Classical Mechanics

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Overview of Topics

Following are the topics covered in my SOS report on Classical Mechanics.

- D'Alembert's Principle
- Euler-Lagrange Equations
- Principle of Least Action
- Hamilton's Principle
- Hamilton-Jacobi Theory
- Classical Field Theories and examples

Classical Field Theory

I will be talking about Classical Field Theory in this small video. Classical Field Theory deals with the application of the Principle of Least Action to continuous fields in the space one is concerned with. The Euler Lagrange Equations for a field have a similar form as the ones for discreet particles.For fields, we have to integrate over the entire space we are concerned with (if we are in 3-space, we have to integrate over d^3x)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) = 0$$

Here, we sum over the terms with the indices (in accordance with Einstein's Summation Convention).

The Electromagnetic Field

To apply the theory to a real field, there is no better candidate than the Electromagnetic Field.
The field is characterised by the 4-vector potential A_μ. From this potential, we can obtain the electric and magnetic fields as well as the Lorentz force law for particles interacting with it. We define a Field tensor as

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

This Field tensor encompasses the electric and magnetic fields within it. The Field tensor helps write Maxwell's Equations in a compact form, thus making operating on the fields less complicated (especially for the Lorentz Transformations of the Fields).

The Electromagnetic Field

To apply the Principle of Least Action, we must first define the Lagrangian for the particles in the field (To obtain the Lorentz Force law) of for the fields (To obtain Maxwell's equations). For a particle, we have the Action defined as

$$S = -\left(\int m d\tau + \int eA_\mu dx^\mu \right)$$

[1] From this, we obtain the Lagrangian as

$$\mathscr{L} = -(m\sqrt{1 - \dot{x}^2} + e(A_0 + \dot{x}^m A_m))$$

When we use the Euler-Lagrange Equations here, we obtain the relativistic Lorentz force law.

[1] τ is the proper time

The Electromagnetic Field

The Lagrangian for the fields themselves are defined as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^{\mu}A_{\nu}$$

From this Lagrangian,we obtain Maxwell's equations as

$$\partial_{\mu}F^{\mu\nu} = j^{\nu}$$

$$\partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} + \partial_{\lambda}F_{\mu\nu} = 0$$

Here, j^{μ} is the 4-current density.[2]

[2]Here, we have set the fundamental constants to 1 for our convenience

More Examples

This method can be applied to more fields to obtain the field equations in different cases. The Principle of Least Action is valid even in the relativistic as well as the quantum realm, making field theory extremely important. The theory can be extended to complex fields as well. The field equations in General Relativity, Quantum Electrodynamics and in many more theories have their own corresponding Lagrangians. The Lagrangians are created by looking at the symmetries of the system (Noether's Theorem is put to work).

Conclusion

The Theory of Classical Mechanics is of paramount important as it not only gives new methods of approaching complicated problems but also gifts us fundamental principles which are applicable all throughout physics. While the calculations might be cumbersome at times, the beauty of the principles more than makes up for it.

Thank You!