Q.01.(A)	**Find Laplace Transform of** $e^{-3t}\sin^2 t$	**04 Marks**

Solution : → To find $L(\underset{②}{e^{-3t}}\,\underset{①}{\sin^2 t})$

* **First Location :**

$$\therefore L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) \quad \ldots\left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= \frac{1}{2}\times L(1-\cos 2t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)$$

$$\therefore \boxed{L(\sin^2 t) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)}$$

* **Second Location :**

→ By theorem - First Shifting

$$\therefore L(e^{-3t}\times\sin^2 t) = \left[\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]_{s\to s+3}$$

$$= \left[\frac{1}{2}\left(\frac{s^2+4-s^2}{s(s^2+4)}\right)\right]_{s\to s+3}$$

$$= \left[\frac{1}{2}\left(\frac{\cancel{s^2}+4-\cancel{s^2}}{s(s^2+4)}\right)\right]_{s\to s+3}$$

$$= \left[\frac{1}{2}\left(\frac{4}{s(s^2+4)}\right)\right]_{s\to s+3}$$

$$= \left[\frac{2}{s(s^2+4)}\right]_{s\to s+3}$$

$$= \frac{2}{(s+3)\left((s+3)^2+4\right)}$$

• **Use** : $(a+b)^2 = a^2 + 2ab + b^2$

$$= \frac{2}{(s+3)\left(s^2+6s+9+4\right)}$$

$$\therefore \boxed{L\left(e^{-3t}\sin^2 t\right) = \frac{2}{(s+3)\left(s^2+6s+13\right)}} \text{....(Ans)}$$

Q.01.(B)	**Find Laplace Transform of** $f(t)=\begin{cases}1, & 0<t<1\\ 0, & 1<t<2\end{cases}$ **where** $f(t)$ **is periodic function of period 2.**	**04 Marks**

Solution : → Given that :

[1] $f(t)=\begin{cases}1, & \boxed{0<t<1} \rightarrow \boxed{0 \text{ to } 1}\\ 0, & \boxed{1<t<2} \rightarrow \boxed{1 \text{ to } 2}\end{cases}$

[2] This function is periodic with period 2. → $\boxed{T=2}$

[3] To find $L\{f(t)\}$

→ We know, L.T. of periodic function is $\boxed{L[f(t)]=\dfrac{1}{1-e^{-s\cdot T}}\int_0^T e^{-st}f(t)\,dt}$

→ Put value of T.

$$\therefore L[f(t)]=\frac{1}{1-e^{-s\cdot 2}}\int_0^2 e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}f(t)\,dt+\int_1^2 e^{-st}f(t)\,dt\right]$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\cdot 1\,dt+\int_1^2 e^{-st}\cdot 0\,dt\right]$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\,dt+0\right]$$

$$=\frac{1}{1-e^{-2s}}\times\int_0^1 e^{-st}\,dt$$

• Use : $\int e^{px}\,dx=\dfrac{e^{px}}{p}$

$$=\frac{1}{1-e^{-2s}}\times\left[\frac{e^{-st}}{-s}\right]_0^1$$

$$=\frac{1}{1-e^{-2s}}\times\left\{\left[\frac{e^{-s\times 1}}{-s}\right]-\left[\frac{e^{-s\times 0}}{-s}\right]\right\}$$

$$=\frac{1}{1-e^{-2s}}\times\left\{\left[\frac{e^{-s}}{-s}\right]-\left[\frac{e^{0}}{-s}\right]\right\}$$

$$=\frac{1}{1-e^{-2s}}\times\left\{-\frac{e^{-s}}{s}+\frac{1}{s}\right\}$$

$$=\frac{1}{1-e^{-2s}}\times\left\{\frac{-e^{-s}+1}{s}\right\}$$

- **Use** : $e^{-2s}=\left(e^{-s}\right)^2$

$$=\frac{1}{1^2-\left(e^{-s}\right)^2}\times\left\{\frac{1-e^{-s}}{s}\right\}$$

- **Use** : $a^2-b^2=(a-b)(a+b)$

$$=\frac{1}{\left(1-e^{-s}\right)\left(1+e^{-s}\right)}\times\left\{\frac{1-e^{-s}}{s}\right\}$$

$$=\frac{1}{\cancel{\left(1-e^{-s}\right)}\left(1+e^{-s}\right)}\times\left\{\frac{\cancel{1-e^{-s}}}{s}\right\}$$

$$\therefore \boxed{L[f(t)]=\frac{1}{s\left(1+e^{-s}\right)}}\text{....(Ans)}$$

Q.01.(C)	**Evaluate using Laplace Transform :** $\int_0^\infty \frac{\cos 4t - \cos 3t}{t} dt$ **04 Marks**

Solution :

$\rightarrow$ To evaluate : $\int_0^\infty \frac{\cos 4t - \cos 3t}{t} dt$, Using L.T..

$\rightarrow$ Let, $I = \int_0^\infty \frac{\cos 4t - \cos 3t}{t} dt$

Step (01) – **To set given integration as a Definition of L.T.**

$\therefore I = \int_0^\infty 1 \times \frac{\cos 4t - \cos 3t}{t} dt$

• **Use** : $e^{-0 \cdot t} = e^0 = 1 \rightarrow \boxed{1 = e^{-0 \cdot t}}$

$\therefore \boxed{I = \int_0^\infty e^{-0 \cdot t} \times \frac{\cos 4t - \cos 3t}{t} dt}$ .... (1)

Step (02) – **Compare equation (1) with Definition of L.T.**

$\rightarrow$ We know, Def^n of L.T. is $\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t) dt}$ .... (2)

$\rightarrow$ Now, we compare equation (1) and (2), we get : $\begin{cases} (1)\ I = L[f(t)] & \text{.... (M)} \\ (2)\ s = 0 & \text{.... (N)} \\ (3)\ f(t) = \frac{\cos 4t - \cos 3t}{t} & \text{.... (O)} \end{cases}$

Step (03) – **We have to find value of Given Integration (i.e. Value of I).**

$\therefore I = L[f(t)]$ ....[$\because$ from (M)]

$\rightarrow$ But, $f(t) = \dfrac{\cos 4t - \cos 3t}{t}$ ....[$\because$ from (O)]

$$\therefore I = L\left[\frac{\cos 4t - \cos 3t}{t}\right] \quad ....(P)$$

Step (04) - Now, to find $L\left[\dfrac{\overset{①}{\cos 4t - \cos 3t}}{\underset{②}{t}}\right]$.

- **First Location :**

$$\therefore L(\cos 4t - \cos 3t) = \frac{s}{s^2+4^2} - \frac{s}{s^2+3^2}$$

$$\therefore \boxed{L(\cos 4t - \cos 3t) = \frac{s}{s^2+16} - \frac{s}{s^2+9}}$$

- **Second Location :**

$\rightarrow$ By theorem - Division by t

$$\therefore L\left(\frac{\cos 4t - \cos 3t}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{2}{2} \times \int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2} \times \int_s^\infty \left(\frac{2s}{s^2+16} - \frac{2s}{s^2+9}\right) ds$$

- **Use** : $\int \dfrac{f'(x)}{f(x)} dx = \log[f(x)]$

$$= \frac{1}{2} \times \left[\log(s^2+16) - \log(s^2+9)\right]_s^\infty$$

- **Use** : $\log\left(\dfrac{a}{b}\right) = \log a - \log b$

$$= \frac{1}{2} \times \left[\log\left(\frac{s^2+16}{s^2+9}\right)\right]_s^\infty$$

$\rightarrow$ Divide (N') & (D') by s^2.

$$=\frac{1}{2}\times\left[\log\left(\frac{\frac{s^2+16}{s^2}}{\frac{s^2+9}{s^2}}\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\times\left[\log\left(\frac{1+\frac{16}{s^2}}{1+\frac{9}{s^2}}\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\times\left\{\left[\log\left(\frac{1+\frac{16}{(\infty)^2}}{1+\frac{9}{(\infty)^2}}\right)\right]-\left[\log\left(\frac{1+\frac{16}{s^2}}{1+\frac{9}{s^2}}\right)\right]\right\}$$

$\rightarrow \frac{16}{(\infty)^2}=\frac{16}{\infty}=0$ and $\frac{9}{(\infty)^2}=\frac{9}{\infty}=0$

$$=\frac{1}{2}\times\left\{\left[\log\left(\frac{1+0}{1+0}\right)\right]-\left[\log\left(\frac{\frac{s^2+16}{s^2}}{\frac{s^2+9}{s^2}}\right)\right]\right\}$$

$$=\frac{1}{2}\times\left\{[\log(1)]-\left[\log\left(\frac{s^2+16}{s^2+9}\right)\right]\right\}$$

• **Use** : $\log 1=0$

$$=\frac{1}{2}\times\left\{[0]-\left[\log\left(\frac{s^2+16}{s^2+9}\right)\right]\right\}$$

$$=\frac{1}{2}\times\left\{-\left[\log\left(\frac{s^2+16}{s^2+9}\right)\right]\right\}$$

• **Use** : $\log\left(\frac{a}{b}\right)=-\log\left(\frac{b}{a}\right)$

$$=\frac{1}{2}\times\left\{-\left[-\log\left(\frac{s^2+9}{s^2+16}\right)\right]\right\}$$

$$\therefore \boxed{\therefore L\left(\frac{\cos 4t-\cos 3t}{t}\right)=\frac{1}{2}\times\log\left(\frac{s^2+9}{s^2+16}\right)}$$

→ Put, this value in equation (P)

$$\therefore \boxed{I=\frac{1}{2}\times\log\left(\frac{s^2+9}{s^2+16}\right)}$$

Step (05) – **Put value of** S, **from equation (N).**

$$\therefore I=\frac{1}{2}\times\log\left(\frac{0^2+9}{0^2+16}\right)$$

$$\therefore I=\frac{1}{2}\times\log\left(\frac{9}{16}\right)$$

$$\therefore I=\frac{1}{2}\times\log\left(\frac{3}{4}\right)^2$$

- **Use** : $\log a^b=b\times\log a$

$$\therefore I=\frac{1}{2}\times 2\times\log\left(\frac{3}{4}\right)$$

$$\therefore \boxed{I=\log\left(\frac{3}{4}\right)}\text{....(Ans)}$$

Q.01.(D) **Find Laplace Transform of $\left(1+2t-3t^2+4t^3\right)H(t-2)$** **04 Marks**

Solution : → To find L.T. of $\left(1+2t-3t^2+4t^3\right)H(t-2)$.

