γ is surjective.

(⇐) Conversely, suppose $v_1, \ldots, v_m$ are linearly dependent. Then there exist scalars $c_1, \ldots, c_m$, not all zero, with

$$\sum_{i=1}^{m} c_i v_i = 0.$$

For every $\varphi \in V'$ we have

$$0 = \varphi\Big(\sum_{i=1}^{m} c_i v_i\Big) = \sum_{i=1}^{m} c_i \varphi(v_i),$$

Hence $\Gamma(V')$ is contained in a proper hyperplane of $\mathbb{F}^m$ and cannot equal all of $\mathbb{F}^m$. Thus Γ is not surjective. This completes the proof.

9. This is essentially the "dual" version of the previous problem. Proof is similar.

10. Let $\dim V = n$ and let $\{\varphi_1, \ldots, \varphi_n\}$ be a basis of V'. We want to find a basis $\{v_1, \ldots, v_n\}$ of V such that $\varphi_i(v_j) = \delta_{ij}$ for all $1 \le i, j \le n$.

    Define a linear map
    $$\Gamma : V \to \mathbb{F}^n, \quad \Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)).$$
    Since $\varphi_1, \ldots, \varphi_n$ span V', part (a) of the previous problem shows that Γ is injective. But $\dim V = n = \dim \mathbb{F}^n$, so an injective linear map between spaces of equal finite dimension is also surjective. Hence Γ is an isomorphism.

    Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{F}^n$, and define
    $$v_j = \Gamma^{-1}(e_j), \quad j = 1, \ldots, n.$$
    Then $\{v_1, \ldots, v_n\}$ is a basis of V, and
    $$\varphi_i(v_j) = \varphi_i(\Gamma^{-1}(e_j)) = e_j(i) = \delta_{ij},$$
    so the dual basis of $\{v_1, \ldots, v_n\}$ is exactly $\{\varphi_1, \ldots, \varphi_n\}$.

11. Let $f, g \in V'$ be linear functionals such that
    $$f(v) = 0 \iff g(v) = 0 \quad \text{for all } v \in V.$$

    Step 1: Pick a vector with $f(v_0) \neq 0$.

    If $f \equiv 0$, then $g \equiv 0$ as well, and the result holds with $\alpha = 0$. Otherwise, choose $v_0 \in V$ such that $f(v_0) \neq 0$.

    Step 2: Define the scalar α.

    Set
    $$\alpha := \frac{g(v_0)}{f(v_0)}.$$
    We claim that $g = \alpha f$.

    Step 3: Show $g(v) = \alpha f(v)$ for all v.

    Let $v \in V$ be arbitrary. Consider the vector
    $$w = f(v_0)v - f(v)v_0.$$
    Then
    $$f(w) = f(v_0)f(v) - f(v)f(v_0) = 0,$$
    so by assumption $g(w) = 0$. But
    $$g(w) = f(v_0)g(v) - f(v)g(v_0) = f(v_0)g(v) - f(v)(\alpha f(v_0)) = f(v_0)(g(v) - \alpha f(v)).$$
    Since $f(v_0) \neq 0$, it follows that $g(v) - \alpha f(v) = 0$, i.e.,
    $$g(v) = \alpha f(v) \quad \text{for all } v \in V.$$

Thus $g = \alpha f$ as claimed.

12. Let $f : V \to \mathbb{F}$ be linear and let $v \in V \setminus N(f)$, where $N(f) = \ker(f)$.

    Step 1: Show $V = N(f) + \{\alpha v : \alpha \in \mathbb{F}\}$.
    Let $w \in V$ be arbitrary. Since $f(v) \neq 0$, set

    $$\alpha := \frac{f(w)}{f(v)} \in \mathbb{F}.$$

    Then $u := w - \alpha v$ satisfies

    $$f(u) = f(w - \alpha v) = f(w) - \alpha f(v) = 0,$$

    so $u \in N(f)$. Therefore,

    $$w = u + \alpha v \in N(f) + \{\alpha v : \alpha \in \mathbb{F}\}.$$

    Since w was arbitrary, we have

    $$V = N(f) + \{\alpha v : \alpha \in \mathbb{F}\}.$$

    Step 2: Show the sum is direct.
    Let $x \in N(f) \cap \{\alpha v : \alpha \in \mathbb{F}\}$. Then $x = \alpha v$ for some $\alpha \in \mathbb{F}$ and $x \in N(f)$, so

