# MS Project

## On consistency of Exponentially Embedded Family (EEF) rule for 1-D and 2-D cisoid model and Interval estimation of model order using residual bootstrapping

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# Outline

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# A Quick Recap

- Model order selection- fundamental problem in signal processing
- A novel rule introduced by Stoica and Babu[3], where model order estimation based on penalizing adaptively the likelihood (PAL)
- PAL rule can be expressed as a function of model order m as follows:

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$$PAL(m) = -2\ln(f_m(y, \hat{\theta}^*_m)) + (3m+1)\ln(3\tilde{m}+1)\frac{\ln(r_m+1)}{\ln(\rho_m+1)} \tag{1}$$

$$\hat{m} = \underset{m\in\{1,2,\ldots,\tilde{m}\}}{\arg\min}\ [PAL(m)] \tag{2}$$

where,

$$r_m = 2\ln\left[\frac{f_{m-1}(y, \hat{\theta}^*_{m-1})}{f_0(y, \hat{\theta}^*_0)}\right] \qquad \rho_m = 2\ln\left[\frac{f_{\tilde{m}}(y, \hat{\theta}^*_{\tilde{m}})}{f_{m-1}(y, \hat{\theta}^*_{m-1})}\right]$$

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$$\hat{m} = \underset{m\in\{1,2,\ldots,\tilde{m}\}}{\arg\min} [PAL(m)] \quad (2)$$

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$$r_m = 2\ln\left[\frac{f_{m-1}(y, \hat{\theta}_{m-1}^*)}{f_0(y, \hat{\theta}_0^*)}\right] \qquad \rho_m = 2\ln\left[\frac{f_{\tilde{m}}(y, \hat{\theta}_{\tilde{m}}^*)}{f_{m-1}(y, \hat{\theta}_{m-1}^*)}\right]$$

- Proved that PAL rule for model order selection 1-D and 2-D cisoid models in white noise is consistent

# Outline

# EEF rule

- Use of exponential embedding of pdfs for model order estimation was introduced by Kay[1]
- Stoica and Babu[2] proposed a Generalised Likelihood Ratio (GLR) based derivation of EEF rule
- Here, we investigate the asymptotic statistical properties of the same for 1-D and 2-D cisoid models
- Simulation examples to compare EEF's performance with other order selection rules for 1-D and 2-D cisoid models

# Outline

# 1-Dimensional Cisoid model

- Building blocks of Digital signal processing

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- Building blocks of Digital signal processing
- $\forall t = 1, 2, ..., n$

$$y_t = \sum_{k=1}^{m} \alpha_k e^{it\omega_k} + \epsilon_t \tag{3}$$

$$y_t = f(t, \theta_m) + \epsilon_t \tag{4}$$

where
$\theta_m = (\alpha_{1_R}, \alpha_{1_C}, \omega_1, ..., \alpha_{m_R}, \alpha_{m_C}, \omega_m)'_{3m\text{x}1}$=unknown signal parameter vector
$\alpha_{j_R}$ and $\alpha_{j_C}$ denotes the real and the imaginary part of $\alpha_j$ for j=1,..., m
$m_o$= true number of components in the the observed signal

# 1-Dimensional Cisoid model-Assumptions

- $\epsilon_t$ are i.i.d complex valued gaussian with zero mean s.t.

$$\epsilon_t = \epsilon_{t_R} + i\epsilon_{t_C} \tag{5}$$
$$\epsilon_{t_R} \sim \mathcal{N}(0, \sigma^2/2) \tag{6}$$
$$\epsilon_{t_C} \sim \mathcal{N}(0, \sigma^2/2) \tag{7}$$

  and $\epsilon_{t_R}$ and $\epsilon_{t_C}$ are independent
- $\forall k = 1, 2, ..., m_o : \omega_k \in (0, 2\pi)$; $\omega_j \neq \omega_k, \forall j \neq k$. Furthermore, $\alpha_k$ are bounded
- The true model parameter vector $\theta_{m_o}$ is an interior point in the parameter space $\Theta \subset \mathbb{R}^{3m_o}$