→ **Here, we use :** $\boxed{\mathrm{L}[f(t)\cdot \mathrm{H}(t-a)]=e^{-as}\cdot \mathrm{L}\left\{[f(t)]_{t\to(t+a)}\right\}}$

$$\therefore \mathrm{L}\left[\left(1+2t-3t^2+4t^3\right)H(t-2)\right]$$

$$=e^{-2s}\cdot \mathrm{L}\left\{\left[1+2t-3t^2+4t^3\right]_{t\to(t+2)}\right\}$$

$$=e^{-2s}\cdot \mathrm{L}\left\{1+2(t+2)-3(t+2)^2+4(t+2)^3\right\}$$

• **Use** : (1) $(a+b)^2=a^2+2ab+b^2$

(2) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$$=e^{-2s}\cdot \mathrm{L}\left\{1+2t+4-3\left(t^2+4t+4\right)+4\left(t^3+3\cdot t^2\cdot 2+3\cdot t\cdot 2^2+2^3\right)\right\}$$

$$=e^{-2s}\cdot \mathrm{L}\left\{1+2t+4-3t^2-12t-12+4\left(t^3+6t^2+12t+8\right)\right\}$$

$$=e^{-2s}\cdot \mathrm{L}\left\{1+2t+4-3t^2-12t-12+4t^3+24t^2+48t+32\right\}$$

$$=e^{-2s}\cdot \mathrm{L}\left\{4t^3-3t^2+24t^2+2t-12t+48t+1+4-12+32\right\}$$

$$=e^{-2s}\cdot \mathrm{L}\left\{4t^3+21t^2+38t+25\right\}$$

$$=e^{-2s}\cdot\left\{4\times \mathrm{L}\left(t^3\right)+21\times \mathrm{L}\left(t^2\right)+38\times \mathrm{L}(t)+25\times \mathrm{L}(1)\right\}$$

$$=e^{-2s}\cdot\left\{4\times\frac{3!}{s^{3+1}}+21\times\frac{2!}{s^{2+1}}+38\times\frac{1!}{s^{1+1}}+25\times\frac{1}{s}\right\} \;....\left[\because \mathrm{L}\left(t^n\right)=\frac{n!}{s^{n+1}}\right]$$

$$=e^{-2s}\cdot\left\{4\times\frac{6}{s^4}+21\times\frac{2}{s^3}+38\times\frac{1}{s^2}+\frac{25}{s}\right\}$$

$$\therefore \boxed{\mathrm{L}\left[\left(1+2t-3t^2+4t^3\right)H(t-2)\right]=e^{-2s}\cdot\left\{\frac{24}{s^4}+\frac{42}{s^3}+\frac{38}{s^2}+\frac{25}{s}\right\}} \;....\text{(Ans)}$$

Q.02.(A)	**Find the inverse Laplace transformation of the function.** $\log\left(1+\dfrac{a^2}{s^2}\right)$ **04 Marks**

Solution :

→ To find the inverse Laplace transform of the function : $\log\left(1+\dfrac{a^2}{s^2}\right)$

→ i.e. to find $L^{-1}\left[\log\left(1+\dfrac{a^2}{s^2}\right)\right]$

→ Let, $L^{-1}\left[\log\left(1+\dfrac{a^2}{s^2}\right)\right]=f(t)$ ....(P)

→ Use theorem : Derivative of $F(s)$

$$\therefore L^{-1}\left\{\frac{d}{ds}\left[\log\left(1+\frac{a^2}{s^2}\right)\right]\right\}=(-1)^1\cdot t^1\cdot f(t)$$

$$\therefore L^{-1}\left\{\frac{d}{ds}\left[\log\left(\frac{s^2+a^2}{s^2}\right)\right]\right\}=(-1)\cdot t\cdot f(t)$$

• **Use** : $\log\left(\dfrac{a}{b}\right)=\log a-\log b$

$$\therefore L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+a^2)-\log s^2\right]\right\}=-t\cdot f(t)$$

• **Use** : $\dfrac{d}{dx}(\log x)=\dfrac{1}{x}$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+a^2)}\times 2s-\frac{1}{s^2}\times 2s\right\}=-t\ f(t)$$

$$\therefore L^{-1}\left\{\frac{2s}{s^2+a^2}-\frac{2}{s}\right\}=-t\cdot f(t)$$

$$\therefore 2\times L^{-1}\left(\frac{s}{s^2+a^2}\right)-2\times L^{-1}\left(\frac{1}{s}\right)=-t\cdot f(t)$$

$$\therefore 2\times \cos at-2\times 1=-t\cdot f(t)$$

$$\therefore 2(\cos at-1)=-t\cdot f(t)$$

$$\therefore \frac{2(\cos at-1)}{-t}=f(t)$$

$$\therefore \boxed{\frac{2(1-\cos at)}{t}=f(t)}$$

→ Put this value in equation (P)

$$\therefore \boxed{L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right]=\frac{2(1-\cos at)}{t}} \text{....(Ans)}$$

Q.02.(B)	By using convolution theorem find $L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$	04 Marks

Solution : → Given that :

[1] Use Convolution Theorem.

[2] To find $L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]$.

Step (01) – (To split given function in multiplication Form and then take their inverse L.T. separately)

→ Here, we split given function as $\boxed{\frac{s}{(s^2+4)(s^2+9)} = \frac{s}{(s^2+4)} \times \frac{1}{(s^2+9)}}$

→ Let, $\boxed{F(s) = \frac{s}{(s^2+4)}}$

→ Take Inverse L.T.

$\therefore L^{-1}[F(s)] = L^{-1}\left[\frac{s}{(s^2+4)}\right]$

$\therefore f(t) = L^{-1}\left[\frac{s}{s^2+2^2}\right]$

$\therefore f(t) = \cos(2 \times t)$

$\therefore \boxed{f(t) = \cos 2t}$

→ Let, $\boxed{G(s) = \frac{1}{(s^2+9)}}$

→ Take Inverse L.T.

$\therefore L^{-1}[G(s)] = L^{-1}\left[\frac{1}{(s^2+9)}\right]$

$\therefore g(t) = L^{-1}\left[\frac{1}{s^2+3^2}\right]$

$\therefore g(t) = \frac{1}{3} \times \sin(3 \times t)$

$\therefore \boxed{g(t) = \frac{1}{3}\sin 3t}$

Step (02) – Use Convolution Theorem

$$\therefore \boxed{L^{-1}[F(s) \times G(s)] = \int_0^t [f(t)]_{t \to t-u} \times [g(t)]_{t \to u} \, du}$$

→ Put values of $F(s)$, $G(s)$, $f(t)$ & $g(t)$ from above.

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)}\times\frac{1}{(s^2+9)}\right]=\int_0^t [\cos 2t]_{t\to t-u}\times\left[\frac{1}{3}\sin 3t\right]_{t\to u} du$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]=\int_0^t \cos 2(t-u)\times\frac{1}{3}\sin 3u\,du$$

$$=\frac{1}{3}\int_0^t \cos(2t-2u)\sin 3u\,du$$

• Use : $2\cos A\cdot\sin B=\sin(A+B)-\sin(A-B)$

$$\therefore \cos A\cdot\sin B=\frac{\sin(A+B)-\sin(A-B)}{2}$$

→ Here, $A=(2t-2u)$, $B=3u$

$$=\frac{1}{3}\int_0^t\left[\frac{\sin(2t-2u+3u)-\sin(2t-2u-3u)}{2}\right]du$$

$$=\frac{1}{3}\times\frac{1}{2}\times\int_0^t[\sin(2t+u)-\sin(2t-5u)]du$$

• Use : $\int\sin px\,dx=\frac{-\cos px}{p}$

$$=\frac{1}{6}\times\left[\frac{-\cos(2t+u)}{1}-\frac{-\cos(2t-5u)}{-5}\right]_0^t$$

$$=\frac{1}{6}\times\left[-\cos(2t+u)-\frac{\cos(2t-5u)}{5}\right]_0^t$$

$$=\frac{1}{6}\times\left\{\left[-\cos(3t)-\frac{\cos(-3t)}{5}\right]-\left[-\cos(2t)-\frac{\cos(2t)}{5}\right]\right\}$$

• Use : $\cos(-\theta)=\cos\theta$

$$=\frac{1}{6}\times\left\{\left[-\cos 3t-\frac{\cos 3t}{5}\right]-\left[-\cos 2t-\frac{\cos 2t}{5}\right]\right\}$$

$$=\frac{1}{6}\times\left\{\left[\frac{-5\cos 3t-\cos 3t}{5}\right]-\left[\frac{-5\cos 2t-\cos 2t}{5}\right]\right\}$$

$$=\frac{1}{6}\times\left\{\left[\frac{-6\cos 3t}{5}\right]-\left[\frac{-6\cos 2t}{5}\right]\right\}$$

$$=\frac{1}{6}\times\left\{\frac{-6\cos 3t}{5}+\frac{6\cos 2t}{5}\right\}$$

$$=\frac{1}{6}\times\frac{6}{5}\times\{-\cos 3t+\cos 2t\}$$

$$=\frac{1}{5}\times(\cos 2t-\cos 3t)$$

$$\therefore \boxed{L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]=\frac{1}{5}\times(\cos 2t-\cos 3t)}$$ ....(Ans)

Q.02.(C) Find the inverse Laplace transformation of the function.

$$\frac{5s^2-15s-11}{(s+1)(s-2)^2}$$

04 Marks

Solution : → To find $L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right\}$, Using partial fraction method.

