# Robotics – Großübung #1

Homogeneous transformations, Denavit-Hartenberg and forward kinematics

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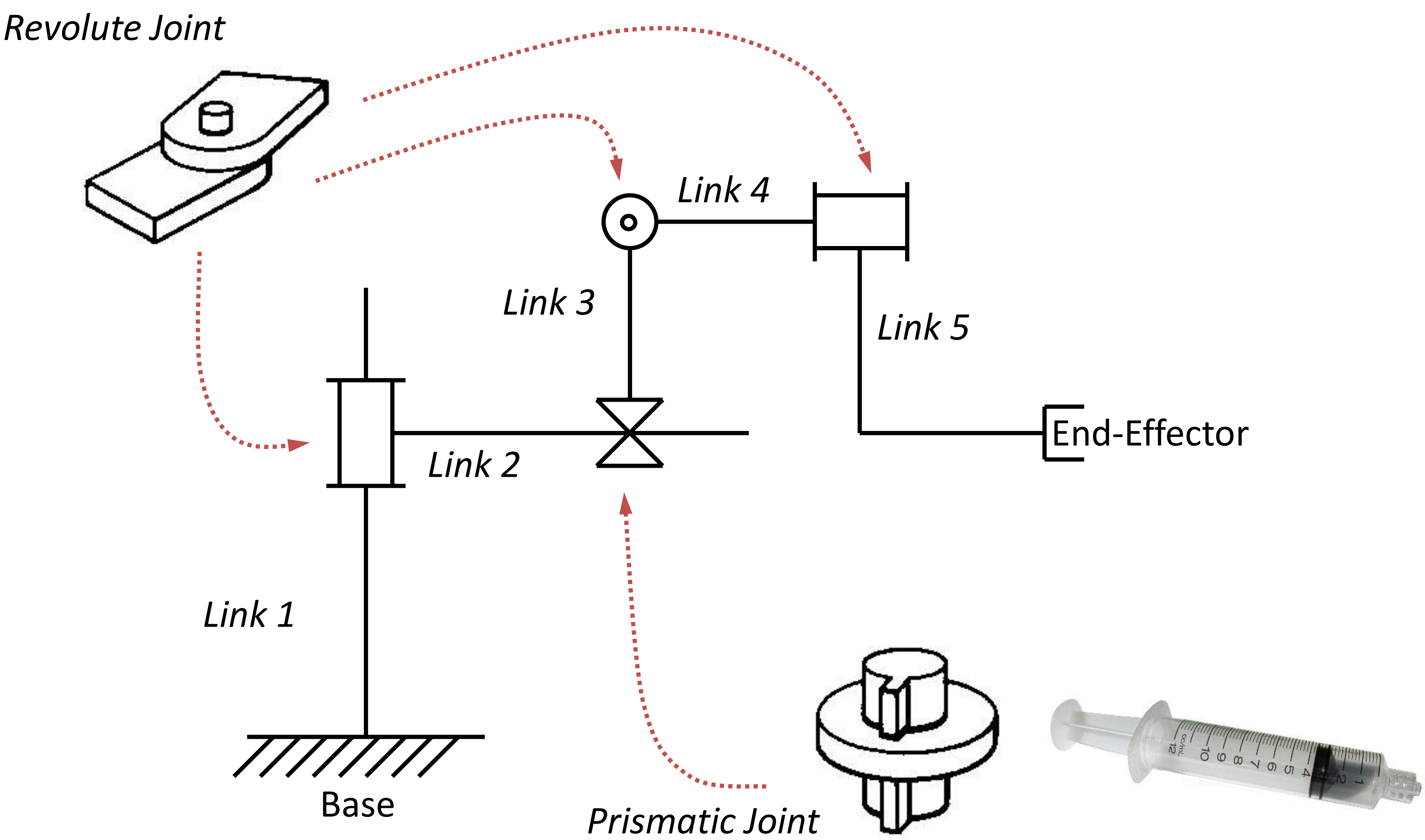
# Main Problem

