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Differential Equations

Lorenz Systems (Chaos Theory)

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Contents

Chapter 1

Abstract

1.1 Objectives

To understand the theoretical aspects of Lorenz Systems and mathematics involved. Important properties and Analysis is presented.

To model some real world applications using Lorenz Equations. We have studied and modeled Atmospheric Modeling, A simple Logistic Map, Chua's Circuit and Image Encryption Application using Lorenz Equations.

Simulations for Logistic Map, Lorenz Model and experimentation with parameters is available at `https://github.com/AnuragSahu/Differential-Equations-Project`

1.2 The Equations

Lorenz was always looking for whats the most simple set of equations that exhibit this chaotic property. And he found a set of three equations, a simplified version of equations used to model convection, with only three variables that change over time. The following is the lorenz system

The Equations look like :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$$

Here $(x, y, z) \in R3$ and $\rho, \beta, \sigma > 0$ x, y, z are called as the state variables. There are three positive parameters sigma, beta, and rho, with values originally chosen by Lorenz to be 10, 8/3 and 28, respectively. For these values, Lorenz observed a stable chaotic attractor.

These equations are used to model some dynamical systems. They were initially assumed to represent idealized behavior of the earth's atmosphere.

Chapter 2

Analysis of Lorenz Equations

2.1 Relevant Mathematical Definitions:

Dynamical System

A system in which a function describes the time dependence of a point in a geometrical space. It is defined by a state $x(t)$ and an evolution function Φ.These systems can be discrete or continuous.We want to study the long term qualitative behaviour of some of the dynamical systems which can be modeled using Lorenz equations, and systems in which see chaos in their time-evolution.

The Evolution Rule

In a dynamical system, quantities change over time. It is a function (an implicit relation) that describes what future states follow from the current state.It gives the state of the system in the next time step. To study the time evolution we have to iteratively apply this rule (solving or integrating the system). This function can be deterministic or random, based on which we have two types major types of Dynamical Systems. They are :

- **Deterministic**: For a given time interval only one future state follows from the current state.
- **Random**: There can be more than one future states possible from the current state resulting in a probability distribution of these possible states.

If the system is discrete the evolution function is called as a **map**. The dynamical system is then defined by a **difference equation** (system may have multiple set of these)

$$\Delta x_t = \Phi\left(x_t\right)$$

If the system is continuous it is called as **Flow**. The dynamical system is then defined by a **differential equation** (system may have multiple set of these)

$$\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t} = \Phi\left(x\left(t\right)\right)$$

Eg. 1D - A voltage source with a switch and an inductor, 2D - Pendulum, 3D - Chua's Circuit

Dynamical State and State Space

The entire state of the system can be uniquely defined with a dynamical state. It can be a vector with minimum number of elements which are called **State Variables**. A set of all possible configurations of a system. A state has all the variables which describe the system fully.

The dynamics of a system can be understood by a **state variable equation**. Solving the system of these differential equations - either get the expression for the solution explicity, or given a set of initial values of the state variables, try to approximately tell how are solutions are behaving by numerically solving it. We go with the second method for non-linear systems

Trajectory or Orbit

If the system can be solved, given an initial point it is possible to determine all its future positions. These collection of points is called as a trajectory. These are nothing but the points in the time evolution of the system.

Attractors, Lorenz Attractors and Strange Attractors

Attractor is a set of points to which all neighbouring trajectories converge. Eg. Stable fixed or Equilibrium points, stable limit cycles.

2.2 Properties of the Lorenz Equations

- **Non linearity**: The flow of the dynamical system represented by system of Lorenz equations is nonlinear due to the yz and xz in the second and third equations respectively.

- **Autonomous**: This system of differential equations is autonomous as time does not explicitly appear on the right hand side.As they involve only first order time derivatives so that (with the autonomy) the evolution depends only on the instantaneous value of x,y,z.

- **Symmetry**: The Lorenz equations are invariant under the following transformation:

$$(x, y, z) \longrightarrow (-x, -y, z)$$

$$(-x)\prime = \sigma\,(-y - (-x)) \implies (x)\prime = \sigma\,(y - x)$$

$$(-y)\prime = -x\,(\rho - z) + y = (x)\,(\rho z)\,(y) \implies (y)\prime = x\,(\rho z) - y$$

$$(z)\prime = (-x)\,(-y) - \beta\,(z) \implies (z)\prime = xy - \beta z$$

  Hence if (x(t), y(t), z(t)) is a solution, so is (x(t), y(t), z(t))

- **In-variance of the Z-axis** :
  - From the Lorenz system of equations, we see that, if $x\,(0) = 0$ and $y\,(0) = 0$, then $x = 0$ and $y = 0\ \forall\ t$.
  - Therefore, the z-axis is an orbit on which $z\prime = -\beta z$. Solving it gives $z\,(t) = z\,(0)\exp\{-\beta t\}$, for $x, y = 0$.
  - Therefore, the z-axis is always a part of the stable manifold for the equilibrium at the origin. It implies that all trajectories on the Z-axis remain on the Z-axis, and approach the origin.
  - Since $x = 0, y > 0 => x\prime > 0$ and $x = 0, y < 0 => x\prime < 0$, all trajectories that rotate about the Z-axis must move clockwise with increasing time(looking from above onto the $X - Y$ plane).

- **Dissipative Property and Volume contraction:**
  - A system is dissipative if every orbit eventually moves away from infinity. The Lorenz system is dissipative i.e. volumes in phase-space contract under the flow.
  - The divergence of the system is negative i.e, $-\,(\sigma + \beta + 1)$. The volume of this region will decrease with $\exp\{-\,(\sigma + \beta + 1)\}$.

α and β are the parameters in the Lorenz Equations.

Therefore, the set towards which all the trajectories tend has Zero Volume.

- **Equilibrium Points** :

  Eq points must be real because the state space is real. To obtain equilibrium points, we need to make $\dot{x}, \dot{y}, \dot{z}$ zero in the Lorenz system of equations. We get $x = 0, \pm\sqrt{\beta\,(\rho - 1)}$ Therefore, the equilibrium points are as follows,

$$C_0 = (0, 0, 0)$$

$$C_1 = \left(-\sqrt{\beta\,(\rho-1)}, -\sqrt{\beta\,(\rho-1)}, \rho - 1\right)$$
$$C_2 = \left(\sqrt{\beta\,(\rho-1)}, \sqrt{\beta\,(\rho-1)}, \rho - 1\right)$$

  Now,to compute the stability of these points , lets compute them in the Jacobian Matrix (J) , here we get :

$$J_{C0} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$
$$J_{C1} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -1\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix}$$
$$J_{C2} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -1\sqrt{b(r-1)} & -1\sqrt{b(r-1)} & -b \end{bmatrix}$$