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^2}=\frac{A}{(s+1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2} \quad \text{.... (P)}$$

$$\therefore 5s^2-15s-11=\left[\frac{A}{(s+1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}\right]\times(s+1)(s-2)^2$$

$$\therefore 5s^2-15s-11=A(s-2)^2+B(s+1)(s-2)+C(s+1) \quad \text{.... (Q)}$$

- For A, B & C : Simplify equation (Q).

$$\therefore 5s^2-15s-11=A\left(s^2-4s+4\right)+B\left(s^2+s-2s-2\right)+Cs+C$$

$$\therefore 5s^2-15s-11=As^2-4As+4A+B\left(s^2-s-2\right)+Cs+C$$

$$\therefore 5s^2-15s-11=As^2-4As+4A+Bs^2-Bs-2B+Cs+C$$

$$\therefore 5s^2-15s-11=As^2+Bs^2-4As-Bs+Cs+4A-2B+C$$

$$\therefore 5s^2-15s-11=(A+B)s^2+(-4A-B+C)s+(4A-2B+C)$$

→ Compare both side, we get :

(1) $A+B=5 \rightarrow \boxed{1\cdot A+1\cdot B+0\cdot C=5}$

(2) $-4A-B+C=-15 \rightarrow \boxed{-4\cdot A-1\cdot B+1\cdot C=-15}$

(3) $4A-2B+C=-11 \rightarrow \boxed{4\cdot A-2\cdot B+1\cdot C=-11}$

→ Solve above three equations by calculator, we get :

$$\boxed{A = 1,\ \ B = 4,\ \ C = -7}$$

→ Put Values of A, B & C in equation (P).

$$\therefore \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2} = \frac{1}{(s+1)} + \frac{4}{(s-2)} + \frac{(-7)}{(s-2)^2}$$

→ Apply L^{-1} on both side.

$$\therefore L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}\right] = L^{-1}\left\{\frac{1}{(s+1)} + \frac{4}{(s-2)} - \frac{7}{(s-2)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)}\right\} + 4 \times L^{-1}\left\{\frac{1}{(s-2)}\right\} - 7 \times L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-t} + 4 \times e^{2t} - 7 \times e^{2t} \times L^{-1}\left\{\frac{1}{s^2}\right\} \quad \text{....}\left[\because \text{First Shifting Theorem}\right]$$

$$= e^{-t} + 4e^{2t} - 7e^{2t} \times L^{-1}\left\{\frac{1}{s^{1+1}}\right\} \quad \text{....}\left[\because L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{n!} \times t^n\right]$$

$$= e^{-t} + 4e^{2t} - 7e^{2t} \times \frac{1}{1!} \times t^1$$

$$= e^{-t} + 4e^{2t} - 7e^{2t} \times \frac{1}{1} \times t$$

$$\therefore \boxed{L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}\right] = e^{-t} + 4e^{2t} - 7te^{2t}} \quad \text{....(Ans)}$$

Q.02.(D)	**Solve using Laplace transformation** $y''+3y'+2y=t\delta(t-1)$ **for which** $y'(0)=y(0)=0$ **04 Marks**

Solution :

→ Given that :

[1] Given equtaion : $\boxed{y''+3y'+2y=t\,\delta(t-1)}$

[2] Conditions are given : $y'(0)=y(0)=0$

[3] To solve given equation.

→ To solve given equation means to find value of $y(t)$.

$\boxed{\text{Step (01)}}$ – **Apply L.T. to given equation**

$$\therefore L[y''+3y'+2y]=L[t\,\delta(t-1)]$$

• **Use** : (1) $y''=y''(t)=\dfrac{d^2y}{dt^2}$ (2) $y'=y'(t)=\dfrac{dy}{dt}$ (3) $y=y(t)$

$$\therefore L\left[\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y(t)\right]=e^{-1\cdot s}[t]_{t=1} \quad ....\left\{\because L[f(t)\cdot\delta(t-a)]=e^{-a\cdot s}[f(t)]_{t=a}\right\}$$

$$\therefore L\left[\frac{d^2y}{dt^2}\right]+3\times L\left[\frac{dy}{dt}\right]+2\times L[y(t)]=e^{-s}[1]$$

• **Use** : (1) $L\left[\dfrac{d^2y}{dt^2}\right]=s^2Y(s)-s^1y(0)-s^0y'(0)$

(2) $L\left[\dfrac{dy}{dt}\right]=s^1Y(s)-s^0y(0)$

..... [Theorem - L.T. of Derivatives]

(3) $L[y(t)]=Y(s)$

$$\therefore s^2Y(s)-s^1y(0)-s^0y'(0)+3\times\left[s^1Y(s)-s^0y(0)\right]+2\times Y(s)=e^{-s}$$

$\rightarrow$ But : $y(0)=y'(0)=0$

$$\therefore s^2Y(s)-s\times 0-1\times 0+3\times[sY(s)-1\times 0]+2Y(s)=e^{-s}$$

$$\therefore s^2Y(s)-0-0+3\times[sY(s)-0]+2Y(s)=e^{-s}$$

$$\therefore s^2Y(s)+3sY(s)+2Y(s)=e^{-s}$$

$$\therefore (s^2+3s+2)Y(s)=e^{-s}$$

$$\therefore \boxed{Y(s)=\frac{e^{-s}}{(s^2+3s+2)}}$$

$\boxed{\text{Step (02)}}$ – **Take inverse L.T. of above equation**

$$\therefore L^{-1}[Y(s)]=L^{-1}\left[\frac{e^{-s}}{(s^2+3s+2)}\right]$$

$\rightarrow$ Use Second Shifting Theorem

$$\therefore \boxed{y(t)=\begin{cases}[f(t)]_{t\rightarrow t-a}, & t>a\\ 0, & t<a\end{cases}} \quad ....(P)$$

$\rightarrow$ Here, (1) $a=1$ (2) $F(s)=\dfrac{1}{s^2+3s+2}$ (3) $f(t)=?$

$$\therefore f(t)=L^{-1}[F(s)]$$

$$\therefore f(t)=L^{-1}\left[\frac{1}{s^2+3s+2}\right]$$

$$\rightarrow \text{L.T.}=\frac{(\text{M.T.})^2}{4\times\text{F.T.}}=\frac{(3s)^2}{4\times s^2}=\frac{9s^2}{4s^2}=\frac{9}{4}$$

$\rightarrow$ Add and Subtract L.T. after the M.T.

$$\therefore f(t) = L^{-1}\left[\frac{1}{s^2+3s+\frac{9}{4}-\frac{9}{4}+2}\right]$$

$$\therefore f(t) = L^{-1}\left[\frac{1}{\left(s+\frac{3}{2}\right)^2-\left(\frac{9}{4}-2\right)}\right]$$

$$\therefore f(t) = L^{-1}\left[\frac{1}{\left(s+\frac{3}{2}\right)^2-\frac{1}{4}}\right]$$

$$\therefore f(t) = e^{-\frac{3}{2}t} \times L^{-1}\left[\frac{1}{s^2-\frac{1}{4}}\right] \quad \text{.... [} \because \text{ First Shifting Theorem]}$$

$$\therefore f(t) = e^{-\frac{3t}{2}} \times L^{-1}\left[\frac{1}{s^2-\left(\frac{1}{2}\right)^2}\right]$$

$$\therefore f(t) = e^{-\frac{3t}{2}} \times \frac{1}{\frac{1}{2}} \times \sinh\left(\frac{1}{2}\times t\right) \quad \text{....}\left[\because L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at\right]$$

$$\therefore f(t) = e^{-\frac{3t}{2}} \times 2 \times \sinh\left(\frac{t}{2}\right)$$

$$\therefore \boxed{f(t) = 2e^{-\frac{3t}{2}}\sinh\left(\frac{t}{2}\right)}$$

→ Put value of a, $F(s)$ & $f(t)$ in equation (P).

$$\therefore y(t) = \begin{cases} \left[2e^{-\frac{3t}{2}}\sinh\left(\frac{t}{2}\right)\right]_{t\to t-1}, & t>1 \\ 0, & t<1 \end{cases}$$

$$\therefore \boxed{y(t)=\begin{cases} 2e^{-\frac{3(t-1)}{2}}\sinh\left(\frac{t-1}{2}\right), & t>1 \\ 0, & t<1 \end{cases}} \text{....(Ans)}$$

Q.03.(A) Using Parseval's identity prove that : $\int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{4}$ **04 Marks**

Solution : → Given that :

[1] Use Parseval's Identity.

[2] To prove that : $\int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{4}$

Step (01) – (Assume suitable $f(x)$ & $g(x)$, then take their appropriate Fourier transforms)

→ Let, $f(x) = e^{-ax}$

→ Here, $a = 1$

$\therefore \boxed{f(x) = e^{-x}}$

→ We know that, Fourier Sine Transform of above function is

$\therefore \boxed{f(s) = \frac{s}{s^2+a^2} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}}$

→ Let, $g(x) = e^{-ax}$

→ Here, $a = 1$

$\therefore \boxed{g(x) = e^{-x}}$

→ We know that, Fourier Sine Transform of above function is

$\therefore \boxed{g(s) = \frac{s}{s^2+a^2} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}}$

Step (02) – Use Parseval's Identity

$$\therefore \boxed{\int_0^\infty f(s)\cdot g(s)\,ds = \frac{\pi}{2}\times\int_0^\infty f(x)\cdot g(x)\,dx}$$

→ Put values of $f(s)$, $g(s)$, $f(x)$ & $g(x)$ from above.

$$\therefore \int_0^\infty \frac{s}{s^2+1}\cdot\frac{s}{s^2+1}\,ds = \frac{\pi}{2}\times\int_0^\infty e^{-x}\cdot e^{-x}\,dx$$

• **Use** : $a^m \ a^n = a^{m+n}$

$$\therefore \int_0^\infty \frac{s^2}{(s^2+1)^2}\,ds = \frac{\pi}{2}\times\int_0^\infty e^{-x+(-x)}\,dx$$

$$= \frac{\pi}{2}\times\int_0^\infty e^{-x-x}\,dx$$

$$= \frac{\pi}{2}\times\int_0^\infty e^{-2x}\,dx$$

• **Use** : $\int e^{px}dx = \frac{e^{px}}{p}$

$$= \frac{\pi}{2}\times\left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$= \frac{\pi}{2}\times\left\{\left[\frac{e^{-2\times\infty}}{-2}\right]-\left[\frac{e^{-2\times 0}}{-2}\right]\right\}$$

• **Use** : $2\times\infty = \infty$

$$= \frac{\pi}{2}\times\left\{\left[\frac{e^{-\infty}}{-2}\right]-\left[\frac{e^{0}}{-2}\right]\right\}$$

• **Use** : (1) $e^{-\infty} = 0$

(2) $e^0 = 1$

$$= \frac{\pi}{2}\times\left\{\left[\frac{0}{-2}\right]-\left[\frac{1}{-2}\right]\right\}$$

$$= \frac{\pi}{2}\times\left\{0+\frac{1}{2}\right\}$$

$$= \frac{\pi}{2}\times\frac{1}{2}$$

$$\therefore \boxed{\int_0^\infty \frac{s^2}{(s^2+1)^2}\,ds = \frac{\pi}{4}}$$

→ **Here, we can replace s by x. ....[$\because$ Property of Definite Integration]**

$$\therefore \boxed{\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{4}} \text{....(Ans)}$$

OR → (In above solution, we use formula notation as $f(s)$ & $f(x)$. You can also use formula notation as $f(\lambda)$ & $f(x)$ as below.)