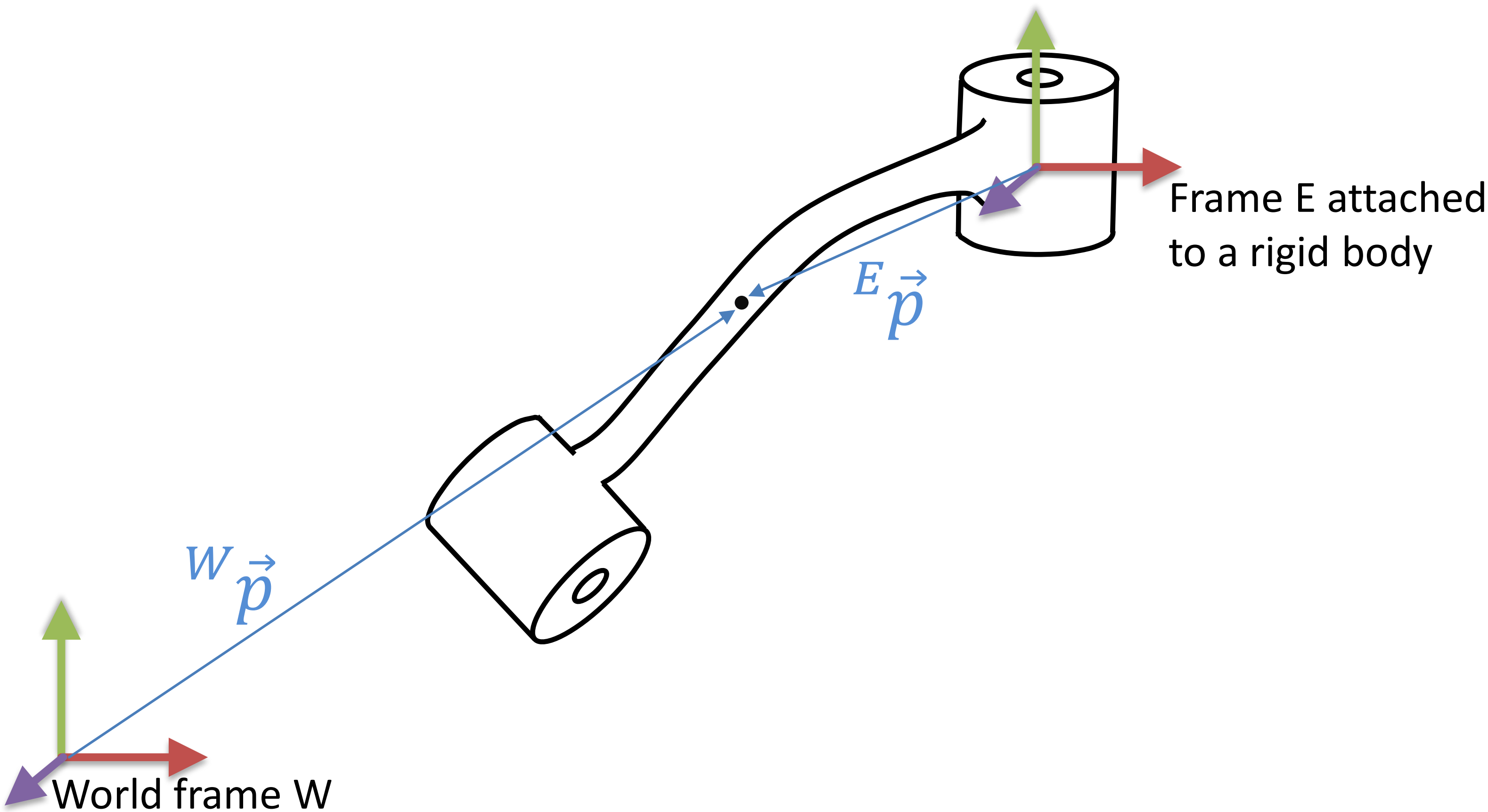
- Given a ***kinematic model*** of the robot and the robot's ***configuration*** – where is the ***end-effector***?

1) Defining a Kinematic Chain
2) Homogeneous transformation
3) Forward kinematics
4) Denavit-Hartenberg Parameters

# Kinematic Schematic of a Robot

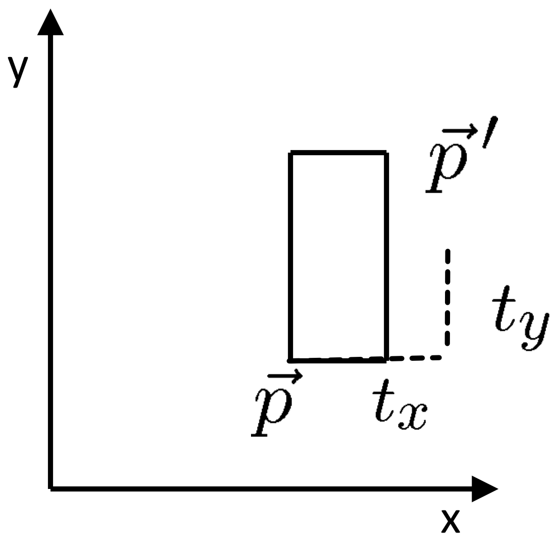


# Frames



# Translation

$$\vec{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \qquad \vec{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

