Technische Universiteit
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Modeling and Control of Quadrotors

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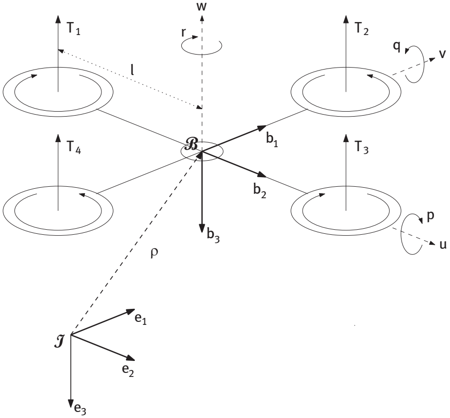
Quadrotor dynamics



Let $\rho = (\mathrm{x}, \mathrm{y}, \mathrm{z})$, $\nu = (\mathrm{u}, \mathrm{v}, \mathrm{w})$, $\Omega = (\mathrm{p}, \mathrm{q}, \mathrm{r}) = (\Omega_1, \Omega_2, \Omega_3)$:

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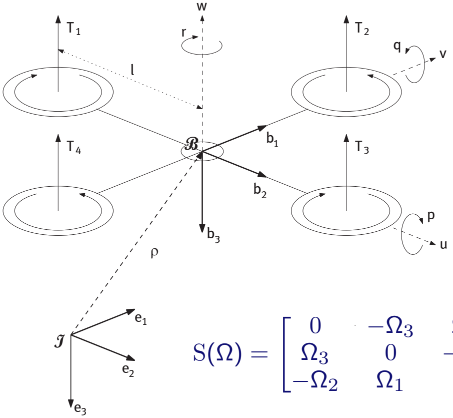

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$$\dot{\rho} = R\nu$$
$$\dot{\nu} = -S(\Omega)\nu + gR^T e_3 - \frac{T}{m} e_3$$
$$\dot{R} = RS(\Omega)$$
$$J\dot{\Omega} = S(J\Omega)\Omega + \tau$$

$$S(\Omega) = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

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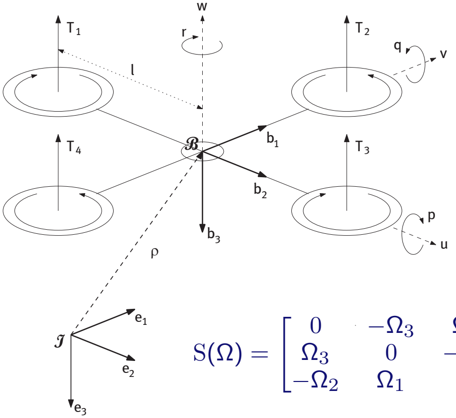

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$$\begin{bmatrix} T \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -l & 0 & l \\ -l & 0 & l & 0 \\ d & -d & d & -d \end{bmatrix} \begin{bmatrix} c\omega_1^2 \\ c\omega_2^2 \\ c\omega_3^2 \\ c\omega_4^2 \end{bmatrix}$$

d: drag coefficient, c rotor parameter.

Rotation. Euler angles (xyz)

Roll angle: $\phi \in \langle -\pi/2, \pi/2 \rangle$, pitch angle $\theta \in \langle -\pi/2, \pi/2 \rangle$, yaw angle $\psi \in \langle -\pi, \pi \rangle$.

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Then $\mathrm{R} = \mathrm{R_z}(\psi)\mathrm{R_y}(\theta)\mathrm{R_x}(\phi)$:

$$\mathrm{R} = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

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Problems for $\theta = \pm\pi/2$ and $\theta = \pm\pi/2$: Gimbal lock

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Quaternion: $\mathrm{q} = \mathrm{q_w} + \mathrm{q_x i} + \mathrm{q_y j} + \mathrm{q_z k}$ (cf. complex number).

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- Pure quaternion: Quaternion with $q_w = 0$

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Rotation matrix for (unit) quaternion $q = q_w + q_x i + q_y j + q_z k$:

$$R = \frac{1}{\|q\|}\begin{bmatrix} \|q\|^2 - 2(q_y^2 + q_z^2) & 2(q_x q_y - q_z q_w) & 2(q_x q_z + q_y q_w) \\ 2(q_x q_y + q_z q_w) & \|q\|^2 - 2(q_x^2 + q_z^2) & 2(q_y q_z - q_x q_w) \\ 2(q_x q_z - q_y q_w) & 2(q_y q_z + q_x q_w) & \|q\|^2 - 2(q_x^2 + q_y^2) \end{bmatrix}$$

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Note that both q and $-q$ yield same rotation matrix.