    $$0 = f(x) = f(\alpha v) = \alpha f(v).$$

    Since $f(v) \neq 0$, it follows that $\alpha = 0$ and hence $x = 0$. Therefore

    $$N(f) \cap \{\alpha v : \alpha \in \mathbb{F}\} = \{0\},$$

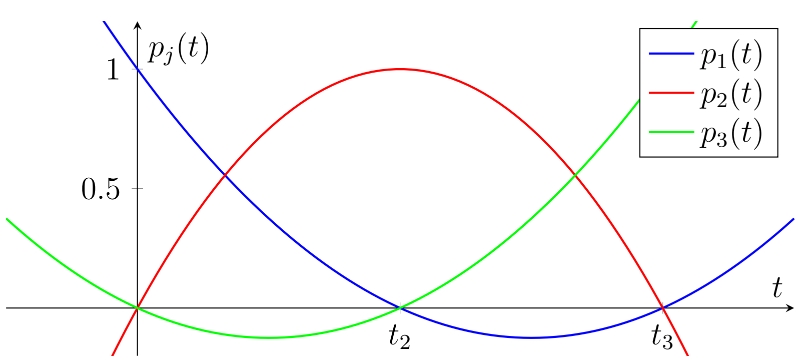
    so the sum is direct.

    Combining Step 1 and Step 2, we conclude

    $$V = N(f) \oplus \{\alpha v : \alpha \in \mathbb{F}\}.$$

13. We illustrate the Lagrange polynomials for $n = 3$ points $t_1 = 0, t_2 = 1, t_3 = 2$:

    $$p_1(t) = \frac{(t-1)(t-2)}{(0-1)(0-2)}, \quad p_2(t) = \frac{(t-0)(t-2)}{(1-0)(1-2)}, \quad p_3(t) = \frac{(t-0)(t-1)}{(2-0)(2-1)}$$



Let $t_1, \dots, t_n \in \mathbb{R}$ be distinct, and consider the space of polynomials $\mathcal{P}_{n-1}(\mathbb{R})$ of degree at most $n-1$. Define linear functionals

$$L_i(p) = p(t_i), \quad i = 1, \dots, n.$$

Define the Lagrange polynomials:

$$p_j(t) = \frac{\prod_{k \neq j}(t - t_k)}{\prod_{k \neq j}(t_j - t_k)}, \quad j = 1, \dots, n.$$

Problem (a): Show that $\{p_1, \dots, p_n\}$ is a basis of $\mathcal{P}_{n-1}(\mathbb{R})$

Step 1: Linear independence

Consider a linear combination

$$c_1 p_1(t) + \cdots + c_n p_n(t) = 0 \quad \text{(zero polynomial)}.$$

Evaluate at t_i:

$$(c_1 p_1 + \cdots + c_n p_n)(t_i) = c_1 p_1(t_i) + \cdots + c_n p_n(t_i) = c_i = 0,$$

since $p_j(t_i) = \delta_{ij}$. Hence all coefficients $c_i = 0$, proving linear independence.

Step 2: Dimension argument

$\dim \mathcal{P}_{n-1}(\mathbb{R}) = n$ and we have n linearly independent polynomials, so $\{p_1, \dots, p_n\}$ is a **basis**.

Problem (b): Show that $\{L_1, \dots, L_n\}$ is a basis of the dual space

The dual space $\mathcal{P}^*_{n-1}(\mathbb{R})$ consists of all linear functionals on $\mathcal{P}_{n-1}(\mathbb{R})$.

Step 1: Linear independence

Suppose

$$c_1 L_1 + \cdots + c_n L_n = 0 \quad \text{(zero functional)}.$$

Then for any polynomial $p \in \mathcal{P}_{n-1}$:

$$(c_1 L_1 + \cdots + c_n L_n)(p) = c_1 p(t_1) + \cdots + c_n p(t_n) = 0.$$

Take $p = p_j$ (the Lagrange polynomial), then

$$0 = c_1 p_j(t_1) + \cdots + c_n p_j(t_n) = c_j \implies c_j = 0.$$

Hence, $\{L_1, \dots, L_n\}$ is linearly independent.