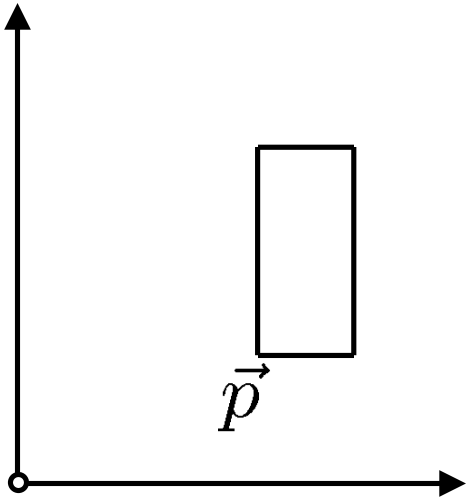



$$\vec{p}' = \vec{p} + \vec{t} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \end{pmatrix}$$

Global Reference Coordinate System = World Frame

# Rotation

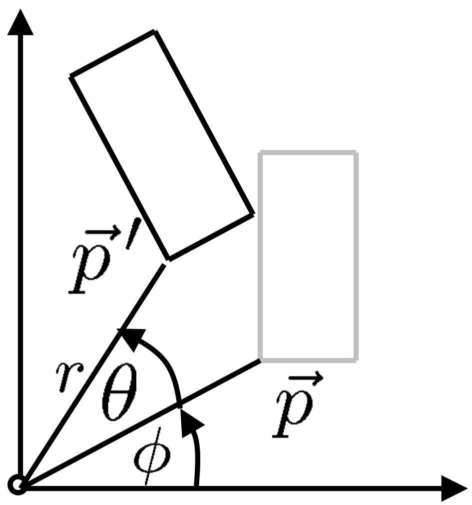
$$\vec{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$




$$\vec{p}\,' = \ ?$$

# Deriving the Rotation Matrix

$$p_x = r \cdot \cos\phi$$
$$p_y = r \cdot \sin\phi$$



$$\begin{aligned} p'_x &= r \cdot \cos(\theta + \phi) \\ &= r \cdot \cos\phi \cdot \cos\theta - r \cdot \sin\phi \cdot \sin\theta \\ &= p_x \cdot \cos\theta - p_y \cdot \sin\theta \\ p'_y &= r \cdot \sin(\theta + \phi) \\ &= r \cdot \cos\phi \cdot \sin\theta + r \cdot \sin\phi \cdot \cos\theta \\ &= p_x \sin\theta + p_y \cos\theta \end{aligned}$$

$$\vec{p}\,' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \vec{p}$$

$$\vec{p}\,' = R(\theta) \cdot \vec{p}$$

# Homogeneous Transformations

$$\vec{p}\,' = \vec{p} + \vec{t} \qquad\qquad \vec{p}\,' = R(\theta) \cdot \vec{p}$$

$$\vec{p}\,' = \underbrace{R(\theta) \cdot \vec{p}}_{\text{Rotation first!}} + \vec{t}$$

$$\begin{pmatrix} p'_x \\ p'_y \\ 1 \end{pmatrix} = \left[ \begin{array}{ccc} \multicolumn{2}{c}{\multirow{2}{*}{$R(\theta)$}} & t_x \\ & & t_y \\ 0 & 0 & 1 \end{array} \right] \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

