Integral Calculus

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1 Definition

Let $\varphi(x)$ and $\psi(x)$ be two functions involving x, such that

$$\frac{d}{dx}(\psi(x)) = \varphi(x)$$

then $\psi(x)$ is the anti – derivative of $\varphi(x)$ with respect to x.

Symbolically,

$$\frac{d}{dx}(\psi(x)) = \varphi(x)$$

$$\Rightarrow d(\psi(x)) = \varphi(x)dx$$

$$\Rightarrow \int d(\psi(x)) = \int \varphi(x)dx$$

$$\Rightarrow \psi(x) = \int \varphi(x)dx$$

$$\Rightarrow \int \varphi(x)dx = \psi(x) + \lambda \ni \lambda \; is \; a \; Constant \; of \; Integration$$

2 Fundamentals

2.1 Formulas

Since,

$$\frac{d}{dx}(\psi(x)) = \varphi(x)$$

$$\Rightarrow \int \varphi(x)dx = \psi(x) + \lambda \ni \lambda \; is \; a \; Constant \; of \; Integration$$

Now, we will be looking at some basic integration formulas.

- $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{\cancel{n+1}}{\cancel{n+1}} \times x^n = x^n \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + \lambda \forall n \neq -1$
- $\frac{d}{dx}(log|x|) = \frac{1}{x} = x^{-1} \Rightarrow \int x^{-1}dx = log|x| + \lambda$

An Exerpt: $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + \lambda \forall n \in R - \{-1\} \\ log|x| + \lambda \forall n = -1 \end{cases}$

- $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + \lambda$
- $\frac{d}{dx}\left(\frac{a^x}{log_e a}\right) = a^x \Rightarrow \int a^x dx = \frac{a^x}{log_e a} + \lambda$
- $\frac{d}{dx}\big(-cos(x)\big) = sin(x) \Rightarrow \int sin(x)dx = -cos(x) + \lambda$
- $\frac{d}{dx}\big(sin(x)\big) = cos(x) \Rightarrow \int cos(x)dx = sin(x) + \lambda$
- $\frac{d}{dx}(tan(x)) = sec^2(x) \Rightarrow \int sec^2(x)dx = tan(x) + \lambda$
- $\frac{d}{dx}(-cot(x)) = cosec^2(x) \Rightarrow \int cosec^2(x)dx = -cot(x) + \lambda$
- $\frac{d}{dx}(sec(x)) = sec(x)tan(x) \Rightarrow \int sec(x)tan(x)dx = sec(x) + \lambda$
- $\frac{d}{dx}(-\,cosec(x)) = cosec(x)\,cot(x) \Rightarrow \int cosec(x)\,cot(x)dx = -\,cosec(x) + \lambda$
- $\frac{d}{dx}(log|sin(x)|) = cot(x) \Rightarrow \int cot(x) = log|sin(x)| + \lambda$

- $\frac{d}{dx}(log|-cos(x)|) = tan(x) \Rightarrow \int tan(x)dx = log|-cos(x)| + \lambda$
- $\frac{d}{dx}(log|sec(x) + tan\,(x)|) = sec(x) \Rightarrow \int sec(x)dx = log|sec(x) + tan\,(x)| + \lambda$
- $\frac{d}{dx}(log|cosec(x) - cot\,(x)|) = cosec(x) \Rightarrow \int cosec(x)dx = log|cosec(x) - cot\,(x)| + \lambda$
- $\frac{d}{dx}\left(sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt[2]{a^2-x^2}} \Rightarrow \int \frac{1}{\sqrt[2]{a^2-x^2}}dx = sin^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt[2]{a^2-x^2}} \Rightarrow \int \frac{-1}{\sqrt[2]{a^2-x^2}}dx = cos^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(\frac{1}{a} \times tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a^2+x^2} \Rightarrow \int \frac{1}{a^2+x^2}dx = \frac{1}{a} \times tan^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(\frac{1}{a} \times cot^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{a^2+x^2} \Rightarrow \int \frac{-1}{a^2+x^2}dx = \frac{1}{a} \times cot^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(\frac{1}{a} \times sec^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{x\sqrt[2]{x^2-a^2}} \Rightarrow \int \frac{1}{x\sqrt[2]{x^2-a^2}}dx = \frac{1}{a} \times sec^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(\frac{1}{a} \times cosec^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{x\sqrt[2]{x^2-a^2}} \Rightarrow \int \frac{-1}{x\sqrt[2]{x^2-a^2}}dx = \frac{1}{a} \times cosec^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(\frac{1}{2a}log_e\left|\frac{x-a}{x+a}\right|\right) = \frac{1}{x^2-a^2} \Rightarrow \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}log_e\left|\frac{x-a}{x+a}\right| + \lambda$
- $\frac{d}{dx}\left(\frac{1}{2a}log_e\left|\frac{a+x}{a-x}\right|\right) = \frac{1}{a^2-x^2} \Rightarrow \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}log_e\left|\frac{a+x}{a-x}\right| + \lambda$
- $\frac{d}{dx}\left(log_e\left|x + \sqrt{x^2+a^2}\right|\right) = \frac{1}{\sqrt{x^2+a^2}} \Rightarrow \int \frac{1}{\sqrt{x^2+a^2}}dx = log_e\left|x + \sqrt{x^2+a^2}\right| + \lambda$
- $\frac{d}{dx}\left(log_e\left|x + \sqrt{x^2-a^2}\right|\right) = \frac{1}{\sqrt{x^2-a^2}} \Rightarrow \int \frac{1}{\sqrt{x^2-a^2}}dx = log_e\left|x + \sqrt{x^2-a^2}\right| + \lambda$
- $\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)\right) = \sqrt{a^2-x^2} \Rightarrow \int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right) + \lambda$
- $\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}log_e\left|x + \sqrt{x^2+a^2}\right|\right) = \sqrt{a^2+x^2} \Rightarrow \int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}log_e\left|x + \sqrt{x^2+a^2}\right| + \lambda$
- $\frac{d}{dx}\left(\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}log_e\left|x + \sqrt{x^2-a^2}\right|\right) = \sqrt{x^2-a^2} \Rightarrow \int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}log_e\left|x + \sqrt{x^2-a^2}\right| + \lambda$

2.2 Some Important Points

- $\int \mu \times \varphi(x)dx = \mu \times \int \varphi(x)dx \ni \mu\ is\ a\ constant$
- $\int \varphi_1(x) \pm \varphi_2(x) \pm \varphi_3(x) + \cdots \pm \varphi_n(x)dx = \int \varphi_1(x)dx \pm \int \varphi_2(x)dx \pm \cdots + \int \varphi_n(x)dx$
- $\int \varphi_1(x) \times \varphi_2(x)dx = \varphi_1(x)\int \varphi_2(x)dx - \int\left(\frac{d}{dx}(\varphi_1(x))(\int \varphi_2(x)dx)\right)dx$, the first and the second functions are chosen as Inverse Logarithmic Algebraic Trigonometric Exponential.

AID TO MEMORY

First function as it is

into

integration of the second

minus

integration of differential coefficient of the first function

into

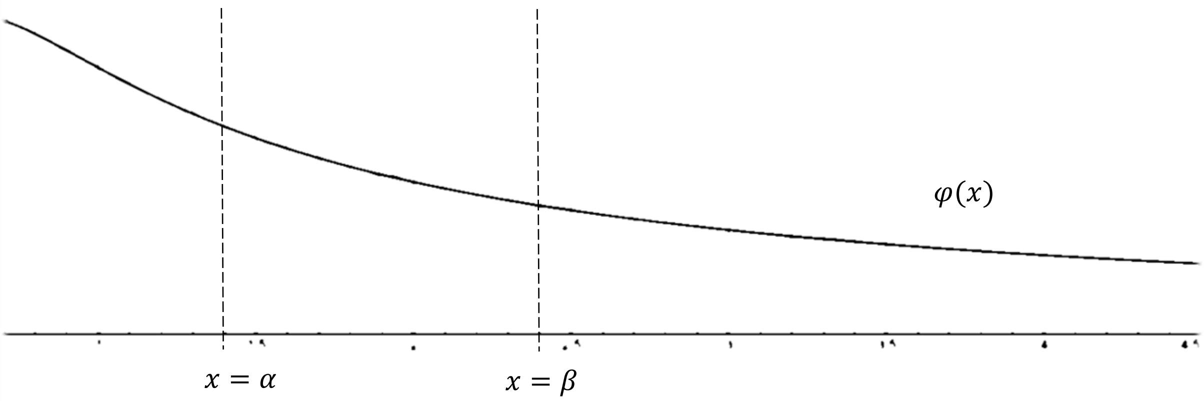
integration of the second

- When there are bounds in an Integration, it is said to be definite in nature. It is denoted as $\int_\alpha^\beta \varphi(x)\,dx$. It is evaluated as $\lim\limits_{x\to\beta}\int \varphi(x)\,dx - \lim\limits_{x\to\alpha}\int \varphi(x)\,dx$
- $\int_\alpha^\beta \varphi(x)\,dx = -\int_\beta^\alpha \varphi(x)\,dx$
- $\int_\alpha^\beta \varphi(x)\,dx = \int_\alpha^\gamma \varphi(x)\,dx + \int_\gamma^\beta \varphi(x)\,dx$, where $\alpha \le \gamma \le \beta$

- $\int_{\alpha}^{\beta} \varphi(x)\,dx = \int_{\alpha}^{\beta} \varphi(\alpha+\beta-x)\,dx$
- $\int_{-\alpha}^{\alpha} \varphi(x)\,dx = \begin{cases} 2 \times \int_{0}^{\alpha} \varphi(x)\,dx & if\ \varphi(-x) = \varphi(x) \\ 0 & if\ \varphi(-x) = -\varphi(x) \end{cases}$
- $\frac{d}{dx}\left(\int \varphi(x)\,dx\right) = \varphi(x)$
- $\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} \varphi(x)\,dx\right) = \left\{\varphi\big(\beta(x)\big) \times \frac{d}{dx}\big(\beta(x)\big)\right\} - \left\{\varphi\big(\alpha(x)\big) \times \frac{d}{dx}\big(\alpha(x)\big)\right\}$

2.3 Geometrical Interpretation

Let we have a curve,



$\int_{\alpha}^{\beta} \varphi(x)$ is the area under the curve, $\varphi(x)$ bounded by the lines.

