1. Find $\int \frac{log_e x}{(1+log_e x)^2} dx$

Solution:

$$I = \int \frac{log_e x}{(1 + log_e x)^2} dx$$

$$\Rightarrow I = \int \frac{log_e x + 1 - 1}{(1 + log_e x)^2} dx$$

$$\Rightarrow I = \int \frac{log_e x + 1}{(1 + log_e x)^2} dx - \int \frac{1}{(1 + log_e x)^2} dx$$

$$\Rightarrow I = \int \frac{(log_e x + 1)}{(1 + log_e x)^2} dx - \int \frac{1}{(1 + log_e x)^2} dx$$

$$\Rightarrow I = \int \frac{1}{(log_e x + 1)} dx - \int \frac{1}{(1 + log_e x)^2} dx$$

Now, $log_e e = 1$

$$\Rightarrow I = \int \frac{1}{(log_e x + log_e e)} dx - \int \frac{1}{(log_e e + log_e x)^2} dx$$

$$\Rightarrow I = \int \frac{1}{(log_e ex)} dx - \int \frac{1}{(log_e ex)^2} dx$$

Integrating by parts,

$$\int \left(\underset{I}{\frac{1}{(log_e ex)}} \quad \times \quad \underset{II}{1} \right) dx = \frac{1}{(log_e ex)} \int dx - \int \frac{d}{dx}\left(\frac{1}{(log_e ex)}\right)\left(\int dx\right) dx$$

$$\Rightarrow \int \left(\underset{I}{\frac{1}{(log_e ex)}} \quad \times \quad \underset{II}{1} \right) dx = \frac{1}{(log_e ex)} x - \int -\left(\frac{1}{x}\right)\left(\frac{1}{(log_e ex)^2}\right) x dx$$

$$\Rightarrow \int \left(\underset{I}{\frac{1}{(log_e ex)}} \quad \times \quad \underset{II}{1} \right) dx = \frac{1}{(log_e ex)} x + \int \frac{1}{(log_e ex)^2} dx$$

$$\Rightarrow I = \int \frac{1}{(log_e ex)} dx - \int \frac{1}{(log_e ex)^2} dx$$

$$\Rightarrow I = \frac{1}{(log_e ex)} x + \int \frac{1}{(log_e ex)^2} dx - \int \frac{1}{(log_e ex)^2} dx$$

$$\Rightarrow I = \frac{1}{(log_e ex)} x + \int \frac{1}{(log_e ex)^2} dx - \int \frac{1}{(log_e ex)^2} dx + \lambda$$

$$\boxed{\Rightarrow I = \frac{1}{(log_e(ex))} x + \lambda}$$

1^{OR}. Find $\int \frac{sin(2x)}{\sqrt{9-cos^4(x)}} dx$

Solution:

$$I = \int \frac{sin(2x)}{\sqrt{9 - cos^4(x)}} dx$$

$$\Rightarrow I = \int \frac{sin(2x)}{\sqrt{3^2 - \left(cos^2(x)\right)^2}} dx$$

Let, $cos^2(x) = t$

$\Rightarrow 1 + cos(2x) = 2t$

Differentiating both sides w.r.t x

$\Rightarrow -sin(2x)dx = dt$

$$\Rightarrow I = \int \frac{-dt}{\sqrt{3^2 - (t)^2}}$$

$\Rightarrow I = sin^{-1}\left({}^{t}/_{3}\right) + \lambda$

$$\boxed{\Rightarrow I = sin^{-1}\left({}^{cos^2(x)}/_{3}\right) + \lambda}$$

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2. Write the sum of degree and order of the DE $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

Solution:

$$f(x, y) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$\boxed{Sum\ of\ Degree\ and\ Order\ =\ 3}$$

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3. If $\hat{a}$ and $\hat{b}$ are two unit vectors, then prove that $\left|\hat{a} + \hat{b}\right| = 2cos\left(\frac{\theta}{2}\right)$, where θ is the angle between them.