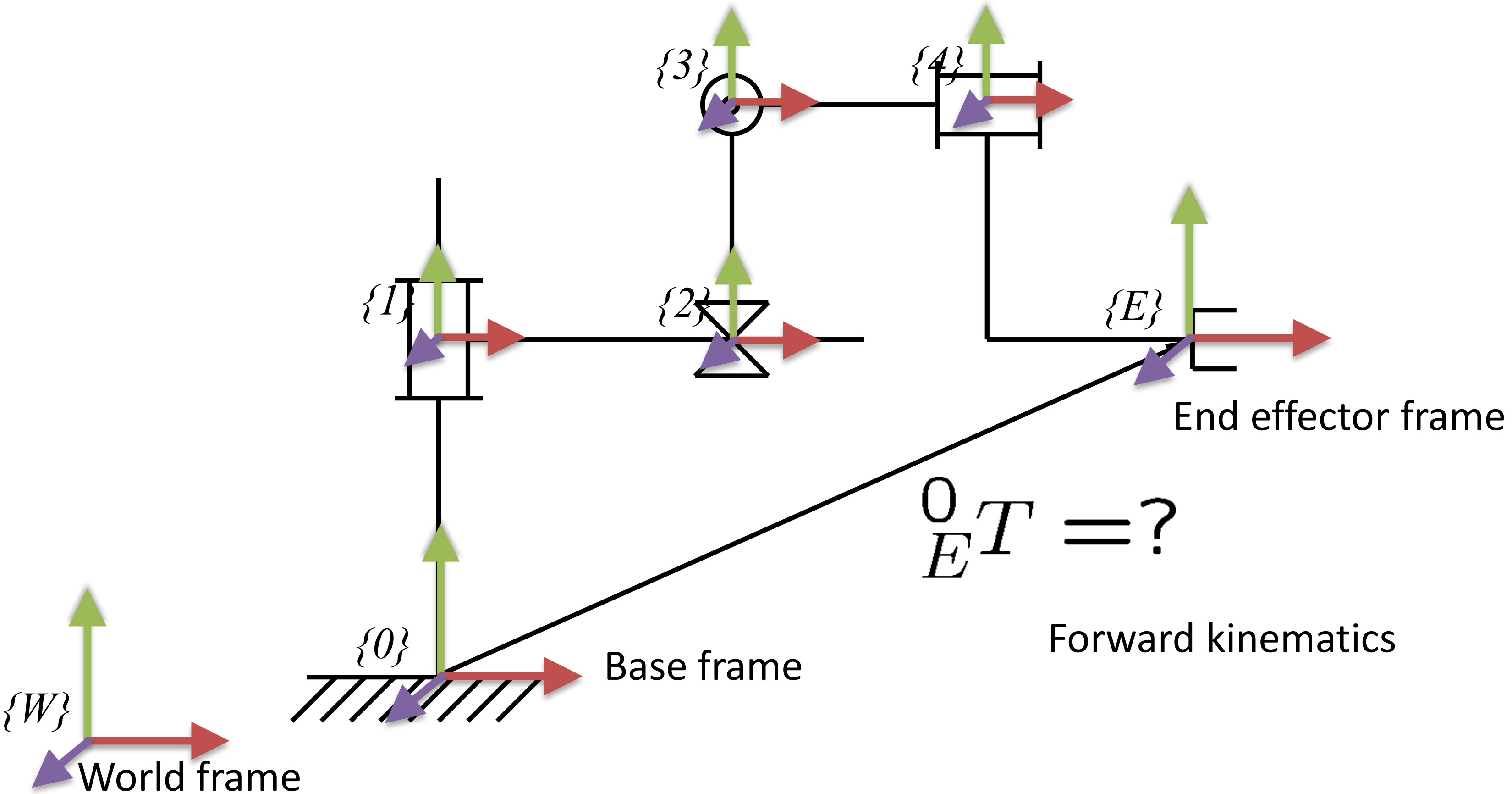
# 3D Homogeneous Transforms

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{pmatrix} = \begin{bmatrix} & R(\theta) & & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

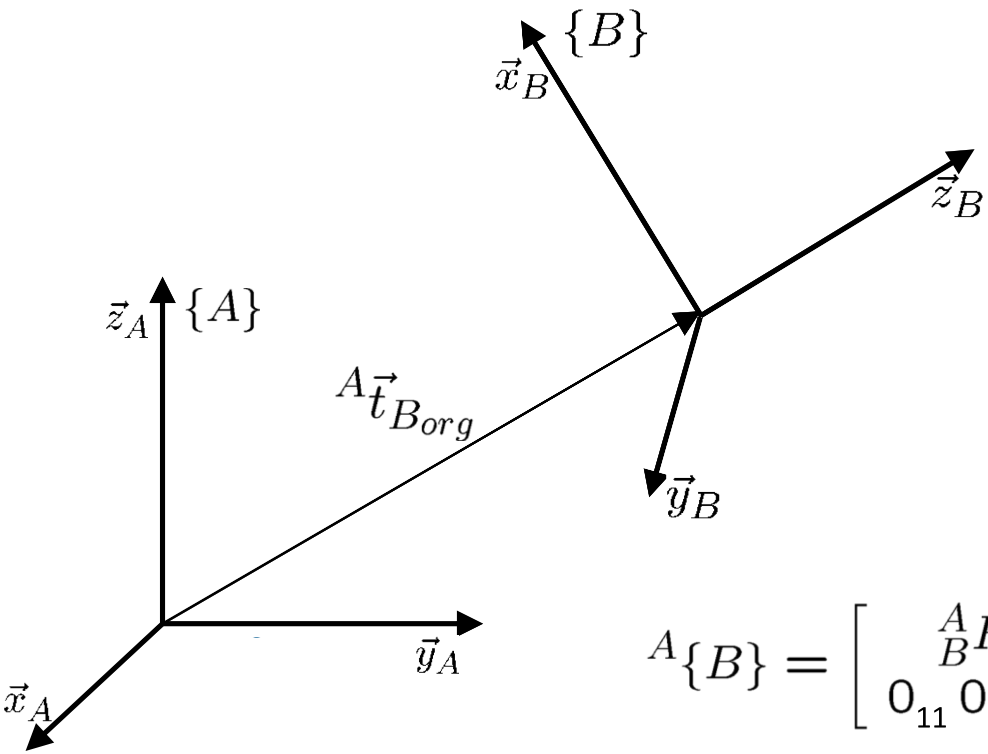
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Kinematic Chains