- $x = \alpha$
- $x = \beta$
- $y = 0$

2.4 Some Daily Life Applications

Function	***Derivative***	***In Symbols***	***Function***	***Integral***	***In Symbols***
Displacement(x)	*Velocity(v)*	$v = \frac{dx}{dt}$	*Velocity(v)*	*Displacement(x)*	$x = \int vdt$
Velocity(v)	*Acceleration(a)*	$a = \frac{dv}{dt}$	*Acceleration(a)*	*Velocity(v)*	$v = \int adt$
Mass(m)	*Linear Density(ρ)*	$\rho = \frac{dm}{dx}$	*Linear Density(ρ)*	*Mass(m)*	$m = \int \rho dx$
Population(P)	*Instantaneous Growth(γ)*	$\gamma = \frac{dP}{dt}$	*Instantaneous Growth(γ)*	*Population(P)*	$P = \int \gamma dt$
Cost(C)	*Marginal Cost(μ)*	$\mu = \frac{dC}{dt}$	*Marginal Cost(μ)*	*Cost(C)*	$C = \int \mu dt$
Revenue(R)	*Marginal Revenue(Ω)*	$\Omega = \frac{dR}{dt}$	*Marginal Revenue(Ω)*	*Revenue(R)*	$R = \int \Omega dt$

Question 1: Evaluate $I = \int tan^2x\, dx$

Solution:

$$I = \int tan^2x\, dx$$

We know, $1 + tan^2x = sec^2x$

So, $tan^2x = sec^2x - 1$

$I = \int tan^2x\, dx$

$\Rightarrow I = \int (sec^2x - 1)\, dx$

$\Rightarrow I = \int sec^2x\, dx - \int dx$

$\Rightarrow I = tanx - x + \lambda$

Question 2: Evaluate $I = \int \frac{dx}{(sin^2x)(cos^2x)}$

Solution:

$$I = \int \frac{dx}{(sin^2x)(cos^2x)}$$

We know, $sin^2x + cos^2x = 1$

$I = \int \frac{sin^2x + cos^2x}{(sin^2x)(cos^2x)} dx$

$\Rightarrow I = \int \frac{\cancel{sin^2x}}{\cancel{(sin^2x)}(cos^2x)} dx + \int \frac{\cancel{cos^2x}}{(sin^2x)\cancel{(cos^2x)}} dx$

$\Rightarrow I = \int sec^2x\, dx + \int cosec^2x\, dx$

$\Rightarrow I = tanx - cotx + \lambda$

Question 3: Evaluate $I = \int \frac{(sin^6x + cos^6x)dx}{(sin^2x)(cos^2x)}$

Solution:

$$I = \int \frac{(sin^6x + cos^6x)dx}{(sin^2x)(cos^2x)}$$

We know, $sin^2x + cos^2x = 1$ and $(\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2 = \alpha^3 + \beta^3 + 3(\alpha\beta)(\alpha + \beta)$

Let $\alpha = sin^2x$ and $\beta = cos^2x$

$I = \int \frac{sin^6x + cos^6x}{(sin^2x)(cos^2x)} dx$

$\Rightarrow I = \int \frac{(sin^2x)^3 + (cos^2x)^3}{(sin^2x)(cos^2x)} dx$

$\Rightarrow I = \int \frac{(sin^2x + cos^2x)^3 - 3(sin^2x)(cos^2x)(sin^2x + cos^2x)}{(sin^2x)(cos^2x)} dx$

$\Rightarrow I = \int \frac{1 - 3(sin^2x)(cos^2x)}{(sin^2x)(cos^2x)} dx$

$\Rightarrow I = \int \frac{dx}{(sin^2 x)(cos^2 x)} - \int 3\, dx$

$\Rightarrow I = \int \frac{dx}{(sin^2 x)(cos^2 x)} - \int 3\, dx$

$\Rightarrow I = tanx - cotx - 3x + \lambda$

Question 4: Evaluate $I = \int \frac{cos\, x - cos\, 2x}{1 - cos\, x} dx$

Solution:

$$I = \int \frac{cos\, x - cos\, 2x}{1 - cos\, x} dx$$

We know, $cos\, 2x = 2cos^2 x - 1$

$I = \int \frac{cos\, x - cos\, 2x}{1 - cos\, x} dx$

$\Rightarrow I = \int \frac{cos\, x - 2cos^2 x + 1}{1 - cos\, x} dx$

$\Rightarrow I = \int \frac{cos\, x - cos^2 x + 1 - cos^2 x}{1 - cos\, x} dx$

$\Rightarrow I = \int \frac{cos\, x\, (\cancel{1 - cos\, x}) + (1 + cos\, x)(\cancel{1 - cos\, x})}{\cancel{1 - cos\, x}} dx$

$\Rightarrow I = \int (2cos\, x + 1) dx$

$\Rightarrow I = \int 2\, cos\, x\, dx + \int dx$

$\Rightarrow I = 2\, sin\, x + x + \lambda$

Question 5: Evaluate $I = \int x \sqrt{\frac{2\, sin(x^2 - 1) - sin2(x^2 - 1)}{2\, sin(x^2 - 1) + sin2(x^2 - 1)}} dx\ \forall x^2 \neq n\pi + 1, n \in \mathbb{N}$

Solution:

$$I = \int x \sqrt{\frac{2\, sin(x^2 - 1) - sin2(x^2 - 1)}{2\, sin(x^2 - 1) + sin2(x^2 - 1)}} dx$$

We know, $sin\, 2x = 2sin\, xcos\, x$

$I = \int x \sqrt{\frac{\cancel{2\, sin(x^2 - 1)} - \cancel{2sin\, (x^2 - 1)}cos\, (x^2 - 1)}{\cancel{2\, sin(x^2 - 1)} + \cancel{2sin\, (x^2 - 1)}cos\, (x^2 - 1)}} dx$

$I = \int x \sqrt{\frac{1 - cos\, (x^2 - 1)}{1 + cos\, (x^2 - 1)}} dx$

Let $x^2 - 1 = t$

Differentiating both sides w.r.t x

$\Rightarrow 2x = \frac{dt}{dx}$

$\Rightarrow x\, dx = \frac{dt}{2}$

$$I = \frac{1}{2}\int \sqrt{\frac{1 - cos\ (t)}{1 + cos\ (t)}}\, dt$$

Multiplying Numerator and Denominator by $1 - cos\ (t)$

$$I = \frac{1}{2}\int \sqrt{\frac{\left(1 - cos\ (t)\right)^2}{1 - cos^2\ (t)}}\, dt$$

$$I = \frac{1}{2}\int \sqrt{\frac{1 + cos^2\ (t) - 2cos\ (t)}{sin^2\ (t)}}\, dt$$

$$I = \frac{1}{2}\int \sqrt{cosec^2\ (t) + cot^2\ (t) - 2cosec(t) \times cot(t)}\, dt$$

$$I = \frac{1}{2}\int \sqrt{\left(cosec(t) - cot(t)\right)^2}\, dt$$

According to Question

$x^2 \neq n\pi + 1$

$\Rightarrow x^2 - 1 \neq n\pi$

$\Rightarrow t \neq n\pi\ \forall n \in \mathbb{N}$

$$I = \frac{1}{2}\int |cosec(t) - cot(t)| dt$$

We know, $1 - cos\ x = 2sin^2 \frac{x}{2}$

$$\Rightarrow I = \frac{1}{2}\int \left|\frac{1 - cos(t)}{sin(t)}\right| dt$$

$$\Rightarrow I = \frac{1}{2}\int \left|\frac{2sin^2 \frac{t}{2}}{2sin\ \frac{t}{2} cos\ \frac{t}{2}}\right| dt$$

$$\Rightarrow I = \frac{1}{2}\int \left|tan\frac{t}{2}\right| dt = \frac{1}{2} \times 2 \times log_e \left|sec\frac{t}{2}\right| + \lambda$$

$$\Rightarrow I = log_e \left|sec\left(\frac{x^2 - 1}{2}\right)\right| + \lambda$$

Question 6: Evaluate $I = \int \frac{x^4}{1+x^2} dx$

Solution:

$$I = \int \frac{x^4}{1 + x^2} dx$$

Let $x = tan(\theta)$

Differentiating both sides w.r.t x

$dx = \ sec^2\theta d\theta$

$$\Rightarrow I = \int \frac{tan^4\theta}{sec^2\theta} sec^2\theta d\theta$$

$$\Rightarrow I = \int tan^4\theta d\theta$$

$$\Rightarrow I = \int (tan^2\theta \times tan^2\theta) d\theta$$

$$\Rightarrow I = \int \left((sec^2\theta - 1) \times tan^2\theta\right) d\theta$$

$$\Rightarrow I = \int (sec^2\theta \times tan^2\theta) d\theta - \int (tan^2\theta) d\theta$$

Let $I_1 = \int (sec^2\theta \times tan^2\theta) d\theta$ and $I_2 = \int (tan^2\theta) d\theta$

$$\Rightarrow I_1 = \int (sec^2\theta \times tan^2\theta) d\theta$$

Let $tan(\theta) = t$

Differentiating both sides w.r.t θ

$sec^2\theta d\theta = dt$

$$\Rightarrow I_1 = \int t^2 dt = \frac{t^3}{3} + \lambda_1 = \frac{tan^3\theta}{3} + \lambda_1$$

$$\Rightarrow I_2 = \int (tan^2\theta) d\theta = tan\theta - \theta + \lambda_2$$

$$\Rightarrow I = \frac{tan^3\theta}{3} + \lambda_1 + tan\theta - \theta + \lambda_2$$

$$\Rightarrow I = \frac{tan^3(tan^{-1}x)}{3} + \lambda_1 + tan(tan^{-1}x) - (tan^{-1}x) + \lambda_2$$

$$\Rightarrow I = \frac{x^3}{3} + \lambda_1 + x - (tan^{-1}x) + \lambda_2$$

$$\Rightarrow I = \frac{x^3}{3} + x - (tan^{-1}x) + \lambda$$

Question 7: Evaluate $I = \int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3} dx$

