MTL104 Linear Algebra and Its Applications
I Semester 2025-26
Practice Sheet I

This Practice Sheet is on Field, Subfield, Linear systems of equations, Vector space, Subspace, Sum and direct sum of subspaces.

1. Let F be a field. Show that
   (a) The additive identity in F is unique.
   (b) The additive inverse of an element of F is unique.
   (c) The multiplicative identity of F is unique.
   (d) The multiplicative inverse of a nonzero element of F is unique.
2. Let F be a field. Using the axioms in the definition of field, prove that $(-1)x = -x$ for all $x \in F$. State which axioms are used in your proof.
3. Let F be a field and let $a, b, c \in F$. Show that
   (a) If $a + b = c + b$, then $a = c$.
   (b) If $ab = cb$ and $b \neq 0$, then $a = c$.
4. Let F be a finite field of characteristic p. Then show that p is prime.
5. Consider the field $\mathbb{R}$ of real numbers with the usual addition and multiplication.
   (a) Prove that any subfield of $\mathbb{R}$ contains $\mathbb{Q}$, the set of rational numbers.
   (b) Let d be a square-free integer (not divisible by the square of any prime). Show that
   $$\mathbb{Q}(\sqrt{d}) := \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}$$
   is a field with the usual addition and multiplication. In particular, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(i)$ (here $i^2 = -1$) are fields.
6. Consider the following system of equations:
   $$\begin{cases} x - y - 3z = 3, \\ 2x + z = 0, \\ 2y + 7z = c. \end{cases}$$
   (a) For what values of c does the system have a solution?
   (b) For the value of c you found in (a), describe the solution set geometrically as a subset of $\mathbb{R}^3$.
   (c) What does part (a) say about the planes
   $$x - y - 3z = 3, \quad 2x + z = 0, \quad 2y + 7z = 4$$
   in $\mathbb{R}^3$?
7. Let $V = \{x \in \mathbb{R} : x > 0\}$ with operations
   $$x \oplus y := xy, \qquad \alpha \odot x := x^{\alpha} \quad (\alpha \in \mathbb{R}).$$
   Show that $(V, \oplus, \odot)$ is a real vector space (that is, vector space over the field $\mathbb{R}$).

8. Let V be the set of all $n \times n$ matrices of real entries. Define an operation of "addition" by

$$A \diamond B = \tfrac{1}{2}(AB + BA)$$

for all $A, B \in V$. Define an operation of "scalar multiplication" by

$$\alpha \star A = 0$$

for all $\alpha \in \mathbb{R}$ and $A \in V$. Under the operations $\diamond$ and $\star$ the set V is not a vector space. Identify all the vector space axioms which fail to hold.

9. The following are excerpts from standard proofs in vector space theory. Unfortunately, due to the age of the manuscript, certain portions are missing. Your task is to fill in the blanks with the correct steps and reasons.

Result 1. For every $x \in V$, one has $0 \cdot x = 0$.

Proof (with missing parts): If $x \in V$, then

$$0 \cdot x = (0 + 0) \cdot x \quad \text{(reason: \_\_\_\_)}$$

$$= 0 \cdot x + 0 \cdot x \quad \text{(reason: \_\_\_\_)}$$

Adding the additive inverse of $0 \cdot x$ on both sides, we obtain

$$0 = 0 \cdot x \quad \text{(reason: \_\_\_\_)}.$$

Result 2. For every $x \in V$, one has $(-1) \cdot x = -x$.

Proof (with missing parts): If $x \in V$, then

$$x + (-1) \cdot x = (1 + (-1)) \cdot x \quad \text{(reason: \_\_\_\_)}$$

$$= 0 \cdot x \quad \text{(reason: \_\_\_\_)}$$

$$= 0 \quad \text{(reason: \_\_\_\_)}.$$

It then follows immediately that

$$(-1) \cdot x = -x.$$

Your Task: Fill in the missing reasons, such as:

- Distributive property of scalar multiplication over field addition,
- Distributive property of scalar multiplication over vector addition,
- Definition of additive inverse in V,
- Result 1 above (that $0 \cdot x = 0$).

