Square Root Evaluation

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1 Introduction

In mathematics, a square root of a number x is a number y such that $y^2 = x$. It is denoted as $\sqrt{x}$.

For example:

$$\sqrt{0} = \pm 0, \sqrt{4} = \pm 2, \sqrt{9} = \pm 3, \sqrt{16} = \pm 4, \ldots, etc$$

Now, it's a point to notice that, for $y^2 = x$, $\sqrt{x} = \pm y$.

This can be justified as follows:

$y^2 = x$

$\Rightarrow y^2 - x = 0$

$\Rightarrow y^2 - \left(\sqrt{x}\right)^2 = 0$

We know, $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

$\Rightarrow \left(y - \sqrt{x}\right)\left(y + \sqrt{x}\right) = 0$

$\Rightarrow \left(y - \sqrt{x}\right) = 0$ and $\left(y + \sqrt{x}\right) = 0$

$\Rightarrow y = +\sqrt{x}$ and $y = -\sqrt{x}$

So, $y = \pm\sqrt{x}$

Now, it's easy to find square roots of numbers that are perfect squares (those numbers which can be represented as product of two same numbers, like 9, 25, 36, 49, etc.).

But, it's comparatively difficult to find square roots of non – perfect squares like 15, 8, etc.

2 Square Root Evaluation for non – perfect squares

Suppose $\varphi(x)$ is a function in x (a mathematical tool that takes an input in x and gives a result after processing).

Now, for a slight change, Δx in the input argument x of the function $\varphi(x)$,

$$\varphi(x + \Delta x) - \varphi(x) = \left(\frac{d}{dx}\left(\varphi(x)\right)\right)\Delta x$$

Now, for Square Root, the function $\varphi(x) = \sqrt{x}\ \forall x \in \mathbb{R}^+$ {Set of all positive real numbers}

and

$$\frac{d}{dx}\left(\varphi(x)\right) = \frac{1}{2\sqrt{x}}$$

So,

$$\sqrt{x + \Delta x} - \sqrt{x} = \left(\frac{1}{2\sqrt{x}}\right)\Delta x$$

Illustration 1: Evaluate $\sqrt{3}$.

Solution:

We know, $\sqrt{3} = \sqrt{4 - 1}$, So, $x = 4$ and $\Delta x = -1$

Now, we know,

$$\sqrt{x+\Delta x}-\sqrt{x}=\left(\frac{1}{2\sqrt{x}}\right)\Delta x$$

$$\Rightarrow \sqrt{4-1}-\sqrt{4}=\left(\frac{1}{2\sqrt{4}}\right)(-1)$$

$$\Rightarrow \sqrt{3}-2=\left(\frac{1}{2\times 2}\right)(-1)$$

$$\Rightarrow \sqrt{3}=2-\frac{1}{4}=2-0.25\approx 1.75$$

Illustration 2: Evaluate $\sqrt{32}$.

Solution:

We know, $\sqrt{32}=\sqrt{36-4}$, So, $x=36$ and $\Delta x=-4$

Now, we know,

$$\sqrt{x+\Delta x}-\sqrt{x}=\left(\frac{1}{2\sqrt{x}}\right)\Delta x$$

$$\Rightarrow \sqrt{36-4}-\sqrt{36}=\left(\frac{1}{2\sqrt{36}}\right)(-4)$$

$$\Rightarrow \sqrt{32}-6=\left(\frac{1}{2\times 6}\right)(-4)$$

$$\Rightarrow \sqrt{32}=6-\frac{1}{3}=6-0.334\approx 5.7$$

Illustration 3: Evaluate $\sqrt{51}$.

Solution:

We know, $\sqrt{51}=\sqrt{49+2}$, So, $x=49$ and $\Delta x=+2$

Now, we know,

$$\sqrt{x+\Delta x}-\sqrt{x}=\left(\frac{1}{2\sqrt{x}}\right)\Delta x$$

$$\Rightarrow \sqrt{51}-\sqrt{49}=\left(\frac{1}{2\sqrt{49}}\right)(+2)$$

$$\Rightarrow \sqrt{51}-7=\left(\frac{1}{2\times 7}\right)(+2)$$

$$\Rightarrow \sqrt{51}=7+\frac{1}{7}=7+0.142857\approx 7.14$$

3 Square Root Evaluation using Babylonian Method

The method that we are going to study in this section was initiated back in Babylonia.

The Algorithm for this method is quite simple, but quite long in big runs.

The steps are as follows:

- Let the number, whose square root is to found is n.
- Start with two number, say, x and y.
- Set $x=n$
- Set $y=1$

- Modify x as $\left(\frac{x+y}{2}\right)$
- Modify y as $\frac{n}{x}$
- Check for $x - y$

If $x - y > \varepsilon$, where ε is the accuracy fraction like 0.0000001, then REPEAT the steps 5, 6, 7 again.

Else, STOP.

Now, $\sqrt{n} = x$

Illustration 4: Evaluate $\sqrt{4}$ with desired accuracy

Solution:

$n = 4$

Let

$\varepsilon = 0.0000000001$

x	y	$x - y$	ε	Is $x - y > \varepsilon$?
4	1	3	0.0000000001	YES
$\frac{4+1}{2} = 2.5$	$\frac{4}{2.5} = 1.6$	0.9	0.0000000001	YES
$\frac{2.5+1.6}{2} = 2.05$	$\frac{4}{2.05} = 1.95$	0.1	0.0000000001	YES
$\frac{2.05+1.95}{2} = \boxed{2}$	2	0	0.0000000001	NO

Now, $\sqrt{4} = 2$