Solution:

$$I = \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$$

$$\Rightarrow I = \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$$

$$\Rightarrow I = \int \frac{2x^{12} + 5x^9}{x^{15}(1 + x^{-2} + x^{-5})^3} dx$$

$$\Rightarrow I = \int \frac{2x^{-3} + 5x^{-6}}{(1 + x^{-2} + x^{-5})^3} dx$$

Let $t = 1 + x^{-2} + x^{-5}$

Differentiating both sides w.r.t to x

$dt = (-2x^{-3} - 5x^{-6}) dx$

$$\Rightarrow I = \int \frac{-1}{t^3} dt$$

$$\Rightarrow I = -\frac{t^{-2}}{-2} + \lambda$$

$\Rightarrow I = \frac{1}{2t^2} + \lambda = \frac{1}{2(1 + x^{-2} + x^{-5})^2} + \lambda$

Question 8: Evaluate $I = \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx$

Solution:

$$I = \int \frac{1}{x^2(x^4 + 1)^{\frac{3}{4}}} dx$$

$\Rightarrow I = \int \frac{1}{x^2(x^4 + 1)^{\frac{3}{4}}} dx$

$\Rightarrow I = \int \frac{1}{x^2 x^{4\times\frac{3}{4}}(1 + x^{-4})^{\frac{3}{4}}} dx$

$\Rightarrow I = \int \frac{x^{-5}}{(1 + x^{-4})^{\frac{3}{4}}} dx$

Let $t = 1 + x^{-4}$

Differentiating both sides w.r.t to x

$dt = -4x^{-5}dx$

$\Rightarrow I = \frac{-1}{4} \int \frac{1}{(t)^{\frac{3}{4}}} dt$

$\Rightarrow I = \frac{-1}{4} \int (t)^{-\frac{3}{4}} dt$

$\Rightarrow I = \frac{-1}{4} \frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4} + 1} + \lambda$

$\Rightarrow I = -t^{-\frac{3}{4}+1} + \lambda$

$\Rightarrow I = -t^{\frac{1}{4}} + \lambda$

$\Rightarrow I = -(1 + x^{-4})^{\frac{1}{4}} + \lambda$

Question 9: Evaluate $I = \sqrt[2]{2} \int \frac{sin(x)}{sin\left(x - \frac{\pi}{4}\right)} dx$

Solution:

$$I = \sqrt[2]{2} \int \frac{sin(x)}{sin\left(x - \frac{\pi}{4}\right)} dx$$

Let $t = x - \frac{\pi}{4}$

Differentiating both sides w.r.t to x

$dt = dx$

$\Rightarrow I = \sqrt[2]{2} \int \frac{sin\left(t + \frac{\pi}{4}\right)}{sin(t)} dt$

$$\Rightarrow I = \sqrt[2]{2}\int \frac{sin(t)\times\frac{1}{\sqrt[2]{2}} + cos(t)\times\frac{1}{\sqrt[2]{2}}}{sin(t)}dt$$

$$\Rightarrow I = \int \frac{sin(t) + cos(t)}{sin(t)}dt$$

$$\Rightarrow I = \int 1 + cot\,(t)dt$$

$$\Rightarrow I = \int dt + \int cot\,(t)dt$$

$$\Rightarrow I = t + log_e|sin\,(t)| + \lambda'$$

$$\Rightarrow I = x + log_e\left|sin\,(x - \frac{\pi}{4})\right| + \lambda$$

Question 10: Evaluate $I = \int |x^n| dx\ \forall n = 2k + 1 \& k \in \mathbb{Z}$

Solution:

$$I = \int |x^n| dx$$

$$\Rightarrow I = \int |x|^n dx$$

Now, $|x| = \begin{cases} +x \forall x \geq 0 \\ -x \forall x < 0 \end{cases}$

Case 1: $x \geq 0$

$$\Rightarrow I = \int |x|^n dx$$

$$\Rightarrow I = \int x^n dx = \frac{x^{n+1}}{n+1} + \lambda$$

Case 2: $x < 0$

$$\Rightarrow I = \int |x|^n dx$$

$$\Rightarrow I = \int (-x)^n dx = -\frac{x^{n+1}}{n+1} + \lambda$$

Question 11: Given $\varphi(x) = \begin{vmatrix} 0 & x^2 - sin\,x & cos\,x - 2 \\ sin\,x - x^2 & 0 & 1 - 2x \\ 2 - cos\,x & 2x - 1 & 0 \end{vmatrix}$. Evaluate $\iiint \iiint \iiint \ldots \infty \iiint \varphi(x)$

Solution:

$$\varphi(x) = \begin{vmatrix} 0 & x^2 - sin\,x & cos\,x - 2 \\ sin\,x - x^2 & 0 & 1 - 2x \\ 2 - cos\,x & 2x - 1 & 0 \end{vmatrix}$$

$$\Rightarrow \varphi(x) = 0(0 - (1 - 2x)(2x - 1)) - (x^2 - sin\,x)\big(0 - (1 - 2x)(2 - cos\,x)\big) \\ + (cos\,x - 2)\big((sin\,x - x^2)(2x - 1)\big)$$

$$\Rightarrow \varphi(x) = 0 - (x^2 - sin\,x)\big(0 - (1 - 2x)(2 - cos\,x)\big) + (cos\,x - 2)\big((sin\,x - x^2)(2x - 1)\big)$$

$$\Rightarrow \varphi(x) = (x^2 - sin\,x)(1 - 2x)(2 - cos\,x) + (cos\,x - 2)(sin\,x - x^2)(2x - 1)$$

$$\Rightarrow \varphi(x) = (x^2 - sin\,x)(1 - 2x)(2 - cos\,x) + \big(-(x^2 - sin\,x)\big)\big(-(1 - 2x)\big)\big(-(2 - cos\,x)\big)$$

$\Rightarrow \varphi(x) = (x^2 - sin\ x)(1-2x)(2-cos\ x) - (x^2 - sin\ x)(1-2x)(2-cos\ x)$

$\Rightarrow \varphi(x) = 0$

$\iiint \iiint \iiint \dots \infty \iiint \varphi(x) = 0$

Question 12: Evaluate $I = \int \frac{e^{3x}+e^{5x}}{e^x+e^{-x}} dx$

Solution:

$$I = \int \frac{e^{3x}+e^{5x}}{e^x+e^{-x}} dx$$

$$\Rightarrow I = \int \frac{e^{3x}+e^{5x}}{e^x+\frac{1}{e^x}} dx$$

$$\Rightarrow I = \int \frac{e^{3x+x}+e^{5x+x}}{e^{2x}+1} dx$$

$$\Rightarrow I = \int \frac{e^{4x}+e^{6x}}{e^{2x}+1} dx$$

$$\Rightarrow I = \int \frac{e^{4x}(e^{2x}+1)}{(e^{2x}+1)} dx$$

$$\Rightarrow I = \int e^{4x} dx$$

Let $t = 4x$

Differentiating both sides w.r.t to x

$dt = 4dx$

$$\Rightarrow I = \int e^t \frac{dt}{4}$$

$$\Rightarrow I = \frac{1}{4}\int e^t dt$$

$$\Rightarrow I = \frac{1}{4}e^t + \lambda$$

$$\Rightarrow I = \frac{1}{4}e^{4x} + \lambda$$

Question 13: Evaluate $I = \iiint \left(dx \big/ \left(\sqrt[2]{x+1} - \sqrt[2]{x}\right)\right)$

Solution:

$$I = \iiint \left(dx \big/ \left(\sqrt[2]{x+1} - \sqrt[2]{x}\right)\right)$$

$$\Rightarrow I = \iiint \left(\frac{dx}{\left(\sqrt[2]{x+1} - \sqrt[2]{x}\right)} \right)$$

$$\Rightarrow I = \iiint \left(\frac{\left(\sqrt[2]{x+1} + \sqrt[2]{x}\right)}{\left(\sqrt[2]{x+1} - \sqrt[2]{x}\right) \times \left(\sqrt[2]{x+1} + \sqrt[2]{x}\right)} dx \right)$$

$\Rightarrow I = \iiint \left(\frac{\left(\sqrt[2]{x+1} + \sqrt[2]{x}\right)}{x+1-x} dx \right)$

$\Rightarrow I = \iiint \left(\frac{\left(\sqrt[2]{x+1} + \sqrt[2]{x}\right)}{1} dx \right)$

$\Rightarrow I = \iiint \sqrt[2]{x+1}\, dx + \iiint \sqrt[2]{x}\, dx$

Let $I_1 = \iiint \sqrt[2]{x+1}\, dx$ and $I_2 = \iiint \sqrt[2]{x}\, dx$

$\Rightarrow I_1 = \iiint \sqrt[2]{x+1}\, dx$

Let $x + 1 = t$

Differentiating both sides w.r.t to x

$dx = dt$

$\Rightarrow I_1 = \iiint \sqrt[2]{t}\, dt$

$\Rightarrow I_1 = \iint \left(\frac{t^{3/2}}{3/2} + \lambda'' \right) dt$

$\Rightarrow I_1 = \int \left(\frac{2}{3} \times \frac{t^{5/2}}{5/2} + \lambda'' t + \lambda' \right) dt$

$\Rightarrow I_1 = \frac{2}{3} \times \frac{2}{5} \times \frac{t^{7/2}}{7/2} + \lambda'' \frac{t^2}{2} + \lambda' t + \lambda$

$\Rightarrow I_1 = \frac{2}{3} \times \frac{2}{5} \times \frac{(x+1)^{7/2}}{7/2} + \lambda'' \frac{(x+1)^2}{2} + \lambda'(x+1) + \lambda$

and

$I_2 = \iiint \sqrt[2]{x}\, dx$

$\Rightarrow I_2 = \frac{2}{3} \times \frac{2}{5} \times \frac{x^{7/2}}{7/2} + \mu'' \frac{x^2}{2} + \mu' x + \mu$