Solution:

$$\left|\hat{a} + \hat{b}\right| = \sqrt{|\hat{a}|^2 + \left|\hat{b}\right|^2 + 2|\hat{a}|\left|\hat{b}\right|cos(\theta)}$$

As, $\hat{a}$ and $\hat{b}$ are two unit vectors, So, $|\hat{a}| = 1$ and $\left|\hat{b}\right| = 1$

$\Rightarrow \left|\hat{a} + \hat{b}\right| = \sqrt{2 + 2cos(\theta)}$

$\Rightarrow \left|\hat{a} + \hat{b}\right| = \sqrt{2(1 + cos(\theta))}$

$$\Rightarrow \left|\hat{a} + \hat{b}\right| = \sqrt{2\left(2cos^2\left({}^{\theta}/_{2}\right)\right)}$$

$$\boxed{\Rightarrow \left|\hat{a} + \hat{b}\right| = \sqrt{4cos^2\left({}^{\theta}/_{2}\right)} = 2cos\left({}^{\theta}/_{2}\right)}$$

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4. Find the direction cosines of the following line $\frac{3-x}{-1} = \frac{2y-1}{2} = \frac{z}{4}$.

Solution:

Given line: $\frac{3-x}{-1} = \frac{2y-1}{2} = \frac{z}{4}$

this can be re – written as $\frac{x-3}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z}{4}$

$$DCs\ = \frac{1}{\sqrt{(1)^2+(1)^2+(4)^2}}, \frac{1}{\sqrt{(1)^2+(1)^2+(4)^2}}, \frac{4}{\sqrt{(1)^2+(1)^2+(4)^2}}$$

$$\boxed{DCs = \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}}$$

5. A bag contains 1 red and 3 white balls. Find the probability distribution of the number of red balls if 2 balls are drawn at random from the bag one-by-one without replacement.

Solution:

$n(R) = 1$ and $n(W) = 3$

So, $n(T) = n(R) + n(W)$

$\rho(R = 2) = {}^{1}/_{4} \times {}^{0}/_{3} = 0$

$\rho(R = 1) = {}^{1}/_{4} \times {}^{3}/_{3} + {}^{3}/_{4} \times {}^{1}/_{3} = {}^{1}/_{2}$

$\rho(R = 0) = {}^{3}/_{4} \times {}^{2}/_{3} = {}^{1}/_{2}$

$$\boxed{\rho(R = \delta) = \begin{cases} 0 & if \quad \delta = 2 \\ {}^{1}/_{2} & if \quad \delta = 1 \\ {}^{1}/_{2} & if \quad \delta = 0 \end{cases}}$$

6. Two cards are drawn at random from a pack of 52 cards one-by-one without replacement. What is the probability of getting first card red and second card Jack?

Solution:

$\rho(E) = \rho(first\ card\ being\ a\ Red\ Jack) + \rho(first\ card\ being\ a\ Red\ NON - Jack)$

$\rho(E) = \left({}^{2}/_{52}\right) \times \left({}^{3}/_{51}\right) + \left({}^{24}/_{52}\right) \times \left({}^{4}/_{51}\right)$

$\rho(E) = \left({}^{6}/_{2652}\right) + \left({}^{96}/_{2652}\right)$

$$\boxed{\rho(E) = \left({}^{102}/_{2652}\right)}$$

7. Find $\int \frac{x+1}{(x^2+1)x} dx$

Solution:

$$I = \int \frac{x + 1}{(x^2 + 1)x} dx$$

$$\Rightarrow I = \int \frac{x + 1}{(x^2 + 1)x} dx$$

$$\Rightarrow I = \int \frac{x}{(x^2 + 1)x} dx + \int \frac{1}{(x^2 + 1)x} dx$$

$$\Rightarrow I = \int \frac{1}{(x^2 + 1)} dx + \int \frac{x^2 + 1 - x^2}{(x^2 + 1)x} dx$$

