Directional Data Analysis

An Introduction to Circular Statistics

What is Circular Statistics?

- **Directional statistics** (also **circular statistics** or **spherical statistics**) is the subdiscipline of statistics that deals with directions, axes or rotations.

Angles

- The fact that 0 degrees and 360 degrees are identical angles, so that
- for example 180 degrees is not a sensible mean of 2 degrees and 358 degrees

Time Periods

- Statistics involving
  - temporal periods (e.g. time of day, week, month, year, etc.),
  - compass directions,
  - dihedral angles in molecules, orientations, rotations and so on.

Conversion

Time can be converted to an angular measurement using the equation:

$$a = \frac{(360^{o})(X)}{k}$$

where *a* is the angular measurement, *X* is the time period, and *k* is the number of time units on the circular measurement scale.

What is the angular measurement of 6:15 a.m. (6.25a.m.)?
(Remember to use a 24hr clock...)

$$a = \frac{(360^{o})(6.25hr)}{24hrs} = 93.75^{o}$$

What is the angular measurement of February 14th?
(Remember to use total days...)

$$a = \frac{(360^{o})(45th\ day)}{365\ days} = 44.38^{o}$$

Calculation

To analyze directional data they must first be transformed into rectangular polar coordinates.

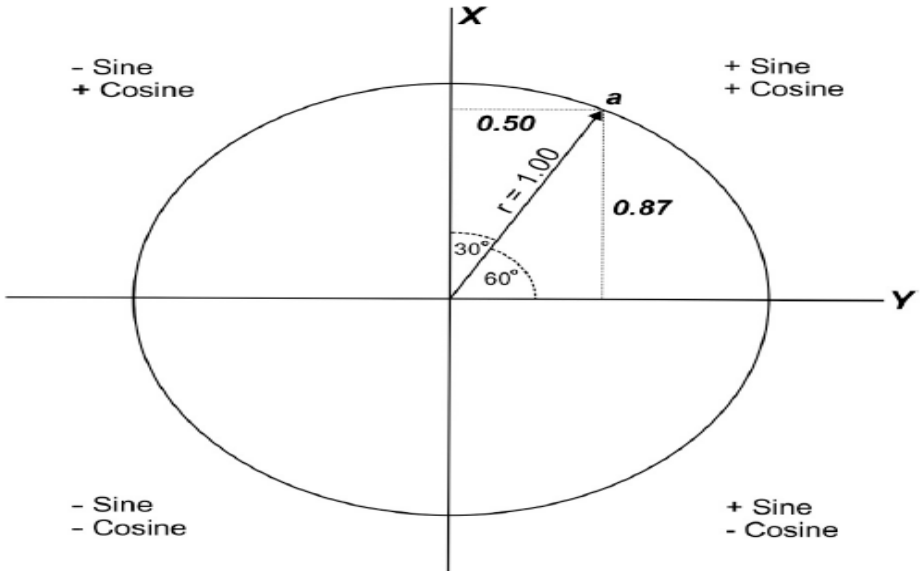
- First, we specify a 'unit circle' that has a radius of 1.
- The polar location is then defined as the angular measurement and its intersection with the unit circle.
- The cosine and sine functions are then used to place this location (based on the angle and unit distance) into a standardized Cartesian space.

$$\cos a = \frac{x}{r} \qquad \sin a = \frac{y}{r}$$

$$\cos 30 = 0.50 \qquad \sin 30 = 0.87$$

$$\cos 60 = 0.50 \qquad \sin 60 = 0.87$$

Note that the coordinates of opposite angles are identical. Also note that the x and y axes are opposite of the typical Cartesian plane.



Mean Angle

The mean angle can not simply be the sum of the angles divided by the sample size, because the mean angle of 359º and 1º (north) would be 180º (south)! Therefore we use the following equations:

$$Y = \frac{\sum_{i=1}^{n} \sin_a}{n} \qquad X = \frac{\sum_{i=1}^{n} \cos_a}{n}$$

