Classical Mechanics

Midsummer Report

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Introduction

Classical Mechanics is one of the most important topics in Physics, not only from a theory standpoint but also from a historical one. The observations of Galileo and the laws given by Newton mark the beginning of the study of physical phenomena with a mathematical basis.
Newton's three laws gave people a surefire method to analyse the evolution and behavior of static and dynamic systems. However, they didn't seem to have explanations of their own.As the study of Physics grew, many observations were made regarding different systems where explanation of the phenomena using Newtons Laws was tedious.
A more mathematical formulation of mechanics would give people a trustworthy method to analyse any system they wanted (Note that all of this was in the mid 18^{th} century to the early-mid 19^{th} century). The work of people like Euler, Lagrange, Hamilton and many others gave us the theory of classical mechanics we have today.
The formulations and principles given in this classical theory [1] are more fundamental than Newton's laws and have found themselves as fundamental principles of modern theories as well.

1 Euler-Lagrange Equations

1.1 The Principle of Least Action

1.1.1 Functionals

To understand the PLA, we need to first understand functionals.

Definition 1.1. A Functional is a mapping from a space S to real/complex numbers

The functional we will deal with will be an integral of the form

$$F = \int_{x_1}^{x_2} f(y, y', x)dx$$

where y is a function of the independent variable x and y' is its derivative.
The condition we want to explore is when the value of the functional becomes stationary i.e. $\delta F = 0$.
We can see from the Calculus of Variations[2]

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{1.1}$$

This expression is of paramount importance in the further discussion.

[1] Here, 'Classical' is used to differentiate it from modern theories of Quantum Mechanics and Relativity

[2] Proof is given in the Appendix

1.1.2 Action

One of the most important quantities we will see in this theory will be the Action. The role this quantity plays is just as glamorous as its name suggests. The Action (denoted by S) is defined as follows

$$S = \int_{t_1}^{t_2} \mathscr{L} dt$$

$\mathscr{L}$ is known as the Lagrangian where

$$\mathscr{L} = T - V$$

where T is the kinetic energy and V is the kinetic energy of the system.
We have $Hamilton's Principle$ as :
The motion of the system from time t_1 to time t_2 is such that the Action has a stationary value for the actual path of motion
In essence, $Hamilton's\ Principle$ tells us that

$$\delta S = 0 \tag{1.2}$$

We see from 1.1 and 1.2 that we get

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{x_i}}\right) - \frac{\partial \mathscr{L}}{\partial x_i} = 0 \tag{1.3}$$

Where x_i is a particular coordinate and $\dot{x_i}$ is its time derivative [3]

1.1.3 Generalised Coordinates

Most of the times, systems are dependent on several independent variables. Each of these variables determines the position of the system. These variables will not necessarily have the dimension of length.(Angles can also be called coordinates if they are pertinent to determining the position of the system)
From here onwards, these generalised coordinates will be denoted as q_i and their time derivatives as $\dot{q}_i$ [4]

1.1.4 The Euler Lagrange Equations

We can see that the Lagrangian will be a function all the q_i, $\dot{q}_i$ and t.
So we will obtain the Euler-Lagrange Equation for each coordinate as

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \tag{1.4}$$

[3] Here, I have referred to the derivatives in regular Cartesian Coordinates. I will state the result for generalised coordinates in the next section

[4] Since all of the q_i are independent, we have $\frac{\partial q_i}{\partial q_j} = \delta_{ij}$ where δ_{ij} is the Kröneckar delta.

1.2 D'Alembert's Principle

1.2.1 Forces of Constraint

A system is almost never free to move however it wants. It has some constraints put on it.
We will call the forces which enforce these constraints as forces of constraint.
We can categorise these constraint forces into categories:

Definition 1.2. (Holonomic Constraints) The constraints which can be expressed as $f(q_i, \dot{q}_i, t) = 0$

Non-Holonomic Constraints are constraints which are not holonomic.

Definition 1.3. (Scleronomous Constraints) Constraints which have no explicit time dependence

On the other hand, Rheonomous constraints have an explicit time dependence
The addition of a constraint reduces the degrees of freedom of the system by 1.