$$\Rightarrow I = \int \frac{1}{(x^2 + 1)} dx + \int \frac{x^2 + 1}{(x^2 + 1)x} dx - \int \frac{x^2}{(x^2 + 1)x} dx$$

$$\Rightarrow I = \int \frac{1}{(x^2 + 1)} dx + \int \frac{x^2 + 1}{(x^2 + 1)x} dx - \int \frac{x}{(x^2 + 1)} dx$$

$$\Rightarrow I = \int \frac{1}{(x^2 + 1)} dx + \int \frac{1}{x} dx - \int \frac{x}{(x^2 + 1)} dx$$

$$\Rightarrow I = tan^{-1}(x) + log_e|x| - \left({}^{1}\!/_{2}\right)\int \frac{2x}{(x^2+1)}dx$$

$$\boxed{\Rightarrow I = tan^{-1}(x) + log_e|x| - \left({}^{1}\!/_{2}\right) log_e|x^2+1| + \lambda}$$

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8. Find the general solution of the following differential equation

$$x\frac{dy}{dx} = y - xsin\left(\frac{y}{x}\right)$$

Solution:

$$x\frac{dy}{dx} = y - xsin\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{dy}{dx} = y - xsin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - \left(\frac{xsin\left(\frac{y}{x}\right)}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - sin\left(\frac{y}{x}\right)$$

Let $y = tx$

Differentiating both sides w.r.t x

$$\frac{dy}{dx} = t + x\left(\frac{dt}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - sin\left(\frac{y}{x}\right)$$

$$\Rightarrow t + x\left(\frac{dt}{dx}\right) = t - sin(t)$$

$$\Rightarrow x\left(\frac{dt}{dx}\right) = -sin(t)$$

$$\Rightarrow -\left(\frac{dt}{sin(t)}\right) = \frac{dx}{x}$$

$$\Rightarrow -\int\left(\frac{dt}{sin(t)}\right) = \int\frac{dx}{x}$$

$$\Rightarrow -\int cosec(t)\,dt = \int\frac{dx}{x}$$

$$\Rightarrow -log_e|cosec(t) - cot(t)| = log_e|x| + \lambda$$

$$\Rightarrow -log_e|cosec(t) - cot(t)| = log_e|x| + \lambda$$

$$\Rightarrow -log_e|cosec(t) - cot(t)| - log_e|x| = \lambda$$

$$\Rightarrow -(log_e|cosec(t) - cot(t)| + log_e|x|) = \lambda$$

$$\Rightarrow -log_e\big|x\big(cosec(t) - cot(t)\big)\big| = \lambda$$

$$\Rightarrow -log_e\left|x\left(cosec\left(\frac{y}{x}\right) - cot\left(\frac{y}{x}\right)\right)\right| = \lambda$$

$$\Rightarrow log_e\left|x\left(cosec\left(\frac{y}{x}\right) - cot\left(\frac{y}{x}\right)\right)\right| = -\lambda$$

$$\Rightarrow e^{\log_e\left|x\left(cosec\left(\frac{y}{x}\right)-cot\left(\frac{y}{x}\right)\right)\right|} = e^{-\lambda}$$

$$\boxed{\Rightarrow x\left(cosec\left(\frac{y}{x}\right) - cot\left(\frac{y}{x}\right)\right) = \lambda'}$$

8^{OR}. Find the particular solution of the following differential equation, given that $y\left(x = \frac{\pi}{4}\right) = 0$

$$\frac{dy}{dx} + ycot(x) = \frac{2}{1 + sin(x)}$$

Solution:

$$\frac{dy}{dx} + ycot(x) = \frac{2}{1 + sin(x)}$$

is of the form

$$\frac{dy}{dx} + y(\phi(x)) = \psi(x)$$

General Solution corresponding to DE of this kind: $y\left(e^{\int \phi(x)dx}\right) = \int \left(\psi(x)\left(e^{\int \phi(x)dx}\right)\right) dx$