# EEF rule for 1-D cisoid

Consider the Generalized likelihood ratio,

$$r_m = 2 \ln \left[ \frac{f_{m-1}(y, \hat{\theta}^*_{m-1})}{f_0(y, \hat{\theta}^*_0)} \right]$$

where,
$\hat{\theta}^*_k = (\hat{\theta}_k, \hat{\sigma}^2_k) =$ M.L.E. of underlying signal and noise parameter vector $\theta^*_k$
$f_0(y, \hat{\theta}^*_0) =$ p.d.f of y when $M_0$ is the model i.e., $f(\theta_0) = 0$

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$f_0(y, \hat{\theta}^*_0) =$ p.d.f of y when $M_0$ is the model i.e., $f(\theta_0) = 0$
The EEF rule can then be defined using the GLR ratio as follows:

$$EEF(m) = \begin{cases} \hat{r}_{m+1} - (3m+1)\left[1 + \ln\left(\frac{\hat{r}_{m+1}}{3m+1}\right)\right] & \text{if } \hat{r}_{m+1} > 3m+1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$\hat{m} = \underset{m \in \{1,2,\dots,\tilde{m}\}}{\arg\max} \, [EEF(m)] \tag{9}$$

where $\tilde{m}$ is the maximum number of cisoid components

# Consistency of EEF rule

## Lemma 1

Under the assumptions A1-A3, $\forall m \leq m_o$,as $n \to \infty$

$$\hat{\sigma}_m^2 = \sigma^2 + \sum_{j=1}^{m_o} \alpha_j^H \alpha_j - \sum_{j=1}^{m} \hat{\alpha}_j^H \hat{\alpha}_j + o(1)\text{a.s.} \tag{10}$$

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## Lemma 2

Under A1-A3, for any integer $k \geq 1$

$$\hat{\sigma}_{m_o+k}^2 = \hat{\sigma}_{m_o}^2 - \sum_{j=1}^{k} \frac{I_\epsilon(\hat{\omega}_{m_o+j})}{n} + o\left(\frac{\ln n}{n}\right) \text{ a.s. as } n \to \infty \tag{11}$$

where $I_\epsilon(\omega)$ corresponds to periodogram of underlying white noise process and $\hat{\omega}_{m_o+1}, \hat{\omega}_{m_o+2}, ..., \hat{\omega}_{m_o+k}$ are the k largest frequencies corresponding to $I_\epsilon(\omega)$

# Consistency of EEF rule-II

**Lemma 3**

Under assumptions A1-A3, $r_m$ satisfies

$$r_m = \begin{cases} 0 & m = 1 \\ O(n) & 2 \leq m \leq \tilde{m} \end{cases} \tag{12}$$

# Consistency of EEF rule-III

## Theorem

Under A1-A3, if $m_o$ is true order($m_o \leq \tilde{m}$) and $\hat{m}$ is the estimated model order using EEF rule then

$$P(\hat{m} \neq m_o) \to 0 \text{ as } \; n \to \infty \tag{13}$$

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**Sketch of the Proof**:Consider sub-cases of Over and Under estimation:
Case I-**Underestimation** $(m \leq m_o)$:
*Sub-case 1-* $\hat{r}_{m+1} > 3m + 1$ ;$\hat{r}_{m_o+1} > 3m_o + 1$: Show that

$$\frac{1}{n}(EEF(m) - EEF(m_o)) \to -2\ln\left(1 + \frac{\sum_{j=m+1}^{m_o} \alpha_j^H \alpha_j}{\sigma^2}\right) \text{ a.s.} \tag{14}$$

as $n \to \infty$ where R.H.S. is a strictly negative and bounded quantity. Use it to show $P(\hat{m} < m_o) \to 0$ as $n \to \infty$.
*Sub-case 2-* $\hat{r}_{m+1} < 3m + 1$ ;$\hat{r}_{m_o+1} > 3m_o + 1$: Show that function of form $g(x) = x - \ln x - 1 \geq 0$ to show that underestimation is not possible.

*Sub-case 3*-$\hat{r}_{m+1} > 3m+1$ ;$\hat{r}_{m_o+1} < 3m_o+1$: From [2], for large samples $EEF(m) \cong \hat{r}_{m+1} - (3m+1)\ln n$. Use this result and Lemma 3 to show

$$\frac{EEF(m) - EEF(m_o)}{n \ln n} \to 0 \text{ as } n \to \infty \tag{15}$$

which can be used to show $P(\hat{m} < m_o) \to 0$ as $n \to \infty$.