The following diagram depicts the nature of flow near the equilibrium points in the Lorenz System (reference is mentioned)

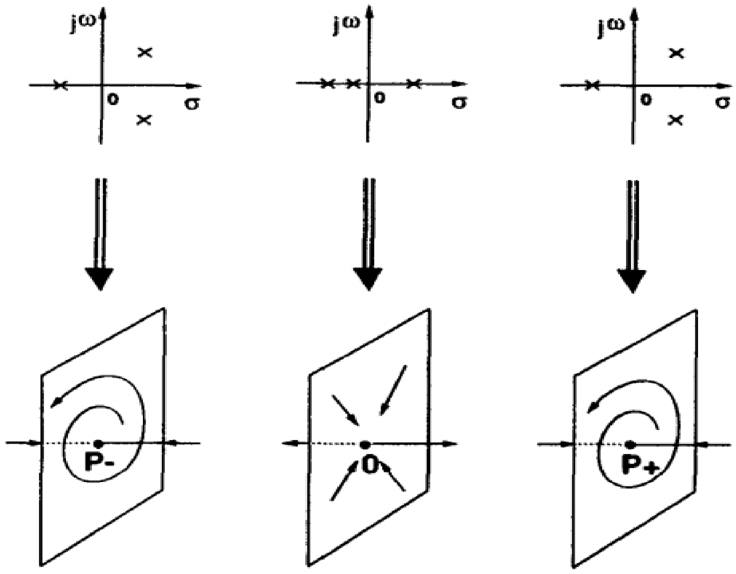


Fig. L.1. A schematic view of the flow near equilibrium points in the Lorenz system. The equilibrium points are obtained at $\sigma = 10$, $b = 8/3$, and $r = 28$.

- **Experimenting with r parameter**

In the above derived equations , we can see how r is an essential parameter , and to understand it's effect , we monitor the variations of $3DLorenzSystem$ plot , while fixing σ and b as following :

$$\sigma = 10.00$$

$$b = 2.67$$

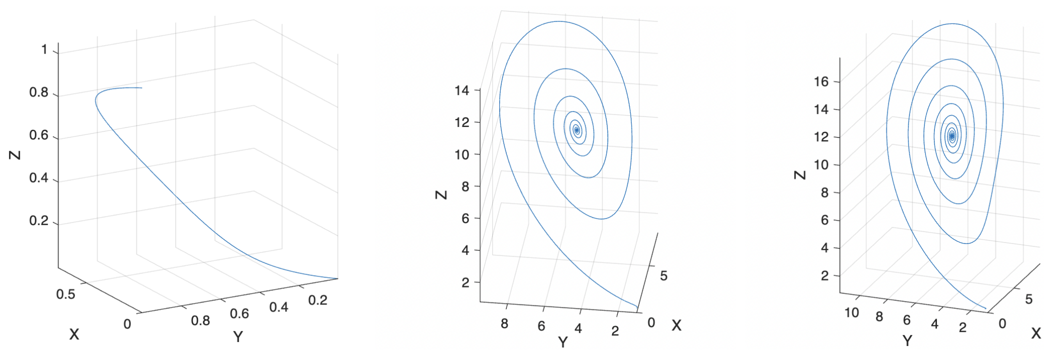


Figure 2.1: (a) r=0.5 (b) r=10 (c) r=12

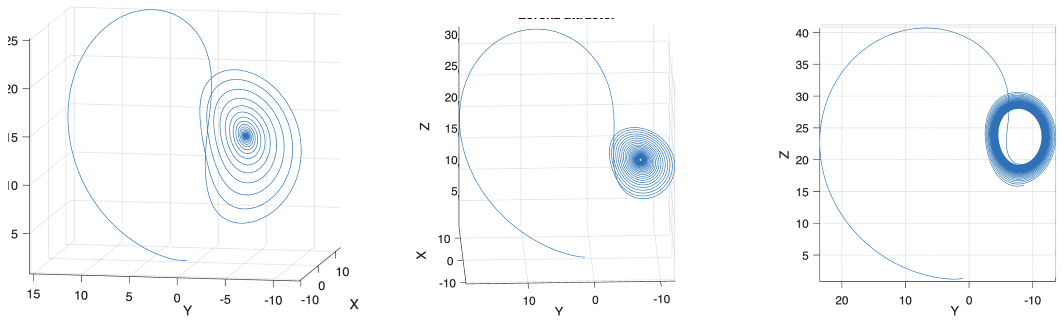


Figure 2.2: (a) r=16 (b) r=10 (c) r=20

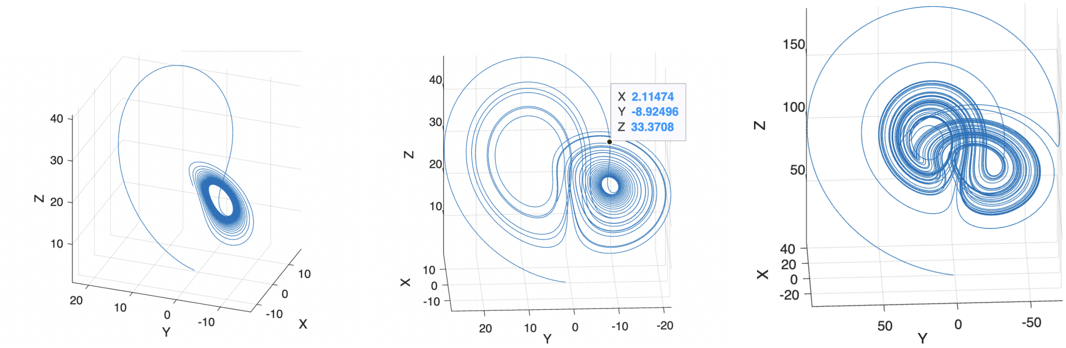


Figure 2.3: (a) r=24.40 (b) r=28.0 (c) r=100

The Code for the plots are available at

`https://github.com/AnuragSahu/Differential-Equations-Project`

- **Sensitivity to Initial Conditions**:

  Two trajectories begin with very close initial conditions;