$$\Rightarrow y\left(e^{\int cot(x)dx}\right) = \int \left(\frac{2}{1 + sin(x)}\left(e^{\int cot(x)dx}\right)\right) dx$$

$$\Rightarrow y\left(e^{\log_e|sin(x)|}\right) = \int \left(\frac{2}{1 + sin(x)}\left(e^{\log_e|sin(x)|}\right)\right) dx$$

$$\Rightarrow ysin(x) = \int \left(\frac{2}{1 + sin(x)} sin(x)\right) dx$$

$$\Rightarrow ysin(x) = 2\int \left(\frac{sin(x) + 1 - 1}{1 + sin(x)}\right) dx$$

$$\Rightarrow ysin(x) = 2\int \left(\frac{sin(x) + 1}{1 + sin(x)}\right) dx - 2\int \left(\frac{1}{1 + sin(x)}\right) dx$$

$$\Rightarrow ysin(x) = 2\int dx - 2\int \left(\frac{(1 - sin(x))}{(1 + sin(x)) \times (1 - sin(x))}\right) dx$$

$$\Rightarrow ysin(x) = 2\int dx - 2\int \left(\frac{1 - sin(x)}{cos^2(x)}\right) dx$$

$$\Rightarrow ysin(x) = 2x - 2\int (sec^2 x - sec(x)tan(x))dx$$

$$\Rightarrow ysin(x) = 2x - 2\int sec^2 x\, dx + 2\int sec(x)tan(x)dx$$

$$\Rightarrow ysin(x) = 2x - 2\int sec^2 x\, dx + 2\int sec(x)tan(x)dx$$

$$\Rightarrow ysin(x) = 2x - 2tan(x) + 2sec(x) + \lambda$$

$$y\left(x = \frac{\pi}{4}\right) = 0$$

$$\Rightarrow 0 = 2\left(\frac{\pi}{4}\right) - 2tan\left(\frac{\pi}{4}\right) + 2sec\left(\frac{\pi}{4}\right) + \lambda$$

$$\Rightarrow 0 = \left(\frac{\pi}{2}\right) - 2 + 2\sqrt{2} + \lambda$$

$\Rightarrow \lambda = -\left(\frac{\pi}{2}\right) + \left(2 - 2\sqrt{2}\right)$

$$\boxed{\Rightarrow y sin(x) = 2x - 2tan(x) + 2sec(x) - \left(\frac{\pi}{2}\right) + \left(2 - 2\sqrt{2}\right)}$$

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9. If $\vec{a} \neq \vec{0}$, $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, then show that $\vec{b} = \vec{c}$.

Solution:

$\vec{a}.\vec{b} = \vec{a}.\vec{c}$

$\Rightarrow |\vec{a}|.\left|\vec{b}\right|.cos(\alpha) = |\vec{a}|.|\vec{c}|.cos(\beta)$

As $\vec{a} \neq \vec{0}$

$\Rightarrow \left|\vec{b}\right|.cos(\alpha) = |\vec{c}|.cos(\beta)\ldots(i)$

$\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

As $\vec{a} \neq \vec{0}$

$\Rightarrow \left|\vec{b}\right|.sin(\alpha) = |\vec{c}|.sin(\beta)\ldots(ii)$

Dividing equation (ii) by (i)

$\Rightarrow tan(\alpha) = tan(\beta)$

So, $\alpha = n\pi + \beta \forall \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Putting value of α in equation (i)

$\Rightarrow \left|\vec{b}\right|.cos(n\pi + \beta) = |\vec{c}|.cos(\beta)$

$\Rightarrow \left|\vec{b}\right|.\left(\pm cos(\beta)\right) = |\vec{c}|.cos(\beta)$

$\Rightarrow \left|\vec{b}\right| = |\vec{c}|$

and as $\alpha = n\pi + \alpha$, where α, β are the angles between vectors.

$\Rightarrow \vec{b} = \vec{c}$