Step 2: Dimension argument

$\dim \mathcal{P}^*_{n-1} = n$ and we have n linearly independent functionals, so $\{L_1, \dots, L_n\}$ is a **basis of the dual space**.

Problem (c): Solve the interpolation problem

Given $a_1, \dots, a_n \in \mathbb{R}$, define

$$p(t) = \sum_{j=1}^{n} a_j p_j(t).$$

Step 1: Verify interpolation

Evaluate at t_i:

$$p(t_i) = \sum_{j=1}^{n} a_j p_j(t_i) = \sum_{j=1}^{n} a_j \delta_{ij} = a_i.$$

Hence $p(t_i) = a_i$ for all i.

Step 2: Uniqueness

Suppose $q(t)$ is another polynomial of degree at most $n-1$ satisfying $q(t_i) = a_i$. Then

$$r(t) = p(t) - q(t)$$

is a polynomial of degree at most $n-1$ with

$$r(t_i) = p(t_i) - q(t_i) = 0, \quad i = 1, \dots, n.$$

A nonzero polynomial of degree at most $n-1$ cannot have n distinct roots, so $r(t) = 0$. Therefore, $p(t) = q(t)$ is unique.

Conclusion:

- $\{p_1, \dots, p_n\}$ is a basis of $\mathcal{P}_{n-1}(\mathbb{R})$.
- $\{L_1, \dots, L_n\}$ is a basis of $\mathcal{P}_{n-1}^*(\mathbb{R})$.
- For any prescribed values $a_1, \dots, a_n$ at $t_1, \dots, t_n$, the unique interpolating polynomial is

$$\boxed{p(t) = \sum_{j=1}^{n} a_j p_j(t)}.$$

Remark: Since the Lagrange polynomials satisfy

$$p_j(t_i) = \delta_{ij}, \quad i, j = 1, \dots, n,$$

it follows that the functionals $L_1, \dots, L_n$, defined by $L_i(p) = p(t_i)$, form the **dual basis** corresponding to the basis $\{p_1, \dots, p_n\}$ of $\mathcal{P}_{n-1}(\mathbb{R})$.

14. (a) Let U, W be subspaces of a finite-dimensional vector space V. Then

$$(U + W)^0 = U^0 \cap W^0 \quad \text{and} \quad (U \cap W)^0 = U^0 + W^0.$$

Step 1: Intersection formula. We first show that

$$(U + W)^0 = U^0 \cap W^0.$$

Forward inclusion: Let $f \in (U+W)^0$. Then $f(v) = 0$ for all $v \in U+W$. In particular, for all $u \in U$ and $w \in W$, we have $f(u) = 0$ and $f(w) = 0$, so $f \in U^0 \cap W^0$. Hence,

$$(U+W)^0 \subseteq U^0 \cap W^0.$$

Reverse inclusion: Let $f \in U^0 \cap W^0$. Then $f(u) = 0$ for all $u \in U$ and $f(w) = 0$ for all $w \in W$. Any $v \in U+W$ can be written as $v = u+w$ with $u \in U, w \in W$. Then

$$f(v) = f(u+w) = f(u) + f(w) = 0 + 0 = 0,$$

so $f \in (U+W)^0$. Hence,

$$U^0 \cap W^0 \subseteq (U+W)^0.$$

Combining both inclusions gives

$$(U+W)^0 = U^0 \cap W^0.$$

Step 2: Sum formula. We now prove that

$$(U \cap W)^0 = U^0 + W^0.$$

Inclusion: Let $f \in U^0 + W^0$, so $f = f_1 + f_2$ with $f_1 \in U^0$ and $f_2 \in W^0$. For any $v \in U \cap W$, we have

$$f(v) = f_1(v) + f_2(v) = 0 + 0 = 0,$$

so $f \in (U \cap W)^0$. Hence,

$$U^0 + W^0 \subseteq (U \cap W)^0.$$

Dimension argument: Since V is finite-dimensional, we have

$$\dim S^0 = \dim V - \dim S \quad \text{for any subspace } S \subseteq V.$$

Also, by the standard dimension formula,

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W).$$

Using Step 1, we know that $(U+W)^0 = U^0 \cap W^0$, so

$$\dim(U^0+W^0) = \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) = (\dim V - \dim U) + (\dim V - \dim W) - \dim(U+$$