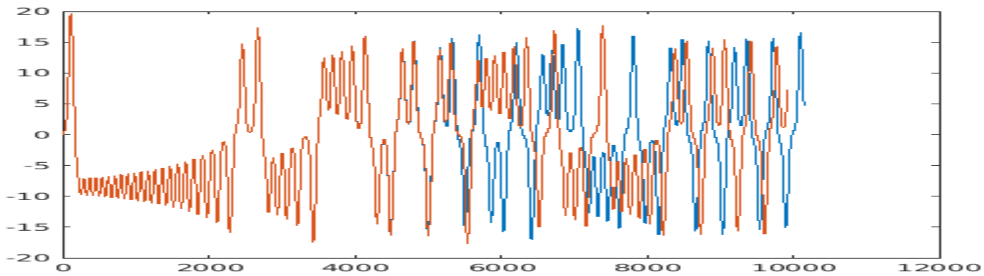
$$[x_0, y_0, z_0]_1 = [0, 1, 1.05]\,;$$

$$[x_0, y_0, z_0]_2 = [0, 1.000001, 1.05]\,;$$

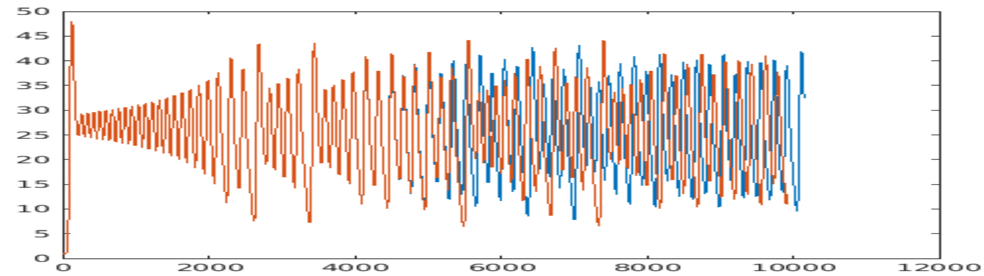
  Simulated for 100 seconds. For around first 60 time units, the two trajectories seem identical. However, after that, they seem completely unrelated to each other.

  Three plots with x-values, y-values, z-values (each with the two numerical solutions of the Lorenz equations) are plotted. These trajectories are plotted with respect to time.

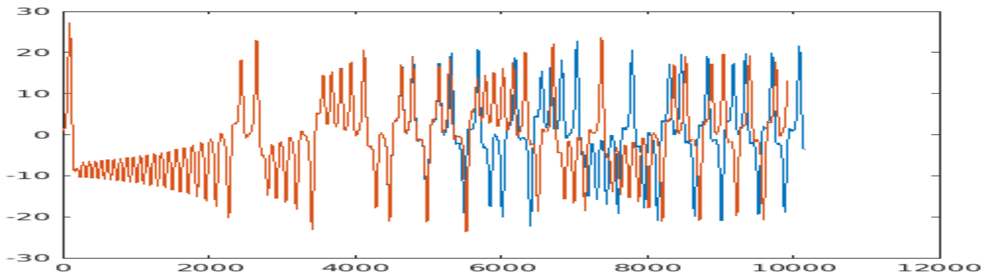
X Vs t



Y Vs t



Z Vs t



The code for plots are available at :

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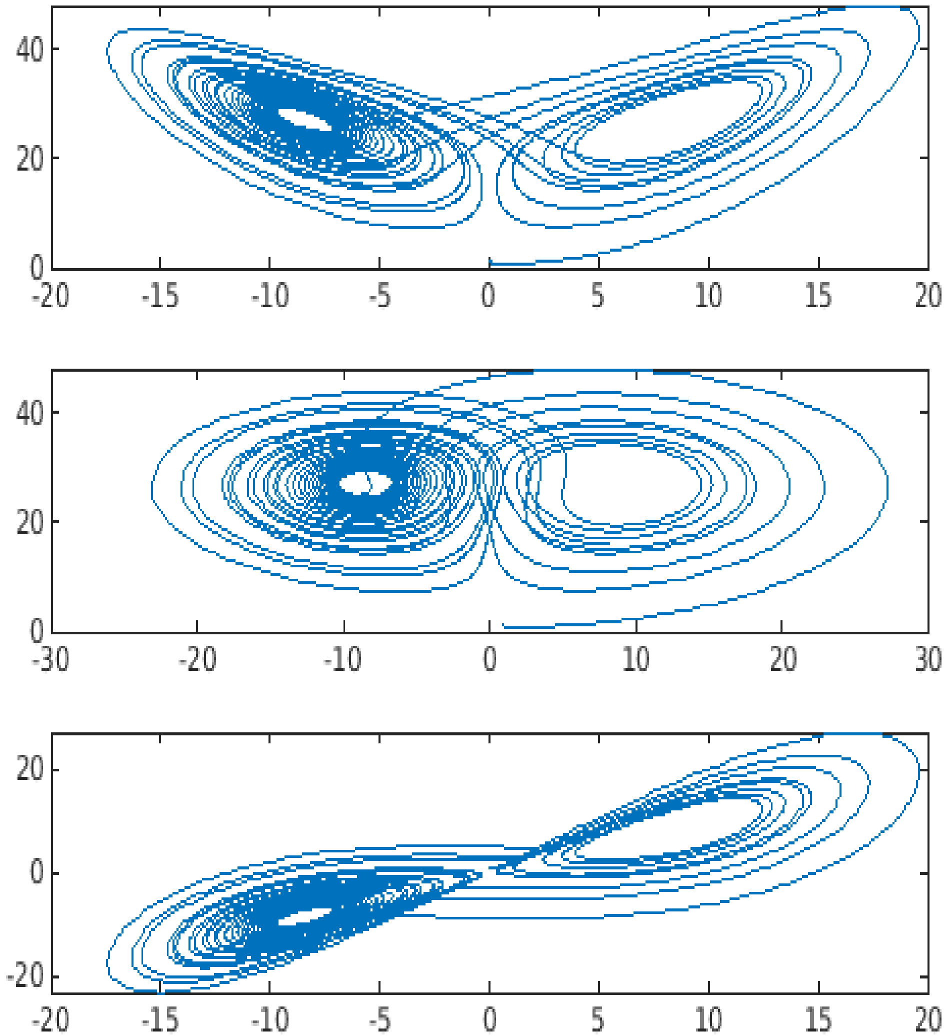
- **Phase Space Structure - Lorenz Model** To visualise the solution of the Lorenz system of equations as trajectory is phase space which gives **Butterfly Wing Pattern**, plot $x\,(t)$ Vs $z\,(t)$

  In the following plot we plotted the 2D projections of the 3D plot of Lorenz system. First is $x\,(t)$ Vs $z\,(t)$ Second is $y\,(t)$ Vs $z\,(t)$ Third is $x\,(t)$ Vs $y\,(t)$

X Vs t



The Code for plots are available at

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In the above plot, we projected the 3D trajectory to 2D space. Therefore, the trajectory appears to cross, but no crossing actually occurs in 3D (due to the uniqueness theorem). That is, the same state never repeats

- **Fixed Points:**
  - A point that does not change with an application of a map, system of differential equations, etc. If a point x_0 is a fixed point of a function $f\,(x$, then $f\,(x_0) = x_0$.
  - It is the simplest kind of an orbit. It is a periodic point with $period = 1$.

- A Lyapunov stable attracting fixed point is said to be a stable fixed point.

$(x^*, y^*, z^*) = (0, 0, 0)$ is a fixed point for all values of the parameters.