# Forward Kinematics

$$ {}^{0}_{n}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \; \cdots \; {}^{n-1}_{n}T $$



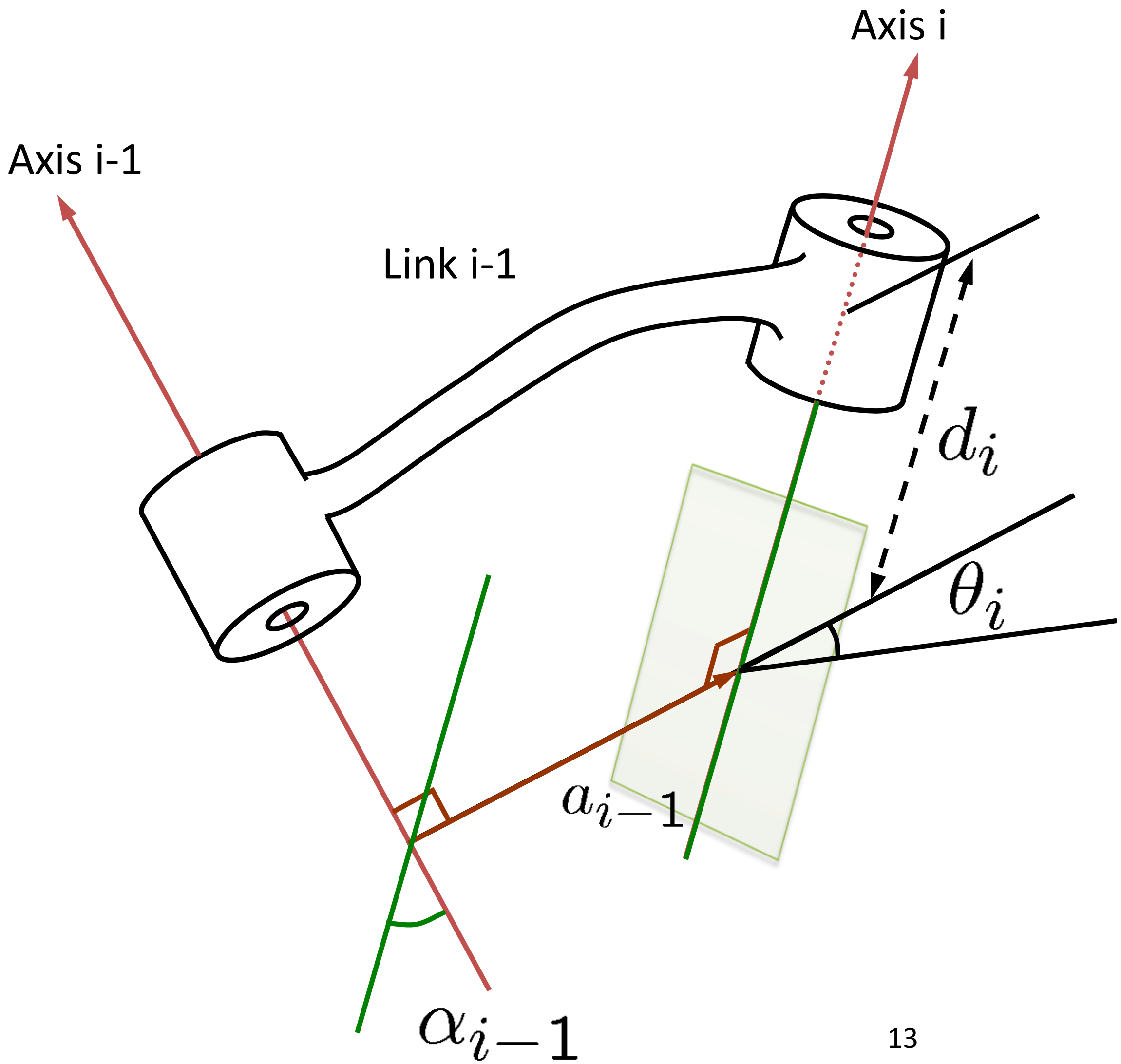

$$ K_F : \mathbb{R}^n \rightarrow \mathbb{R}^6 $$

$$ {}^{A}\{B\} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}\vec{t}_{B_{org}} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix} $$

# Glossary

- Frame
  - Coordinate system attached rigidly to a body
  - Examples: World frame, base frame, end-effector frame
- Link
  - A rigid body that is part of the robot (any object)
- Joint
  - Movable connection between two links
  - Most common: Revolute, prismatic
- Degrees of Freedom
  - Amount of independent position variables the robot has (Usually amount of joints)
- End-effector
  - Free end at the end of the robot manipulator (e.g. a hand)
- Forward kinematics
  - Position and orientation of the end-effector relative to the base frame