1.2.2 D'Alembert's Principle

[5]We will now be looking at a system of particles upon which an external force acts along with the constraint forces. We have

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \tag{1.5}$$

If we add a negative of the rate of change of momentum to both sides of the above equation , we see that if the system was acted upon a force of the form $\mathbf{F}_i - \dot{\mathbf{p}}_i$, we would have the system to be in equilibrium.
So, the virtual work done by this force would be 0. So, we have

$$(\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \tag{1.6}$$

We can decompose the net force into the applied forces and the constraint forces. The virtual work done by the constraint forces would be zero. So, we have arrived at $D'Alembert's\ Principle$:

$$(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \tag{1.7}$$

where $\mathbf{F}_i^{(a)}$ is the applied force.

1.2.3 Euler-Lagrange Equations

We need to move on to obtain the ELE's from D'Alembert's Principle. Since virtual displacements hold time constant, we have

$$\delta\mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_i}\delta q_i \tag{1.8}$$

[5]The Einstein Summation convention will be used here onwards unless otherwise specified

Also,

$$\mathbf{v}_i = \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \tag{1.9}$$

Using these two equations, we can manipulate 1.7 to obtain

$$\left(\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] - Q_j\right)\delta q_j = 0 \tag{1.10}$$

Where Q_j is the generalized force given by $\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$ and T is the net kinetic energy of the system given by the traditional formula of $\frac{1}{2}m_i v_i^2$.
We can choose the q_i to be independent and their variations to be in compliance with the constraints. Thus, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{1.11}$$

If the $\mathbf{F}_i^{(a)}$ are derivable from a potential as $-\nabla_i V$, we get

$$Q_i = -\frac{\partial V}{\partial q_i}$$

Further,if V is independent of the generalised velocities, we get

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \tag{1.12}$$

Where $\mathscr{L}$ is the Lagrangian as we defined before.
It is interesting to note that the Lagrangian is not necessarily a unique quantity. Even if we add a term of the form $\frac{dF(q_i,t)}{dt}$ to the Lagrangian, the ELE's remain unchanged.[6]

1.2.4 Generalised Potentials

In the last section, $F_i^{(a)}$ was assumed to be the gradient of a scalar potential. Unfortunately, this assumption leaves out several important forces, most notably the Lorentz Force.
We can slightly generalise the concept of potential as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{1.13}$$

This allows us to retain our definition of the Lagrangian as $T - U$. For the Lorentz force, we can show that

$$U = q\phi - q\mathbf{A}\cdot\mathbf{v} \tag{1.14}$$

[6]Proof in appendix

Where ϕ is the scalar potential for the fields and $\mathbf{A}$ is the vector potential for the fields.
Even after generalising the concept of the potential, we still miss out on some important forces like friction which we cannot express in the required way.
Most of the times, we have the magnitude of the frictional force in a particular direction to be proportional to the velocity of the particle in that direction.

$$F_{(fric)x} = -kv_x$$

For these kinds of forces, we can define a Dissapation function such that

$$\mathbf{F}_{fric} = -\nabla_v \mathscr{F} \tag{1.15}$$

Thus, we can also obtain

$$Q_j = -\frac{\partial \mathscr{F}}{\partial \dot{q}_j} \tag{1.16}$$

So, setting our $\mathscr{F}$ according to the systems frictional forces and setting our $\mathscr{L}$ as $T - U$ where U is the generalised potential, we can write the ELE's as

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} + \frac{\partial \mathscr{F}}{\partial \dot{q}_i} = 0 \tag{1.17}$$

2 Conservation Laws, Symmetry and Noether's Theorem

We have seen the extension of regular coordinates to a generalised set of coordinates. With the benefit of foresight, we will find it useful to define a quantity called 'conjugate momentum'.as

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \tag{2.1}$$

Substituting this in the ELE's, we obtain

$$\dot{p}_i = \frac{\partial \mathscr{L}}{\partial q_i} \tag{2.2}$$

Definition 2.1. (Cyclic Coordinates) Coordinates upon which the Lagrangian doesn't explicitly depend.

If q_i is a cyclic coordinate, we have

$$\frac{\partial \mathscr{L}}{\partial q_i} = 0$$

Thus, if p_i is the conjugate momentum corresponding to the cyclic coordinate q_i, we have

$$\dot{p}_i = 0 \tag{2.3}$$

Thus, p_i is time invariant i.e. it is conserved.
A coordinate being cyclic means that the system is symmetric about that coordinate. For any number of symmetries, we have the exact same number of conserved quantities.