$\Rightarrow I = \frac{2}{3} \times \frac{2}{5} \times \frac{(x+1)^{7/2}}{7/2} + \lambda'' \frac{(x+1)^2}{2} + \lambda'(x+1) + \lambda + \frac{2}{3} \times \frac{2}{5} \times \frac{x^{7/2}}{7/2} + \mu'' \frac{x^2}{2} + \mu' x + \mu$

$\Rightarrow I = \frac{2}{3} \times \frac{2}{5} \times \frac{(x+1)^{7/2}}{7/2} + \lambda'' \frac{(x+1)^2}{2} + \lambda'(x+1) + \frac{2}{3} \times \frac{2}{5} \times \frac{x^{7/2}}{7/2} + \mu'' \frac{x^2}{2} + \mu' x + \psi$

Question 14: Evaluate $I = \int \frac{1}{sin(x-\alpha)cos(x-\beta)} dx$

Solution:

$$I = \int \frac{1}{sin(x-\alpha)cos(x-\beta)} dx$$

We know,

$cos(\alpha - \beta) = sin(\alpha)sin(\beta) + cos(\alpha)cos(\beta)$

Now,

$cos(\alpha-\beta) = cos\big((x-\beta)-(x-\alpha)\big) = sin(x-\beta)sin(x-\alpha) + cos(x-\beta)cos(x-\alpha)$

$$\Rightarrow I = \frac{cos(\alpha-\beta)}{cos(\alpha-\beta)}\int \frac{1}{sin(x-\alpha)cos(x-\beta)}dx$$

$$\Rightarrow I = \frac{1}{cos(\alpha-\beta)}\int \frac{cos(\alpha-\beta)}{sin(x-\alpha)cos(x-\beta)}dx$$

$$\Rightarrow I = \frac{1}{cos(\alpha-\beta)}\int \frac{sin(x-\beta)sin(x-\alpha) + cos(x-\beta)cos(x-\alpha)}{sin(x-\alpha)cos(x-\beta)}dx$$

$$\Rightarrow I = \frac{1}{cos(\alpha-\beta)}\int tan(x-\beta) + cot(x-\alpha)dx$$

$$\Rightarrow I = \frac{1}{cos(\alpha-\beta)}\int tan(x-\beta)\,dx + \frac{1}{cos(\alpha-\beta)}\int cot(x-\alpha)\,dx$$

Let $I_1 = \frac{1}{cos(\alpha-\beta)}\int tan(x-\beta)\,dx$ and $I_2 = \frac{1}{cos(\alpha-\beta)}\int cot(x-\alpha)\,dx$

$$\Rightarrow I_1 = \frac{1}{cos(\alpha-\beta)}\int tan(x-\beta)\,dx$$

Let $(x-\beta) = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I_1 = \frac{1}{cos(\alpha-\beta)}\int tan(t)\,dt$$

$$\Rightarrow I_1 = \frac{1}{cos(\alpha-\beta)}log_e|-cos\,(t)| + \lambda'$$

$$\Rightarrow I_1 = \frac{1}{cos(\alpha-\beta)}log_e|-cos\,(x-\beta)| + \lambda'$$

$$\Rightarrow I_2 = \frac{1}{cos(\alpha-\beta)}\int cot(x-\alpha)\,dx$$

Let $(x-\alpha) = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I_2 = \frac{1}{cos(\alpha-\beta)}\int cot(t)\,dt$$

$$\Rightarrow I_2 = \frac{1}{cos(\alpha-\beta)}log_e|sin\,(t)| + \lambda''$$

$$\Rightarrow I_2 = \frac{1}{cos(\alpha-\beta)}log_e|sin(x-\alpha)| + \lambda''$$

$$\Rightarrow I = \frac{1}{cos(\alpha-\beta)}log_e|-cos(x-\beta)| + \lambda' + \frac{1}{cos(\alpha-\beta)}log_e|sin(x-\alpha)| + \lambda''$$

$$\Rightarrow I = \frac{1}{cos(\alpha-\beta)}log_e|-cos(x-\beta)| + \frac{1}{cos(\alpha-\beta)}log_e|sin(x-\alpha)| + \lambda$$

$$\Rightarrow I = \frac{1}{cos(\alpha - \beta)} log_e \left|\frac{sin(x-\alpha)}{cos(x-\beta)}\right| + \lambda$$

Question 15: Evaluate $I = \int \frac{1}{cos(x-\alpha)cos(x-\beta)} dx$

Solution:

$$I = \int \frac{1}{cos(x-\alpha)cos(x-\beta)} dx$$

We know,

$sin(\alpha - \beta) = sin(\alpha)cos(\beta) - cos(\alpha)sin(\beta)$

Now,
$sin(\alpha - \beta) = sin\big((x-\beta) - (x-\alpha)\big) = sin(x-\beta)cos(x-\alpha) - cos(x-\beta)sin(x-\alpha)$

$$\Rightarrow I = \frac{sin(\alpha-\beta)}{sin(\alpha-\beta)} \int \frac{1}{cos(x-\alpha)cos(x-\beta)} dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha-\beta)} \int \frac{sin(\alpha-\beta)}{cos(x-\alpha)cos(x-\beta)} dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha-\beta)} \int \frac{sin(x-\beta)cos(x-\alpha) - cos(x-\beta)sin(x-\alpha)}{cos(x-\alpha)cos(x-\beta)} dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha-\beta)} \int tan(x-\beta) - tan(x-\alpha) dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha-\beta)} \int tan(x-\beta)\, dx - \frac{1}{sin(\alpha-\beta)} \int tan(x-\alpha)\, dx$$

Let $I_1 = \frac{1}{sin(\alpha-\beta)} \int tan(x-\beta)\, dx$ and $I_2 = \frac{1}{sin(\alpha-\beta)} \int tan(x-\alpha)\, dx$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha-\beta)} \int tan(x-\beta)\, dx$$

Let $(x-\beta) = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha-\beta)} \int tan(t)\, dt$$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha-\beta)} log_e|-cos\,(t)| + \lambda'$$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha-\beta)} log_e|-cos\,(x-\beta)| + \lambda'$$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha-\beta)} \int tan(x-\alpha)\, dx$$

Let $(x-\alpha) = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha-\beta)} \int tan(t)\, dt$$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha - \beta)} log_e|-cos(t)| + \lambda''$$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha - \beta)} log_e|-cos(x - \alpha)| + \lambda''$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} log_e|- cos(x - \beta)| - \lambda' + \frac{1}{sin(\alpha - \beta)} log_e|-cos(x - \alpha)| + \lambda''$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} log_e|- cos(x - \beta)| - \frac{1}{sin(\alpha - \beta)} log_e|-cos(x - \alpha)| + \lambda$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} log_e \left|\frac{cos(x - \alpha)}{cos(x - \beta)}\right| + \lambda$$

Question 16: Evaluate $I = \int \frac{1}{sin(x-\alpha)sin(x-\beta)} dx$

Solution:

$$I = \int \frac{1}{sin(x - \alpha)sin(x - \beta)} dx$$

We know,

$sin(\alpha - \beta) = sin(\alpha)cos(\beta) - cos(\alpha)sin(\beta)$

Now,

$sin(\alpha - \beta) = sin\big((x - \beta) - (x - \alpha)\big) = sin(x - \beta)cos(x - \alpha) - cos(x - \beta)sin(x - \alpha)$

$$\Rightarrow I = \frac{sin(\alpha - \beta)}{sin(\alpha - \beta)} \int \frac{1}{sin(x - \alpha)sin(x - \beta)} dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} \int \frac{sin(\alpha - \beta)}{sin(x - \alpha)sin(x - \beta)} dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} \int \frac{sin(x - \beta)cos(x - \alpha) - cos(x - \beta)sin(x - \alpha)}{sin(x - \alpha)sin(x - \beta)} dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} \int cot(x - \beta) - cot(x - \alpha)dx$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} \int cot(x - \beta)\, dx - \frac{1}{sin(\alpha - \beta)} \int cot(x - \alpha)\, dx$$

Let $I_1 = \frac{1}{sin(\alpha-\beta)} \int cot(x - \beta)\, dx$ and $I_2 = \frac{1}{sin(\alpha-\beta)} \int cot(x - \alpha)\, dx$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha - \beta)} \int cot(x - \beta)\, dx$$

Let $(x - \beta) = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha - \beta)} \int cot(t)\, dt$$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha - \beta)} log_e|sin\,(t)| + \lambda'$$

$$\Rightarrow I_1 = \frac{1}{sin(\alpha - \beta)} log_e|sin\,(x - \beta)| + \lambda'$$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha - \beta)} \int cot(x - \alpha)\, dx$$

Let $(x - \alpha) = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha - \beta)} \int cot(t)\, dt$$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha - \beta)} log_e|sin(t)| + \lambda''$$

$$\Rightarrow I_2 = \frac{1}{sin(\alpha - \beta)} log_e|sin(x - \alpha)| + \lambda''$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} log_e|sin(x - \beta)| - \lambda' + \frac{1}{sin(\alpha - \beta)} log_e|sin(x - \alpha)| + \lambda''$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} log_e|sin(x - \beta)| - \frac{1}{sin(\alpha - \beta)} log_e|sin(x - \alpha)| + \lambda$$

$$\Rightarrow I = \frac{1}{sin(\alpha - \beta)} log_e \left|\frac{sin(x - \beta)}{sin(x - \alpha)}\right| + \lambda$$