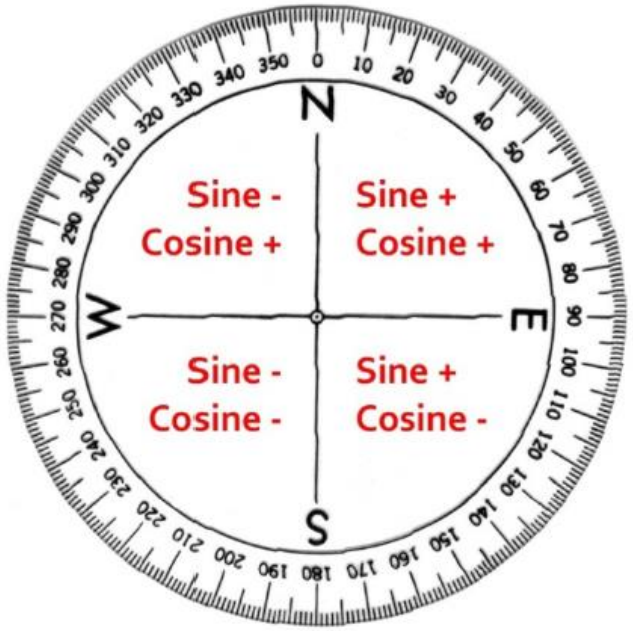
$$r = \sqrt{X^2 + Y^2}$$

$$\cos \bar{a} = \frac{X}{r} \qquad \sin \bar{a} = \frac{Y}{r} \qquad \theta_r = \arctan\left(\frac{\sin \bar{a}}{\cos \bar{a}}\right)$$

where *X* and *Y* are the rectangular coordinates of the mean angle, and *r* is the mean vector.

Determining the Quadrant

- Sin +, Cos + : the mean angle is computed directly.
- Sin +, Cos - : the mean angle = $180 - \theta_r$.
- Sin -, Cos - : the mean angle = $180 + \theta_r$.
- Sin -, Cos + : the mean angle = $360 - \theta_r$.



Calculations

Rocks Vectors		Sin (Azimuth)	Cos (Azimuth)
341		-0.32557	0.94552
330		-0.50000	0.86603
301		-0.85717	0.51504
299		-0.87462	0.48481
9		0.15643	0.98769
7		0.12187	0.99255
359		-0.01745	0.99985
334		-0.43837	0.89879
353		-0.12187	0.99255
15		0.25882	0.96593
27		0.45399	0.89101
28		0.46947	0.88295
25		0.42262	0.90631
23		0.39073	0.92050
350		-0.17365	0.98481
30		0.50000	0.86603
26		0.43837	0.89879
22		0.37461	0.92718
8		0.13917	0.99027
356		-0.06976	0.99756
	∑	0.34763	17.91415

- First take the sine and cosine of the angles (azimuths) and sum them.
- In Excel the formula is*: =sin(radians(cell #))* and *=cos(radians(cell #))*

Calculations

$$n = 20$$

$$\sum \sin_a = 0.34763 \qquad \sum \cos_a 17.91415$$

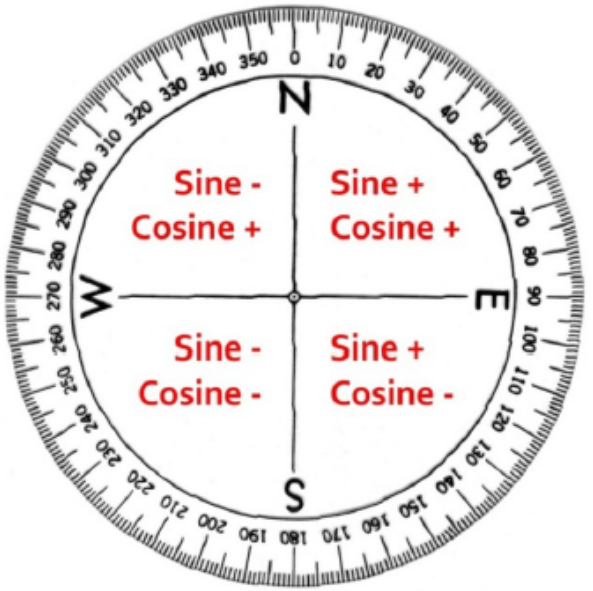
$$Y = \frac{0.34763}{20} = 0.01738 \qquad X = \frac{17.91415}{20} = 0.89571$$

$$r = \sqrt{0.01738^2 + 089571^2} = \sqrt{0.00030 + 0.80229} = 0.8959$$

$$\sin \bar{a} = \frac{0.01738}{0.8959} = 0.0194 \qquad \cos \bar{a} = \frac{0.89571}{0.8959} = 0.9998$$

$$\theta_r = \arctan\left(\frac{0.0194}{0.9998}\right) = 1.11$$

Ignore the sign.



Angular Dispersion

The value of *r* is also a measure of angular dispersion, similar to the standard deviation with a few exceptions:

- Unlike the standard deviation it ranges from 0 – 1.
- A value of 0 means uniform dispersion.
- A value of 1 means complete concentration in one direction.

Some Circular probability distributions

- von Mises circular distribution
- Circular uniform distribution
- Wrapped normal distribution
- Wrapped Cauchy distribution
- Wrapped Lévy distribution

For more info: https://en.wikipedia.org/wiki/Circular_distribution

Some Directional Hypothesis Tests

- Rayleigh z Test
- Watson's U^2 test.