We also have *Noether's Theorem* as :
With every continuous symmetry of the Lagrangian, there exists one conserved quantity associated with it
Note that the above theorem makes no reference to cyclic coordinates, rather it refers to symmetry under a particular transformation.
Let $\mathscr{L}(q_i, \dot{q}_i, t)$ be the Lagrangian of the system.
Let $q_i \to q_i(s)$ be the transformation we make the system undergo. We have

$$\frac{d\mathscr{L}}{ds} = 0 \tag{2.4}$$

We can show that the quantity $p_i \frac{dq_i(s)}{ds}$ [7]is conserved. For N such symmetries, we can show that there exist N conserved quantities. For a cyclic coordinate, $q_i(s)$ simply reduces to $q_i + c$ where c is a constant.

3 Hamiltonian formalism

Hamiltonian formalism is a form of expressing the equations of motion in terms of a new and more convenient variable as well as the canonical variables (p, q).

3.1 Legendre transformation

Let

$$df = udx + vdy$$

where

$$\frac{\partial f}{\partial x} = u, \frac{\partial f}{\partial y} = v$$

If we want to change the basis from (x,y) to (u,y) , we introduce a new variable g defined as

$$g = f - ux$$

Thus , we see that

$$dg = vdy - xdu$$

and

$$v = \frac{\partial g}{\partial y}, x = -\frac{\partial g}{\partial u}$$

[7]Note that here, there is a sum over i. We sum over all possible $q_i(s)$ and the corresponding p_i

Now, for the lagrangian, we have

$$d\mathscr{L} = \frac{\partial \mathscr{L}}{\partial q_i} dq_i + \frac{\partial \mathscr{L}}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial \mathscr{L}}{\partial t} dt$$

To eliminate $\dot{q}_i$ as the independent variable, we can do the following:

$$H = p_i \dot{q}_i - \mathscr{L} \tag{3.1}$$

Thus, we get

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial \mathscr{L}}{\partial t} dt \tag{3.2}$$

3.2 Canonical Equations of Hamilton

From we did in the last section, we can easily see that we have the following relations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{3.3}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{3.4}$$

$$\frac{\partial \mathscr{L}}{\partial t} = -\frac{\partial H}{\partial t} \tag{3.5}$$

These equations constitute $The\ Canonical\ Equations\ of\ Hamilton$
We can further derive

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{3.6}$$

Further, if V is velocity independent and T is a homogenous quadratic function of the velocities, we can show that

$$H = T + V$$

This form is identical to our expression for the total energy of the system. Thus,if H has no explicit time dependence, H is a time-conserved quantity. So for the special systems where H is the total energy, we have shown Energy Conservation to hold.
Generally, if H has an explicit time dependence, there is another system exchanging energy with the one under consideration.
We can also obtain the canonical equations from the variational principle as we had done before.
We again use Hamilton's principle and we substitute $\dot{q}_i p_i - H$ in place of the Lagrangian. Then, again obtain the Euler Lagrange Equations , this time in terms of the canonical variables and the canonical equations of motion pop out.

4 Canonical Transformations

Often, the process of solving the problem at hand can be simplified by some variables becoming cyclic. However, things aren't always so simple. Fortunately, we can change coordinates such that several variables become cyclic, thus simplifying our task. We can express these transformations as

$$Q_i = Q_i(q, p, t)$$

and

$$P_i = P_{i(q,p,t)}$$

These will act as our new set of variables. We will also have our new Hamiltonian in this set of coordinates:$K(Q, P, t)$.
In this new system as well, we have the set of Hamilton's equations:

$$\frac{\partial K}{\partial Q_i} = -P_i \tag{4.1}$$

$$\frac{\partial K}{\partial P_i} = Q_i$$

Along with these, we still expect the action integrals to be stationary. So, we have

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H) dt = 0 \tag{4.2}$$

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K) dt = 0$$

From these, we can conclude that the integrands are related as :

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{dF}{dt} \tag{4.3}$$

where F is a function depending on the phase space coordinates with continuous second derivatives. We call a transformation as a canonical transformation if we have $\lambda = 1$ i.e.