Solution : → Given that :

[1] Use Parseval's Identity.

[2] To prove that : $\int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{4}$

Step (01) – (Assume suitable $f(x)$ & $g(x)$, then take their appropriate Fourier transforms)

→ Let, $f(x) = e^{-ax}$

→ Here, $a = 1$

$\therefore \boxed{f(x) = e^{-x}}$

→ We know that, Fourier Sine Transform of above function is

$\therefore \boxed{f(\lambda) = \frac{\lambda}{\lambda^2 + a^2} = \frac{\lambda}{\lambda^2 + 1^2} = \frac{\lambda}{\lambda^2 + 1}}$

→ Let, $g(x) = e^{-ax}$

→ Here, $a = 1$

$\therefore \boxed{g(x) = e^{-x}}$

→ We know that, Fourier Sine Transform of above function is

$\therefore \boxed{g(\lambda) = \frac{\lambda}{\lambda^2 + a^2} = \frac{\lambda}{\lambda^2 + 1^2} = \frac{\lambda}{\lambda^2 + 1}}$

Step (02) – Use Parseval's Identity

$$\therefore \boxed{\int_0^\infty f(\lambda)\cdot g(\lambda)\, d\lambda = \frac{\pi}{2} \times \int_0^\infty f(x)\cdot g(x)\, dx}$$

→ Put values of $f(\lambda)$, $g(\lambda)$, $f(x)$ & $g(x)$ from above.

$$\therefore \int_0^\infty \frac{\lambda}{\lambda^2+1}\cdot\frac{\lambda}{\lambda^2+1}\, d\lambda = \frac{\pi}{2} \times \int_0^\infty e^{-x}\cdot e^{-x}\, dx$$

• **Use** : $a^m \cdot a^n = a^{m+n}$

$$\therefore \int_0^\infty \frac{\lambda^2}{(\lambda^2+1)^2}\, d\lambda = \frac{\pi}{2} \times \int_0^\infty e^{-x+(-x)}\, dx$$

$$= \frac{\pi}{2} \times \int_0^\infty e^{-x-x}\, dx$$

$$= \frac{\pi}{2} \times \int_0^\infty e^{-2x}\, dx$$

• **Use** : $\int e^{px} dx = \frac{e^{px}}{p}$

$$= \frac{\pi}{2} \times \left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$= \frac{\pi}{2} \times \left\{\left[\frac{e^{-2\times\infty}}{-2}\right] - \left[\frac{e^{-2\times 0}}{-2}\right]\right\}$$

• **Use** : $2\times\infty = \infty$

$$= \frac{\pi}{2} \times \left\{\left[\frac{e^{-\infty}}{-2}\right] - \left[\frac{e^{0}}{-2}\right]\right\}$$

• **Use** : (1) $e^{-\infty} = 0$

(2) $e^0 = 1$

$$= \frac{\pi}{2} \times \left\{\left[\frac{0}{-2}\right] - \left[\frac{1}{-2}\right]\right\}$$

$$= \frac{\pi}{2} \times \left\{0 + \frac{1}{2}\right\}$$

$$= \frac{\pi}{2} \times \frac{1}{2}$$

$$\therefore \boxed{\int_0^{\infty} \frac{\lambda^2}{(\lambda^2+1)^2}\, d\lambda = \frac{\pi}{4}}$$

→ Here, we can replace λ by x. ....[$\because$ Property of Definite Integration]

$$\therefore \boxed{\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\, dx = \frac{\pi}{4}}$$ ....(Ans)

Q.03.(B)	**Find the Fourier transform of** $f(x)=\begin{cases}1-x^2, & \lvert x\rvert\le 1\\ 0, & \lvert x\rvert>1\end{cases}$	**04 Marks**

Solution :

→ Given that :

[1] Function : $f(x)=\begin{cases}1-x^2, & \boxed{\lvert x\rvert\le 1} \rightarrow 0 \text{ to } 1\\ 0, & \boxed{\lvert x\rvert>1} \rightarrow 1 \text{ to } \infty\end{cases}$

[2] To find fourier transform.

→ Here, in problem, it is not given that which transform we can use.

→ So, we first check given function.

→ So, replace x by $-x$.

$$\therefore f(-x)=\begin{cases}1-(-x)^2, & \lvert x\rvert\le 1\\ 0, & \lvert x\rvert>1\end{cases}$$

$$\therefore f(-x)=\begin{cases}1-x^2, & \lvert x\rvert\le 1\\ 0, & \lvert x\rvert>1\end{cases}$$

$$\therefore \boxed{f(-x)=f(x)}$$

→ This shows that : function is even.

$\boxed{\textbf{Part (01)}}$ – **Fourier transform of even function is**

$$\therefore f(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot\cos(sx)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left\{\int_0^1 (1-x^2)\cdot\cos(sx)\,dx+\int_1^\infty 0\cdot\cos(sx)\,dx\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[+(1-x^2)\left(\frac{\sin(sx)}{s}\right)-(-2x)\left(\frac{-\cos(sx)}{s^2}\right)+(-2)\left(\frac{-\sin(sx)}{s^3}\right)\right]_0^1+0\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[0-\frac{2\cos s}{s^2}+\frac{2\sin s}{s^3}\right]-[0-0+0]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{2\cos s}{s^2}+\frac{2\sin s}{s^3}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-2s\cos s+2\sin s}{s^3}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2(-s\cos s+\sin s)}{s^3}\right]$$

$$\therefore \boxed{f(s)=\sqrt{\frac{2}{\pi}}\times\left[\frac{2(\sin s-s\cos s)}{s^3}\right]}$$

→ This is our required Fourier transform of given function.

OR $\rightarrow$ (In above solution, we use formula notation as $f(s)$ & $f(x)$. You can also use formula notation as $f(\lambda)$ & $f(x)$ as below.)

Solution : $\rightarrow$ Given that :

[1] Function : $f(x)=\begin{cases}1-x^2, & \boxed{|x|\le 1} \rightarrow 0 \text{ to } 1\\ 0, & \boxed{|x|>1} \rightarrow 1 \text{ to } \infty\end{cases}$

[2] To find fourier transform.

$\rightarrow$ Here, in problem, it is not given that which transform we can use.

$\rightarrow$ So, we first check given function.

$\rightarrow$ So, replace x by $-x$.

$$\therefore f(-x)=\begin{cases}1-(-x)^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

$$\therefore f(-x)=\begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

$$\therefore \boxed{f(-x)=f(x)}$$

$\rightarrow$ This shows that : function is even.

Part(01) – **Fourier transform of even function is**

$$\therefore f(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\lambda x)dx$$

$$=\sqrt{\frac{2}{\pi}}\left\{\int_0^1(1-x^2)\cdot\cos(\lambda x)dx+\int_1^\infty 0\cdot\cos(\lambda x)dx\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[+(1-x^2)\left(\frac{\sin(\lambda x)}{\lambda}\right)-(-2x)\left(\frac{-\cos(\lambda x)}{\lambda^2}\right)+(-2)\left(\frac{-\sin(\lambda x)}{\lambda^3}\right)\right]_0^1+0\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[0-\frac{2\cos\lambda}{\lambda^2}+\frac{2\sin\lambda}{\lambda^3}\right]-[0-0+0]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{2\cos\lambda}{\lambda^2}+\frac{2\sin\lambda}{\lambda^3}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-2\lambda\cos\lambda+2\sin\lambda}{\lambda^3}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2(-\lambda\cos\lambda+\sin\lambda)}{\lambda^3}\right]$$

$$\therefore \boxed{f(\lambda)=\sqrt{\frac{2}{\pi}}\times\left[\frac{2(\sin\lambda-\lambda\cos\lambda)}{\lambda^3}\right]}$$

→ This is our required Fourier transform of given function.

Q.03.(C)	Find the Fourier Sine transform $e^{-ax}, a>0$	04 Marks

Solution : → Given that :

[1] Function : e^{-ax} → Let, $\boxed{f(x)=e^{-ax}}$

[2] To find fourier sine transform.

$\boxed{\text{Part (01)}}$ **– Fourier sine transform is**

$$\therefore f(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\sin(sx)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cdot\sin(sx)dx$$

• **Use** : $\int e^{ax}\sin(bx)dx=\frac{e^{ax}}{a^2+b^2}[a\sin(bx)-b\cos(bx)]$

→ Here, $a=-a,\quad b=s$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{(-a)^2+s^2}[-a\sin(sx)-s\cos(sx)]\right]_0^{\infty}$$

• **Use** : $e^{-\infty}=0$

$$=\sqrt{\frac{2}{\pi}}\left\{[0]-\left[\frac{1}{a^2+s^2}[-a\times 0-s\times 1]\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{-\left[\frac{1}{a^2+s^2}[0-s]\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{-\left[\frac{-s}{a^2+s^2}\right]\right\}$$

$$\therefore \boxed{f(s) = \sqrt{\frac{2}{\pi}} \times \frac{s}{a^2 + s^2}} \text{....(Ans)}$$

→ This is required Fourier Sine Transform.

OR → (In above solution, we use formula notation as $f(s)$ & $f(x)$. You can also use formula notation as $f(\lambda)$ & $f(x)$ as below.)

Solution : → Given that :

[1] Function : e^{-ax} → Let, $\boxed{f(x)=e^{-ax}}$

[2] To find fourier sine transform.

Part (01) – **Fourier sine transform is**

$$\therefore f(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot\sin(\lambda x)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cdot\sin(\lambda x)dx$$

• Use : $\int e^{ax}\sin(bx)dx=\frac{e^{ax}}{a^2+b^2}[a\sin(bx)-b\cos(bx)]$

→ Here, $a=-a$, $b=\lambda$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{(-a)^2+\lambda^2}[-a\sin(\lambda x)-s\cos(\lambda x)]\right]_0^\infty$$

• Use : $e^{-\infty}=0$

$$=\sqrt{\frac{2}{\pi}}\left\{[0]-\left[\frac{1}{a^2+\lambda^2}[-a\times0-\lambda\times1]\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{-\left[\frac{1}{a^2+\lambda^2}[0-\lambda]\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{-\left[\frac{-\lambda}{a^2+\lambda^2}\right]\right\}$$

$$\therefore \boxed{f(\lambda)=\sqrt{\frac{2}{\pi}}\times\frac{\lambda}{a^2+\lambda^2}} \quad ....\textbf{(Ans)}$$

→ This is required Fourier Sine Transform.

