# TECHNIQUES FOR DERIVING SDE MODELS

## PH549 Course Project

**Anurag Abhijit Pendse**
200260008
IIT Bombay
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# Contents

# 1 Introduction

SDEs are a potent tool for analyzing random processes. However, setting them up, given a system, is not always trivial. In this report, I explore one of the techniques used for setting up SDEs given a (physical system).

SDEs, in general, are ubiquitous in the field of random processes. However, setting up the SDE is not trivial at all. In some cases, intuition allows us to set up the equation, but as is the case with most systems, they are too non-trivial to be tackled by intuition. Thus, we must proceed in a systematic way and set up a proper technique to do the same.

We must start by understanding what is known. In order to proceed with our analysis, we need to know all the possible states one can go to from a particular state in a single time-step (in our case, this time-step is infinitesimal). Furthermore, we need to know the probabilities of all those transitions. Using this information, we can set up the SDE. In the following sections, we will see how exactly to use this information to set up our SDE.

# 2 The Procedure

Rather than dealing with a single stochastic variable, we can deal with the general case of a multi-component system.

## 2.1 Probabilities

Let us assume a system with $k$ components whose position is denoted by a position

$$X(t) = \begin{pmatrix} X_1(t) & X_2(t) & \dots & X_k(t) \end{pmatrix}^T \tag{1}$$

We can now define a set of allowed transitions from this position by

$$\Delta X = \begin{pmatrix} \delta^{(1)} & \dots & \delta^{(m)} \end{pmatrix} \tag{2}$$

Here, we have each $\delta^{(i)}$ to be a $k$ component column vector. Thus, we have

$$\Delta X \in \mathbb{R}^{m \times k} \tag{3}$$

Now, we can define the probabilities of each transition as

$$p_i = P_i(X, t)\Delta t \tag{4}$$

We also have another probability, which is of no transition.

$$p_{m+1} = 1 - \sum_{i=1}^{m} p_i \tag{5}$$

Before moving forward, it is necessary toclarify the SDE's form. We will have a drift term which is deterministic and a purely Brownian term. We will write

$$dX = \mu(X, t)dt + B(X, t)dW(t) \tag{6}$$

Here, $W(t)$ is a column vector with $k$ independent Weiner processes as its components. We can now find the expectation values of different powers of $\Delta X$ to determine the quantities relevant to setting up the SDE.

## 2.2 Moments

We can now look at the first moment of $\Delta X$ and we will get

$$E[\Delta X] = \sum_{i=1}^{m} p_i \delta^{(i)} \tag{7}$$

Now, we can find the drift term from this expression by using 4.

$$\frac{E[\Delta X]}{\Delta t} = \sum_{i=1}^{m} P_i \delta^{(i)} \tag{8}$$

This is precisely the drift term. Thus, we have obtained

$$\mu(X,t) = \sum_{i=1}^{m} P_i \delta^{(i)} \tag{9}$$

We can now move our focus to the Brownian term. We need to compute the second moment of $\Delta X$ in order to move forward. We will obtain

$$E[(\Delta X)(\Delta X)^T] = \sum_{i=1}^{m} P_i(\delta^{(i)})(\delta^{(i)})^T \Delta t \tag{10}$$

We can now get $V = \frac{E[(\Delta X)(\Delta X)^T]}{\Delta t}$

$$V = \sum_{i=1}^{m} P_i(\delta^{(i)})(\delta^{(i)})^T \tag{11}$$

We can now calculate the coefficient $B(X,t)$ as

$$B(X,t) = \sqrt{V(X,t)} \tag{12}$$

We now have an Ito SDE corresponding to our system.

## 2.3 Alternate Formulation

While the method explained above gives us an Ito SDE, it is not the only way to obtain the SDE corresponding to the system. In the previous formulation, we used a Weiner column vector of size $k \times 1$. However, we can go about the problem alternatively by looking at a set of $m$ independent Weiner processes i.e., our column vector in the SDE is of size $m \times 1$. As long as our covariance matrix $V$ is obtained from the SDE we write, the SDE is valid. All such formulations are equivalent. The only difference is differing computational complexity during simulations.

One such alternative formulation is as follows:

Let us define $C$ as

$$C_{ij} = \delta_i^{(j)} \sqrt{P_j} \tag{13}$$

We now have $C \in \mathbb{R}^{k \times m}$. Our SDE looks like

$$dX = \mu(X,t)dt + C(X,t)dW^* \tag{14}$$

Note that $CC^T = v$.

We also have a similar form of equation where $C$ is replaced by the lower triangular matrix $G$, which shows up in the Cholesky factorization of $V$.

I conclude this section with a definition.

**Definition 2.1.** An Ito SDE defined as given in section 1.2 from a given set of $(\delta, P) \in \mathbb{R}^{k \times m} \times \mathbb{R}^k$ is defined to be a $(\delta, p)$-SDE for the system.