For $\rho > 1$ there is also a pair of fixed points $C^{\pm}$ at $x^* = y^* = \pm\sqrt{\beta(\rho - 1)}, z^* = \rho - 1$

- **Removing the two non linear terms:** Linearization of the original equations about the origin gives the following equations. Note that the z-motion is decoupled.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = \rho x - y$$

$$\dot{z} = -\beta z$$

In the matrix form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ \rho & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Trace is $\tau = -\sigma - 1 < 0$

Determinant is $\Delta = \sigma(1 - \rho)$

- For $\rho > 1, \Delta < 0$. Therefore, origin is a Saddle point
- For $\rho > 1$, $\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - \rho) = (\sigma - 1)^2 + 4\sigma\tau > 0 \implies$ a stable node. Therefore, origin is a sink
- For $\rho < 1$, every trajectory approaches the origin as $t \to \infty$ the origin is globally stable.
- Note : **Limit cycle** is an isolated closed trajectory seen in some non-linear systems. At least one other trajectory spirals into it either as $t \to \infty$ or as $t \to -\infty$. There is another trajectory which spirals out of it. All oscillatory systems will settle into a stable limit cycle.

- **Boundedness of the Solutions:** The easiest way to see that the solution is bounded in time is by looking at the motion of the solution in phase space, (x, y, z), as the flow of a fluid, with velocity $(\dot{x}, \dot{y}, \dot{z})$. The divergence of this flow is given by

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z},$$

and measures how the volume of a fluid particle or parcel changes – a positive divergence means that the fluid volume is increasing locally, and a negative volume means that the fluid volume is shrinking locally . Looking at the Lorenz equations, the divergence comes out to be:

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + b + 1).$$

Since σ and b are both positive, real constants, the divergence is a negative number, $-(\sigma + b + 1) < -1$. Therefore, each small volume shrinks to zero as the time $t \to \infty$, at a rate which is independent of x, y and z.

2.3 Behaviour of Solutions - Variation of Parameters

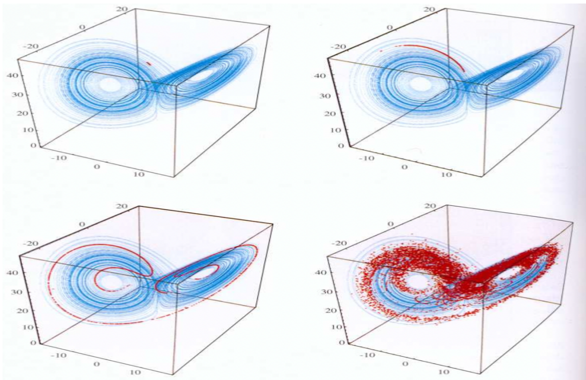
Eigen Values of the Jacobian:

- We have 3 eigen values for the Lorenz system.
- If 3 eigen values are real and negative, then the equilibrium point is a sink
- If 3 eigen values are real and positive, then the equilibrium point is a source
- One eigen values is positive real and 2 are negative real, then The equilibrium point is a saddle point
- The negative eigen vectors plane is a stable subspace, positive eigen value direction is unstable.
- 2 complex conjugate and 1 real positive: A plane can be associated with complex eigen values. The positive real part results in a vector or a direction with an outgoing spiral. In this direction, the equilibrium point is unstable.
- 2 complex conjugate and 1 real negative: A plane can be associated with complex eigen values. The negative real part results in a vector or a direction with an inward going spiral. In this direction, the equilibrium point is stable.
- By varying the parameter ρ, we will get all the above cases and At $\rho = 1$, the origin becomes unstable, and two stable equilibrium points (the other two equilibrium) emerge, C_1, C_2 or $C^{\pm}$.

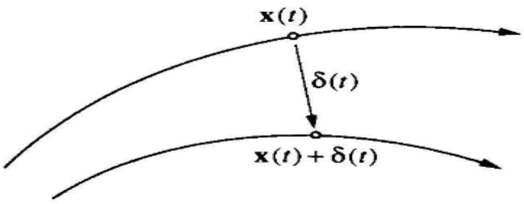
2.4 Until what time can we predict (weather) correctly?

The entire trajectory of the attractor exhibits high sensitive dependence on initial conditions . Two trajectories starting very close together will rapidly diverge from each other , and can have different futures .The practical implication is that the long-term prediction becomes impossible , due to small uncertainties giving rise to fast paced massive amplifications .Here's a diagram that shows the above statement , and the initial values differ by 0.000001 only .

"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" If this theory is correct, yes!This "sensitive dependence on initial conditions" found by Lorenz is known affectionately as Lorenz's "butterfly effect"



Suppose we let transients decay so that the trajectory is "on" the attractor. Suppose x(t) is a point on the attractor at time t, and consider a nearby point, say $x(t) + \delta(t)$, where δ is a tiny separation vector of initial length $||\delta(0)|| = 10^{-15}$



In numerical studies of the Lorenz attractor, one finds that

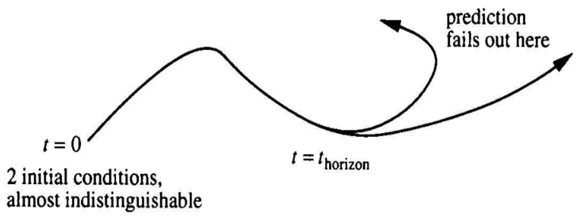
$$||\delta(t)|| = ||\delta(0)|| * \exp^{\lambda * t}, \text{ where } \lambda = 0.9$$

Hence, the neighbouring trajectories separate our exponentially fast !

Note: The number λ is called the Lyapunov Exponent .There are actually n different Lyapunov exponents for an n-dimensional system. This factor varies throughout the orbit and average of this factor helps us to know the rate of divergence and to what time can be reasonably predict some variable (say atmosphere pressure, etc).

Suppose we measure the initial conditions of an experimental system very accurately. Of course no measurement is perfect - there is always some error $||\delta(0)||$ between our estimate and the true initial state. After a time t the discrepancy grows to

$$||\delta(t)|| = ||\delta(0)|| \exp^{\lambda(t)}.$$



Reference research paper is mentioned in the references.

2.5 Phase Space - Lorenz Model

To construct and analyse solutions of a dynamical system. The solution is a curve(trajectory or an orbit)in the phase space, parametrized by time. 'The state diagram would go on till the sun melts the wax in our wings' - Subryamanyam Chabdrashekhar

By substituting the current (x,y,z) in the set of differential equations, we will know how far and which direction to go from the current state to the next state.

With the state space diagram, we basically defined at every point where to go and how much to go (DEs have defined arrows at every point, the state evolves along the arrows, like if u drop a leaf it flows with that flow).