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When using for controller design: ambiguity. Need to make sure that $u(q) = u(-q)$!

Euler angles

Singularities in representation (gimbal lock)

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Both problems are overcome by working with the rotation matrix R. I.e., consider dynamics on SO(3)

$$\mathrm{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \,\middle|\, R^T R = R R^T = I, \det R = 1 \right\}$$

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Even better: consider dynamics on SE(3):

$$\mathrm{SE}(3) = \left\{ \mathrm{A} \,\middle|\, \mathrm{A} = \begin{bmatrix} \mathrm{R} & \mathrm{r} \\ 0 & 1 \end{bmatrix}, \mathrm{R} \in \mathbb{R}^{3\times 3}, \mathrm{r} \in \mathbb{R}^{3}, \mathrm{R}^{\mathrm{T}}\mathrm{R} = \mathrm{R}\mathrm{R}^{\mathrm{T}} = \mathrm{I}, |\mathrm{R}| = 1 \right\}$$

Performance comparisons

Storage requirements:

Quaternion	4
Rotation matrix	9

Rotation chaining:	Multiplies	Add/subtracts	Total
Quaternion	16	12	28
Rotation matrix	27	18	45

Vector Rotation:	Multiplies	Add/subtracts	Total
Quaternion	12+9	12+6	39
Rotation matrix	9	6	15

Summary

- Dynamics:

$$
\begin{aligned}
\dot{\rho} &= \mathrm{R}\nu \\
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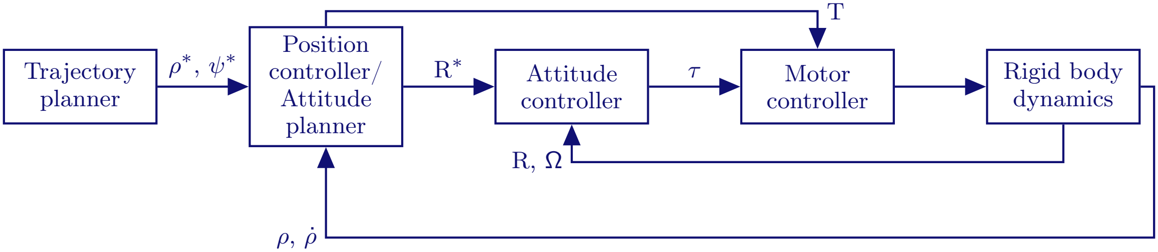
- Controller design using R/on SE(3): no gimbal lock or ambiguity.
- Implementation using quaternions

Typical control scheme

- Fast inner-loop for controlling attitude
- Slower outer-loop for controlling position (considering attitude as input)



Warning: We are controlling a nonlinear system!

Example

- Inner-loop: $x_2 = -k_2 x_2$ with scalar x_2 and $k_2 > 0$
- Outer-loop: $x_1 = -k_1 x_1$ with scalar x_1 and $k_1 > 0$

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Nonlinear dynamics, so closed-loop error dynamics:

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\begin{aligned}
x_1 &= -k_1 x_1 \;+\; x_1^2\, x_2 & \qquad x_1(0) &= x_{10} \\
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Solution:

$$x_1(t) = \frac{x_{10} e^{-k_1 t}}{1 - \frac{x_{10} x_{20}}{k_1 + k_2}\left[1 - e^{-(k_1+k_2)t}\right]} \qquad x_2(t) = x_{20} e^{-k_2 t}$$

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Problem when $x_{10} x_{20} > k_1 + k_2$: denominator is 0 for $t = \frac{\log[x_{10}x_{20}] - \log[x_{10}x_{20} - (k_1+k_2)]}{k_1+k_2}$.
Finite escape time!