Case II-**Overestimation** ($m > m_o$):For *Sub-case 1*, Using the asymptotic theory of likelihood ratios(see [4,5]), $\frac{1}{\ln(n)}\left(2n\ln\left(\frac{\hat{\sigma}_m^2}{\hat{\sigma}_{m_o}^2}\right)\right) = o_p(1)$

Use this to show:

$$P\left(\frac{1}{\ln n}\Big(EEF(m) - EEF(m_o)\Big) < 0\right) \to 1 \text{ as } n \to \infty \tag{16}$$

*Sub-case 2 and Sub-case 3* goes through as the underestimation case.

Thus as $n \to \infty$,

$$P(\hat{m} > m_o) = P(EEF(m) > EEF(m_o) \text{ for some } m > m_o) \to 0 \tag{17}$$

This implies

$$P(\hat{m} \neq m_o) \to 0 \text{ as } n \to \infty \tag{18}$$

## Example

Simulation example

$$y_t = \sum_{k=1}^{3} \alpha_k e^{it\omega_k} + \epsilon_t$$

$$\alpha_1 = 3 + i2, \quad \alpha_2 = 2 + i1.66, \quad \alpha_3 = 1.75 + i$$

$$\omega_1 = 0.8\pi, \quad \omega_2 = 1.2\pi, \quad \omega_3 = 1.4\pi$$

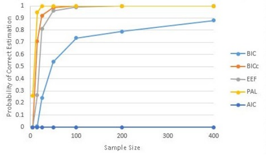
$\epsilon_t$ are i.i.d complex valued gaussian with zero mean s.t.

$$\epsilon_t = \epsilon_{t_R} + i\epsilon_{t_C} \tag{19}$$

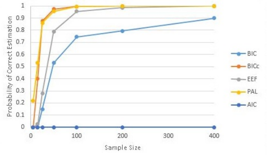
$$\epsilon_{t_R} \sim \mathcal{N}(0, \sigma^2/2) \tag{20}$$

$$\epsilon_{t_C} \sim \mathcal{N}(0, \sigma^2/2) \tag{21}$$

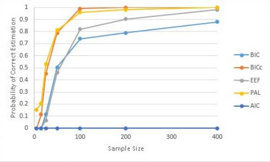
$\epsilon_{t_R}$ and $\epsilon_{t_C}$ are independent



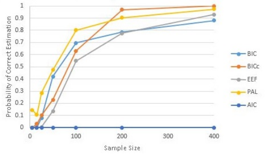

(a) $\sigma^2 = 1$



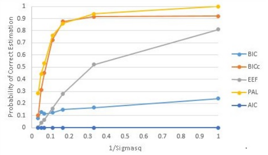

(b) $\sigma^2 = 6$



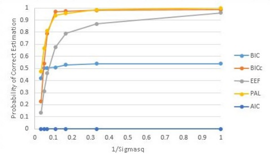

(c) $\sigma^2 = 15$




(d) $\sigma^2 = 30$




(e) $Samplesize = 25$




(f) $Samplesize = 50$

Figure: (a)-(d):Plot of the probability of correct estimation of model order against different sample size for fixed values of variance of noise; (e)-(f) plot the probability of correct estimation with different values of $\sigma^2$ for fixed sample sizes of 25 and 50

# Outline

# 2-Dimensional Cisoid Model

- Widespread application in wireless communication

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- Widespread application in wireless communication
- $\forall s, t$ such that$1 \leq s \leq S; 1 \leq t \leq T$

$$y(s,t) = \sum_{k=1}^{m} \alpha_k e^{i(s\beta_k + t\omega_k)} + \epsilon(s,t) \tag{22}$$

$$y(s,t) = f(s,t,\theta_m) + \epsilon(s,t) \tag{23}$$

$\theta_m = (\alpha_{1_R}, \alpha_{1_C}, \beta_1, \omega_1, ..., \alpha_{m_R}, \alpha_{m_C}, \beta_m, \omega_m)'_{4m\times1}$: vector of unknown signal parameters
$m_o$=True number of components in the the observed signal.