# Link Description and Connections




$a_{i-1}$

**link length**

perpendicular to both axes

not unique for parallel axes

$\alpha_{i-1}$

**link twist**

Angle measured about vector $a_{i-1}$

$\theta_i$

**joint angle**

measured about axis i

joint variable for revolute

fixed for prismatic

$d_i$

**link offset**

measured along axis i

joint variable for prismatic

fixed for revolute



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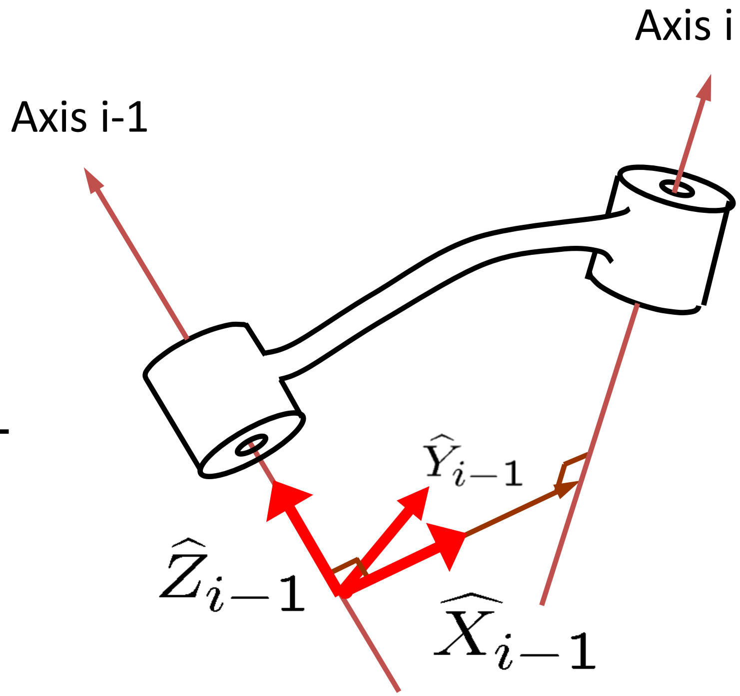
# Frame Attachment




1. Identify joint axes; consider *i* and *i*+1
2. Identify common perpendicular
3. Label frame origin at perpendicular (or intersection)
4. Assign Z axis *i* along joint axis
5. Assign X axis *i* along perpendicular; if joint axes intersect, orthogonal to the axes plane
6. Complete frame by adding Y axis (right-hand-rule)
7. Assign {0} to match {1}
8. Choose end-effector frame {*n*}