# 3 Viability and Limitations

**Definition 3.1.** The positive cone in $\mathbb{R}^k$ is defined as

$$K^+ = \{y \in \mathbb{R}^k : y_i \geq 0, i = 1, 2, \ldots, k\}$$

## 3.1 Non-Negativity

**Definition 3.2.** We say that a stochastic system

$$\begin{aligned} dX &= f(t, X)dt + G(t, X)dW \\ X(t_0) &= X_0 \end{aligned}$$

where $f : [t_0, \infty) \times \mathbb{R}^k \to \mathbb{R}^k$, $G : [t_0, \infty) \times \mathbb{R}^k \to \mathbb{R}^{k \times m}$ and $W = (W_1, \ldots, W_m)^T$, served non-negativity if for every initial condition $X_0 \in K^+$, the corresponding solution $X(t)$ for $t > t_0$ satisfies

$$P(\{X(t) \in K^+, t \in [t_0, \infty)\}) = 1$$

i.e. the equation always attains non-negative values.

We now have a theorem as follows

**Theorem 3.3.** Let $(\delta, p) \in \mathbb{R}^{m \times k} \times \mathbb{R}^k$. Then the $(\delta, p)$-SDE model preserves non-negativity iff

$$P_i(t, y)(\delta_j^{(i)})^2 = 0 \qquad y \in K^+, y_j = 0, t \geq 0$$

for all $i = 1, \ldots, m$. In other words, for all $1 \leq j \leq k$, $1 \leq 1 \leq m$ such that $\delta_j^{(i)} \neq 0$, it follows that

$$P_i(t, y) = 0 \qquad y \in K^+, y_j = 0, t \geq 0$$

**Proof:**

Before moving to the proof, we need to look at a few definitions and a lemma. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right continuous increasing family $F = (\mathcal{F}_t)_{t \geq 0}$ of sub-$\sigma$ fields of $\mathcal{F}$ each containing all sets of $P$ of measure 0. We consider Ito SDE's of the form

$$dX = f(t, X)dt + g(t, X)dW \tag{15}$$

and initial condition

$$X(t_0) = X_0 \tag{16}$$

Where $f = (f_i)_{1 \leq i \leq k} : [0, \infty) \times \mathbb{R}^k \to \mathbb{R}^k$ is a Borel measureable function and $g = (g_{ij})_{1 \leq i \leq k, 1 \leq j \leq r} : [0, \infty) \times \mathbb{R}^k \to \mathbb{R}^{k \times r}$ is a Borel measurable mapping into the set of all $\mathbb{R}^{k \times r}$. $W : [0, \infty) \times \Omega \to \mathbb{R}^r$ is an r-dimensional $\mathcal{F}$ adpated Weiner process and $dW$ is its corresponding Ito differential. Assume a non-negative initial time. Thus, we have

$$X(t) = X_0 + int_{t_0}^t f(s, X(s))ds + int_{t_0}^t g(s, X(s))dW(s) \tag{17}$$

**Definition 3.4.** We call the subset $K \subset \mathbb{R}^k$ invariant for the stochastic system if for every initial data $X_0 \in K$ and initial time $t_0 \geq 0$, the corresponding solution satisfies

$$P(\{X(t) \in K, t \in (t_0, \infty)\}) = 1$$

**Theorem 3.5.** Let $I \subset \{1, \ldots, k\}$ be a non-empty subset and $a_i, b_i \in \mathbb{R} \bigcup \{\infty\}$ be such that $b_i > a_i, i \in I$. Then, the set

$$K := \{x \in \mathbb{R}^k : a_i \leq x_i \leq b_i, i \in I\}$$

is invariant for the stochastic system $(f, g)$ iff

$$\begin{aligned} f_i(t, x) &\geq 0 \quad \forall x \in K, x_i = a_i \\ f_i(t, x) &\leq 0 \quad \forall x \in K, x_i = b_i \\ g_{ij}(t, x) &= 0 \forall x \in K, x_i \in \{a_i, b_i\}, j = 1, \ldots, r \end{aligned}$$

for all $t \geq 0$ and $i \in I$. This result is independent of Ito's or Stratnovich's interpretation.

**Corollary 3.6.** Let $I \subset \{1, \ldots, k\}$ be a non-empty subset. Then, the set

$$K^+ := \{x \in \mathbb{R}^k : 0 \leq x_i, i \in I\}$$

is invariant for the stochastic system $(f, g)$ iff

$$\begin{aligned} f_i(t, x) &\geq 0 \quad \forall x \in K^+, x_i = 0 \\ g_{ij}(t, x) &= 0 \forall x \in K^+, x_i = 0, j = 1, \ldots, r \end{aligned}$$

for all $t \geq 0$ and $i \in I$. This result is independent of Ito's or Stratnovich's interpretation.