Question 17: Evaluate $I = \int \frac{sin(4x)}{sin(x)} dx$

Solution:

$$I = \int \frac{sin(4x)}{sin(x)} dx$$

$$\Rightarrow I = \int \frac{sin(2 \times 2x)}{sin(x)} dx$$

$$\Rightarrow I = \int \frac{2sin(2x)\cos(2x)}{sin(x)} dx$$

$$\Rightarrow I = 2 \int \frac{2sin(x)cos(x)cos(2x)}{sin(x)} dx$$

$$\Rightarrow I = 4 \int \frac{sin(x)cos(x)cos(2x)}{sin(x)} dx$$

$$\Rightarrow I = 4 \int cos(x)cos(2x)\, dx$$

$$\Rightarrow I = 4 \int cos(x)(cos^2 x - sin^2 x)\, dx$$

$$\Rightarrow I = 4 \int cos^3 x - cos(x)sin^2 x\, dx$$

$$\Rightarrow I = 4 \int cos^3 x\, dx - 4 \int cos(x)sin^2 x\, dx$$

$$\Rightarrow I = 4 \int cos^3 x\, dx - 4 \int cos(x)(cos^2 x - 1)\, dx$$

$$\Rightarrow I = 4 \int cos^3 x\, dx - 4 \int \left(cos^3 x - cos(x)\right) dx$$

$$\Rightarrow I = 4\int cos^3x\,dx - 4\int cos^3x\,dx + 4\int cos(x)\,dx$$

$$\Rightarrow I = 4\int cos(x)\,dx$$

$$\Rightarrow I = 4sin(x) + \lambda$$

Question 18: Evaluate $I = \int \gamma^{\delta}\delta^{\gamma}\,d\gamma$

Solution:

$$I = \int \gamma^{\delta}\delta^{\gamma}\,d\gamma$$

Here, γ is the variable but δ is the constant.

$$\Rightarrow I = \int \underset{\mathrm{I}}{\gamma^{\delta}}\quad \underset{\mathrm{II}}{\delta^{\gamma}}\,d\gamma$$

$$\Rightarrow I = \gamma^{\delta}\int \delta^{\gamma}\,d\gamma - \int \left(\delta\gamma^{\delta-1}\right)\left(\int \delta^{\gamma}\,d\gamma\right)d\gamma$$

$$\Rightarrow I = \gamma^{\delta}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \int \left(\delta\gamma^{\delta-1}\right)\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right)d\gamma$$

$$\Rightarrow I = \gamma^{\delta}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \frac{\delta}{log_e\,\delta}\int \left(\gamma^{\delta-1}\right)(\delta^{\gamma})d\gamma$$

Let $I_1 = \int \underset{\mathrm{I}}{\gamma^{\delta-1}}\quad \underset{\mathrm{II}}{\delta^{\gamma}}\,d\gamma$

$$I_1 = \gamma^{\delta-1}\int \delta^{\gamma}\,d\gamma - \int \left((\delta-1)\left(\gamma^{\delta-2}\right)\right)\left(\int \delta^{\gamma}\,d\gamma\right)d\gamma$$

$$I_1 = \gamma^{\delta-1}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \int \left((\delta-1)\left(\gamma^{\delta-2}\right)\right)\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right)d\gamma$$

$$I_1 = \gamma^{\delta-1}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \frac{\delta-1}{log_e\,\delta}\int \left(\gamma^{\delta-2}\right)(\delta^{\gamma})d\gamma + \lambda_1$$

Let $I_2 = \int \underset{\mathrm{I}}{\gamma^{\delta-2}}\quad \underset{\mathrm{II}}{\delta^{\gamma}}\,d\gamma$

$$I_2 = \gamma^{\delta-2}\int \delta^{\gamma}\,d\gamma - \int \left((\delta-2)\left(\gamma^{\delta-3}\right)\right)\left(\int \delta^{\gamma}\,d\gamma\right)d\gamma$$

$$I_2 = \gamma^{\delta-2}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \int \left((\delta-2)\left(\gamma^{\delta-3}\right)\right)\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right)d\gamma$$

$$I_2 = \gamma^{\delta-2}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \frac{\delta-2}{log_e\,\delta}\int \left(\gamma^{\delta-3}\right)(\delta^{\gamma})d\gamma + \lambda_2$$

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So, $I_n = \int \underset{\mathrm{I}}{\gamma^{\delta-n}}\quad \underset{\mathrm{II}}{\delta^{\gamma}}\,d\gamma = \gamma^{\delta-n}\left(\frac{\delta^{\gamma}}{log_e\,\delta}\right) - \frac{\delta-n}{log_e\,\delta}\int\left(\gamma^{\delta-n-1}\right)(\delta^{\gamma})d\gamma + \lambda_n$

This is continued till $n = \delta$

So, $I_\delta = \int \underset{\text{I}}{\gamma^{\delta-\delta}} \quad \underset{\text{II}}{\delta^\gamma}\, d\gamma = \gamma^{\delta-\delta}\left(\frac{\delta^\gamma}{log_e\,\delta}\right) - \frac{\delta-\delta}{log_e\,\delta}\int\left(\gamma^{\delta-\delta-1}\right)(\delta^\gamma)d\gamma = \frac{\delta^\gamma}{log_e\,\delta} + \lambda_\delta$

$\Rightarrow I = \gamma^\delta\left(\frac{\delta^\gamma}{log_e\,\delta}\right) - \frac{\delta}{log_e\,\delta}\left(\gamma^{\delta-1}\left(\frac{\delta^\gamma}{log_e\,\delta}\right) - \frac{\delta-1}{log_e\,\delta}\left(\gamma^{\delta-2}\left(\frac{\delta^\gamma}{log_e\,\delta}\right) - \frac{\delta-2}{log_e\,\delta}(\dots) + \lambda_2\right) + \lambda_1\right)$

Question 19: Evaluate $I = \int sin(mx)cos(nx)\, dx$

Solution:

$$I = \int sin(mx)cos(nx)\, dx$$

We know, $sin(mx)cos(nx) = \frac{1}{2}\{sin(m-n)x + sin(m+n)x\}$

$\Rightarrow I = \int \frac{1}{2}\{sin(m-n)x + sin(m+n)x\}\, dx$

$\Rightarrow I = \frac{1}{2}\int \{sin(m-n)x + sin(m+n)x\}\, dx$

$\Rightarrow I = \frac{1}{2}\int sin(m-n)x\, dx + \frac{1}{2}\int sin(m+n)x\, dx$

$\Rightarrow I = \frac{1}{2}\int sin(m-n)x\, dx + \frac{1}{2}\int sin(m+n)x\, dx$

Let $I_1 = \frac{1}{2}\int sin(m-n)x\, dx$ and $I_2 = \frac{1}{2}\int sin(m+n)x\, dx$

Let $(m-n)x = t$

Differentiating both sides w.r.t to x

$(m-n)dx = dt$

$\Rightarrow I_1 = \frac{1}{2}\int sin(m-n)x\, dx$

$\Rightarrow I_1 = \frac{1}{2(m-n)}\int sin(t)\, dt$

$\Rightarrow I_1 = \frac{-cos\left((m-n)x\right)}{2(m-n)} + \lambda'$

$\Rightarrow I_2 = \frac{-cos\left((m+n)x\right)}{2(m+n)} + \lambda''$

$\Rightarrow I = \frac{-cos(m-n)x}{2(m-n)} - \frac{cos(m+n)x}{2(m+n)} + \lambda$

Question 20: Evaluate $I = \int cos(mx)sin(nx)\, dx$

Solution:

$$I = \int cos(mx)sin(nx)\, dx$$

We know, $cos(mx)sin(nx) = \frac{1}{2}\{sin(m+n)x - sin(m-n)x\}$

$\Rightarrow I = \int \frac{1}{2}\{sin(m+n)x - sin(m-n)x\}\, dx$

$\Rightarrow I = \frac{1}{2}\int \{sin(m+n)x - sin(m-n)x\}\, dx$

$$\Rightarrow I = \frac{1}{2}\int sin(m+n)x\,dx - \frac{1}{2}\int sin(m-n)x\,dx$$

$$\Rightarrow I = \frac{-\,cos(m+n)\,x}{2(m+n)} + \frac{cos(m-n)\,x}{2(m-n)} + \lambda$$

$$\Rightarrow I = \frac{cos(m-n)\,x}{2(m-n)} - \frac{cos(m+n)\,x}{2(m+n)} + \lambda$$

Question 21: Evaluate $I = \int sin(mx)sin(nx)\,dx$

Solution:

$$I = \int sin(mx)sin(nx)\,dx$$

We know, $sin(mx)sin(nx) = \frac{1}{2}\{cos(m-n)x - cos(m+n)x\}$

$$\Rightarrow I = \int \frac{1}{2}\{cos(m-n)x - cos(m+n)x\}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int \{cos(m-n)x - cos(m+n)x\}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int cos(m-n)x\,dx - \frac{1}{2}\int cos(m+n)x\,dx$$

$$\Rightarrow I = \frac{sin(m-n)\,x}{2(m-n)} - \frac{sin(m+n)\,x}{2(m+n)} + \lambda$$

Question 22: Evaluate $I = \int cos(mx)cos(nx)\,dx$

Solution:

$$I = \int cos(mx)cos(nx)\,dx$$

We know, $cos(mx)cos(nx) = \frac{1}{2}\{cos(m-n)x + cos(m+n)x\}$

$$\Rightarrow I = \int \frac{1}{2}\{cos(m-n)x + cos(m+n)x\}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int \{cos(m-n)x + cos(m+n)x\}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int cos(m-n)x\,dx + \frac{1}{2}\int cos(m+n)x\,dx$$

$$\Rightarrow I = \frac{sin(m-n)\,x}{2(m-n)} + \frac{sin(m+n)\,x}{2(m+n)} + \lambda$$

Question 23: Evaluate $I = \int cos(mx)cos(nx)cos(ox)\,dx$

Solution:

$$I = \int cos(mx)cos(nx)cos(ox)\,dx$$

We know, $cos(mx)cos(nx) = \frac{1}{2}\{cos(m-n)x + cos(m+n)x\}$

$$cos(mx)cos(nx)cos(ox) = \left[\frac{1}{2}\{cos(m-n)x + cos(m+n)x\}\right] \times cos(ox)$$

$$cos(mx)cos(nx)cos(ox) = \left[\frac{1}{2}\{cos((m-n)x)cos(ox) + cos((m+n)x)cos(ox)\}\right]$$

$$cos(mx)cos(nx)cos(ox)$$
$$= \frac{1}{2}\left[\frac{1}{2}\{cos(m-n-o)x + cos(m-n+o)x\} + \frac{1}{2}\{cos(m+n-o)x + cos(m+n+o)x\}\right]$$

$$cos(mx)cos(nx)cos(ox)$$
$$= \frac{1}{2^2}[\{cos(m-n-o)x + cos(m-n+o)x\} + \{cos(m+n-o)x + cos(m+n+o)x\}]$$

$$\Rightarrow I = \int \frac{1}{2^2}[\{cos(m-n-o)x + cos(m-n+o)x\} + \{cos(m+n-o)x + cos(m+n+o)x\}]\,dx$$

$$\Rightarrow I = \frac{1}{2^2}\int [\{cos(m-n-o)x + cos(m-n+o)x\} + \{cos(m+n-o)x + cos(m+n+o)x\}]\,dx$$

$$\Rightarrow I = \frac{1}{2^2}\left(\frac{sin(m-n-o)x}{(m-n-o)}\right) + \frac{1}{2^2}\left(\frac{sin(m-n+o)x}{(m-n+o)}\right) + \frac{1}{2^2}\left(\frac{sin(m+n-o)x}{(m+n-o)}\right) + \frac{1}{2^2}\left(\frac{sin(m+n+o)x}{(m+n+o)}\right) + \lambda$$