But $\dim(U+W)^0 = \dim V - \dim(U+W)$, hence

$$\dim(U^0+W^0) = (\dim V - \dim U) + (\dim V - \dim W) - (\dim V - \dim(U+W)) = \dim V - \dim(U \cap W$$

That is,

$$\dim(U^0 + W^0) = \dim(U \cap W)^0.$$

Since we already have the inclusion $U^0 + W^0 \subseteq (U \cap W)^0$ and their dimensions are equal, it follows that

$$(U \cap W)^0 = U^0 + W^0.$$

(b) Let U and W be subspaces of a finite-dimensional vector space V. Prove that if $V = U \oplus W$, then

$$U' \simeq W^0, \quad W' \simeq U^0, \quad \text{and} \quad V' = U^0 \oplus W^0.$$

Step 1: Show $U' \simeq W^0$.

Define a map $\phi : W^0 \to U'$ by restricting functionals:

$$\phi(f) := f|_U, \quad f \in W^0.$$

- **Injective:** If $\phi(f) = 0$, then $f(u) = 0$ for all $u \in U$. Also, $f(w) = 0$ for all $w \in W$ because $f \in W^0$. Since $V = U \oplus W$, any $v \in V$ can be written as $v = u + w$, so $f(v) = f(u) + f(w) = 0$. Hence $f = 0$.
- **Surjective:** For any $g \in U'$, define $\tilde{g} \in V'$ by

  $$\tilde{g}(u + w) := g(u), \quad u \in U, w \in W.$$

  Then $\tilde{g} \in W^0$ and $\phi(\tilde{g}) = g$.

Thus, ϕ is an isomorphism and $U' \simeq W^0$.

Step 2: Show $W' \simeq U^0$.

By symmetry, swapping U and W in Step 1 gives $W' \simeq U^0$.

Step 3: Show $V' = U^0 \oplus W^0$.

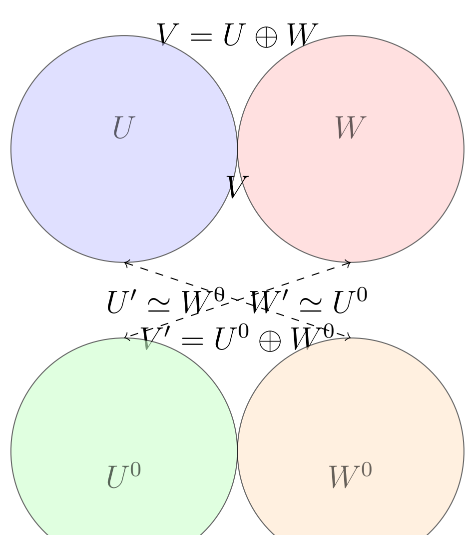
- **Direct sum:** If $f \in U^0 \cap W^0$, then $f(u) = 0$ for all $u \in U$ and $f(w) = 0$ for all $w \in W$. Hence $f(v) = 0$ for all $v = u + w \in V$, so $f = 0$.
- **Dimension argument:** Let $\dim V = n$, $\dim U = k$, $\dim W = n - k$. Then

  $$\dim U^0 = n - k, \quad \dim W^0 = k, \quad \dim(U^0 \oplus W^0) = n = \dim V'.$$

  So $U^0 \oplus W^0$ spans V'.

Hence, we conclude:

$$U' \simeq W^0, \quad W' \simeq U^0, \quad V' = U^0 \oplus W^0.$$



(c) Let U be a subspace of a finite-dimensional vector space V. Show that the restriction of the natural isomorphism

$$T : V \to V'', \quad (Tv)(g) = g(v), \ g \in V'$$

to U is an isomorphism from U to U^{00}, where

$$U^{00} := \{F \in V'' : F(f) = 0 \ \forall f \in U^0\}.$$

Step 1: Well-defined map into U^{00}
For $u \in U$, consider $Tu \in V''$:

$$(Tu)(f) = f(u), \quad \forall f \in V'.$$

If $f \in U^0$, then $f(u) = 0$ by definition of the annihilator. Hence

$$(Tu)(f) = 0, \quad \forall f \in U^0,$$

so $Tu \in U^{00}$. Therefore, the restriction

$$T|_U : U \to U^{00}, \quad u \mapsto Tu$$

is well-defined.