Q.03.(D) Find the Fourier cosine transform of the function

$$f(y)=\begin{cases}\cos y, & 0<y<a\\ 0, & y>a\end{cases}$$

04 Marks

Solution : → Given that :

[1] Function : $\boxed{f(y)=\begin{cases}\cos y, & 0<y<a\\ 0, & y>a\end{cases}} \leftrightarrow \boxed{f(x)=\begin{cases}\cos x, & 0<x<a\\ 0, & x>a\end{cases}}$

[2] To find fourier cosine transform.

Part (01) – **Fourier cosine transform is**

$$\therefore f(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\cos(sx)dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a} f(x)\cdot\cos(sx)dx+\int_a^{\infty} f(x)\cdot\cos(sx)dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a}\cos x\cdot\cos(sx)dx+\int_a^{\infty} 0\cdot\cos(sx)dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a}\cos x\cdot\cos(sx)dx+0\right]$$

• Use : $2\cos A\cdot\cos B=\cos(A+B)+\cos(A-B)$

$$\therefore \cos A\ \cos B=\frac{\cos(A+B)+\cos(A-B)}{2}$$

→ Here, $A=x,\quad B=sx$

$$=\sqrt{\frac{2}{\pi}}\times\int_0^{a}\frac{\cos(x+sx)+\cos(x-sx)}{2}\,dx$$

$$=\sqrt{\frac{2}{\pi}}\times\frac{1}{2}\times\int_0^{a}\left[\cos(1+s)x+\cos(1-s)x\right]dx$$

• **Use** : $\int \cos px\, dx = \dfrac{\sin px}{p}$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left[\frac{\sin(1+s)x}{(1+s)} + \frac{\sin(1-s)x}{(1-s)} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left\{ \left[\frac{\sin(1+s)a}{(1+s)} + \frac{\sin(1-s)a}{(1-s)} \right] - \left[\frac{\sin 0}{(1+s)} + \frac{\sin 0}{(1-s)} \right] \right\}$$

• **Use** : $\sin 0 = 0$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left\{ \left[\frac{\sin(1+s)a}{(1+s)} + \frac{\sin(1-s)a}{(1-s)} \right] - \left[\frac{0}{(1+s)} + \frac{0}{(1-s)} \right] \right\}$$

$$\therefore \boxed{f(s) = \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left\{ \left[\frac{\sin(1+s)a}{(1+s)} + \frac{\sin(1-s)a}{(1-s)} \right] \right\}} \text{....(Ans)}$$

→ This is our required Fourier Cosine Transform.

OR $\rightarrow$ (In above solution, we use formula notation as $f(s)$ & $f(x)$. You can also use formula notation as $f(\lambda)$ & $f(x)$ as below.)

Solution : $\rightarrow$ Given that :

[1] Function : $\boxed{f(y)=\begin{cases}\cos y, & 0<y<a\\ 0, & y>a\end{cases}} \leftrightarrow \boxed{f(x)=\begin{cases}\cos x, & 0<x<a\\ 0, & x>a\end{cases}}$

[2] To find fourier cosine transform.

Part (01) – **Fourier cosine transform is**

$$\therefore f(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\cos(\lambda x)dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a} f(x)\cdot\cos(\lambda x)dx+\int_a^{\infty} f(x)\cdot\cos(\lambda x)dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a} \cos x\cdot\cos(\lambda x)dx+\int_a^{\infty} 0\cdot\cos(\lambda x)dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a} \cos x\cdot\cos(\lambda x)dx+0\right]$$

- **Use** : $2\cos A\cdot\cos B=\cos(A+B)+\cos(A-B)$

$$\therefore \cos A\cdot\cos B=\frac{\cos(A+B)+\cos(A-B)}{2}$$

$\rightarrow$ Here, $A=x$, $B=sx$

$$=\sqrt{\frac{2}{\pi}}\times\int_0^{a}\frac{\cos(x+\lambda x)+\cos(x-\lambda x)}{2}\,dx$$

$$=\sqrt{\frac{2}{\pi}}\times\frac{1}{2}\times\int_0^{a}\left[\cos(1+\lambda)x+\cos(1-\lambda)x\right]dx$$

• Use : $\int \cos px \, dx = \frac{\sin px}{p}$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left[\frac{\sin(1+\lambda)x}{(1+\lambda)} + \frac{\sin(1-\lambda)x}{(1-\lambda)} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left\{ \left[\frac{\sin(1+\lambda)a}{(1+\lambda)} + \frac{\sin(1-\lambda)a}{(1-\lambda)} \right] - \left[\frac{\sin 0}{(1+\lambda)} + \frac{\sin 0}{(1-\lambda)} \right] \right\}$$

• Use : $\sin 0 = 0$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left\{ \left[\frac{\sin(1+\lambda)a}{(1+\lambda)} + \frac{\sin(1-\lambda)a}{(1-\lambda)} \right] - \left[\frac{0}{(1+\lambda)} + \frac{0}{(1-\lambda)} \right] \right\}$$

$$\therefore \boxed{f(\lambda) = \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \times \left\{ \left[\frac{\sin(1+\lambda)a}{(1+\lambda)} + \frac{\sin(1-\lambda)a}{(1-\lambda)} \right] \right\}} \quad \text{....(Ans)}$$

→ This is our required Fourier Cosine Transform.

Q.04.(A) **Form the partial differential equation by eliminating arbitrary constants from $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$** **04 Marks**

Solution : → Given that :

[1] Equation : $\boxed{(x-a)^2+(y-b)^2=z^2\cot^2\alpha}$ ....(01)

[2] To eliminate arbitrary constants from above equation and form P.D.E.

→ Here, in given equation :

[1] z is dependent variable.

[2] x, y are independent variable.

[3] a, b are arbitrary constants.

[4] α is fix constant.

→ We have to remove arbitrary constants a and b.

- <u>So, take P.D. of equation (01) w.r.t. x:</u>

$$\therefore \frac{\partial}{\partial x}\left[(x-a)^2+(y-b)^2\right]=\frac{\partial}{\partial x}\left(z^2\cot^2\alpha\right)$$

$$\therefore 2(x-a)^{2-1}\times(1-0)+0=\cot^2\alpha\times 2z\times\frac{\partial z}{\partial x}$$

$$\therefore 2(x-a)=2z\cdot\cot^2\alpha\cdot\frac{\partial z}{\partial x}$$

$$\therefore \boxed{(x-a)=z\cdot\cot^2\alpha\cdot\frac{\partial z}{\partial x}} \quad ....(P)$$

- <u>Also, take P.D. of equation (01) w.r.t. y:</u>

$$\therefore \frac{\partial}{\partial y}\left[(x-a)^2+(y-b)^2\right]=\frac{\partial}{\partial y}\left(z^2\cot^2\alpha\right)$$

$$\therefore 0+2(y-b)^{2-1}\times(1-0)=\cot^2\alpha\times 2z\times\frac{\partial z}{\partial y}$$

$$\therefore 2(y-b) = 2z \cdot \cot^2\alpha \cdot \frac{\partial z}{\partial y}$$

$$\therefore \boxed{(y-b) = z \cdot \cot^2\alpha \cdot \frac{\partial z}{\partial y}} \quad ....(Q)$$

→ Put value of $(x-a)$ and $(y-b)$ in equation (01).

$$\therefore \left[z \cdot \cot^2\alpha \cdot \frac{\partial z}{\partial x}\right]^2 + \left[z \cdot \cot^2\alpha \cdot \frac{\partial z}{\partial y}\right]^2 = z^2 \cot^2\alpha \quad .... [\because \text{Equation (P) \& (Q)}]$$

$$\therefore z^2 \cdot \cot^4\alpha \cdot \left(\frac{\partial z}{\partial x}\right)^2 + z^2 \cdot \cot^4\alpha \cdot \left(\frac{\partial z}{\partial y}\right)^2 = z^2 \cot^2\alpha$$

$$\therefore z^2 \cdot \cot^4\alpha \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = z^2 \cot^2\alpha$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{z^2 \cot^2\alpha}{z^2 \cdot \cot^4\alpha}$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\cancel{z^2}\ \cancel{\cot^2}\alpha}{\cancel{z^2}\ \cot^{\cancel{4}}\alpha}$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{\cot^2\alpha}$$

$$\therefore \boxed{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \tan^2\alpha}$$

→ We know: $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$

$$\therefore \boxed{p^2 + q^2 = \tan^2\alpha} \text{ --(Ans)}$$

→ This is our required P.D.E.

Q.04.(B)	Solve the Partial differential equation $x(y-z)p+y(z-x)q=z(x-y)$	04 Marks

Solution :

→ Given that :

[1] Partial Differential Equation : $\boxed{x(y-z)p+y(z-x)q=z(x-y)}$

[2] To solve this P.D.E. $\left[\text{i.e. to find } \phi(a,b)=0\right]$

→ Since, given P.D.E. Is Lagrange's Linear Equation.