Question 24: Evaluate $I = \int \frac{1}{ax^2+bx+c}dx$

Solution:

$$I = \int \frac{1}{ax^2 + bx + c}dx$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + 2x\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right)$$
$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^2\right)$$

$$\Rightarrow I = \int \frac{1}{ax^2 + bx + c}dx$$

$$\Rightarrow I = \int \frac{1}{a\left(\left(x + \frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^2\right)}dx$$

Let $x + \frac{b}{2a} = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I = \int \frac{1}{a\left((t)^2 + \left(\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^2\right)}dt$$

$$\Rightarrow I = \frac{1}{a}\int \frac{1}{\left((t)^2 + \left(\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^2\right)}dt$$

$$\Rightarrow I = \frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}tan^{-1}\left(\frac{t}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}\right)\right) + \lambda$$

$$\Rightarrow I = \frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}} tan^{-1}\left(\frac{x+\frac{b}{2a}}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}}\right)\right)+\lambda$$

Question 25: Evaluate $I = \int \frac{1}{\sqrt{ax^2+bx+c}} dx$

Solution:

$$I = \int \frac{1}{\sqrt{ax^2+bx+c}} dx$$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(x^2+2x\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right) = a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)$$

$$\Rightarrow I = \int \frac{1}{\sqrt{ax^2+bx+c}} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{a\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}} dx$$

Let $x+\frac{b}{2a} = t$

Differentiating both sides w.r.t to x

$dx = dt$

$$\Rightarrow I = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left((t)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}} dt$$

$$\Rightarrow I = \frac{1}{\sqrt{a}} log_e\left|t+\sqrt{t^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|+\lambda$$

$$\Rightarrow I = \frac{1}{\sqrt{a}} log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|+\lambda$$

Question 26: Evaluate $I = \int \sqrt{ax^2+bx+c}\, dx$

Solution:

$$I = \int \sqrt{ax^2+bx+c}\, dx$$

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+2x\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)=a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$
$$=a\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)$$

$$\Rightarrow I=\int\sqrt{ax^2+bx+c}\,dx$$

$$\Rightarrow I=\int\sqrt{a\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}dx$$

$$\Rightarrow I=\sqrt{a}\int\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}dx$$

Let $x+\frac{b}{2a}=t$

Differentiating both sides w.r.t to x

$dx=dt$

$$\Rightarrow I=\sqrt{a}\int\sqrt{\left((t)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}dt$$

$$\Rightarrow I=\sqrt{a}\left(\frac{t}{2}\sqrt{\left((t)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}+\frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|t+\sqrt{t^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)+\lambda$$

$$\Rightarrow I=\sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}\right.$$
$$\left.+\frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)+\lambda$$

Question 27: Evaluate $I=\int\frac{px+q}{ax^2+bx+c}dx$

Solution:

$$I=\int\frac{px+q}{ax^2+bx+c}dx$$

Let

$$px+q=\alpha\frac{d}{dx}(ax^2+bx+c)+\beta$$

$\Rightarrow px + q = \alpha(2ax + b) + \beta$

$\Rightarrow px + q = 2\alpha ax + b\alpha + \beta$

Comparing, we get

$\Rightarrow p = 2\alpha a$ and $\Rightarrow q = b\alpha + \beta$

$\Rightarrow p = 2\alpha a$ and $q = b\alpha + \beta$

$\Rightarrow \alpha = \frac{p}{2a}$ and $\beta = q - b\left(\frac{p}{2a}\right)$

Further, on solving by substituting the values of α and β, we can get the

$$\Rightarrow I = \int \frac{\alpha(2ax + b) + \beta}{ax^2 + bx + c} dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{2a}\right)(2ax + b) + q - b\left(\frac{p}{2a}\right)}{ax^2 + bx + c} dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{2a}\right)(2ax + b)}{ax^2 + bx + c} dx + \int \frac{q - b\left(\frac{p}{2a}\right)}{ax^2 + bx + c} dx$$

$$\Rightarrow I = \left(\frac{p}{2a}\right) \int \frac{(2ax + b)}{ax^2 + bx + c} dx + \left(q - b\left(\frac{p}{2a}\right)\right) \int \frac{1}{ax^2 + bx + c} dx$$

We know,

$$\int \frac{1}{ax^2 + bx + c} dx = \frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}} tan^{-1}\left(\frac{x + \frac{b}{2a}}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}\right)\right) + \lambda$$

$$\Rightarrow I = \left(\frac{p}{2a}\right) \int \frac{(2ax + b)}{ax^2 + bx + c} dx + \left(q - b\left(\frac{p}{2a}\right)\right)\left(\frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}} tan^{-1}\left(\frac{x + \frac{b}{2a}}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}\right)\right)\right)$$

$$\Rightarrow I = \left(\frac{p}{2a}\right) \times log_e|ax^2 + bx + c| + \left(q - b\left(\frac{p}{2a}\right)\right) \times \left(\frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}} tan^{-1}\left(\frac{x + \frac{b}{2a}}{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}\right)\right)\right) + \lambda$$

Question 28: Evaluate $I = \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Solution:

$$I = \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$$

Let

$$px + q = \alpha \frac{d}{dx}(ax^2 + bx + c) + \beta$$

$\Rightarrow px + q = \alpha(2ax + b) + \beta$

$\Rightarrow px + q = 2\alpha ax + b\alpha + \beta$

Comparing, we get

$\Rightarrow p = 2\alpha a$ and $\Rightarrow q = b\alpha + \beta$

$\Rightarrow p = 2\alpha a$ and $q = b\alpha + \beta$

$\Rightarrow \alpha = \frac{p}{2a}$ and $\beta = q - b\left(\frac{p}{2a}\right)$

Further, on solving by substituting the values of α and β, we can get the

$$\Rightarrow I = \int \frac{\alpha(2ax+b)+\beta}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{2a}\right)(2ax+b)+q-b\left(\frac{p}{2a}\right)}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{2a}\right)(2ax+b)+q-b\left(\frac{p}{2a}\right)}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{2a}\right)(2ax+b)}{\sqrt{ax^2+bx+c}}dx + \int \frac{q-b\left(\frac{p}{2a}\right)}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \left(\frac{p}{2a}\right)\int \frac{(2ax+b)}{ax^2+bx+c}dx + \left(q-b\left(\frac{p}{2a}\right)\right)\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

We know,

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|+\lambda$$

$$\Rightarrow I = \left(\frac{p}{2a}\right)\int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx + \left(q-b\left(\frac{p}{2a}\right)\right)\left(\frac{1}{\sqrt{a}}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)$$

$$\Rightarrow I = \left(\frac{p}{2a}\right)\left(2\sqrt{ax^2+bx+c}\right) + \left(q-b\left(\frac{p}{2a}\right)\right)\left(\frac{1}{\sqrt{a}}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)+\lambda$$

Question 29: Evaluate $I = \int(px+q)\sqrt{ax^2+bx+c}\,dx$

Solution:

$$I = \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

Let

$$px+q = \alpha\frac{d}{dx}(ax^2+bx+c)+\beta$$

$\Rightarrow px+q = \alpha(2ax+b)+\beta$

$\Rightarrow px+q = 2\alpha ax + b\alpha + \beta$

Comparing, we get

$\Rightarrow p = 2\alpha a$ and $\Rightarrow q = b\alpha + \beta$

$\Rightarrow p = 2\alpha a$ and $q = b\alpha + \beta$

$\Rightarrow \alpha = \frac{p}{2a}$ and $\beta = q - b\left(\frac{p}{2a}\right)$

Further, on solving by substituting the values of α and β, we can get the

$\Rightarrow I = \int (px+q)\sqrt{ax^2+bx+c}\,dx$

$\Rightarrow I = \int \left(\left(\frac{p}{2a}\right)(2ax+b) + q - b\left(\frac{p}{2a}\right)\right)\sqrt{ax^2+bx+c}\,dx$

$\Rightarrow I = \int \left(\left(\frac{p}{2a}\right)(2ax+b)\right)\sqrt{ax^2+bx+c}\,dx + \int \left(q - b\left(\frac{p}{2a}\right)\right)\sqrt{ax^2+bx+c}\,dx$

$\Rightarrow I = \left(\frac{p}{2a}\right)\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \left(q - b\left(\frac{p}{2a}\right)\right)\int \sqrt{ax^2+bx+c}\,dx$

We know,

$$\int \sqrt{ax^2+bx+c}\,dx = \sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)} + \frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right) + \sqrt{\left(x+\frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right) + \lambda$$

$$\Rightarrow I = \left(\frac{p}{2a}\right)\left(\frac{2}{3}(ax^2+bx+c)^{\frac{3}{2}}\right) + \left(q - b\left(\frac{p}{2a}\right)\right)\left(\sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)} + \frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right) + \sqrt{\left(x+\frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)\right) + \lambda$$