Step 2: Injectivity
If $Tu = 0$ in U^{00}, then

$$(Tu)(f) = f(u) = 0, \quad \forall f \in V'.$$

Since V' separates points in finite dimensions, this implies $u = 0$. Thus, $T|_U$ is injective.

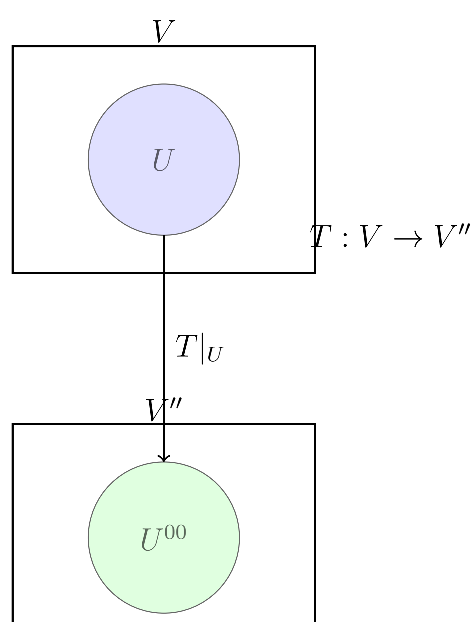
Step 3: Surjectivity
In finite dimensions,

$$\dim U^{00} = \dim U.$$

Since $T|_U : U \to U^{00}$ is injective and the dimensions are equal, it is also surjective.

Conclusion: $T|_U : U \to U^{00}$ is a linear isomorphism, i.e.

$$\boxed{U \simeq U^{00}}.$$



(d) Let U and W be subspaces of a finite-dimensional vector space V. Then

$$U = W \iff U^0 = W^0,$$

where $U^0 := \{f \in V' : f(u) = 0 \ \forall u \in U\}$.
Step 1: $U = W \implies U^0 = W^0$
If $U = W$, then by definition of annihilator,

$$U^0 = \{f \in V' : f(u) = 0 \ \forall u \in U\} = \{f \in V' : f(w) = 0 \ \forall w \in W\} = W^0.$$

Step 2: $U^0 = W^0 \implies U = W$
Let $T : V \to V''$ be the natural isomorphism defined by

$$(Tv)(f) := f(v), \quad \forall f \in V', v \in V.$$

Define the double annihilator

$$U^{00} := \{F \in V'' : F(f) = 0 \ \forall f \in U^0\} \subset V''.$$

The restriction $T|_U : U \to U^{00}$ is an isomorphism. Similarly, $T|_W : W \to W^{00}$ is an isomorphism.
If $U^0 = W^0$, then

$$U^{00} = W^{00}.$$

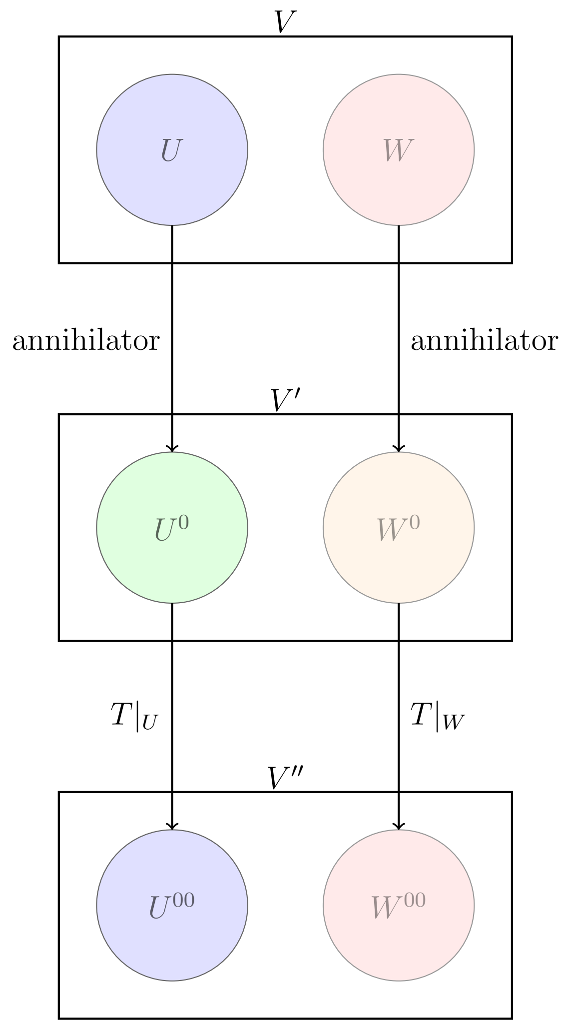
Hence, under the natural isomorphism T, we identify