To study the system well in a numerical approach, we want to see all the initial conditions and see what the system will do (ideally for all ICs). Where do we start? Equilibrium points. Then we go very close to these equilibrium points because here non-linearity will vanish. Jacobian matrix is used to obtain the local linearisation. We try to make the system work around these equilibrium points.

Eigen Directions and Stability:

In a 2d system there are 2 directions such that if the initial vector is in that direction, the resultant vec is also in the same direction. This is an eigen direction, vector along this direction is eigen vector). We will try to examine these directions.

For $\dot{X} = Ax, \dot{X}$ will be on the same line as X. We study the 2 eigen directions and take linear combination of the solution to get final solution of the system. We need to understand the systems behaviour in all cases of eigen values. Same can be extended to 3D (Lorenz Systems) with X being 3x1 vector.

Vector space behaves like magnetic lines of force except at poles (at equilibrium points with magnitude 0, that is the initial conditions), because at every point in the phase space, there is a unique arrow.

- **Phase Portrait:** A geometric representation of the trajectories of a dynamical system in the phase space (with no time parametrization). They reveal if an **attractor**(stable point or sink), a **repeller**(an unstable point or source) or **limit cycle** is present for the chosen parameter value.

- **Periodic Points** - Points that come back to the same value after a finite number of iterations of the function are called periodic points.

  Trajectories are indicated with arrows , and stable steady states with dots.Unstable steady states with circles The axes are of state variables.

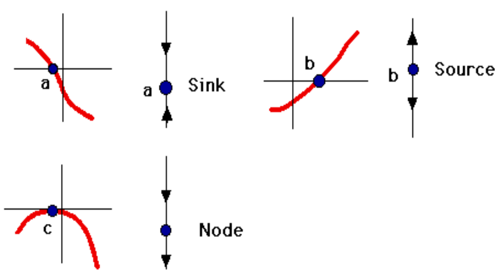
- **Source, Sink and Node:**

  - **Source** - An equilibrium point 'p' is called a source if $f\prime(p) > 0$. Nearby solutions diverge from here. ALl eigen values are negative.

  - **Sink** - An equilibrium point 'p' is called a sink if $f\prime(p) < 0$. Nearby solutions nearby solutions converge to this point. All eigen values are positive.

  - **Node** - An equilibrium point 'p' is called a node If $f\prime(p) = 0$. Then we get no information. All points other than source and sink.

  - **Saddle point or Minimax point-** A stationary point that is not a local extremum.

    Some eigen values are positive, some are negative.

- **Stability**

Unstable	Most of the system's solutions tend towards ∞ over time
Asymptotically stable	All of the system's solutions tend to 0 over time
Neutrally stable	None of the system's solutions tend towards ∞ over time, but most solutions do not tend towards 0 either

Stable and Unstable Equilibrium Points - A fixed point or an equilibrium is considered stable if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable.

2.6 Chaos - Deterministic Chaos

If the evolution of a system depends on the initial conditions, then it is said to be chaotic as seen in complex nonlinear dynamic systems.A chaotic system is deterministic but harbours chaotic behaviour. It has a few simple equations and randomness is not involved in the development of the future states of the system.The systems that appear to be chaotic may have some underlying order.They are unstable as they significantly react to outside disturbances.

2.7 Characteristics of a chaotic system (Deterministic):

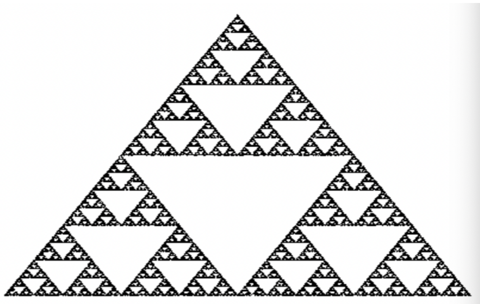
- **Main factors:**
- Non Periodic Behaviour or Aperiodic Nature (doubling periods are observed in nature).
- Sensitive Dependence on initial conditions
- Bounded Nature of the solutions or Trajectories
- **Other factors:**
- Motion (long term prediction) is almost impossible to predict, due to extreme divergence of the series.
- Non-linear system.
- At least one Lyapunov exponent is positive (for divergence of trajectories)

Unpredictability in Deterministic Chaos

Each individual trajectory is unique and evolves distinguish-ably in the system.If we do not the exact initial data, we can completely predict a single trajectory for all past or future times.If too many individual trajectories (infinite) are possible, the probability of exactly knowing the initial conditions for one of these trajectories is usually equal zero.

2.8 Fractals - "The Chaos Pattern"

Fractals are geometric objects are repeated on all scales by itself . That is , on scaling a fractal object it will look similar or exactly like the original shape . For example :



The property of self-similarity or scaling is closely related to the notion of dimension. In fact, the name "fractal" comes from property that fractal objects have fractional dimension. Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over in an ongoing feedback loop. Driven by recursion, fractals are images of dynamic systems – the pictures of Chaos. Geometrically, they exist in between our familiar dimensions. Fractal patterns are extremely familiar, since nature is full of fractals. So, a fractal image is a visual representation of a strange attractor (or fractal space) that defines the orbit of a deterministic system that behaves chaotically.
Reference for Images : Google Images

Chapter 3

The Logistic Map - A Simple Chaotic Model

Consider the model of population growth one of the most famous dynamical systems, the logistic map. The Logistic map is a one-dimensional discrete time map that despite its formal simplicity, exhibits an unexpected degree of complexity. The logistic map is deterministic chaos which is determined by the following equation:

$$x_{n+1} = \lambda x_n(1 - x_n) \ with \ n = 0, 1, 2, 3, 4...$$

Given a starting value $0 <= x_0 <= 1$ and a positive parameter $0 < \lambda < 4$ the map produces a sequence of values

$$x_0, x_1, x_2, ..$$

that we get by iterating.

Why is this so important?

Let's say have a population of a species in which individuals replicate. Lets say for simplicity that in every generation n = 0,1,2,.. an individual has on average λ offspring before it dies. This could be any positive number. It doesn't have to be an integer number of off springs. this λ is called reproduction rate.

Now let's say the population at time n=0 has U_n individuals. A generation later the population size will be

$$U_{n+1} = \lambda U_n$$

Therefore, If we start with an initial population U_0, we get

$$U_n = \lambda^n U_0$$

So depending on the value of λ we can have two different scenarios.

- if $0 < \lambda < 1$ then $In\lambda < 0$ and thus $U_{n+1} < U_n$,the population dies out.
- if $\lambda > 1$ then $In\lambda > 0$ and thus $U_{n+1} > U_n$,the population grows indefinitely and exponentially.

so if $\lambda > 1$ we expect exponential growth. Although this can't go on forever. Other factors will come into play if the population gets too big, e.g. limiting resources, competition, limited space, etc. One way to account for this is by saying that the reproduction rate λ effectively depends in some way on the size of the population.