Question 30: Evaluate $I = \int x\sqrt{1+x-x^2}\,dx$

Solution:

$$I = \int x\sqrt{1+x-x^2}\,dx$$

Let,

$x = \alpha\frac{d}{dx}(1+x-x^2) + \beta$

$\Rightarrow x = \alpha(1-2x) + \beta$

$\Rightarrow x = \alpha - \alpha 2x + \beta$

Comparing, we get

$x = -\alpha 2x$ and $0 = \alpha + \beta$

$\Rightarrow \alpha = -\frac{1}{2}$ and $\beta = -\alpha = \frac{1}{2}$

$$\Rightarrow I = \int (\alpha(1-2x) + \beta)\sqrt{1+x-x^2}\,dx$$

$$\Rightarrow I = \int \left(\left(-\frac{1}{2}\right)(1-2x) + \frac{1}{2}\right)\sqrt{1+x-x^2}\,dx$$

$$\Rightarrow I = \int \left(\left(-\frac{1}{2}\right)(1-2x)\right)\sqrt{1+x-x^2}\,dx + \int \frac{1}{2}\sqrt{1+x-x^2}\,dx$$

$$\Rightarrow I = \left(-\frac{1}{2}\right)\int (1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$$

Let $I_1 = \left(-\frac{1}{2}\right)\int (1-2x)\sqrt{1+x-x^2}\,dx$ and $I_2 = \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$

$$\Rightarrow I_1 = \left(-\frac{1}{2}\right)\int (1-2x)\sqrt{1+x-x^2}\,dx$$

Let $1 + x - x^2 = t$

Differentiating both sides w.r.t x

$(1-2x)dx = dt$

$$\Rightarrow I_1 = \left(-\frac{1}{2}\right)\int \sqrt{t}\,dt$$

$$\Rightarrow I_1 = \left(-\frac{1}{2}\right)\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + \lambda_1$$

$$\Rightarrow I_1 = \left(-\frac{1}{2}\right)\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + \lambda_1$$

$$\Rightarrow I_1 = \left(-\frac{1}{2}\right)\left(\frac{(1+x-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right) + \lambda_1$$

$$\Rightarrow I_1 = \left(-\frac{1}{3}\right)(1+x-x^2)^{\frac{3}{2}} + \lambda_1$$

$$\Rightarrow I_2 = \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$$

$$1 + x - x^2 = -x^2 + x + 1 = -(x^2 - x - 1) = -\left(x^2 - 2x\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 1\right) = -\left(\left(x - \frac{1}{2}\right)^2 - \left(\sqrt{\frac{5}{4}}\right)^2\right)$$

$$= \left(\left(\sqrt{\frac{5}{4}}\right)^2 - \left(x - \frac{1}{2}\right)^2\right)$$

$$\Rightarrow I_2 = \frac{1}{2}\int \sqrt{\left(\left(\sqrt{\frac{5}{4}}\right)^2 - \left(x - \frac{1}{2}\right)^2\right)}\, dx$$

Let $\left(x - \frac{1}{2}\right) = t$

Differentiating both sides w.r.t x

$dx = dt$

$$\Rightarrow I_2 = \frac{1}{2}\int \sqrt{\left(\left(\sqrt{\frac{5}{4}}\right)^2 - t^2\right)}\, dt$$

$$\Rightarrow I_2 = \frac{1}{2}\left(\frac{t}{2}\sqrt{\left(\left(\sqrt{\frac{5}{4}}\right)^2 - t^2\right)} + \frac{\left(\sqrt{\frac{5}{4}}\right)^2}{2} sin^{-1}\left(\frac{t}{\sqrt{\frac{5}{4}}}\right)\right) + \lambda_2$$

$$\Rightarrow I = \left(-\frac{1}{3}\right)(1 + x - x^2)^{\frac{3}{2}} + \lambda_1 + \frac{1}{2}\left(\frac{t}{2}\sqrt{\left(\left(\sqrt{\frac{5}{4}}\right)^2 - t^2\right)} + \frac{\left(\sqrt{\frac{5}{4}}\right)^2}{2} sin^{-1}\left(\frac{t}{\sqrt{\frac{5}{4}}}\right)\right) + \lambda_2$$

$$\Rightarrow I = \left(-\frac{1}{3}\right)(1 + x - x^2)^{\frac{3}{2}} + \frac{1}{2}\left(\frac{t}{2}\sqrt{\left(\left(\sqrt{\frac{5}{4}}\right)^2 - t^2\right)} + \frac{\left(\sqrt{\frac{5}{4}}\right)^2}{2} sin^{-1}\left(\frac{t}{\sqrt{\frac{5}{4}}}\right)\right) + \lambda$$

$$\Rightarrow I = \left(-\frac{1}{3}\right)(1 + x - x^2)^{\frac{3}{2}} + \frac{1}{2}\left(\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{\left(\left(\sqrt{\frac{5}{4}}\right)^2 - \left(x - \frac{1}{2}\right)^2\right)} + \frac{\left(\sqrt{\frac{5}{4}}\right)^2}{2} sin^{-1}\left(\frac{\left(x - \frac{1}{2}\right)}{\sqrt{\frac{5}{4}}}\right)\right) + \lambda$$

Question 31: Evaluate $I = \int \frac{px^2+qx+r}{(ax^2+bx+c)} dx$

Solution:

$$I = \int \frac{px^2 + qx + r}{(ax^2 + bx + c)} dx$$

Let,

$$px^2 + qx + r = \alpha(ax^2 + bx + c) + \beta\frac{d}{dx}(ax^2 + bx + c) + \gamma$$

$$\Rightarrow px^2 + qx + r = \alpha(ax^2 + bx + c) + \beta(2ax + b) + \gamma$$

Comparing, we get

$\Rightarrow px^2 = \alpha ax^2$, $qx = \alpha bx + 2\beta ax$, and $r = \alpha c + \beta b + \gamma$

$\Rightarrow p = \alpha a$, $q = \alpha b + 2\beta a$, and $r = \alpha c + \beta b + \gamma$

$\Rightarrow \alpha = \left(\frac{p}{a}\right), \beta = \frac{q-\left(\frac{p}{a}\right)b}{2a}$, and $\gamma = r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)$

$$\Rightarrow I = \int \frac{\alpha(ax^2+bx+c)+\beta(2ax+b)+\gamma}{(ax^2+bx+c)}dx$$

$$\Rightarrow I = \int \frac{\left(\left(\frac{p}{a}\right)(ax^2+bx+c) + \frac{q-\left(\frac{p}{a}\right)b}{2a}(2ax+b) + r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)}{(ax^2+bx+c)}dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{a}\right)(ax^2+bx+c)}{(ax^2+bx+c)}dx + \int \frac{\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)(2ax+b)}{(ax^2+bx+c)}dx + \int \frac{r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)}{(ax^2+bx+c)}dx$$

$$\Rightarrow I = \left(\frac{p}{a}\right)\int 1\,dx + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)\int \frac{(2ax+b)}{(ax^2+bx+c)}dx + \left(r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\int \frac{1}{(ax^2+bx+c)}dx$$

We know,

$$\int \frac{1}{ax^2+bx+c}dx = \frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}}tan^{-1}\left(\frac{x+\frac{b}{2a}}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}}\right)\right)+\lambda$$

$$\Rightarrow I = \left(\frac{p}{a}\right)x + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)(log_e|ax^2+bx+c|)$$

$$+\left(r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\left(\frac{1}{a}\left(\frac{1}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}}tan^{-1}\left(\frac{x+\frac{b}{2a}}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}}\right)\right)\right)+\lambda$$

Question 32: Evaluate $I = \int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx$

Solution:

$$I = \int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx$$

Let,

$$px^2+qx+r = \alpha(ax^2+bx+c) + \beta\frac{d}{dx}(ax^2+bx+c) + \gamma$$

$$\Rightarrow px^2+qx+r = \alpha(ax^2+bx+c) + \beta(2ax+b) + \gamma$$

Comparing, we get

$\Rightarrow px^2 = \alpha ax^2$, $qx = \alpha bx + 2\beta ax$, and $r = \alpha c + \beta b + \gamma$

$\Rightarrow p = \alpha a$, $q = \alpha b + 2\beta a$, and $r = \alpha c + \beta b + \gamma$

$$\Rightarrow \alpha = \left(\frac{p}{a}\right), \beta = \frac{q-\left(\frac{p}{a}\right)b}{2a}, \text{ and } \gamma = r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)$$

$$\Rightarrow I = \int \frac{\alpha(ax^2+bx+c)+\beta(2ax+b)+\gamma}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \int \frac{\left(\left(\frac{p}{a}\right)(ax^2+bx+c) + \frac{q-\left(\frac{p}{a}\right)b}{2a}(2ax+b) + r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \int \frac{\left(\frac{p}{a}\right)(ax^2+bx+c)}{\sqrt{ax^2+bx+c}}dx + \int \frac{\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)(2ax+b)}{\sqrt{ax^2+bx+c}}dx + \int \frac{r-\left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \int \left(\frac{p}{a}\right)\sqrt{ax^2+bx+c}\,dx + \int \frac{\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)(2ax+b)}{\sqrt{ax^2+bx+c}}dx + \int \frac{r-\left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow I = \left(\frac{p}{a}\right)\int \sqrt{ax^2+bx+c}\,dx + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)\int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx$$

$$+\left(r-\left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

We know,

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|+\lambda$$

$$\int \sqrt{ax^2+bx+c}\,dx$$

$$= \sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}\right.$$

$$\left.+\frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)+\lambda$$

$$\Rightarrow I = \left(\frac{p}{a}\right)\left(\sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}\right.\right.$$

$$\left.\left.+\frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)\right)$$

$$+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)\left(2\left(\sqrt{ax^2+bx+c}\right)\right)$$

$$+\left(r-\left(\left(\frac{p}{a}\right)c+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\left(\frac{1}{\sqrt{a}}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)+\lambda$$

Question 33: Evaluate $I=\int(px^2+qx+r)\sqrt{ax^2+bx+c}\,dx$

Solution:

$$I=\int(px^2+qx+r)\sqrt{ax^2+bx+c}\,dx$$

Let,

$px^2+qx+r=\alpha(ax^2+bx+c)+\beta\frac{d}{dx}(ax^2+bx+c)+\gamma$

$\Rightarrow px^2+qx+r=\alpha(ax^2+bx+c)+\beta(2ax+b)+\gamma$

Comparing, we get

$\Rightarrow px^2=\alpha ax^2$, $qx=\alpha bx+2\beta ax$, and $r=\alpha c+\beta b+\gamma$