We can now use these results to prove our original theorem.
We have the SDE system to preserve non-negativity iff

$$\mu_i(t, y) \geq 0, \qquad B_{ij}(t, y) = 0, y \in K^+, y_i = 0 \tag{18}$$

for $i = 1, \ldots k$, $j = 1 \ldots m$, $t \geq 0$. Since we have $B = \sqrt{V}$, we get

$$V_{ij}(t, y) = \sum_{l=1}^{k} B_{il}(t, y) B_{lj}(t, y) \tag{19}$$

This implies that

$$V_{ij}(t,y) = 0, \quad y \in K^+, y_i = 0 \tag{20}$$

Now, using the form of $V$, we get the condition given in the theorem.
Starting with the conditions, it is trivial to show that the SDE system preserves non-negativity. This completes the proof $\square$
We see from this that

**Corollary 3.7.** For a SDE system to preserve non-negativity, we need

$$\mu_i(t,y) = 0 \quad y \in K^+, y_i = 0, t \geq 0$$

The theorems described for the positive cone can be extended to any rectangular subset of $\mathbb{R}^k$.

## 3.2 Applications

### 3.2.1 Scalar Growth Models

We have the following

| i | $\delta^{(i)}$ | $p_i$ |
|---|---|---|
| 1 | 1 | $\alpha X \Delta t$ |
| 2 | -1 | $\beta X \Delta t$ |
| 3 | 1 | $\gamma \Delta t$ |

We thus have the SDE systems

$$dX = ((\alpha - \beta)X + \gamma)dt + \sqrt{(\alpha + \beta)X + \gamma}dW \tag{21}$$

or

$$dX = ((\alpha - \beta)X + \gamma)dt + \sqrt{\alpha X}dW_1 + \sqrt{\beta X}dW_2 + \sqrt{\gamma}dW_3 \tag{22}$$

depending on the method used for construction. It is clear that this construction preserves non-negativity. Furthermore, no other SDE extension derived by Allen's method can possess this property.

### 3.2.2 Logistic Growth Model

We can approach this via two probability definitions.

| i | $\delta^{(i)}$ | $p_i$ | $p_i'$ |
|---|---|---|---|
| 1 | 1 | $\alpha X \Delta t$ | $\alpha X(1 - \frac{X}{2\beta})\Delta t$ |
| 2 | -1 | $\frac{\alpha X^2}{\beta}\Delta t$ | $\frac{\alpha X^2}{2\beta}\Delta t$ |

These lead to the SDE models

$$dX = (\alpha X(1 - \frac{X}{\beta}))dt + \sqrt{\alpha X(1 + \frac{X}{\beta})}dW \tag{23}$$

and

$$dX = (\alpha X(1 - \frac{X}{\beta}))dt + \sqrt{\alpha X}dW \tag{24}$$

While these give the required deterministic parts, the interval $[0, \beta]$ is notinvariant as we require. This can be rectified by defining $p_1 = 2\alpha X(1 - \frac{X}{\beta})$ and $p_2 = \alpha X(1 - \frac{X}{\beta})$. This leads to the SDE

$$dX = (\alpha X(1 - \frac{X}{\beta}))dt + \sqrt{3\alpha X(1 - \frac{X}{\beta})}dW \tag{25}$$

Using the Theorem in section 3, we can see that the interval $[0, beta]$ is invariant. Note that there are several probability definitions that can give this result as well. However, whether or not there are alternate SDE extensions using Allen's procedure is dependent on each probability distribution.

### 3.2.3 Epidemic Dynamics

This is a case of two interacting populations. We denote them by $S$, who are the susceptible population, and $I$, who are the infected population. If we look at the deterministic part of the system, the equations to model the system are

$$\begin{aligned} \frac{dS}{dt} &= -\alpha\frac{SI}{N} + \gamma I \\ \frac{dI}{dt} &= \alpha\frac{SI}{N} - \gamma I \end{aligned} \tag{26}$$

Thus, we can see from the equation that the probability of reduction in $S$, i.e., the infection rate is $\alpha\frac{SI}{N}$. The recovery rate is $\gamma I$. Thus, we can use the method we obtained in section 2 to get

$$dS = \left(-\alpha\frac{SI}{N} + \gamma I\right)dt + \sqrt{\frac{1}{2}\left(\alpha\frac{SI}{N} + \gamma I\right)}(dW_1 - dW_2) \tag{27}$$

$$dI = \left(\alpha\frac{SI}{N} - \gamma I\right)dt + \sqrt{\frac{1}{2}\left(\alpha\frac{SI}{N} + \gamma I\right)}(-dW_1 + dW_2) \tag{28}$$

While the deterministic equation preserves non-negativity, the SDE does not.