Cascade result (Panteley, Loría, SCL, 1998)

Consider

$$\begin{aligned} \dot{z}_1 &= f_1(t, z_1) \quad + \quad g(t, z_1, z_2)\, z_2 & \qquad z_1(0) &= z_{10} \\ \dot{z}_2 &= f_2(t, z_2) & \qquad z_2(0) &= z_{20} \end{aligned}$$

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If both $\dot z_1 = f_1(t, z_1)$ and $\dot z_2 = f_2(t, z_2)$ are uniformly globally asymptotically stable, and solutions for the cascaded system are globally uniformly bounded, then the cascade is uniformly globally asymptotically stable

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Useful corollary of more general result in the above mentioned paper

- $\dot{z}_2 = f_2(t, z_2)$ uniformly locally exponentially stable,
- and $\|g(t, z_1, z_2)\| \leq k_1(\|z_2\|) + k_2(\|z_2\|)\|z_1\|$,

then solutions of cascade are globally uniformly bounded.

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Similar remark holds for designing observer (to filter out noise and reconstruct unmeasured signals). No certainty equivalence holds, so observer design should be integrated in controller design/ stability proof needs more work.

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Measurements

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- (GPS: $\xi_{\mathrm{GPS}} = \xi$.)
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All with (constant) bias and noise.

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All with (constant) bias and noise. Observers can be designed (see e.g., work of Mahony, Hamel).

Trajectory generation

Recall dynamics

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Associated rotation is not uniquely defined: arbitrary yaw ψ is possible.

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Assume $\rho(t)$ is given. Note that $\ddot{\rho} = ge_3 - \frac{T}{m}Re_3$, so assuming $T > 0$, we have

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Associated rotation is not uniquely defined: arbitrary yaw ψ is possible.
Either also prescribe $\psi(t)$, or make particular choice for ψ.

Trajectory generation

Recall dynamics

$$\dot{\rho} = \mathrm{R}\nu \qquad \dot{\nu} = -\mathrm{S}(\Omega)\nu + \mathrm{g}\mathrm{R}^{\mathrm{T}}\mathrm{e}_3 - \frac{\mathrm{T}}{\mathrm{m}}\mathrm{e}_3 \qquad \dot{\mathrm{R}} = \mathrm{R}\mathrm{S}(\Omega) \qquad \mathrm{J}\dot{\Omega} = \mathrm{S}(\mathrm{J}\Omega)\Omega + \tau$$

Assume $\rho(\mathrm{t})$ is given. Note that $\ddot{\rho} = \mathrm{g}\mathrm{e}_3 - \frac{\mathrm{T}}{\mathrm{m}}\mathrm{R}\mathrm{e}_3$, so assuming $\mathrm{T} > 0$, we have

$$\text{Thrust:} \quad \mathrm{T} = \|\mathrm{T}\mathrm{R}\mathrm{e}_3\| = \mathrm{m}\|\mathrm{g}\mathrm{e}_3 - \ddot{\rho}\| \qquad \text{Direction:} \quad \mathrm{R}\mathrm{e}_3 = \frac{\mathrm{g}\mathrm{e}_3 - \ddot{\rho}}{\|\mathrm{g}\mathrm{e}_3 - \ddot{\rho}\|} = \begin{bmatrix} \mathrm{r}_{13} \\ \mathrm{r}_{23} \\ \mathrm{r}_{33} \end{bmatrix}$$

Associated rotation is not uniquely defined: arbitrary yaw ψ is possible.
Either also prescribe $\psi(\mathrm{t})$, or make particular choice for ψ.
Rotation from e_3 to $\mathrm{R}\mathrm{e}_3$ in plane spanned by e_3 and $\mathrm{R}\mathrm{e}_3$ is given by

$$\mathrm{R} = \begin{bmatrix} 1 - \frac{\mathrm{r}_{13}^2}{1+\mathrm{r}_{33}} & -\frac{\mathrm{r}_{13}\mathrm{r}_{23}}{1+\mathrm{r}_{33}} & \mathrm{r}_{13} \\ -\frac{\mathrm{r}_{13}\mathrm{r}_{23}}{1+\mathrm{r}_{33}} & 1 - \frac{\mathrm{r}_{23}^2}{1+\mathrm{r}_{33}} & \mathrm{r}_{23} \\ -\mathrm{r}_{13} & -\mathrm{r}_{23} & \mathrm{r}_{33} \end{bmatrix} \qquad \mathrm{q} = \pm \begin{bmatrix} \frac{1}{2}\sqrt{2 + 2\mathrm{r}_{33}} \\ -\frac{\mathrm{r}_{23}}{\sqrt{2+2\mathrm{r}_{33}}} \\ \frac{\mathrm{r}_{13}}{\sqrt{2+2\mathrm{r}_{33}}} \\ 0 \end{bmatrix}$$

