

CGS698C, Assignment 01

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Part 1: Probability

1.1 In an experiment, two coins are tossed simultaneously.

(A coin toss gives you either heads (denoted by H) or tails (denoted by T).)

- (a) What is the sample space of the two-coin-tossing experiment?
- (b) What is the event space of the experiment? List all the possible events.
- (c) Suppose that all the elementary events in the sample space have equal probabilities.
(For example, if there are three possible outcomes in the sample space Ω such that $\Omega = \{x_1, x_2, x_3\}$, then $P(\{x_1\}) = P(\{x_2\}) = P(\{x_3\})$)
 - i. What is the probability of occurrence of each elementary event?
 - ii. What is the probability of the event that at least one head appears?
 - iii. What is the probability of the event that exactly one head appears?

Part 2: Discrete random variables

2.1 In a visual word recognition experiment, a participant has to recognize whether the word shown on screen is a meaningful word (e.g., "book") or a non-word (e.g., "bktr"). The participant is asked to answer "yes" if the shown string is a meaningful word, and "no" if it is a meaningless non-word. Suppose a participant is shown 50 words, what is the probability that the participant will correctly recognize 45 out of 50 words? You are given that the probability of correctly recognizing a word is 0.9.

You are given a probability assigner function $f(k, n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$, where k is number of correctly recognized words, n is total number of words, and p is underlying probability of correctly recognizing a word.

2.2 Suppose in a small city, 10 road accidents happen on average in a single day. The probability of k number of road accidents in a day is given by the probability mass function $f(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$ where $\lambda = 10$.

- (a) What is the probability that zero road accidents in a day?
- (b) What is the probability of occurrence of more than 7 but less than 10 road accidents in a day?
- (c) Draw the graph of the probability mass function $f(x)$.
(Hint: Choose a range of possible values for X say 0,1,2,...,50; compute the probabilities for each value of X using the formula for $P(X = x)$; and then, plot the graph with probabilities on y -axis and X on x -axis.)

Part 3: Continuous random variables

3.1 Suppose a random variable X is normally distributed. The probability density function of the

normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

- (a) Find the probability density of obtaining $x = 0$, given that $\mu = 1, \sigma = 1$.
- (b) Find the probability density of obtaining $x = 1$, given that $\mu = 0, \sigma = 1$.
- (c) You are given

The probability that the outcome occurs between x_1 and x_2 : $P(x_1 \leq X \leq x_2) = 0.3$

The probability that the outcome occurs between x_1 and x_3 : $P(x_1 \leq X \leq x_3) = 0.45$

Find the probability that the outcome occurs between x_2 and x_3 .

Part 4: The likelihood function

Suppose a random variable X has the probability density function $f(x, \theta)$ where θ is a parameter of the probability density function and x is a value of the random variable X . You can write:

$$X \sim f(x, \theta)$$

The probability density function (PDF) $f(x, \theta)$ tells you the probability density of generating an outcome x when the value of θ is known/assumed. For example, if you know/assume $\theta = 2$, you can calculate the probability density for different values of the random variable such as $X = 5$, $X = 3$, $X = 100$. Basically, the PDF is viewed as a function of x when θ is fixed.

However, you can also view the PDF in a different manner. You can calculate the probability density of obtaining a given, fixed outcome x for different values of θ . That is, the PDF can be viewed as a function of θ when x is fixed. This alternative characterization of the PDF is called **the likelihood function**.

The likelihood function is a function that maps the values of the parameter θ to probability densities, when the sample x is taken as a fixed, observed quantity.

In sum, the PDF is a function of x where θ is assumed to be fixed; the likelihood function is a function of parameter θ when the sample x is fixed/known.

The likelihood function is often represented by $\mathcal{L}(\theta|x)$:

$$\mathcal{L}(\theta|x) = f(x, \theta) \text{ when } x \text{ is fixed}$$

Use the above information to do the following exercise.

4.1 In a visual word recognition experiment, a participant has to recognize whether a string shown on screen is a meaningful word (e.g., "book") or a non-word (e.g., "bktr"). The participant is asked to answer "yes" if the shown string is a meaningful word, and "no" if it is a meaningless non-word. Suppose a participant is shown 5 strings on the screen one by one. The time taken by the participants to recognize each string is shown below (in milliseconds):

Recognition time for 5 strings: 303, 443, 220, 560, 880

Suppose the random variable X represents the string recognition times.

A researcher proposes a hypothesis that the string recognition times are generated by the probability density functions $f(x, \mu)$:

$$X \sim f(x, \mu)$$

such that,

$$f(x, \mu) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2}}$$

- (a) Plot the graph of the likelihood function with respect to values of μ , assuming that x is fixed to 220.

(Hint: Choose a range of values for μ and plug those values in the function $f(x, \mu)$ along with $x = 220$ to get probability densities; plot a graph with μ on x-axis and the corresponding values of $f(x, \mu)$ on y-axis.)

- (b) Graph the likelihood function when x is (fixed as) the observed sample of recognition times i.e., 303.25, 443, 220, 560, 880.

(Hint: The probability density for any $\mu = \mu_1$ and $x = [x_1, x_2, x_3, \dots, x_n]$ will be $f(x_{1:n}, \mu_1) = \frac{1}{(\prod_{i=1}^n x_i) (\sqrt{2\pi})^n} e^{-\frac{\sum_{i=1}^n (\log x_i - \mu_1)^2}{2}}$.)

- (c) For what value of μ , the likelihood (probability density) of obtaining the observed sample 303, 443, 220, 560, 880 is maximum? You do not need to be precise, you can tell the approximate value.