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \tag{4.4}$$

4.1 Harmonic Oscillator using Canonical Transformations

Before starting off with the actual analysis of the oscillator, it would be beneficial to investigate the nature of F. We can consider cases where F only a couple of the q's,p's,Q's and P's.F is of paramount importance and it is the generating function of the transformation.
If F is only a function of q,Q and t, we get

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial Q_i} \dot{Q}_i + \frac{\partial F}{\partial t} \tag{4.5}$$

Putting this into (4.4), we get

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t} \tag{4.6}$$

In this, we require coefficients of $\dot{q}_i$ and $\dot{Q}_i$ to be equal(Independence of the two velocities). Thus, we get

$$\frac{\partial F}{\partial q_i} = p_i$$

and

$$\frac{\partial F}{\partial Q_i} = -P_i$$

We are also left with

$$H + \frac{\partial F}{\partial t} = K$$

Now, moving on to the Harmonic Oscillator, we have the Hamiltonian defined as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{4.7}$$

Looking at the form of this equation, we can try out a transformation of the form

$$q = f(P)cosQ$$

$$p = \frac{f(P)}{m\omega}sinQ$$

where f(P) is some function of P alone. Trying out a function of the form F(q,Q) as the generating function. Trying out

$$F = \frac{m\omega q^2}{2}cotQ$$

From the equations we got, we get

$$p = m\omega q cotQ$$

$$P = \frac{m\omega q^2}{2sin^2 Q}$$

So, we get

$$q = \sqrt{\frac{2P}{m\omega}}sinQ \tag{4.8}$$

So, we get

$$f(P) = \sqrt{\frac{2P}{m\omega}}$$

and

$$p = \sqrt{2Pm\omega}cosQ$$

We get

$$H = \omega P \tag{4.9}$$

H is cyclic in Q. So,P is a constant. We get

$$P = \frac{E}{\omega}$$

and

$$\dot{Q} = \omega$$

This has the clear solution

$$Q = \omega t + \alpha \tag{4.10}$$

We get a solution for p and q as:

$$q = \sqrt{\frac{2E}{m\omega^2}} sin(\omega t + \alpha) \tag{4.11}$$

$$p = \sqrt{2mE} cos(\omega t + \alpha)$$

In p-q phase space, the plot is an ellipse.

4.2 Poisson Brackets

Poisson Brackets are mathematical operators with respect to the canonical variables acting on two functions as

$$[F, G] = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \tag{4.12}$$

This definition gives several identities

$$[A, B + C] = [A, B] + [A, C]$$

$$[kA, B] = k[A, B]$$

$$[AB, C] = [A, C]B + [B, C]A$$

$$[A, B] = -[B, A]$$

Another beautiful identity is Jacobi's Identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Poisson Bracket's themselves have several interesting formulations in symmetries.However we will look at the Poisson Brackets involving the Hamiltonian. We have

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t}$$

Using Hamilton's equations , we get

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \tag{4.13}$$

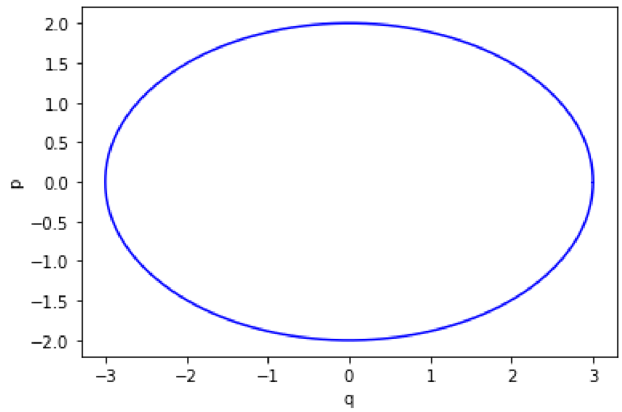

Figure 1: Phase Plot For a Harmonic Oscillator

From this, we can see that if u is a constant of motion , we have

$$[H, u] = 0$$

Further, using Jacobi's identity we get

$$[H, [u, v]] = 0$$

for two constants of motion u and v. Thus, we can see that [u,v] is also a constant of motion.

4.3 Liouville's Theorem

4.3.1 Phase Plots

Phase plots are plots in q-p space which will be 2N dimensional(N is the number of q and corresponding p) These show the nature of the motion described by Hamilton's equations.
We saw in the case of the Harmonic Oscillator, we obtained an ellipse in the q-p space (Figure 1). These phase plots are relevant to the study of Hamilton's equations and their consequences as we can directly study the nature of the motion using these $2N$ dimensional plots. The $2N$ dimensional objects defined by Hamilton's equations possess interesting properties which will be the subject of Liouville's theorem.

4.3.2 Liouville's Theorem

Lioville's Theorem is applicable to statistical systems where an ensemble of states exists. It will also provide interesting results along the way. First off, let us define some quantities.
ρ is the 'density' of particles in $2N$ dimensional phase space.