Trajectory generation

Dynamics

$$\dot{\rho} = R\nu \qquad \dot{\nu} = -S(\Omega)\nu + gR^T e_3 - \frac{T}{m} e_3 \qquad \dot{R} = RS(\Omega) \qquad J\dot{\Omega} = S(J\Omega)\Omega + \tau$$

For given $\rho(t)$ and particular choice for $\psi(t)$ we have

$$T = m\|ge_3 - \ddot{\rho}\| \qquad R = \begin{bmatrix} 1 - \frac{r_{13}^2}{1+r_{33}} & -\frac{r_{13}r_{23}}{1+r_{33}} & r_{13} \\ -\frac{r_{13}r_{23}}{1+r_{33}} & 1 - \frac{r_{23}^2}{1+r_{33}} & r_{23} \\ -r_{13} & -r_{23} & r_{33} \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = \frac{ge_3 - \ddot{\rho}}{\|ge_3 - \ddot{\rho}\|}$$

Trajectory generation

Dynamics

$$\dot{\rho} = R\nu \qquad \dot{\nu} = -S(\Omega)\nu + gR^{T}e_3 - \frac{T}{m}e_3 \qquad \dot{R} = RS(\Omega) \qquad J\dot{\Omega} = S(J\Omega)\Omega + \tau$$

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Using dynamics we also have

$$\nu = R^{T}\dot{\rho} \qquad \Omega = \begin{bmatrix} -\dot{r}_{23} + \frac{r_{23}\dot{r}_{33}}{1+r_{33}} \\ \dot{r}_{13} - \frac{r_{13}\dot{r}_{33}}{1+r_{33}} \\ \frac{r_{23}\dot{r}_{13} - r_{13}\dot{r}_{23}}{1+r_{33}} \end{bmatrix} \qquad \tau = J\dot{\Omega} - S(J\Omega)\Omega$$

Trajectory generation

Dynamics

$$\dot{\rho} = R\nu \qquad \dot{\nu} = -S(\Omega)\nu + gR^T e_3 - \frac{T}{m} e_3 \qquad \dot{R} = RS(\Omega) \qquad J\dot{\Omega} = S(J\Omega)\Omega + \tau$$

For given $\rho(t)$ and particular choice for $\psi(t)$ we have

$$T = m\|ge_3 - \ddot{\rho}\| \qquad R = \begin{bmatrix} 1 - \frac{r_{13}^2}{1+r_{33}} & -\frac{r_{13}r_{23}}{1+r_{33}} & r_{13} \\ -\frac{r_{13}r_{23}}{1+r_{33}} & 1 - \frac{r_{23}^2}{1+r_{33}} & r_{23} \\ -r_{13} & -r_{23} & r_{33} \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = \frac{ge_3 - \ddot{\rho}}{\|ge_3 - \ddot{\rho}\|}$$

Using dynamics we also have

$$\nu = R^T \dot{\rho} \qquad \Omega = \begin{bmatrix} -\dot{r}_{23} + \frac{r_{23}\dot{r}_{33}}{1+r_{33}} \\ \dot{r}_{13} - \frac{r_{13}\dot{r}_{33}}{1+r_{33}} \\ \frac{r_{23}\dot{r}_{13} - r_{13}\dot{r}_{23}}{1+r_{33}} \end{bmatrix} \qquad \tau = J\dot{\Omega} - S(J\Omega)\Omega$$

So all signals can be expressed a functions of ρ, $\dot{\rho}$, $\ddot{\rho}$, $\dddot{\rho}$, $\ddddot{\rho}$, or similarly $\ddddot{\rho}(t)$, $\dddot{\rho}(t_0)$, $\ddot{\rho}(t_0)$, $\dot{\rho}(t_0)$, $\rho(t_0)$.

Presentation of possible nonlinear controller

E. Lefeber, S.J.A.M. van den Eijnden, H. Nijmeijer, Almost global tracking control of a quadrotor UAV on SE(3), In: Proceedings of the 56th IEEE Conference on Decision and Control, Melbourne, Australia, p.1175–1180, December 2017.

Unmodelled second order aerodynamical effects

Horizonal:

- blade flapping,
- parasitic drag,
- aerodynamic drag

Vertical:

- ground effect,
- inflow damping,
- translational lift

Other:

- vortex states,
- flow variation due to local environment