$$U = W \quad \text{via} \quad T(U) = T(W) = U^{00} = W^{00}.$$

Conclusion: Combining the two directions,

$$U = W \iff U^0 = W^0.$$

$\square$



15.

Definition 0.1 (Invariant Subspace)**.** *Let V be a vector space over $\mathbb{R}$ and $T \in L(V)$. A subspace $W \subseteq V$ is said to be* invariant *under T if*

$$T(W) \subseteq W.$$

Remark 0.1. *In the finite-dimensional case, the adjoint of an operator T, denoted T', and the transpose of T, denoted T^t, are used interchangeably. That is, $T' = T^t$ when V is finite-dimensional.*

Theorem 0.1. *Let V be a finite-dimensional real vector space and $T \in L(V)$. If $W \subseteq V$ is invariant under T, then its annihilator $W^0 \subseteq V'$ is invariant under the adjoint operator T' (or transpose T^t). That is,*

$$T'(W^0) \subseteq W^0.$$

Proof. Let $f \in W^0$. By definition,

$$f(w) = 0 \quad \forall w \in W.$$

Consider $(T'f)(w)$ for any $w \in W$. By definition of the adjoint (or transpose),

$$(T'f)(w) = f(T(w)).$$

Since W is T-invariant, $T(w) \in W$.

Therefore,

$$f(T(w)) = 0.$$

Hence,

$$(T'f)(w) = 0 \quad \forall w \in W,$$

which shows that $T'f \in W^0$. Since f was arbitrary, we conclude

$$T'(W^0) \subseteq W^0,$$

proving the claim. $\square$

16. - In finite-dimensional spaces, the natural map

$$V \longrightarrow V'', \quad v \mapsto F_v$$

  is an isomorphism, so every element of V'' corresponds to some $v \in V$.

- This example illustrates that the canonical embedding $V \hookrightarrow V''$ is **not, in general, surjective** in infinite-dimensional vector spaces. That is, the given infinite dimensional space V is not reflexive.

17. **(a)** Let f be a nonzero linear functional on a vector space V. Define

$$N(f) = \{v \in V : f(v) = 0\}.$$

Step 1. $N(f)$ is a subspace of V.

Indeed, $f(0) = 0$, so $0 \in N(f)$. If $u, v \in N(f)$, then

$$f(u + v) = f(u) + f(v) = 0 + 0 = 0,$$

and for any scalar α,

$$f(\alpha u) = \alpha f(u) = \alpha \cdot 0 = 0.$$

Hence $N(f)$ is a subspace of V.

Step 2. $N(f)$ is a proper subspace.

Since f is nonzero, there exists $w \in V$ such that $f(w) \neq 0$. Thus $w \notin N(f)$, implying that $N(f) \neq V$.

Step 3. Maximality of $N(f)$.

Let U be a subspace of V such that $N(f) \subseteq U \subseteq V$. We claim that either $U = N(f)$ or $U = V$.

Suppose $U \neq N(f)$. Then there exists $w \in U \setminus N(f)$, so $f(w) \neq 0$. We now show that every $v \in V$ belongs to U.

For any $v \in V$, write

$$v = \left(v - \frac{f(v)}{f(w)}w\right) + \frac{f(v)}{f(w)}w.$$

Then

$$f\left(v - \frac{f(v)}{f(w)}w\right) = f(v) - \frac{f(v)}{f(w)}f(w) = 0,$$

so $v - \frac{f(v)}{f(w)} w \in N(f) \subseteq U$, and $w \in U$. Hence $v \in U$. Since v was arbitrary, $U = V$.