To this end we produce a capacity C, the largest population size which if reached reduces the reproduction to zero or a very small number. and reproduction at some maximum value λ_0 if $U << C$. so the reproduction rate can be written as

$$\lambda = \lambda_0(1 - U/C)$$

and from this we get

$$U_{n+1} = \lambda_0(1 - U_n/C)U_n$$

which can iterated for a chosen set of parameters λ and C and an initial population $U << C$. We can this by expressing the population as a fraction of the capacity C.

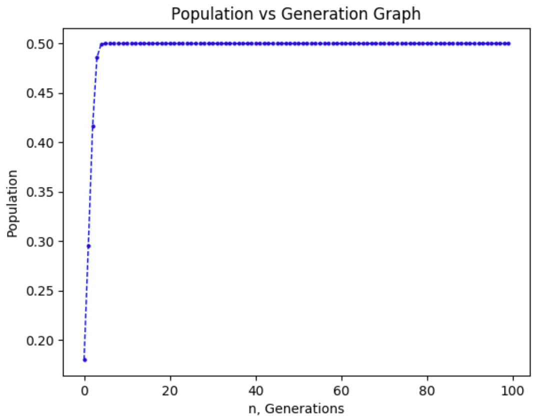
$$x_b = U_n/C$$

and using this we get:

$$x_{n+1} = \lambda_0(1 - x_0)x_0.$$

3.1 Experimenting with λ:

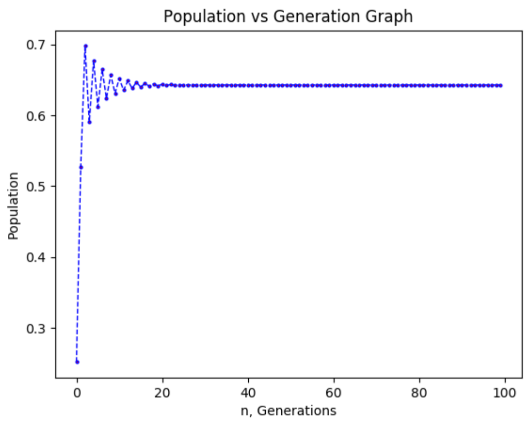
If we select a particular value of lambda, It can be seen that the value of x maxout at some value shown as by the below diagrams sometimes they oscillate between different values.



X = 0.1

Lambda = 2

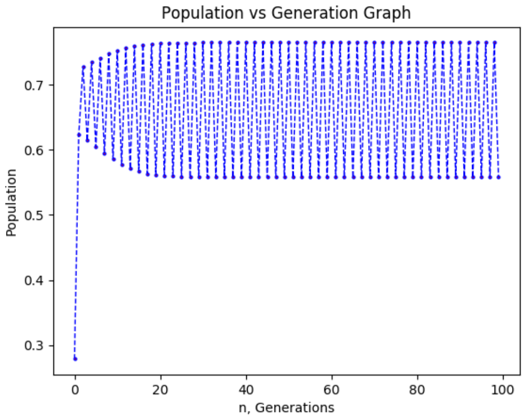
Maxs out at 1 point



X = 0.2

Lambda = 2.8

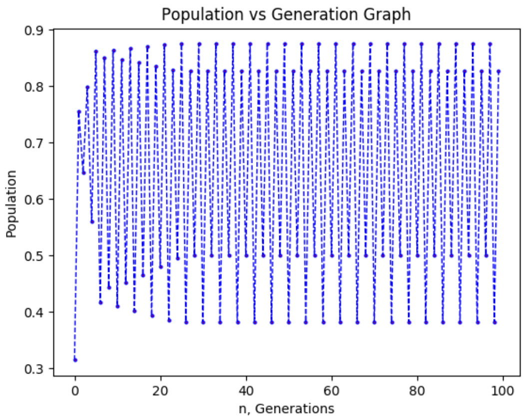
Maxs out at 1 point



X = 0.2

Lambda = 3.1

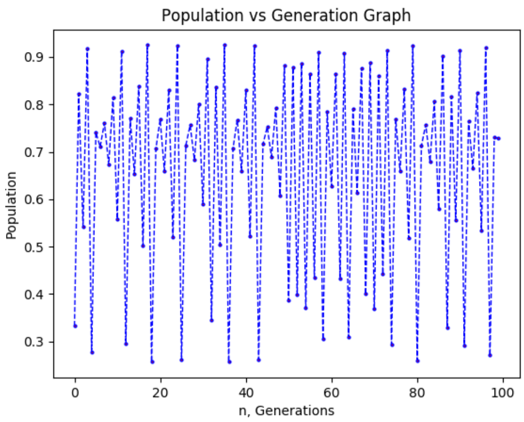
Maxs out at 2 points



X = 0.3

Lambda = 3.5

Maxs out at 4 points



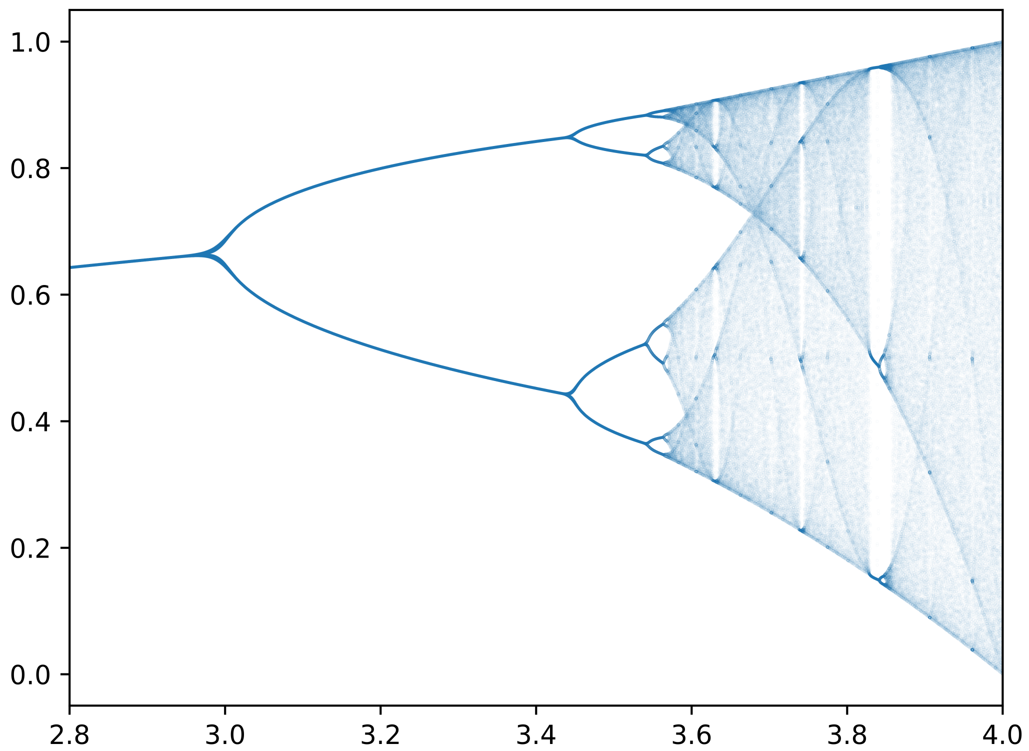
X = 0.35

Lambda = 3.7, Maxs out at 8 points

The Code for plots are available at

`https://github.com/AnuragSahu/Differential-Equations-Project`

For various values of the lambda we find the values at which the values at which the population max outs and plot the lambda values in x-axis vs the values at which the population saturates as y axis. By this we obtain the bifurcation diagram. Chaos can be seen in the bifurcation diagram.



