MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet IV

This practice sheet focuses on characteristic and minimal polynomials, invariant subspaces, diagonalization, the primary decomposition theorem, and basic concepts of inner product spaces.

1. Let T be a linear operator on a finite-dimensional vector space V over $\mathbb{C}$. Then T is diagonalizable if and only if there exists a polynomial $p(x) \in \mathbb{C}[x]$ with *distinct* roots that annihilates T, that is, $p(T) = 0$.

2. Let T be the integral operator defined by
$$(Tf)(x) = \int_0^x f(t)\,dt,$$
on the space of continuous functions on the interval $[0, 1]$.

   Determine whether the following subspaces are invariant under T:

   (a) the space of polynomial functions,
   (b) the space of differentiable functions,
   (c) the space of functions which vanish at $x = \frac{1}{2}$.

3. Let V be a finite-dimensional vector space.

   (a) What is the minimal polynomial for the identity operator on V?
   (b) What is the minimal polynomial for the zero operator on V?

4. Find a 3×3 matrix whose minimal polynomial is x^2.

5. Consider the polynomial space $V = \mathcal{P}_2(\mathbb{R})$ and its standard basis $\mathcal{B} = 1, x, x^2$. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be a linear tranformation defined by
$$T(1) = 2 - 3x^2, \quad T(x) = 4 - 4x - 7x^2, \quad T(x^2) = -2 + 3x + 4x^2.$$
   Find

   (a) the characteristic polynomial of T.
   (b) the minimal polynomial of T.
   (c) a basis $\mathcal{B}'$ for V such that $[T]_{\mathcal{B}'}$ is diagonal.

6. Let V be a finite dimensional vector space over an algebraically closed field F, let $T\colon V \to V$ be linear, and let $f \in F[x]$ be a polynomial. Show that a scalar $c \in F$ is a characteristic value of $f(T)$ if and only if there exists a characteristic value t of T such that $f(t) = c$.

7. Let T be a linear operator on a vector space V. If every subspace of V is invariant under T, prove that T is a scalar multiple of the identity operator.

8. Let $V = M_{n\times n}(F)$ be the vector space of all $n \times n$ matrices over a field F. Fix $A \in M_{n\times n}(F)$ and define a linear operator $T : V \to V$ by
$$T(B) = AB.$$
   Show that the minimal polynomial of the linear operator T equals the minimal polynomial of the matrix A.

9. If E is a projection and f is a polynomial, then show that

$$f(E) = aI + bE.$$

What are a and b in terms of the coefficients of f?

10. Let V be an n-dimensional vector space over a field F and $T \in L(V)$. For $\lambda \in F$, let the *eigenspace* be $\ker(T - \lambda I)$. Its dimension is defined as the *geometric multiplicity* of λ. On the other hand, the *algebraic multiplicity* of λ is defined as its multiplicity as a root of the characteristic polynomial $\chi_T(x) := \det(T - xI) = \det([T]_{\mathcal{B}} - xI)$. Prove or disprove the following claims.

    (a) For each eigenvalue λ of T the geometric multiplicity of λ is at most its algebraic multiplicity.

    (b) T is diagonalizable if and only if for every eigenvalue λ the algebraic multiplicity equals the geometric multiplicity.

11. Let T be a linear operator on a vector space V such that $\operatorname{rank}(T) = 1$. Prove that either T is diagonalizable or T is nilpotent, but not both.

12. Let T be a linear operator on a finite-dimensional vector space V over a field $\mathbb{F}$. Suppose the characteristic and minimal polynomials of T are given by

$$f(x) = \prod_{i=1}^{k} (x - \lambda_i)^{d_i} \quad \text{and} \quad p(x) = \prod_{i=1}^{k} (x - \lambda_i)^{r_i},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T.

Define

$$W_i = \ker\big((T - \lambda_i I)^{r_i}\big).$$

    (a) Prove that W_i is the set of all vectors $a \in V$ such that

$$(T - \lambda_i I)^m a = 0 \quad \text{for some positive integer } m \text{ (which may depend upon } a).$$

    (b) Prove that the dimension of W_i is d_i.

13. Let V be an inner product space. Prove the following identities:

    (a) *Parallelogram identity*: For all $x, y \in V$,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

    Look at $\mathbb{R}^2$, and interpret this identity geometrically, that justifies its name.

    (b) *Polarization identity*:

$$\langle u, v\rangle = \frac{1}{4}\Big(\|u + v\|^2 - \|u - v\|^2 + i\big(\|u + iv\|^2 - \|u - iv\|^2\big)\Big), \qquad \forall\, u, v \in V.$$

14. (a) Suppose a, b, c, d are positive numbers. Prove that

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16.$$

(b) Prove that for all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$,

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right).$$

15. Let $V = \mathcal{P}(\mathbb{R})$ and define

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx \text{ for all } p, q \in \mathcal{P}(\mathbb{R}).$$

(a) Verify that $\langle , \rangle$ is indeed an inner product on V.

(b) Apply the Gram–Schmidt procedure to the basis $\{1, x, x^2\}$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

16. If we apply the Gram–Schmidt orthogonalization process to a set of vectors that is not known a priori to be linearly independent, what will happen?

17. Suppose V is finite-dimensional, $T \in L(V)$, and U is a subspace of V. Prove that

$$U \text{ and } U^\perp \text{ are both invariant under } T \iff P_U T = T P_U,$$

where P_U denotes the orthogonal projection of V onto U.

18. In $\mathbb{R}^4$, let

$$U = \operatorname{span}\{(1, 1, 0, 0), (1, 1, 1, 2)\}.$$

Find $u \in U$ such that $\|u - (1, 2, 3, 4)\|$ is as small as possible.

19. Let $\{a_1, \dots, a_n\}$ be an orthogonal set of nonzero vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$. For any $x \in V$ we have

$$\sum_{k=1}^{n} \frac{|\langle x, a_k\rangle|^2}{\|a_k\|^2} \le \|x\|^2,$$

called Bessel's inequality.

20. Let $V = C([-1, 1])$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let $W = \{f \in V : \ f(-t) = -f(t) \text{ for all } t\}$ be the subspace of odd functions. Show that

$$W^\perp = \{g \in V : \ g(-t) = g(t) \text{ for all } t\},$$

the subspace of even functions.