Lyapunov Exponent

Lyapunov Exponent is a mathematical characteristic that defines the separation between trajectories of the orbits of certain dynamical systems.

where, $\Delta^n = \left|f^n(x_0+\delta) - f^n(x_0)\right|$ is the separation after n iterations

Initially, the separation was $\Delta^0 = |x_0 + \delta - x_0| = \delta$

Lyapunov showed that $\left|f^n(x_0+\delta) - f^n(x_0)\right| = \delta e^{n\lambda}$

where, λ is the Lyapunov exponent

Further,

$$\frac{\left|f^n(x_0+\delta) - f^n(x_0)\right|}{\delta} = e^{n\lambda}$$

taking log_e on both sides to bring down the Lyapunov exponent

$$log_e \frac{\left|f^n(x_0+\delta) - f^n(x_0)\right|}{\delta} = n\lambda$$

applying limits on the values of n and δ, we get

$$\lambda = \lim_{n\to\infty,\delta\to 0} \left(\frac{1}{n} log_e \frac{\left| f^n(x_0+\delta) - f^n(x_0) \right|}{\delta} \right)$$

Now, this $\frac{\left| f^n(x_0+\delta) - f^n(x_0) \right|}{\delta}$ is nothing but $\left(\frac{df}{dx} \Big|_{x=x_0} \right)$

$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n} log_e \left(\frac{df}{dx} \Big|_{x=x_0} \right) \right)$$

Now, we can apply the chain rule of differentiation, like

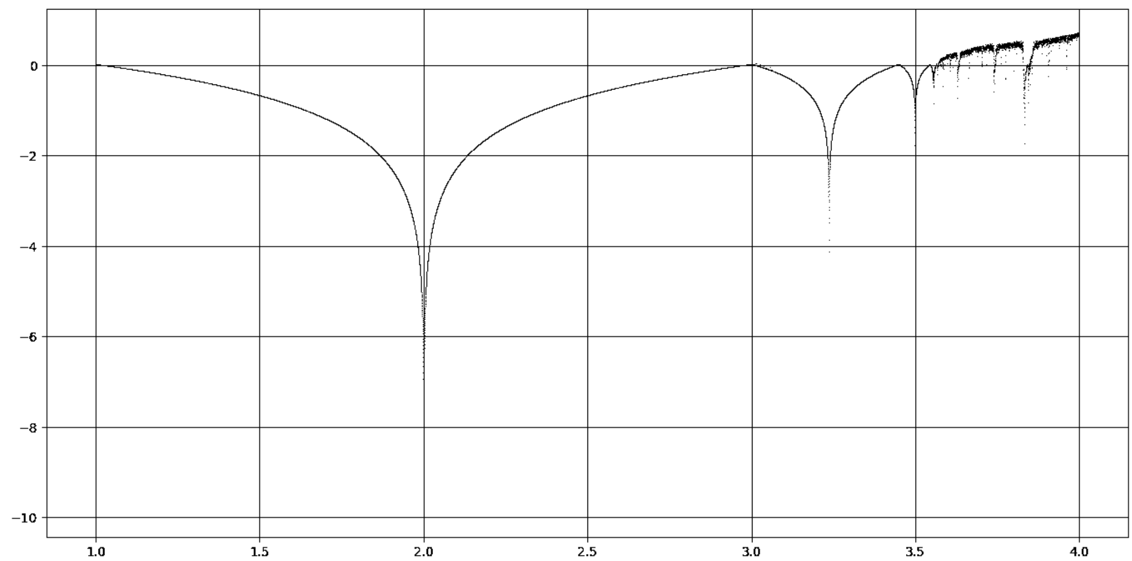
$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n} log_e \left(\left(\frac{df}{dx} \Big|_{x=x_{n-1}=f^{n-1}(x_0)} \right) + \left(\frac{df}{dx} \Big|_{x=x_{n-2}=f^{n-2}(x_0)} \right) + \cdots + \left(\frac{df}{dx} \Big|_{x=x_0=f^0(x_0)} \right) \right) \right)$$

which could be reformatted as

$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n} log_e \left(\sum_{k=0}^{n-1} \left(\frac{df}{dx} \Big|_{x=x_k=f^k(x_0)} \right) \right) \right)$$

We have applied the same algorithm and have developed a python script to calculate the LEs of Chaotic Maps.

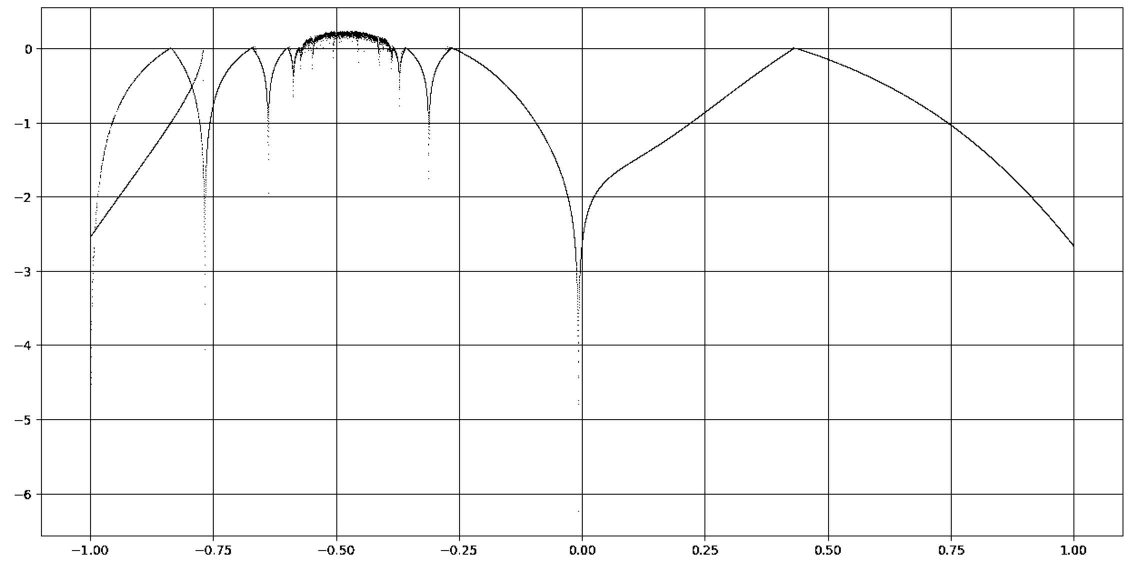
1. **Logistic Maps**



Here, for the Logistic Map, we have considered the r in the range $[0,4]$.

Also, we have iterated the map 100 times.

2. **Gauss Maps**



Here, for the Gauss Map, we have considered the β in the range $[-1,1]$, and $\alpha = 4.90$.

Also, we have iterated the map 100 times.