Probability&RV Assignment-09

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https://github.com/Anuradha−Uggi/Assignments−AI5002−Probability−and−Random−Variables/blob/main/Prob_ass09/rvsp_9.tex

I. QUESTION(UGC NET 2019,Q-108)

Suppose $X_i = X_1, X_2, ..., X_n$ are i.i.d Uniform $(\theta, 2\theta), \theta > 0$. Let $X_{(1)}$=min$\{X_1, X_2, ..., X_n\}$ and $X_{(n)}$=max$\{X_1, ..., X_n\}$.then which of the following statements are correct.

1) ($X_{(1)}$,$X_{(n)}$) is jointly sufficient and complete for θ
2) ($X_{(1)}$,$X_{(n)}$) is jointly sufficient but not complete for θ
3) $\frac{X_{(n)}}{2}$ is maximum likelihood estimate for θ
4) $X_{(1)}$ is maximum likelihood estimate for θ

II. Basic Definitions

Sufficient Statistic:

Given X i.i.d Data conditioned on an unknown parameter θ, T(X) is called sufficient statistic for θ if its values contains all the information needed to compute any estimate of the parameter (Maximum likelihood estimate). according to Fisher−Neymen Factorization PDF is

$$f(X; \theta) = h(X)S(\theta, T(X)) \tag{1}$$

where h(X) is a constant and S(θ,T(X)) is a function through which θ will interact to X only through T(X).

Statistic Completeness:

T(X) is said to be complete for θ if for every measurable function g; if

$$E_\theta(g(T)) = 0 \tag{2}$$

for all θ then

$$P_\theta(g(T) = 0) = 1 \tag{3}$$

for all θ.

III. SOLUTION

Given

$$X_{(1)} = \min\{X_1, X_2, ..., X_n\} = \min\{X_i\} \tag{4}$$

$$X_{(n)} = \max\{X_1, X_2, ..., X_n\} = \max\{X_i\} \tag{5}$$

$$X_i = \{X_1, X_2, ..., X_n\} \sim U(\theta, 2\theta) \tag{6}$$

$$P(X_i) = \frac{1}{\theta} \tag{7}$$

Statistic

$$T(X) = (\min\{X\}, \max\{X\}) \tag{8}$$

X are i.i.d so Likelihood or joint PDF is simple product of all marginal PDFs given by

$$f_X(x) = \frac{1}{\theta^n} 1_{\{\theta \le x_1 \le 2\theta\}} 1_{\{\theta \le x_2 \le 2\theta\}} ... 1_{\{\theta \le x_n \le 2\theta\}} \tag{9}$$

Indicator Function($\mathbf{1}_{\{\theta \le X_i \le 2\theta\}}$):
let we have

$$A \subset X \tag{10}$$

Indicator Function (or) Characteristic Function in Mathematics Indicates membership of elements in set X,having value 1 for elements of X in A and 0 for those of X not in A.its denoted by Symbol 1 or I with a subscript.

$$1_{(A)} : X \to \{0, 1\} \tag{11}$$

$$1_{(A(x)):} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{12}$$

we know that all random samples from Random Variables $X_1, X_2, ...X_n$ lies in the range $(\theta, 2\theta)$ with a probability

$$\Pr(X = x) = \frac{1}{\theta} 1_{\{\theta \leq x \leq 2\theta\}} \tag{13}$$

Now the joint PDF can be expressed as

$$f_X(x) = \frac{1}{\theta^n} 1_{\{\theta \leq \min\{X_i\}} 1_{\max\{X_i\} \leq 2\theta} \tag{14}$$

above equation implies that all samples $x_1, x_2, ..x_n$ fall in θ and 2θ.from above equation

$$h(X) = 1 \tag{15}$$

which is constant and

$$S_{(\theta, 2\theta)}(X) = \frac{1}{\theta^n} \tag{16}$$

which is function of only θ.
therefore $T(\min\{X_i\}, \max\{X_i\})$ is jointly sufficient to define θ thus Sufficient statistic.

Let

$$g(T) = \max\{X_i\} - \min\{X_i\} \tag{17}$$

$$E[g(T)] = E[\max\{X_i\} - \min\{X_i\}] = c \tag{18}$$

$$E[\max\{X_i\} - \min\{X_i\} - c] = 0 \tag{19}$$

$$E[g(T) - c] = \int (\max\{X_i\} - \min\{X_i\} - c) \frac{1}{\theta^n} dx \tag{20}$$

from equation (20) its clear that

$$\max\{X_i\} - \min\{X_i\} - c = 0 \tag{21}$$

for all θ therefore

$$P(\max\{X_i\} - \min\{X_i\} - c = 0) = 1 \tag{22}$$

for all θ therefore $T(X_{(n)}, X_{(1)})$ is Jointly sufficient and complete for θ.

Maximum Likelihood Estimate(MLE) :

Likelihood can be written as

$$f_\theta(X_1, X_2, .., X_n) = \frac{1}{\theta^n} I\left(\frac{\max\{X_i\}}{2} \leq \theta \leq \min\{X_i\}\right) \tag{23}$$

the MLE is the statistic that maximizes the liklihood.from equation (7) liklihood is a decreasing function of θ.therefore MLE of θ is

$$\theta = \frac{X_{(n)}}{2} = \frac{\max\{X_i\}}{2} \tag{24}$$

IV. CONCLUTION

From above observations option (1) and option (3) holds.