$\boxed{\text{Step (01)}}$ – **To find P, Q, R**

→ We know, standard form of Lagrange's Linear Equation is $\boxed{Pp+Qq=R}$

→ Compare given P.D.E. with this standard form, we get :

(1) $P=x(y-z)$

(2) $Q=y(z-x)$

(3) $R=z(x-y)$

$\boxed{\text{Step (02)}}$ – **Write the Auxiliary Equation**

→ Auxiliary Equation is $\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$

→ Put values of P,Q,R.

$$\therefore \boxed{\underbrace{\frac{dx}{x(y-z)}}_{\text{First Ratio}}=\underbrace{\frac{dy}{y(z-x)}}_{\text{Second Ratio}}=\underbrace{\frac{dz}{z(x-y)}}_{\text{Third Ratio}}}$$

$\boxed{\text{Step (03)}}$ – **First Solution**

→ For first solution, operation as below :

→ Using the set of multipliers 1,1,1 :

$$\therefore \text{Each Ratio} = \frac{1\cdot dx + 1\cdot dy + 1\cdot dz}{x(y-z)+y(z-x)+z(x-y)}$$

$$\therefore \text{Each Ratio} = \frac{dx+dy+dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$$

$$\therefore \text{Each Ratio} = \frac{dx+dy+dz}{0} \rightarrow \boxed{dx+dy+dz=0}$$

→ Take integration on both side

$$\therefore \boxed{x+y+z=a}$$

$\boxed{\text{Step (04)}}$ – **Second Solution**

→ For second solution, operation as below :

→ Using the set of multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$:

$$\therefore \text{Each Ratio} = \frac{\frac{1}{x}\cdot dx + \frac{1}{y}\cdot dy + \frac{1}{z}\cdot dz}{\frac{1}{x}\times x(y-z)+\frac{1}{y}\times y(z-x)+\frac{1}{z}\times z(x-y)}$$

$$\therefore \text{Each Ratio} = \frac{\frac{1}{x}\cdot dx + \frac{1}{y}\cdot dy + \frac{1}{z}\cdot dz}{(y-z)+(z-x)+(x-y)}$$

$$\therefore \text{Each Ratio} = \frac{\frac{1}{x}\cdot dx + \frac{1}{y}\cdot dy + \frac{1}{z}\cdot dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$\therefore$ Each Ratio $= \dfrac{\frac{1}{x}\cdot dx + \frac{1}{y}\cdot dy + \frac{1}{z}\cdot dz}{0} \rightarrow \boxed{\frac{1}{x}\cdot dx + \frac{1}{y}\cdot dy + \frac{1}{z}\cdot dz = 0}$

→ Take integration on both side

$\therefore \log x + \log y + \log z = c$

$\therefore \log(x\cdot y\cdot z) = c$

$\therefore x\cdot y\cdot z = e^c$

→ Let, $e^c = b$

$\therefore \boxed{xyz = b}$

Step (05) – **General Solution**

→ The required general solution is $\boxed{\phi(a,b)=0}$

→ Put value of a and b.

$\therefore \boxed{\phi(x+y+z,\ xyz)=0}$ ....(Ans)

Q.04.(C)	Use the method of separation of variables to solve $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$, given that $u(x,0)=6e^{-3x}$	04 Marks

Solution : → Given that :

[1] P.D.E. : $\boxed{\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u}$

[2] Condition : $u(x,0)=6e^{-3x}$

[3] To solve give P.D.E. $\left[\text{i.e. to find } u(x,t)\right]$

[4] Use method of separation of variables

Step (01) – **Assume Solution of given P.D.E**

→ Let, solution of given PD.E. is $\boxed{u=\mathrm{X}\cdot\mathrm{T}}$ ....(P)

→ Where,

(1) X is a function of 'x' only

(2) T is a function of 't' only

Step (02) – **To find Partial Derivatives present in Given P.D.E.**

→ i.e. to find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

[1] <u>To find $\frac{\partial u}{\partial x}$</u> :

→ We have, $u=\mathrm{X}\cdot\mathrm{T}$

→ Take P.D. w.r.t x

$$\therefore \frac{\partial}{\partial x}(u)=\frac{\partial}{\partial x}(\mathrm{X}\cdot\mathrm{T})$$

$$\therefore \boxed{\frac{\partial u}{\partial x}=\mathrm{T}\cdot\mathrm{X}'} \quad ....(1)$$

[2] To find $\frac{\partial u}{\partial t}$:

→ We have, $u = X \cdot T$

→ Take P.D. w.r.t t

$$\therefore \frac{\partial}{\partial t}(u) = \frac{\partial}{\partial t}(X \cdot T)$$

$$\therefore \boxed{\frac{\partial u}{\partial t} = X \cdot T'} \quad ....(2)$$

Step (03) – (**Put these values of Partial Derivatives & value of u in give P.D.E.. Also, to separate the variables.**)

$$\therefore X' \cdot T = 2X \cdot T' + X \cdot T \quad .... [\because \text{From equation } (1),(2) \,\&\, (P)]$$

→ Now, to separate the variables.

$$\therefore X' \cdot T - X \cdot T = 2X \cdot T'$$

$$\therefore (X' - X)T = 2X \cdot T'$$

$$\therefore \boxed{\frac{(X' - X)}{X} = \frac{2 \cdot T'}{T}}$$

→ Here, varibles X and T are separated.

Step (04) – (**Take any constant k in thier equality. And from this find out two relations.**)

→ Let, $\boxed{\frac{(X' - X)}{X} = \frac{2 \cdot T'}{T} = k}$

→ From this relation, we get two relations as :

(1) $\boxed{\frac{(X' - X)}{X} = k}$ and (2) $\boxed{\frac{2 \cdot T'}{T} = k}$

$\boxed{\text{Step (05)}}$ - Solve above two relations, we get, X & T.

[1] $\dfrac{(X' - X)}{X} = k$

→ Take integration on both side

$\therefore \int \dfrac{(X' - X)}{X}\, dx = \int k\, dx + c$

$\therefore \int \left(\dfrac{X'}{X} - \dfrac{X}{X}\right) dx = k \cdot \int 1\, dx + c$

• **Use** : $\int 1\, dx = x$

$\therefore \int \left(\dfrac{X'}{X} - 1\right) dx = k \cdot x + c$

$\therefore \int \dfrac{X'}{X}\, dx - \int 1\, dx = k\,x + c$

• **Use** : $\int \dfrac{f'(x)}{f(x)}\, dx = \log[f(x)]$

$\therefore \log X - x = k\,x + c$

$\therefore \log X = k\,x + x + c$

$\therefore \log X = (k+1)\,x + c$

• **Use** : $\log a = b \rightarrow a = e^{b}$

$\therefore X = e^{(k+1)x + c}$

• **Use** : $a^{m} \times a^{n} = a^{m+n}$

$\therefore X = e^{(k+1)x} \times e^{c}$

→ Let, $\boxed{e^{c} = A}$ ....[$\because$ New Constant]

$\therefore X = e^{(k+1)x} \times A$

$\therefore \boxed{X = Ae^{(k+1)x}}$

[2] $\frac{2 \cdot T'}{T} = k$

→ Take integration on both side

$\therefore \int \frac{2 \cdot T'}{T}\, dt = \int k\, dt + c$

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$\therefore 2 \times \int \frac{T'}{T}\, dt = k \cdot \int 1\, dt + c$

• **Use** : (1) $\int \frac{f'(x)}{f(x)} dx = \log[f(x)]$

(2) $\int 1\, dx = x$

$\therefore 2 \times \log T = k \cdot t + c$

$\therefore \log T = \frac{kt + c}{2}$

• **Use** : $\log a = b \rightarrow a = e^{b}$

$\therefore T = e^{\frac{kt+c}{2}}$

$\therefore T = e^{\frac{kt}{2} + \frac{c}{2}}$

• Use : $a^m \times a^n = a^{m+n}$

$\therefore T = e^{\frac{kt}{2}} \times e^{\frac{c}{2}}$

→ Let, $e^{\frac{c}{2}} = B$ .... [∵ New Constant]

$\therefore T = e^{\frac{kt}{2}} \times B$

$\therefore \boxed{T = Be^{\frac{kt}{2}}}$

Step(06) – Put these values of X & T in equation (P)

$\therefore u = Ae^{(k+1)x} \cdot Be^{\frac{kt}{2}}$

• Use : $a^m \times a^n = a^{m+n}$

$\therefore u = A \cdot B \cdot e^{(k+1)x + \frac{kt}{2}}$

→ Let, $\boxed{A \cdot B = C}$ .... [∵ New Constant]

$\therefore \boxed{u = C \cdot e^{(k+1)x + \frac{kt}{2}}}$ ....(Q)

Step(07) – To find value of constant (C) by using condition.

→ Condition is $u(x,0) = 6e^{-3x}$

→ i.e. when $t = 0$ then $u = 6e^{-3x}$

→ Put these two values in above equation.

$\therefore 6e^{-3x} = C \cdot e^{(k+1)x+0}$

$\therefore 6e^{-3x} = C \cdot e^{(k+1)x}$

→ Compare both side, we get :

(1) $\boxed{C=6}$

(2) $(k+1) = -3 \rightarrow k = -3-1 \rightarrow \boxed{k=-4}$

→ Put value of C and k in equation (Q).

$$\therefore u = 6 \cdot e^{(-4+1)x + \frac{-4t}{2}}$$

$$\therefore u = 6 \cdot e^{-3x-2t}$$

$$\therefore \boxed{u = 6 \cdot e^{-(3x+2t)}}$$

→ This is our required solution of given P.D.E..

Q.04.(D) A bar with insulated at its ends is initially at temperature 0^0C throughout. The end $x=0$ is kept at 0^0C for all times and the heat is suddenly applied so that $\frac{\partial u}{\partial x}=10$ at $x=l$ for all time. Find the temperature function $u(x,t)$. **04 Marks**

Solution : → Given that :

[1] Assume one dimensional heat equation : $\boxed{\frac{\partial u}{\partial t}=k^2\frac{\partial^2 u}{\partial x^2}}$

[2] Conditions : $\begin{cases}(1)\ u(0,t)=0\\(2)\left(\frac{\partial u}{\partial x}\right)=10 \text{ at } x=l\\(3)\ u(x,0)=0\end{cases}$

[3] To find the solution of one dimensional heat equation.

Step (01) – **Solution of One Dimensional Heat Flow Equation**

→ We know, solution of given equation is

$$\therefore \boxed{u(x,t)=\left[a\cos(px)+b\sin(px)\right]\times\left[ce^{-k^2p^2t}\right]} \quad ....(1)$$

→ Also, solution of given equation (if $p=0$) is

$$\therefore \boxed{u(x,t)=d+e\cdot x} \quad ....(2)$$

→ Now, combine above two solutions, we get :

$$\therefore \boxed{u(x,t)=\left[a\cos(px)+b\sin(px)\right]\times\left[ce^{-k^2p^2t}\right]+[d+e\cdot x]} \quad ....(3)$$

→ Where, a,b,c,d,e,p are constants.