$\mathbf{J}$ is the current density of the particles moving in the space.
$\mathbf{v}$ is the velocity of the particles in the space.
We have

$$\mathbf{J} = \rho\mathbf{v} \tag{4.14}$$

Also, we have the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{4.15}$$

where $\nabla\cdot$ is defined as

$$\nabla \cdot \mathbf{F} := \frac{\partial F_{q_i}}{\partial q_i} + \frac{\partial F_{p_i}}{\partial p_i}$$

Using this definition, we can proceed. First, we must find out $\nabla \cdot \mathbf{v}$. From the above definition,

$$\nabla \cdot \mathbf{v} = \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}$$

Putting in Hamilton's equation's for $\dot{q}_i$ and $\dot{p}_i$, we get

$$\nabla \cdot \mathbf{v} = \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0$$

Thus, we get

$$\nabla \cdot \mathbf{v} = 0 \tag{4.16}$$

We have

$$\nabla \cdot (\mathbf{v}\rho) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho \tag{4.17}$$

Thus, we get

$$\nabla \cdot (\mathbf{v}\rho) = \mathbf{v} \cdot \nabla \rho \tag{4.18}$$

We have

$$\mathbf{v} \cdot \nabla \rho = \dot{q}_i \frac{\partial \rho}{\partial q_i} + \dot{p}_i \frac{\partial \rho}{\partial p_i}$$

Combining this with the continuity equation, we get

$$\dot{q}_i \frac{\partial \rho}{\partial q_i} + \dot{p}_i \frac{\partial \rho}{\partial p_i} + \frac{\partial \rho}{\partial t} = 0$$

So,we obtain

$$\frac{d\rho}{dt} = 0 \tag{4.19}$$

The flow is incompressible.
Liouville's theorem states that: *The density of states in an ensemble of many identical states with different initial conditions is constant along every trajectory in phase space.*

5 Appendix

5.1 The Stationary Value of a Functional

We have
$$F = \int_{x_1}^{x_2} f(y, y', x)dx \tag{5.1.1}$$
and
$$\delta F = 0 \tag{5.1.2}$$
Thus,
$$\delta \int_{x_1}^{x_2} f(y, y', x)dx = 0 \tag{5.1.3}$$
Let $y(x, 0)$ be the path for which the functional attains a stationary value.
We want to consider a path which is slightly deviated from this one; so we take $y(x, \alpha) = y(x, 0) + \alpha\eta(x)$ [8].
Every path we choose must have the same value at the endpoints i.e. at x_1 and x_2. This tells is that $\eta(x_1)$=$\eta(x_2)$=0
Let us denote
$$F(\alpha) = \int_{x_1}^{x_2} f(y(x, \alpha), y'(x, \alpha), x)dx \tag{5.1.4}$$
We see that
$$\left[\frac{dF}{d\alpha}\right]_{\alpha=0} = 0 \tag{5.1.5}$$
We have
$$\int_{x_1}^{x2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx = \frac{dF}{d\alpha} \tag{5.1.6}$$
After substituting the suitable expression for y, we see that
$$\frac{dF}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) \eta(x)dx \tag{5.1.7}$$
At α=0, We must have the integral to be zero.
Furthermore, the integrand has no explicit dependence on α.
Since η is an arbitrary function only restricted by the boundary value, we can see that
$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \tag{5.1.8}$$

5.2 Invariance of the Lagragian upon addition of certain terms

Let $\mathscr{L}$ be the Lagrangian of a system. Consider $\mathscr{L}'$ to be $\mathscr{L} + \frac{dF}{dt}$ where F doesn't explicitly depend on the generalised velocities.

[8] $\eta(x)$ is some function and α is a constant

We have
$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\left[\frac{dF}{dt}\right]\right)-\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)=\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)-\frac{\partial \mathscr{L}}{\partial q_i} \tag{5.2.1}$$

We also have
$$\frac{dF}{dt}=\frac{\partial F}{\partial q_j}\dot{q}_j+\frac{\partial F}{\partial t}$$

And
$$\frac{\partial F}{\partial \dot{q}_i}=0$$

Thus, using these in 5.2.1 we get
$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}_i}\right)-\frac{\partial \mathscr{L}'}{\partial q_i}=\frac{\partial^2 F}{\partial q_i \partial q_j}-\frac{\partial^2 F}{\partial q_j \partial q_i} \tag{5.2.2}$$

The RHS of the above equation is zero for F in the domain we are concerned with (Simply Connected). Thus, we obtain
$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}_i}\right)-\frac{\partial \mathscr{L}'}{\partial q_i}=0$$

6 Bibliography/References

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2. 4 July-11 July:
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