# 2-Dimensional Cisoid Model-Assumptions

- $\epsilon(s,t)$ are i.i.d complex valued gaussian with zero mean s.t.

$$\epsilon(s,t) = \epsilon_R(s,t) + i\epsilon_C(s,t) \tag{24}$$

$$\epsilon_R(s,t) \sim \mathcal{N}(0, \sigma^2/2) \tag{25}$$

$$\epsilon_C(s,t) \sim \mathcal{N}(0, \sigma^2/2) \tag{26}$$

  and $\epsilon_R(s,t)$ and $\epsilon_C(s,t)$ are independent.
- $\forall k = 1, 2, ..., m_o : (\beta_k, \omega_k) \in (0, 2\pi)$x$(0, 2\pi)$;
  where $(\beta_k, \omega_k)$ are pairwise different i.e. $\omega_j \neq \omega_k$ or $\beta_j \neq \beta_k, \forall j \neq k$.
  Furthermore, $\forall k = 1, 2..., m_o : \alpha_k$ are bounded
- The true model parameter vector $\theta_{m_o}$ is an interior point in the parameter space $\Theta \subset \mathbb{R}^{4m_o}$

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$$\hat{m} = \underset{m \in \{1,2,\ldots,\tilde{m}\}}{\arg\max} \left[EEF(m)\right] \tag{28}$$

where $\tilde{m}$ is the maximum number of components in the model

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Remark:Let $\{\Psi_i\}$ be a sequence of rectangles s.t.

$$\Psi = \{(s,t) \in \mathbb{Z}^2 | 1 \leq s \leq S_i, 1 \leq t \leq T_i\} \tag{29}$$

Then, sequence of subsets $\{\Psi_i\}$ is said to tend to infinity as $i \to \infty$ if

$$\lim_{i\to\infty} min(S_i, T_i) \to \infty \text{ and } 0 < \lim_{i\to\infty}\left(\frac{S_i}{T_i}\right) < \infty \tag{30}$$

# Consistency of EEF rule

**Lemma 1**

Under the assumptions A1-A3, $\forall m \leq m_o$,as $\Psi(S, T) \rightarrow \infty$

$$\hat{\sigma}_m^2 = \sigma^2 + \sum_{j=1}^{m_o} \alpha_j^H \alpha_j - \sum_{j=1}^{m} \hat{\alpha}_j^H \hat{\alpha}_j + o(1) \text{ a.s.} \tag{31}$$

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## Lemma 2

Under A1-A3, for any integer $k \geq 1$,as $\Psi(S, T) \to \infty$

$$\hat{\sigma}_{m_o+k}^2 = \hat{\sigma}_{m_o}^2 - \sum_{j=1}^{k} \frac{I_\epsilon(\hat{\beta}_{m_o+j}, \hat{\omega}_{m_o+j})}{ST} + o\left(\frac{(\ln(ST)\ln S \ln T)^{1/2}}{ST}\right) \text{ a.s.} \tag{32}$$

where $I_\epsilon(\beta, \omega)$ corresponds to periodogram of underlying white noise process and $(\hat{\beta}_{m_o+1}, \hat{\omega}_{m_o+1}), (\hat{\beta}_{m_o+2}, \hat{\omega}_{m_o+2}), ..., (\hat{\beta}_{m_o+k}, \hat{\omega}_{m_o+k})$ are the k largest frequencies corresponding to $I_\epsilon(\beta, \omega)$

# Consistency of EEF rule-II

**Lemma 3**

Under assumptions A1-A3, $r_m$ satisfies

$$r_m = \begin{cases} 0 & m = 1 \\ O(ST) & 2 \le m \le \tilde{m} \end{cases} \tag{33}$$

# Consistency of EEF rule-III

**Theorem**

Under A1-A3, if $m_o$ is true order($m_o \leq \tilde{m}$) and $\hat{m}$ is the estimated model order using EEF rule then

$$P(\hat{m} \neq m_o) \rightarrow 0 \text{ as } \Psi(S, T) \rightarrow \infty \tag{34}$$

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Under A1-A3, if $m_o$ is true order($m_o \leq \tilde{m}$) and $\hat{m}$ is the estimated model order using EEF rule then

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**Sketch of the Proof**:Consider sub-cases of Over and Under estimation:
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*Sub-case 1-* $\hat{r}_{m+1} > 4m+1$ ;$\hat{r}_{m_o+1} > 4m_o+1$: Show that

$$\frac{1}{ST}(EEF(m) - EEF(m_o)) \to -2\ln\left(1 + \frac{\sum_{j=m+1}^{m_o} \alpha_j^H \alpha_j}{\sigma^2}\right) \text{ a.s.} \quad (35)$$

as $\Psi(S,T) \to \infty$ where R.H.S. is a strictly negative and bounded quantity. Use it to show $P(\hat{m} < m_o) \to 0$ as $\Psi(S,T) \to \infty$.
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$$\frac{EEF(m) - EEF(m_o)}{ST \ln ST} \to 0 \text{ as } \Psi(S,T) \to \infty \tag{36}$$

which can be used to show $P(\hat{m} < m_o) \to 0$ as $\Psi(S,T) \to \infty$.