Step (02) – **To find values of these constants by using conditions.**

• Condition (01) : $u(0,t)=0$

→ i.e. when $x=0$ then $u=0$

→ Put these values in equation (1).

$$\therefore 0=[a\cos(0)+b\sin(0)]\times[ce^{-k^2p^2t}]$$

$$\therefore \frac{0}{[ce^{-k^2p^2t}]}=[a\times1+b\times0]$$

$$\therefore 0=a+0$$

$$\therefore \boxed{0=a}$$

→ Also, put condition (1) in equation (2).

$$\therefore 0=d+e\cdot 0$$

$$\therefore 0=d+0$$

$$\therefore \boxed{0=d}$$

→ Put, $a=d=0$ in equation (3).

$$\therefore u(x,t)=[0+b\sin(px)]\times[ce^{-k^2p^2t}]+[0+e\cdot x]$$

$$\therefore u(x,t)=b\sin(px)\times ce^{-k^2p^2t}+e\cdot x$$

$$\therefore u(x,t)=b\times c\times\sin(px)e^{-k^2p^2t}+e\cdot x$$

→ Let, $b\times c=f$ .... [∵ New Constant]

$$\therefore \boxed{u(x,t)=f\times\sin(px)e^{-k^2p^2t}+e\cdot x} \quad ....(4)$$

• <u>Condition(02)</u>: $\left(\frac{\partial u}{\partial x}\right)=10$ at $x=l$

→ So, take P.D. of equation (4) w.r.t. x

$$\therefore \frac{\partial u}{\partial x} = f \times \cos(px) \times p \times e^{-k^2p^2t} + e$$

→ Now, put above condition.

$$\therefore 10 = f \times \cos(pl) \times p \times e^{-k^2p^2t} + e$$

→ If we take $\boxed{\cos(pl) = 0}$ then we get $\boxed{e = 10}$

→ Also, we know, $\boxed{\cos\left[\frac{(2n+1)\pi}{2}\right] = 0}$

→ Therefore, $pl = \frac{(2n+1)\pi}{2}$

→ i.e., $\boxed{p = \frac{(2n+1)\pi}{2l}}$

→ Put the value of e and p in equation (4).

→ And also add all the solutions from $n = 0$ to ∞.

$$\therefore \boxed{u(x,t) = \sum_{n=0}^{\infty} f \times \sin\left(\frac{(2n+1)\pi}{2l} \cdot x\right) e^{-k^2\left(\frac{(2n+1)\pi}{2l}\right)^2 t} + 10 \cdot x} \quad ....(5)$$

- <u>**Condition (03)**</u> : $u(x,0) = 0$

→ i.e. when $t = 0$ then $u = 0$

→ Put these values in equation (5).

$$\therefore 0 = \sum_{n=0}^{\infty} f \times \sin\left(\frac{(2n+1)\pi x}{2l}\right) e^{0} + 10 \cdot x$$

$$\therefore \boxed{-10x = \sum_{n=0}^{\infty} f \times \sin\left(\frac{(2n+1)\pi x}{2l}\right)}$$

→ But, R.H.S. is Half Range Fourier Sine Series.

$$\therefore f = b_n = \frac{2}{L}\int_0^l f(x)\,\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx$$

→ Here, $L = l$ and $f(x) = -10x$

$$= \frac{2}{l}\int_0^l (-10x)\cdot\sin\left(\frac{(2n+1)\pi x}{2l}\right)dx$$

$$= \frac{2}{l}\times(-10)\int_0^l x\cdot\sin\left(\frac{(2n+1)\pi x}{2l}\right)dx$$

$$= \frac{2}{l}\times(-10)\left[+(x)\cdot\frac{-\cos\left(\frac{(2n+1)\pi x}{2l}\right)}{\left(\frac{(2n+1)\pi}{2l}\right)} - (1)\cdot\frac{-\sin\left(\frac{(2n+1)\pi x}{2l}\right)}{\left(\frac{(2n+1)\pi}{2l}\right)^2}\right]_0^l$$

$$= \frac{-20}{l}\left\{\left[-0+\frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{\left(\frac{(2n+1)\pi}{2l}\right)^2}\right]-[-0+0]\right\}$$

$$= \frac{-20}{l}\left\{\frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{\frac{(2n+1)^2\pi^2}{4l^2}}\right\}$$

$$= \frac{-20}{l}\times\frac{4l^2\times\sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)^2\pi^2}$$

$$\therefore \boxed{f = b_n = \frac{-80l\sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)^2\pi^2}}$$

→ Put this value in equation (5)

$$\therefore u(x,t)=\sum_{n=0}^{\infty}\frac{-80l\sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)^2\pi^2}\times\sin\left(\frac{(2n+1)\pi x}{2l}\right)e^{-k^2\left(\frac{(2n+1)\pi}{2l}\right)^2 \cdot t}+10x$$

$$\therefore \boxed{u(x,t)=10x-\frac{-80l}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\times\sin\left(\frac{(2n+1)\pi}{2}\right)\times\sin\left(\frac{(2n+1)\pi x}{2l}\right)e^{-k^2\left(\frac{(2n+1)\pi}{2l}\right)^2 t}}$$

....(Ans)

Q.05.(A)	Determine k such that the function $f(z)=e^x\cos y+ie^x\sin ky$ is analytic.	04 Marks

Solution :

→ Given that :

[1] $f(z)=e^x\cos y+ie^x\sin ky$

[2] This function is analytic.

[3] To determine value of k.

→ We know, if function is analytic then it satisfy C-R equations.

→ So, C-R equations are :

[1] First C-R Equation : $\boxed{\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}}$

$$\therefore \frac{\partial}{\partial x}(u)=\frac{\partial}{\partial y}(v)$$

→ Here: $\begin{cases} u=\text{R.P.}=e^x\cos y \\ v=\text{I.P.}=e^x\sin ky \end{cases}$ .... [$\because$ From given $f(z)$]

$$\therefore \frac{\partial}{\partial x}(e^x\cos y)=\frac{\partial}{\partial y}(e^x\sin ky)$$

$$\therefore \boxed{\cos y\times e^x=e^x\times\cos ky\times k}$$

→ Compare on both side, we get : $\boxed{k=1}$ ....(Ans)

Q.05.(B)	**Show that $u = x^2 - y^2 - 2xy - 2x + 3y$ is a harmonic function and hence determine the analytic function $f(z)$ in terms of z.** **04 Marks**

Solution : → Given that :

[1] Given function, $u = x^2 - y^2 - 2xy - 2x + 3y$

[2] To show that u is harmonic.

[3] Also, to find analytic function.

- **To prove that u is harmonic :**

→ We know, Laplace Equation is $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ ....(P)

→ So, we find $\frac{\partial^2 u}{\partial x^2}$ & $\frac{\partial^2 u}{\partial y^2}$.

[1] $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(u)\right]$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2 - y^2 - 2xy - 2x + 3y)\right]$$

$$= \frac{\partial}{\partial x}(2x - 0 - 2y \times 1 - 2 + 0)$$

$$= \frac{\partial}{\partial x}(2x - 2y - 2) \quad ....(1)$$

$$= 2 \times 1 - 0 - 0$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial x^2} = 2}$$

[2] $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(u)\right]$

$$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^2 - y^2 - 2xy - 2x + 3y\right)\right]$$

$$= \frac{\partial}{\partial y}(0 - 2y - 2x \times 1 - 0 + 3 \times 1)$$

$$= \frac{\partial}{\partial x}(-2y - 2x + 3) \quad ....(2)$$

$$= 0 - 2 \times 1 + 0$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial y^2} = -2}$$

→ Put these values of $\frac{\partial^2 u}{\partial x^2}$ & $\frac{\partial^2 u}{\partial y^2}$ in equation (P).

$$\therefore (2) + (-2) = 0$$

$$\therefore 2 - 2 = 0$$

$$\therefore \boxed{0 = 0}$$

$$\therefore \boxed{\text{L.H.S.} = \text{R.H.S.}}$$

→ This shows that, u satisfies Laplace Equation.

→ i.e. u is harmonic function.

- <u>**To find Analytic Function**</u> :

→ So, use Thomson's Method formula.

$$\therefore \boxed{f(z) = \int \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]_{x=z,\, y=0} dz}$$

→ We convert above v in terms of u. Because in our problem,

there is given value of u only, not v.

→ So, C-R Equations : (1) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & (2) $\boxed{\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$

$$\therefore \boxed{f(z)=\int\left[\frac{\partial u}{\partial x}+i\left(-\frac{\partial u}{\partial y}\right)\right]_{x=z,\, y=0} dz} \quad ....(Q)$$

→ Now, there is no need of to calculate : $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$.

→ Because, we already calculated $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$. [∵ Equation (1) & (2)]

→ <u>So, from equation (1)</u>:

$$\therefore \frac{\partial u}{\partial x}=2x-2y-2$$

$$\therefore \boxed{\left.\frac{\partial u}{\partial x}\right|_{x=z,\, y=0}=2z-2\times 0-2=2z-2}$$

→ <u>Also, from equation (2)</u>:

$$\therefore \frac{\partial u}{\partial y}=-2y-2x+3$$

$$\therefore \boxed{\left.\frac{\partial u}{\partial y}\right|_{x=z,\, y=0}=-2\times 0-2z+3=-2z+3}$$

→ Put thwse values in equation (Q).

$$\therefore f(z)=\int\{(2z-2)+i[-(-2z+3)]\}dz$$

$$=\int\{2z-2+i(2z-3)\}dz$$

$$=\int\{2z-2+2iz-3i\}dz$$

$$=\int\{2z+2iz-2-3i\}dz$$

$$=\int\{2(1+i)z-(2+3i)\}dz$$

$$\therefore f(z) = 2(1+i)\times\frac{z^2}{2} - (2+3i)\times z$$

$$\therefore \boxed{f(z) = (1+i)z^2 - (2+3i)z}$$

→ This is our analytic function.

Q.05.(C)	Determine the pole of the function $f(z)=\dfrac{2z-1}{z(z+1)(z-3)}$ and also find the residue at each pole & sum of all residues. **04 Marks**

Solution :

→ Given that :

[1] Function : Let, $f(z)=\dfrac{2z-1}{z(z+1)(z-3)}$

[2] To find pole.

[3] To find residue at each pole.

[4] To find sum of all residue.