Therefore, $N(f)$ is a maximal proper subspace of V.

Conclusion. $N(f)$ is a hyperspace in V.

(b) Let $H \subset V$ be a hyperspace. Choose $v_0 \in V \setminus H$. Since H is maximal, we have

$$H + \text{span}\{v_0\} = V.$$

Define a linear functional $f : V \to \mathbb{F}$ by

$$f(v + \alpha v_0) := \alpha, \quad v \in H, \ \alpha \in \mathbb{F}.$$

Clearly, f is linear and $f(v) = 0$ for all $v \in H$, so $H \subseteq N(f)$. Also, $f(v_0) = 1 \neq 0$, so $N(f) \neq V$.

By part (a), $N(f)$ is a hyperspace. Since $H \subseteq N(f)$ and both are hyperspaces, we must have

$$H = N(f).$$

(c) The linear functional f in part (b) need not be unique.

For example, let $V = \mathbb{R}^2$ and consider the hyperspace $H = \{(x, 0) : x \in \mathbb{R}\}$. - Define $f_1(x, y) = y$, then $N(f_1) = H$. - Define $f_2(x, y) = 2y$, then $N(f_2) = H$ as well.

Thus, multiple functionals can have the same null space.

18.

Theorem 0.2. *If V is an infinite-dimensional vector space over a field $\mathbb{F}$, then*

$$\dim_{\mathbb{F}}(V) < \dim_{\mathbb{F}}(V'),$$

where $V' = L(V, \mathbb{F})$ is the (algebraic) dual space of V.

In particular, replacing V with V', we have

$$\dim_{\mathbb{F}}(V) < \dim_{\mathbb{F}}(V') < \dim_{\mathbb{F}}(V''),$$

and hence $V \not\cong V''$ as vector spaces over $\mathbb{F}$.

This theorem depends on the Axiom of Choice, which is equivalent to the existence of a basis for arbitrary vector spaces.

Proof. **Step 1. V'** is infinite-dimensional. Let $B = \{e_i : i \in I\}$ be a basis of V. For each $i \in I$, define $\phi_i \in V'$ by

$$\phi_i(e_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Then for any $v = \sum_{j \in I} c_j e_j \in V$ (with finitely many nonzero c_j), we have $\phi_i(v) = c_i$. Hence, $\{\phi_i : i \in I\}$ is linearly independent in V', so V' is infinite-dimensional.

Step 2. If W is an infinite-dimensional $\mathbb{F}$-vector space, then

$$|W| = \max\{|\mathbb{F}|, \dim_{\mathbb{F}}(W)\}.$$

Indeed, let $\{e_i : i \in J\}$ be a basis of W. Each $w \in W$ can be uniquely expressed as

$$w = \sum_{i \in J} c_i e_i, \quad c_i \in \mathbb{F}, \text{ finitely many } c_i \neq 0.$$

We can associate to each w the finite set $\{(i, c_i) : c_i \neq 0\} \subseteq J \times \mathbb{F}$, giving an embedding

$$W \hookrightarrow \mathrm{Fin}(J \times \mathbb{F}),$$

where $\mathrm{Fin}(A)$ denotes the set of finite subsets of A. If A is infinite, then

$$|\mathrm{Fin}(A)| = |A|.$$

This is proved in the following.

Lemma 0.1. *If A is an infinite set, then $|\mathrm{Fin}(A)| = |A|$, where $\mathrm{Fin}(A)$ denotes the set of all finite subsets of A.*

Proof. For each $n \in \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A. Then

$$\mathrm{Fin}(A) = \bigcup_{n=0}^{\infty} [A]^n.$$

Since each $[A]^n$ is in bijection with the set of n-tuples of distinct elements of A, we have $|[A]^n| \leq |A|^n = |A|$, because A is infinite. Hence

$$|\mathrm{Fin}(A)| \leq \sum_{n=0}^{\infty} |[A]^n| \leq \aleph_0 \cdot |A| = |A|.$$

On the other hand, the map $a \mapsto \{a\}$ gives an injection $A \hookrightarrow \mathrm{Fin}(A)$, so $|A| \leq |\mathrm{Fin}(A)|$. Combining both inequalities yields

$$|\mathrm{Fin}(A)| = |A|.$$

□

Hence

$$|W| \leq |J \times \mathbb{F}| = \max\{|J|, |\mathbb{F}|\} = \max\{\dim_{\mathbb{F}}(W), |\mathbb{F}|\}.$$

The reverse inequalities $|\mathbb{F}| \leq |W|$ and $\dim_{\mathbb{F}}(W) \leq |W|$ are immediate, as follows.