Case II-**Overestimation** ($m > m_o$):For *Sub-case 1*, Using the asymptotic theory of likelihood ratios(see [4,5]), $\frac{1}{\ln(ST)}\left(2ST\ln\left(\frac{\hat{\sigma}^2_m}{\hat{\sigma}^2_{m_o}}\right)\right) = o_p(1)$

Use this to show:

$$P\left(\frac{1}{\ln(ST)}\Big(EEF(m) - EEF(m_o)\Big) < 0\right) \to 1 \text{ as } \Psi(S,T) \to \infty \tag{37}$$

*Sub-case 2 and Sub-case 3* goes through as the underestimation case. Thus as $\Psi(S,T) \to \infty$,

$$P(\hat{m} > m_o) = P(EEF(m) > EEF(m_o) \text{ for some } m > m_o) \to 0 \tag{38}$$

This implies

$$P(\hat{m} \neq m_o) \to 0 \text{ as } \Psi(S,T) \to \infty \tag{39}$$

# Numerical examples

## Example

Simulation example

$$y(s,t) = \sum_{k=1}^{2} \alpha_k e^{i(s\beta_k + t\omega_k)} + \epsilon(s,t)$$

$$\alpha_1 = 1 + \sqrt{2}i, \quad \beta_1 = 0.26\pi, \quad \omega_1 = 0.26\pi$$

$$\alpha_2 = 2 + 2i, \quad \beta_2 = 0.62\pi, \quad \omega_2 = 0.62\pi$$

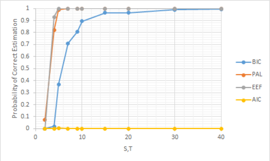
$\epsilon(s,t)$ are i.i.d complex valued gaussian with zero mean s.t.

$$\epsilon(s,t) = \epsilon_R(s,t) + i\epsilon_C(s,t) \tag{40}$$

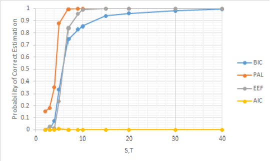
$$\epsilon_R(s,t) \sim \mathcal{N}(0, \sigma^2/2) \tag{41}$$

$$\epsilon_C(s,t) \sim \mathcal{N}(0, \sigma^2/2) \tag{42}$$

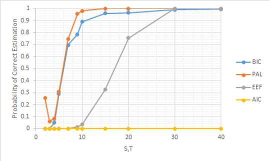
and $\epsilon_R(s,t)$ and $\epsilon_C(s,t)$ are independent



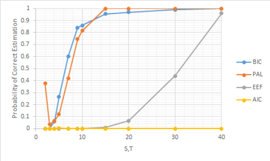

(a) $\sigma^2 = 0.5$



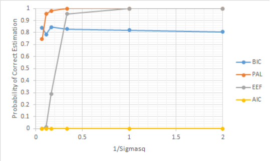

(b) $\sigma^2 = 3$



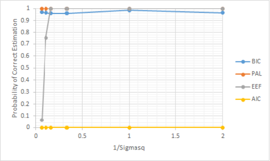

(c) $\sigma^2 = 9$




(d) $\sigma^2 = 15$




(e) $S, T = 9$




(f) $S, T = 20$

Figure: (a)-(d):Plot of the probability of correct estimation of model order against different S, T values for fixed values of variance of noise; (e)-(f) plot the probability of correct estimation with different values of $\sigma^2$ for fixed S,T values of 9 and 20 each

# Outline

# A discussion

- Model order estimation rules like PAL and EEF given only a point estimate
- Interval estimates would provide a range of values with a known probability of capturing the model order
- Propose an algorithm based on residual bootstrapping to generate confidence for model order using PAL rule