$\boxed{\textbf{Step (01)}}$ – **To find out factors of (D^r).**

→ Here, $D^r=\dfrac{2z-1}{z(z+1)(z-3)}$

→ This D^r is already in factor Form.

$$\therefore f(z)=\frac{2z-1}{z(z+1)(z-3)} \quad \text{....(1)}$$

$\boxed{\textbf{Step (02)}}$ – **To find Poles and Its Order**

$\therefore D^r=0$

$\therefore z(z+1)(z-3)=0$

$\therefore \boxed{z=0}$

→ Order : $\boxed{n=1}$

$\therefore (z+1)=0$

$\therefore \boxed{z=-1}$

→ Order : $\boxed{n=1}$

$\therefore (z-3)=0$

$\therefore \boxed{z=3}$

→ Order : $\boxed{n=1}$

→ So our poles are $\boxed{z=0,\ z=-1,\ z=3}$ ....(Ans)

$\boxed{\textbf{Step (03)}}$ – **To find Position of Pole**

→ Here, in problem path is not given.

→ So, we assume that, all the pole lies inside of the path.

Step (04) - **Write Cauchy's Residue Formula**

→ For algebraic Poles, formula is

$$\therefore \boxed{\text{Residue at } (z=a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n \times f(z)\right]\right\}_{z=a}} \quad \text{....(P)}$$

→ In this problem, from eq^n(1) : $\boxed{f(z) = \frac{2z-1}{z(z+1)(z-3)}}$

(1) Residue at $(z=0)$:

→ Here, $\boxed{a=0, \quad n=1, \quad f(z) = \frac{2z-1}{z(z+1)(z-3)}}$

→ Put these values in equation (P).

$$\therefore \text{Residue at } (z=0) = \frac{1}{(1-1)!}\left\{\frac{d^{1-1}}{dz^{1-1}}\left[(z-0)^1 \times \frac{2z-1}{z(z+1)(z-3)}\right]\right\}_{z=0}$$

$$= \frac{1}{0!}\left\{\frac{d^0}{dz^0}\left[z \times \frac{2z-1}{z(z+1)(z-3)}\right]\right\}_{z=0}$$

$$= \frac{1}{1} \times \left[\frac{2z-1}{(z+1)(z-3)}\right]_{z=0}$$

$$= \frac{2\times 0-1}{(0+1)(0-3)}$$

$$= \frac{0-1}{(1)(-3)}$$

$$= \frac{-1}{-3}$$

$$\therefore \boxed{\text{Residue at } (z=0) = \frac{1}{3}} \text{...(Ans)}$$

(2) Residue at $(z=-1)$:

→ Here, $\boxed{a=-1, \quad n=1, \quad f(z)=\frac{2z-1}{z(z+1)(z-3)}}$

→ Put these values in equation (P).

$$\therefore \text{Residue at } (z=-1)=\frac{1}{(1-1)!}\left\{\frac{d^{1-1}}{dz^{1-1}}\left[(z+1)^1\times\frac{2z-1}{z(z+1)(z-3)}\right]\right\}_{z=-1}$$

$$=\frac{1}{0!}\left\{\frac{d^0}{dz^0}\left[(z+1)\times\frac{2z-1}{z(z+1)(z-3)}\right]\right\}_{z=-1}$$

$$=\frac{1}{1}\times\left[\frac{2z-1}{z(z-3)}\right]_{z=-1}$$

$$=\frac{2\times(-1)-1}{(-1)(-1-3)}$$

$$=\frac{-2-1}{(-1)(-4)}$$

$$\therefore \boxed{\text{Residue at } (z=-1)=\frac{-3}{4}} \text{....(Ans)}$$

(3) Residue at $(z=3)$:

→ Here, $\boxed{a=3, \quad n=1, \quad f(z)=\frac{2z-1}{z(z+1)(z-3)}}$

→ Put these values in equation (P).

$$\therefore \text{Residue at } (z=3)=\frac{1}{(1-1)!}\left\{\frac{d^{1-1}}{dz^{1-1}}\left[(z-3)^1\times\frac{2z-1}{z(z+1)(z-3)}\right]\right\}_{z=3}$$

$$=\frac{1}{0!}\left\{\frac{d^0}{dz^0}\left[(z-3)\times\frac{2z-1}{z(z+1)(z-3)}\right]\right\}_{z=3}$$

$$=\frac{1}{1}\times\left[\frac{2z-1}{z(z+1)}\right]_{z=3}$$

$$= \frac{2 \times 3 - 1}{3(3+1)}$$

$$= \frac{6-1}{3(4)}$$

$\therefore$ $\boxed{\text{Residue at } (z=3) = \frac{5}{12}}$ ....(Ans)

Step (05) – **Take Sum of all Residue.**

$\therefore$ Sum of all residue $= \frac{1}{3} - \frac{3}{4} + \frac{5}{12} = \frac{4-9+5}{12} = \frac{9-9}{12} = \frac{0}{12} = 0$

$\therefore$ $\boxed{\text{Sum of all residue} = 0}$ ....(Ans)

Q.05.(D)	Evaluate : $\oint_C \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)} dz$, where C is the circle $\lvert z \rvert = 4$.
	04 Marks

Solution : → Given that :

[1] To evaluate : $\oint_C \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)} dz$

[2] Path : circle $\lvert z \rvert = 4$

→ Let, $I = \oint_C \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)} dz$

→ To solve this, Use Cauchy's Residue Formula & Theorem.

Step (01) – **To find out factors of (D^r).**

→ Here, $D^r = (z-1)^2 (z-2)$

→ This D^r is already in factor Form.

$\therefore I = \oint_C \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)} dz$ ....(1)

Step (02) – **To find Poles and Its Order**

$\therefore D^r = 0$

$\therefore (z-1)^2 (z-2) = 0$

$\therefore (z-1)^2 = 0$	$\therefore (z-2) = 0$
$\therefore (z-1) = 0$	$\therefore \boxed{z=2}$
$\therefore \boxed{z=1}$	→ Order : $\boxed{n=1}$
→ Order : $\boxed{n=2}$	

→ These are our poles with order 2 & 1 respectively.

$\boxed{\text{Step (03)}}$ – **To find Position of Pole**

→ We have path : $|z|=4$

(1) First Pole is $(z=1)$:

→ Put, this pole in path.

$\therefore |1|=4$

$\therefore \boxed{1<4}$

→ This shows that, this pole lies inside of the path.

(2) Second Pole is $(z=2)$:

→ Put, this pole in path.

$\therefore |2|=4$

$\therefore \boxed{2<4}$

→ This shows that, this pole lies inside of the path.

$\boxed{\text{Step (04)}}$ – **Write Cauchy's Residue Formula**

→ For algebraic Poles, formula is

$$\therefore \boxed{\text{Residue at } (z=a)=\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n\times f(z)\right]\right\}_{z=a}} \quad \text{....(P)}$$

→ Here, we find residues at those poles which are inside of the path.

→ In this problem, from eq^n(1) : $\boxed{f(z)=\dfrac{\sin \pi z^2+2z}{(z-1)^2(z-2)}}$

(1) Residue at $(z=1)$:

→ Here, $\boxed{a=1,\quad n=2,\quad f(z)=\dfrac{\sin \pi z^2+2z}{(z-1)^2(z-2)}}$

→ Put these values in equation (P).

$$\therefore \text{Residue at } (z=1) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2 \times \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)}\right]\right\}_{z=1}$$

$$= \frac{1}{1!}\left\{\frac{d^1}{dz^1}\left[\cancel{(z-1)^2} \times \frac{\sin \pi z^2 + 2z}{\cancel{(z-1)^2} (z-2)}\right]\right\}_{z=1}$$

$$= \frac{1}{1}\left\{\frac{d}{dz}\left[\frac{\sin \pi z^2 + 2z}{(z-2)}\right]\right\}_{z=1}$$

$$= \left\{\frac{(z-2)\cdot\left(\cos \pi z^2 \cdot 2\pi z + 2\times 1\right) - \left(\sin \pi z^2 + 2z\right)\cdot(1-0)}{(z-2)^2}\right\}_{z=1}$$

$$= \left\{\frac{(z-2)\cdot\left(2\pi z \cos \pi z^2 + 2\right) - \left(\sin \pi z^2 + 2z\right)}{(z-2)^2}\right\}_{z=1}$$

$$= \frac{(-1)\cdot(2\pi \cos \pi + 2) - (\sin \pi + 2)}{(-1)^2}$$

- **Use** : (1) $\cos \pi = -1$

  (2) $\sin \pi = 0$

$$= \frac{(-1)\cdot(-2\pi + 2) - (0+2)}{1}$$

$$= 2\pi - 2 - 2$$

$$\therefore \boxed{\text{Residue at } (z=1) = 2\pi - 4}$$

(2) Residue at $(z=2)$:

$\rightarrow$ Here, $\boxed{a=2, \quad n=1, \quad f(z) = \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)}}$

$\rightarrow$ Put these values in equation (P).

$$\therefore \text{Residue at } (z=2) = \frac{1}{(1-1)!}\left\{\frac{d^{1-1}}{dz^{1-1}}\left[(z-2)^1 \times \frac{\sin \pi z^2 + 2z}{(z-1)^2 (z-2)}\right]\right\}_{z=2}$$

$$=\frac{1}{0!}\left\{\frac{d^0}{dz^0}\left[\cancel{(z-2)}\times\frac{\sin\pi z^2+2z}{(z-1)^2\cancel{(z-2)}}\right]\right\}_{z=2}$$

$$=\frac{1}{1}\left\{\frac{\sin\pi z^2+2z}{(z-1)^2}\right\}_{z=2}$$

$$=\frac{\sin(4\pi)+4}{(1)^2}$$

• **Use** : (1) $\sin 4\pi = 0$

$$=\frac{0+4}{1}$$

$\therefore$ $\boxed{\text{Residue at } (z=2)=4}$

$\boxed{\textbf{Step (05)}}$ **– Use Cauchy's Residue Theorem**

$\therefore I = 2\pi i\times[\text{sum of the all residue}]$

$= 2\pi i\times[(2\pi-4)+(4)]$

$= 2\pi i\times[2\pi-4+4]$

$= 2\pi i\times[2\pi]$

$\therefore$ $\boxed{I = 4\pi^2 i}$

→ But, $I=\oint_c \frac{\sin\pi z^2+2z}{(z-1)^2(z-2)}dz$

$\therefore$ $\boxed{\oint_c \frac{\sin\pi z^2+2z}{(z-1)^2(z-2)}dz = 4\pi^2 i}$ ....(Ans)

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