Bifurcation Plot

The Code for plots are available at

`https://github.com/AnuragSahu/Differential-Equations-Project`

Chapter 4

Atmosphere Model - The First Lorenz Model:

4.1 Atmosphere Modeling

So taking a look into the earth's atmosphere, we have the surface of the earth towards the bottom of the earth the surface is warm, it being heated, crust is absorbing the energy from the sun and as we go up in the atmosphere it gets colder. And thats the kinds of the setup we have here its warmer at the bottom and colder at the top. Lorenz equations are looking at how these two things interact with each other, The change in Temperature ΔT, for small ΔT we get a kind of a linear variation in temperature but it doesn't actually affect the motion of the fluid. But if we make ΔT larger we get some steady convection currents and as we increase ΔT the more convection currents increase. [Refer to figure 3.1].

So at the bottom the earth is warm making the air warmer, this warmer air gets lifeted up and at the top, it gets cold and the colder air startes to fall down, this cold air is again heated up by the earths surface warmth and this cycle goes on and on and on.

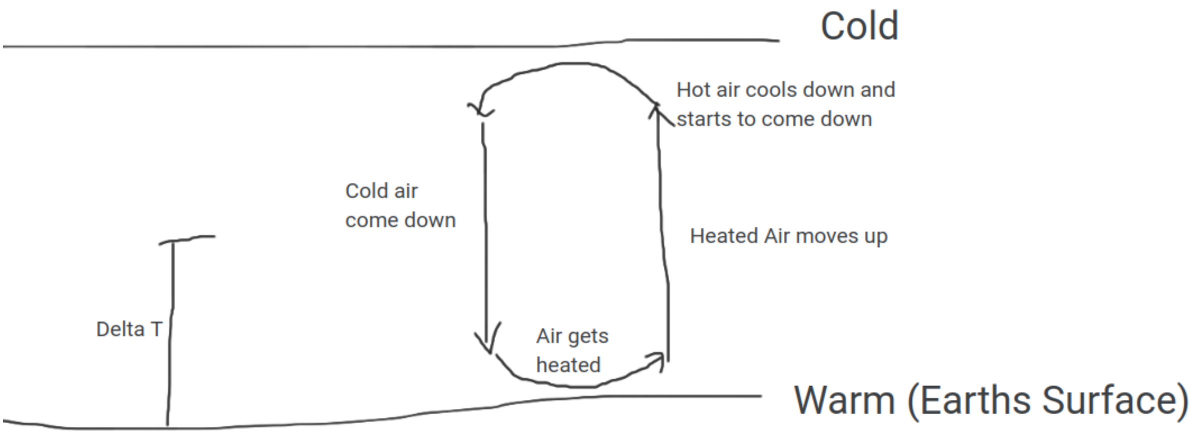


Figure 4.1: Reference Figure

And this was modeled by Lorenz Equations. The Equations look like :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$$

where x corresponded to the intensity of the fluid motion (how fast or slow it is moving). y and z are the horizontal and vertical temperature variations. And then we had three parameters σ, β and ρ, on earth it was found that σ was found to be 10 and β was found to be 8/3. The parameter σ is called the Prandtl number, r is called the Rayleigh number and β scaling (aspect-ratio) coefficient. These are typical quantities to describe the properties of fluids in general.

Then we also have ρ, ρ was found to change with temperature, and we want to see how ρ changes with Temperature. So if we know the change in the Temperature, we can link to back to ρ and then we can put this ρ in the equation and get how the fluids (the weather) will change.

4.2 Solving the ODE

First we need to find the critical points or the equilibrium points.

We will set each one of the equation equal to zero,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

This gives us :

$$\sigma(y - x) = 0 \text{ - } (3.1)$$

$$x(\rho - z) - y = 0 \text{ - } (3.2)$$

$$xy - \beta z = 0 \text{ - } (3.3)$$

From (3.1) :

$$x = y \text{ (since } \sigma = 10) \; x^2 - (8 * z)/3 = 0 \text{ (since } \beta = 8/3)$$

Let $x = 0$, $y = 0$ (since $x = y$) this implies $z = 0$ from the above equation.

so $\boldsymbol{P_1 = (0, 0, 0)}$

let, $z = r - 1$, $x = \pm\sqrt{8/3(r - 1)}$, similarly $y = \pm\sqrt{8/3(r - 1)}$

so $\boldsymbol{P_2 = (\sqrt{8/3(r - 1)}, \sqrt{8/3(r - 1)}, r - 1)}$,
$\boldsymbol{P_3 = (-\sqrt{8/3(r - 1)}, -\sqrt{8/3(r - 1)}, r - 1)}$

here P_1 is valid if $r < 0$ and P_2, P_3 are valid if $r \geq 1.$ constraints from equation (3.2).So P_1, P_2, P_3 are the three critical points.

For simplicity let's write :

$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$ as f,

$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y$ as g and

$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$ as h.

Close to the equilibrium point we can consider the non linear case just as a linear case, so :

$$\frac{\mathrm{d}(x, y, z)}{\mathrm{d}t} = J * (x, y, z)$$

where J is Jacobian matrix,

$$J = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

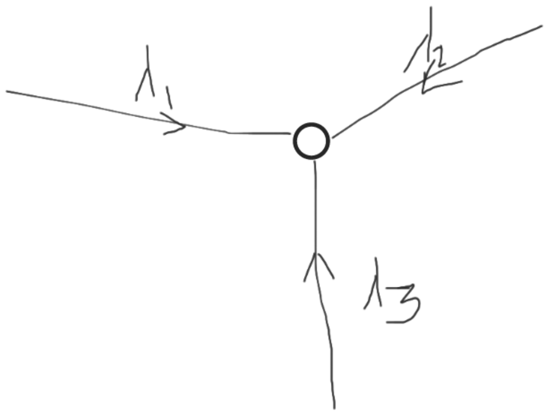
Here the Jacobian matrix for P_1 comes out to be : $\begin{bmatrix} -10 & 10 & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

now we compute the eigen values of this Jacobian Matrix :

$$J - \lambda I = \begin{bmatrix} -10-\lambda & 10 & 0 \\ \rho & -1-\lambda & 0 \\ 0 & 0 & -8/3-\lambda \end{bmatrix}$$

The values comes out to be $\lambda_1 = -8/3$, $\lambda_2 = \frac{-11-\sqrt{81+40r}}{2}$, $\lambda_3 = \frac{-11+\sqrt{81+40r}}{2}$.

if $r < 1$, all $\lambda_{1,2,3}$ are negative, therefore all eigen values in the phase portrait are coming towards the point P1. So its a Proper Node.



if $r = 1$, $\lambda_3 = 0$, So here we have two eigen values coming in one eigen value going out which makes the point unstable.

and if $r > 1$, $\lambda_3 > 0$ Here just like the above case the node becomes unstable.