10. Can a nonzero vector space have finite cardinality if it is defined over a finite field? What about if it is defined over an infinite field? Justify.

11. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers are as usual, and for $t \in \mathbb{R}$ define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$t + \infty = \infty + t = \infty + \infty = \infty, \qquad t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$

$$\infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

12. Let V be the set of all complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$f(-t) = \overline{f(t)} \qquad \text{for all } t \in \mathbb{R},$$

where the bar denotes complex conjugation.

Define the operations:

$$(f + g)(t) = f(t) + g(t), \qquad (cf)(t) = cf(t) \quad (c \in \mathbb{R}).$$

(a) Show that V, with these operations, is a vector space over the field of real numbers $\mathbb{R}$.

(b) Give an example of a function in V which is not real-valued.

13. Let V be the set of all fifth-degree polynomials with standard operations. Is it a vector space? Justify your answer.

14. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that

$$f'(-1) = 3f(2)$$

is a subspace of $\mathbb{R}^{(-4,4)}$, the space of all functions $f : (-4, 4) \to \mathbb{R}$.

15. Let $C([0, 1])$ be the vector space of all continuous real-valued functions defined on $[0, 1]$.

Suppose $b \in \mathbb{R}$. Show that the set

$$W_b = \{\, f \in C([0, 1]) : \int_0^1 f(x)\, dx = b \,\}$$

is a subspace of $C([0, 1])$ if and only if $b = 0$.

16. Prove or disprove: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses, then U is a subspace of $\mathbb{R}^2$.

17. Prove that the intersection of every collection of subspaces of V is a subspace of V. Is the result true for union? Justify.

18. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

19. Prove or disprove

    (a) If V_1, V_2, U are subspaces of V with $V_1 + U = V_2 + U$, then $V_1 = V_2$.
    (b) If V_1, V_2, U are subspaces of V with $V = V_1 \oplus U = V_2 \oplus U$, then $V_1 = V_2$.

20. Let A be a fixed $m \times n$ matrix with entries from a field F. The set

$$W := \{X \in M_{n\times 1}(F) : AX = 0\}$$

    is a subspace of the vector space $F^{n\times 1}$.

21. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Let

$$V_e = \{f \in V : \ f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$$

    be the subset of even functions, and

$$V_o = \{f \in V : \ f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$$

    the subset of odd functions.

    (a) Prove that V_e and V_o are subspaces of V.
    (b) Prove that $V_e + V_o = V$.
    (c) Prove that $V_e \cap V_o = \{0\}$.
    (d) What can you conclude from the above two results?

22. Let $\mathbb{R}^\infty$ denote the vector space of all real sequences (addition and scalar multiplication defined coordinatewise). For each subset below write Yes if it is a subspace of $\mathbb{R}^\infty$ and No if it is not; give a short reason.

    (a) Sequences that have infinitely many zeros (for example, $(1, 1, 0, 1, 1, 0, 1, 1, 0, \dots)$).
    (b) Sequences which are eventually zero (there exists N with $x_n = 0$ for all $n \geq N$).
    (c) Absolutely summable sequences ($\sum_{k=1}^{\infty} |x_k| < \infty$).
    (d) Bounded sequences (there exists M with $|x_k| \leq M$ for all k).
    (e) Decreasing sequences ($x_{n+1} \leq x_n$ for every n).
    (f) Convergent sequences.
    (g) Arithmetic progressions (sequences of the form $a, a+k, a+2k, \dots$ for some $a, k \in \mathbb{R}$).
    (h) Geometric progressions (sequences of the form $a, ak, ak^2, ak^3, \dots$ for some $a, k \in \mathbb{R}$).