# First and Last Link (0 and n)

- Frame 0 is reference (world) frame
  - Origin and Z axis coincide with frame 1
  - $\alpha_0 = a_0 = 0$
  - $d_i$ or $\theta_i = 0$ depending on joint type
- Frame n is end-effector frame
  - X axis is aligned with X axis of frame n-1
  - $d_n = 0$
  - $d_i$ or $\theta_i = 0$ depending on joint type
- Maximize zeros in DH parameters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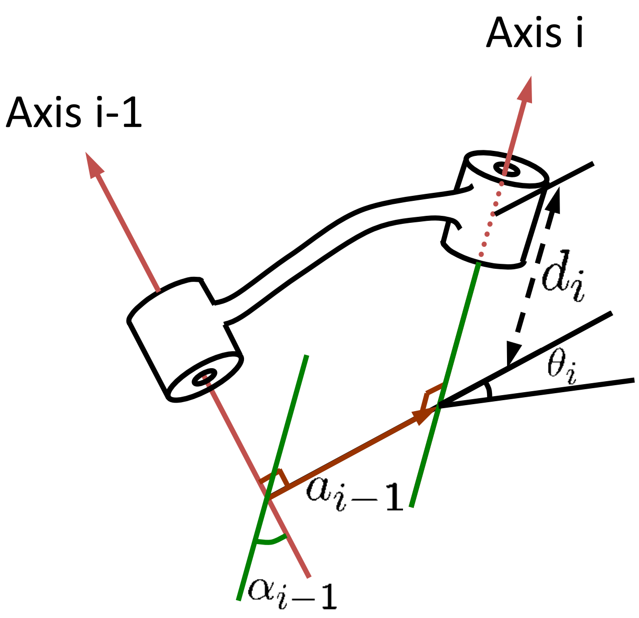
# Denavit-Hartenberg (DH) parameters




$\alpha_i$ = the angle between $\hat{Z}_i$ and $\hat{Z}_{i+1}$ measured about $\widehat{X}_i$

$a_i$ = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\widehat{X}_i$

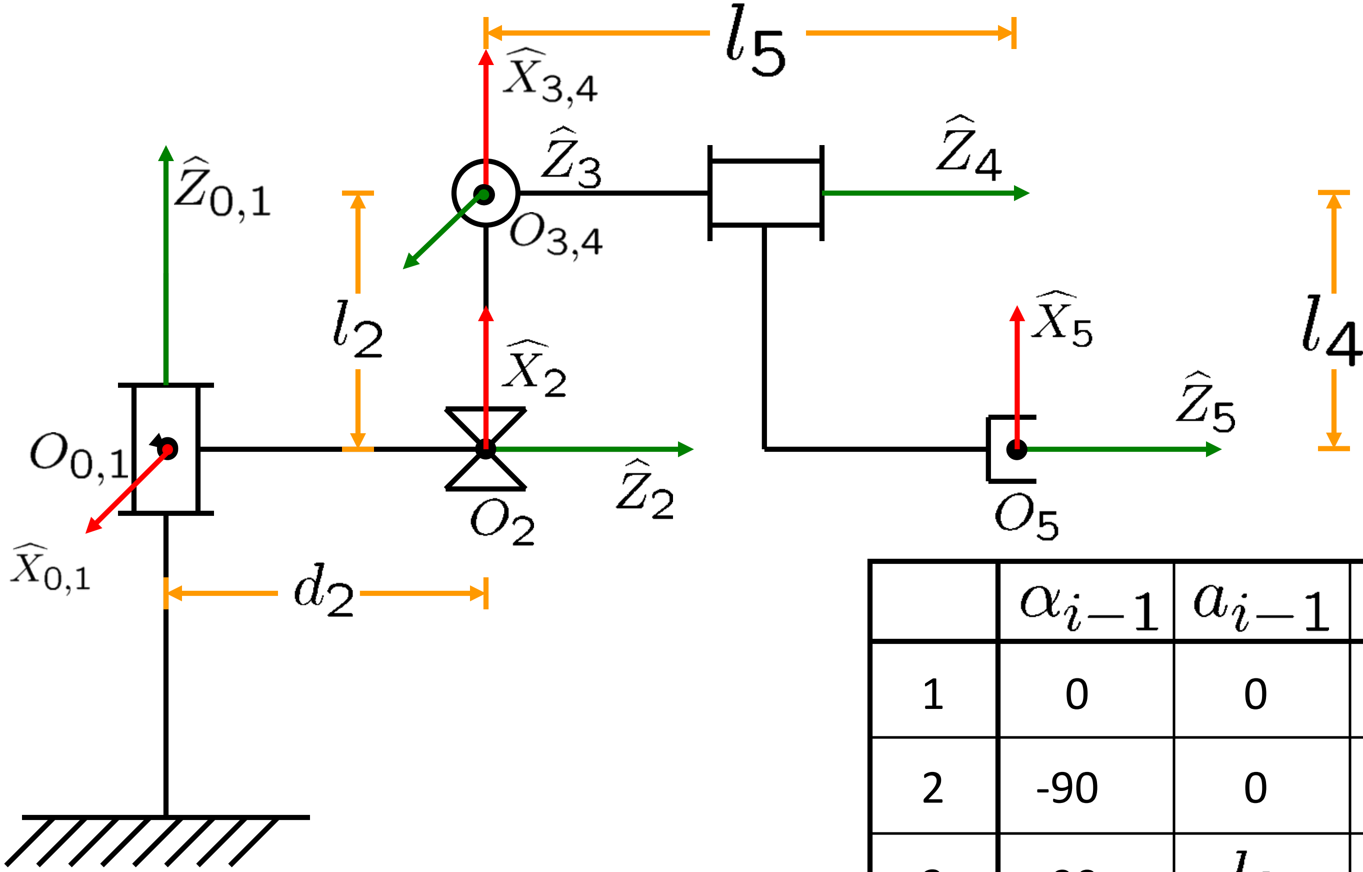
$d_i$ = the distance from $\widehat{X}_{i-1}$ to $\widehat{X}_i$ measured along $\hat{Z}_i$