Remark 0.2. *We claim that the reverse inequalities*

$$|\mathbb{F}| \leq |W| \quad \textit{and} \quad \dim_{\mathbb{F}}(W) \leq |W|$$

hold. This is immediate:

(a) *Since W is nonzero, pick $v \in W$, $v \neq 0$. Then the map*

$$\mathbb{F} \longrightarrow W, \quad a \mapsto av$$

is injective, showing $|\mathbb{F}| \leq |W|$.

(b) Let $\mathcal{B}$ be a basis of W. Since $\mathcal{B} \subseteq W$, we have

$$\dim_{\mathbb{F}}(W) = |\mathcal{B}| \leq |W|.$$

Combining this with the inequality $|W| \leq \max\{\dim_{\mathbb{F}}(W), |\mathbb{F}|\}$ gives equality:

$$|W| = \max\{\dim_{\mathbb{F}}(W), |\mathbb{F}|\}.$$

Step 3. $\dim_{\mathbb{F}}(V') \geq |\mathbb{F}|$. Let $B = \{e_i : i \in I\}$ be a basis of V. Since V is infinite-dimensional, it contains a countably infinite subset $E = \{e_n : n \in \mathbb{N}\} \subseteq B$. For each $a \in \mathbb{F}$, define $\phi_a \in V'$ by

$$\phi_a(e_n) = a^n \quad (n \in \mathbb{N}), \qquad \phi_a(e_i) = 0 \quad (e_i \notin E).$$

Then $\phi_a \neq 0$ for all a, and the map $a \mapsto \phi_a$ is injective since $\phi_a(e_1) = a$. To see that $\{\phi_a : a \in \mathbb{F}\}$ is linearly independent, suppose

$$\sum_{r=1}^{n} \alpha_r \phi_{a_r} = 0,$$

where $a_1, \ldots, a_n$ are distinct and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. Evaluating at e_k for $k = 0, 1, \ldots, n-1$ gives

$$\begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0, \\ \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0, \\ \qquad \vdots \\ \alpha_1 a_1^{n-1} + \alpha_2 a_2^{n-1} + \cdots + \alpha_n a_n^{n-1} = 0. \end{cases}$$

The corresponding Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$$

has nonzero determinant $\prod_{1 \leq j < k \leq n}(a_k - a_j)$, so all $\alpha_r = 0$. Thus $\{\phi_a : a \in \mathbb{F}\}$ is linearly independent, and hence $\dim_{\mathbb{F}}(V') \geq |\mathbb{F}|$.

Step 4. $\dim_{\mathbb{F}}(V') = |V'|$. From Step 1, V' is infinite-dimensional. Applying Step 2 with $W = V'$ and using Step 3, we get

$$|V'| = \max\{|\mathbb{F}|, \dim_{\mathbb{F}}(V')\} = \dim_{\mathbb{F}}(V').$$

Step 5. $\dim_{\mathbb{F}}(V) < \dim_{\mathbb{F}}(V')$. Let $B = \{e_i : i \in I\}$ be a basis of V, so $\dim_{\mathbb{F}}(V) = |I|$. Each $f \in V'$ is determined by its values on B, which can be chosen arbitrarily in $\mathbb{F}$. Thus

$$|V'| = |\mathbb{F}^I|.$$

Since $|\mathbb{F}| \geq 2$, we have

$$|V'| \geq |\{0,1\}^I| \cong |\mathcal{P}(I)|,$$

where $\mathcal{P}(I)$ is the power set of I. By Cantor's theorem, $|\mathcal{P}(I)| > |I|$ since I is infinite. Therefore,

$$|V'| > |I| = \dim_{\mathbb{F}}(V).$$

Using Step 4, we conclude

$$\dim_{\mathbb{F}}(V') = |V'| > \dim_{\mathbb{F}}(V),$$

as desired. $\square$