$\Rightarrow p=\alpha a$, $q=\alpha b+2\beta a$, and $r=\alpha c+\beta b+\gamma$

$\Rightarrow \alpha=\left(\frac{p}{a}\right)$, $\beta=\frac{q-\left(\frac{p}{a}\right)b}{2a}$, and $\gamma=r-\left(\left(\frac{p}{a}\right)c+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)$

$\Rightarrow I=\int(\alpha(ax^2+bx+c)+\beta(2ax+b)+\gamma)\left(\sqrt{ax^2+bx+c}\right)dx$

$\Rightarrow I=\int\alpha(ax^2+bx+c)\left(\sqrt{ax^2+bx+c}\right)dx+\int\beta(2ax+b)\left(\sqrt{ax^2+bx+c}\right)dx+\int\gamma\left(\sqrt{ax^2+bx+c}\right)dx$

$\Rightarrow I=\int\alpha(ax^2+bx+c)\left(\sqrt{ax^2+bx+c}\right)dx+\int\beta(2ax+b)\left(\sqrt{ax^2+bx+c}\right)dx+\int\gamma\left(\sqrt{ax^2+bx+c}\right)dx$

$$\Rightarrow I=\int\left(\frac{p}{a}\right)(ax^2+bx+c)\left(\sqrt{ax^2+bx+c}\right)dx+\int\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)(2ax+b)\left(\sqrt{ax^2+bx+c}\right)dx$$

$$+\int\left(r-\left(\left(\frac{p}{a}\right)c+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\left(\sqrt{ax^2+bx+c}\right)dx$$

$$\Rightarrow I=\left(\frac{p}{a}\right)\int(ax^2+bx+c)\left(\sqrt{ax^2+bx+c}\right)dx+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)\left(\frac{2}{3}(ax^2+bx+c)^{\frac{3}{2}}\right)$$

$$+\left(r-\left(\left(\frac{p}{a}\right)c+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\int\left(\sqrt{ax^2+bx+c}\right)dx$$

We know,

$$\int\sqrt{ax^2+bx+c}\,dx$$

$$=\sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}\right.$$

$$\left.+\frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)+\lambda$$

$$\Rightarrow I=\left(\frac{p}{a}\right)\int(ax^2+bx+c)^{\frac{3}{2}}dx+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)\left(\frac{2}{3}(ax^2+bx+c)^{\frac{3}{2}}\right)$$

$$+\left(r\right.$$

$$\left.-\left(\left(\frac{p}{a}\right)c+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\left(\sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)}\right.\right.$$

$$\left.\left.+\frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)\right)$$

$$\Rightarrow I_1=\int(ax^2+bx+c)^{\frac{3}{2}}dx$$

$$\Rightarrow I_1=\int\left(\left(\sqrt{a}x\right)^2+2\left(\sqrt{a}x\right)\left(\frac{b}{2\sqrt{a}}\right)+\left(\frac{b}{2\sqrt{a}}\right)^2-\left(\frac{b}{2\sqrt{a}}\right)^2+c\right)^{\frac{3}{2}}dx$$

$$\Rightarrow I_1=\int\left(\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2-\left(\frac{b}{2\sqrt{a}}\right)^2+c\right)^{\frac{3}{2}}dx$$

$$\Rightarrow I_1=\frac{1}{8a^{\frac{3}{2}}}\int((2ax+b)^2+4ac-b^2)^{\frac{3}{2}}dx$$

Let $t=2ax+b$

Differentiating both sides w.r.t x

$dt = dx$

$\Rightarrow I_1 = \frac{1}{8a^{\frac{3}{2}}} \int ((t)^2 + 4ac - b^2)^{\frac{3}{2}} \, dt$

Let $t = \sqrt{4ac - b^2} tan(u)$

$\Rightarrow u = tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)$

Differentiating both sides w.r.t x

$dt = \sqrt{4ac - b^2}\left(sec^2(u)\right) du$

$\Rightarrow I_1 = \frac{1}{8a^{\frac{3}{2}}} \int \left((4ac - b^2)tan^2(u) + 4ac - b^2\right)^{\frac{3}{2}} \left(\sqrt{4ac - b^2}\left(sec^2(u)\right)\right) du$

$\Rightarrow I_1 = \frac{1}{8a^{\frac{3}{2}}} \int \left((4ac - b^2)(tan^2(u) + 1)\right)^{\frac{3}{2}} \left(\sqrt{4ac - b^2}\left(sec^2(u)\right)\right) du$

$\Rightarrow I_1 = \frac{1}{8a^{\frac{3}{2}}} \int \left((4ac - b^2)\left(sec^2(u)\right)\right)^{\frac{3}{2}} \left(\sqrt{4ac - b^2}\left(sec^2(u)\right)\right) du$

$\Rightarrow I_1 = \frac{(4ac - b^2)^{\frac{7}{2}}}{8a^{\frac{3}{2}}} \int sec^5(u) \, du$

Let

$\Rightarrow I_2 = \int \underset{I}{sec^3(u)} \quad \underset{II}{sec^2(u)} \, du$

$\Rightarrow I_2 = sec^3(u) \int sec^2(u) \, du - \int \left(\left(\frac{d}{dx}\left(sec^3(u)\right) \right) \int sec^2(u) \, du \right) du$

$\Rightarrow I_2 = sec^3(u)\left(tan(u)\right) - \int \left(\left(3sec^2(u)sec(x)tan(x)\right)\left(tan(u)\right)\right) du$

$\Rightarrow I_2 = sec^3(u)\left(tan(u)\right) - 3 \int sec^3(u)(sec^2(u) - 1) \, du$

$\Rightarrow I_2 = sec^3(u)\left(tan(u)\right) - 3 \int sec^5(u) \, du + 3 \int sec^3(u) \, du$

$\Rightarrow I_2 = sec^3(u)\left(tan(u)\right) - 3I_2 + 3 \int sec^3(u) \, du$

$\Rightarrow 4I_2 = sec^3(u)\left(tan(u)\right) + 3 \int sec^3(u) \, du$

Let

$\Rightarrow I_3 = \int \underset{I}{sec(u)} \quad \underset{II}{sec^2(u)} \, du$

$\Rightarrow I_3 = sec(u) \int sec^2(u) \, du - \int \left(sec(u)tan(u) \int sec^2(u) \, du\right) du$

$\Rightarrow I_3 = sec(u)tan(u) - \int \left(sec(u)tan(u)tan(u)\right) du$

$$\Rightarrow I_3 = sec(u)tan(u) - \int sec^3(u)\,du + \int sec(u)\,du$$

$$\Rightarrow I_3 = sec(u)tan(u) - I_3 + \int sec(u)\,du$$

$$\Rightarrow 2I_3 = sec(u)tan(u) + \int sec(u)\,du$$

$$\Rightarrow 2I_3 = sec(u)tan(u) + log_e|tan(u) + sec(u)|$$

$$\Rightarrow I_3 = \frac{sec(u)tan(u) + log_e|tan(u) + sec(u)|}{2} + some\ constant$$

So,

$$\Rightarrow 4I_2 = sec^3(u)\big(tan(u)\big) + \frac{3}{2}(sec(u)tan(u) + log_e|tan(u) + sec(u)|)$$

$$\Rightarrow I_2 = \frac{sec^3(u)\big(tan(u)\big) + \frac{3}{2}(sec(u)tan(u) + log_e|tan(u) + sec(u)|)}{4} + some\ constant$$

So,

$$\Rightarrow I_1 = \frac{(4ac - b^2)^{\frac{7}{2}}}{8a^{\frac{3}{2}}}\left(\frac{sec^3(u)\big(tan(u)\big) + \frac{3}{2}(sec(u)tan(u) + log_e|tan(u) + sec(u)|)}{4}\right) + some\ constant$$

As

$$u = tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)$$

$$\Rightarrow I_1 = \frac{(4ac - b^2)^{\frac{7}{2}}}{8a^{\frac{3}{2}}}\left(\frac{sec^3\left(tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)\right)tan\left(tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)\right)}{4} + \frac{3sec\left(tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)\right)tan\left(tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)\right)}{8} + \frac{log_e\left|tan\left(tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)\right) + sec\left(tan^{-1}\left(\frac{t}{\sqrt{4ac - b^2}}\right)\right)\right|}{4}\right) + some\ constant$$

As

$$t = 2ax + b$$

$$\Rightarrow I_1 = \frac{(4ac-b^2)^{\frac{7}{2}}}{8a^{\frac{3}{2}}}\left(\frac{sec^3\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)tan\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)}{4} + \frac{3sec\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)tan\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)}{8} + \frac{log_e\left|tan\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)+sec\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)\right|}{4}\right) + some\ constant$$

$$\Rightarrow I = \left(\frac{p}{a}\right)\frac{(4ac-b^2)^{\frac{7}{2}}}{8a^{\frac{3}{2}}}\left(\frac{sec^3\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)tan\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)}{4} + \frac{3sec\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)tan\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)}{8} + \frac{log_e\left|tan\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)+sec\left(tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right)\right|}{4}\right)$$

$$+\left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)\left(\frac{2}{3}(ax^2+bx+c)^{\frac{3}{2}}\right)$$

$$+\left(r - \left(\left(\frac{p}{a}\right)c + \left(\frac{q-\left(\frac{p}{a}\right)b}{2a}\right)b\right)\right)\left(\sqrt{a}\left(\frac{\left(x+\frac{b}{2a}\right)}{2}\sqrt{\left(\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2\right)} + \frac{\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}{2}log_e\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\right)^2}\right|\right)\right)+\lambda$$

Question 34: Evaluate $I = \int_0^{4\pi} sin(x)\,dx$

Solution:

$$I = \int_0^{4\pi} sin(x)\,dx$$

$$\Rightarrow I = \int_0^{4\pi} sin(x)\,dx = [-cos(x)]_0^{4\pi} = -cos(4\pi) - \left(-cos(0)\right) = -1+1 = 0$$