$\theta_i$ = the angle between $\widehat{X}_{i-1}$ and $\widehat{X}_i$ measured about $\hat{Z}_i$

# DH: Non-uniqueness

- Convention does not result in a unique frame assignment:
  - When aligning $\widehat{Z}_i$ with axis i, there are two choices with respect to the direction (+ / -)
  - If joint axes intersect, there are two choices with respect to the direction of $\widehat{X}_i$
  - When {i} and {i+1} are parallel, the origin location for {i} is arbitrary (though usually chosen such that $d_i = 0$)
  - When prismatic joints are present, there is some freedom in frame assignment

# Example with n=4




| | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | -90 | 0 | $d_2$ | -90 |
| 3 | -90 | $l_2$ | 0 | $\theta_3$ |
| 4 | 90 | 0 | 0 | $\theta_4$ |
| 5 | 0 | $-l_4$ | $l_5$ | 0 |

$\alpha_i$ = the angle between $\hat{Z}_i$ and $\hat{Z}_{i+1}$ measured about $\widehat{X}_i$

$a_i$ = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\widehat{X}_i$

$d_i$ = the distance from $\widehat{X}_{i-1}$ to $\widehat{X}_i$ measured along $\hat{Z}_i$

$\theta_i$ = the angle between $\widehat{X}_{i-1}$ and $\widehat{X}_i$ measured about $\hat{Z}_i$

# Frame to Frame Transformation

$$
{}^{i-1}_{\ \ i}T = R_X(\alpha_{i-1})\, D_X(a_{i-1})\, R_Z(\theta_i)\, D_Z(d_i)
$$

$$
= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix}
\alpha_i & = & \text{the angle between } \hat{Z}_i \text{ and } \hat{Z}_{i+1} \text{ measured about } \widehat{X}_i \\
a_i & = & \text{the distance from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured along } \widehat{X}_i \\
d_i & = & \text{the distance from } \widehat{X}_{i-1} \text{ to } \widehat{X}_i \text{ measured along } \hat{Z}_i \\
\theta_i & = & \text{the angle between } \widehat{X}_{i-1} \text{ and } \widehat{X}_i \text{ measured about } \hat{Z}_i
\end{bmatrix}
$$

RBO

| | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | -90 | 0 | $d_2$ | -90 |
| 3 | -90 | $l_2$ | 0 | $\theta_3$ |
| 4 | 90 | 0 | 0 | $\theta_4$ |
| 5 | 0 | $-l_4$ | $l_5$ | 0 |

$$
\begin{aligned}
{}^{i-1}_{i}T &= R_X(\alpha_{i-1})\ D_X(a_{i-1})\ R_Z(\theta_i)\ D_Z(d_i) \\
&= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

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$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^{1}_{2}T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^{2}_{3}T = \begin{bmatrix} c_3 & -s_1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^{4}_{5}T = \begin{bmatrix} 1 & 0 & 0 & -l_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

# Summary

- Homogoneous transformation:
  - Describing positions and orientations in space and how to map between them
- Forward kinematics:
  - Where is the robot‘s i-th link or end-effector given the robots configuration
- Denavit-Hartenberg parameters
  - Uniform procedure for defining the robot‘s kinematics (in order to derive forward kinematics)