**Algorithm 1** Confidence interval estimation of model order

Define the number of components $m_o$ and model parameter values $\alpha_k, \beta_k \forall k = 1 \dots m_o$
$n \leftarrow$ Sample size
**for** `isim=1 to nsim` **do**
    $y \leftarrow$ generate-data$(n, m_o, \alpha, \beta)$
    Estimate model order for $y$ using PAL rule
    Estimate $\alpha_k, \beta_k \; \forall k = 1 \dots m$
    Calculate $\hat{y}_i$ and use it to Calculate $\hat{\epsilon}_i = y_i - \hat{y}_i \; \forall i = 1, \dots, n$
    **for** `b=1 to bootsim` **do**
        Draw a bootstrap sample $\hat{\epsilon}_b$ of size n from $\hat{\epsilon}$
        Calculate $y_b = \hat{y} + \hat{\epsilon}_b$
        Estimate model order for $y_b$ using PAL rule
    **end for**
    Find the empirical interval with $\alpha$ level of confidence using Bootsim number of estimated model orders
**end for**
Calculate average over confidence interval for nsim simulations

## Example

Simulation example

$$y_t = \sum_{k=1}^{m} \alpha_k e^{it\omega_k} + \epsilon_t$$

$$\alpha_1 = 3 + i2, \quad \alpha_2 = 2 + i1.66, \quad \alpha_3 = 1.75 + i$$

$$\omega_1 = 0.8\pi, \quad \omega_2 = 1.2\pi, \quad \omega_3 = 1.4\pi$$

$\epsilon_t$ are i.i.d complex valued gaussian with zero mean s.t.

$$\epsilon_t = \epsilon_{t_R} + i\epsilon_{t_C} \tag{43}$$

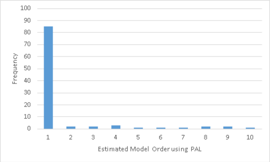
$$\epsilon_{t_R} \sim \mathcal{N}(0, \sigma^2/2) \tag{44}$$

$$\epsilon_{t_C} \sim \mathcal{N}(0, \sigma^2/2) \tag{45}$$

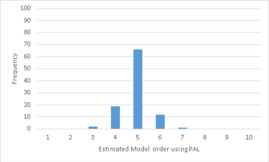
$\epsilon_{t_R}$ and $\epsilon_{t_C}$ are independent

Table: Average quantiles from 100 bootstrap samples of different sample sizes

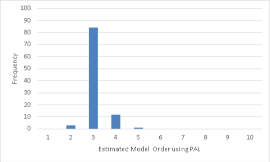
| | Quantile values | | | | |
|---|---|---|---|---|---|
| Sample Size | 0.025 | 0.05 | 0.5 | 0.95 | 0.975 |
| 15 | 1.51 | 1.725 | 2.2 | 7.45 | 8.5 |
| 25 | 1.18 | 1.265 | 1.905 | 3.62 | 4.34 |
| 50 | 1.56 | 1.79 | 2.775 | 3.995 | 4.35 |
| 100 | 2.39 | 2.6 | 3.06 | 4.125 | 4.35 |
| 200 | 2.95 | 3 | 3.05 | 3.915 | 4.15 |




(a) $Samplesize = 15$




(b) $Samplesize = 50$




(c) $Samplesize = 200$

Figure: (a)-(c):Plot of the Histogram for estimated model order over 100 bootstrap samples for a simulation for different sample size at a fixed noise variance $\sigma^2$=30

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- Proposed a residual bootstrapping based algorithm for confidence interval estimation of model order
- Outlook
  - PAL and EEF rule consistency for chirp signal model
  - Consistency of PAL based order selection rule and EEF rule in presence of coloured noise

# References


1. Kay, Steven. "Exponentially embedded families-new approaches to model order estimation." IEEE Transactions on Aerospace and Electronic Systems 41.1 (2005): 333-345.
2. Stoica, Petre, and Prabhu Babu. "On the exponentially embedded family (EEF) rule for model order selection." IEEE Signal Processing Letters 19.9 (2012): 551-554.
3. Stoica, Petre, and Prabhu Babu. "Model order estimation via penalizing adaptively the likelihood (PAL)." Signal processing 93.11 (2013): 2865-2871.
4. Wilks, Samuel S. "Sample criteria for testing equality of means, equality of variances, and equality of covariances in a normal multivariate distribution." The Annals of Mathematical Statistics (1946): 257-281.
5. Lehmann, Erich L., and Joseph P. Romano. "Testing Statistical Hypotheses (Springer Texts in Statistics)." (2005).