Similarly for P_2

$$J = \begin{bmatrix} -10 & -10 & 0 \\ 1 & -1 & \sqrt{\frac{8}{3}(r-1)} \\ \sqrt{\frac{8}{3}(r-1)} & \sqrt{\frac{8}{3}(r-1)} & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

computing $\det(J - \lambda I) = 0$, we get

$$3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1) = 0$$

if $1 < r < 1.3456$: we get $\lambda_{1,2,3} < 0$, its a proper node which is Asymptotically stable,

then if $1.3456 < r < 24.737$ we get $\lambda_1 < 0$ and $\lambda_{2,3}$ Becomes complex but their real parts are still negative. so its Asymptotically stable,

then if $1.3456 < r < 24.737$ we get $\lambda_1 < 0$ and $\lambda_{2,3}$ Becomes complex but their real parts are positive. so it Asymptotically unstable.

And the same cases happens with the node P3.

4.3 Summary

So summarizing things up with P1, P2 and P3:

r value	P1	P2	P3
$r < 1$	Asymptotically Stable	Do not exist yet	Do not exist yet
$1 < r < 1.3456$	Unstable	Asymptotically Stable	Asymptotically Stable
$1.3456 < r < 24.737$	Unstable	Asymptotically Stable	Asymptotically Stable
$24.737 < r$	Unstable	Unstable	Unstable

So basically as we increase the r, we go through the time, We come in towards P1 but P2 and P3 dont exist yet for values which is less than one and as soon as we hit 1, P2 and P3 come into existance, as we go beyond 24.737 all three points become unstable.

A qualitative change in behaviour of the solution when a parameter is varied is called a bifurcation. Bifurcations occur at:

$\rho = 1$, when the origin switches from stable to unstable, and two more stationary points appear.

$\rho = \rho^*$, where the remaining two stationary points switch from being stable to unstable.

Chapter 5

Chua's Circuit - Lorenz Model

5.1 Chua's circuit:

A simple, real and physical electronic oscillator circuit which exhibits chaotic oscillations, bifurcation phenomenon, modeled using Lorenz equations. It was invented in 1993 by Leon Chua. It is extensively used to study chaos.

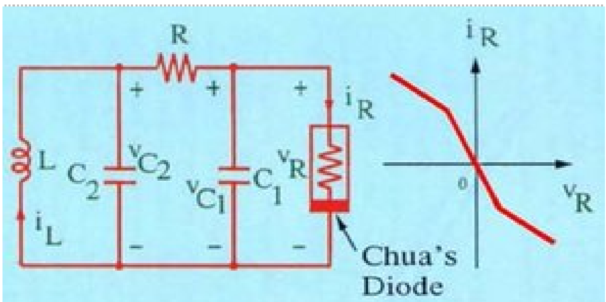
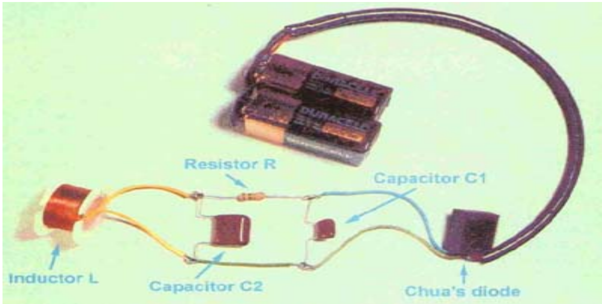


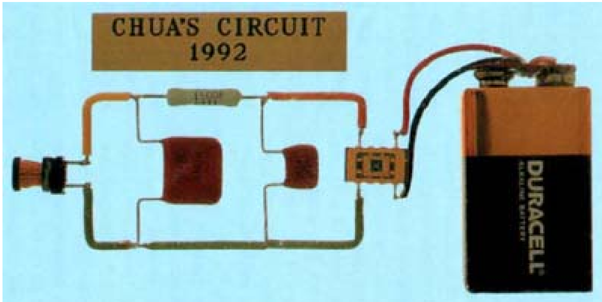
figure 1

It contains 5 circuit elements. The first four elements on the left are linear passive electrical components (relation between V and I is linear and they do not need power supply) Inductance $L > 0$, Resistance $R > 0$, and 2 Capacitances $C1 > 0$ and $C2 > 0$.

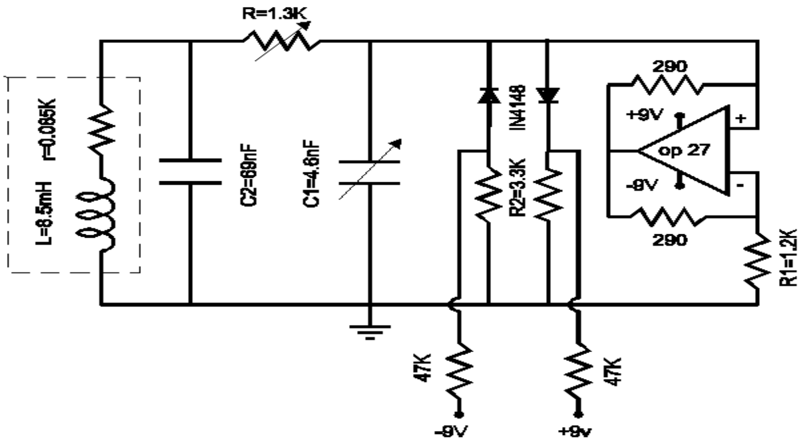
At least one locally active nonlinear element, powered by a battery, such as the **Chua diode** is necessary for chaotic behaviour.

Chua's Diode: A two-terminal, nonlinear active resistor which can be described with piecewise-linear equations. It is implemented as a voltage-controlled, nonlinear negative resistor (increase in voltage results in a decrease in electric current).





Op amps and other devices can be used to realise the negative resistance (which can provide some power gain and therefore, can be called active, then I-V graph slope if negative).



I vs V is a nonlinear function $i_R = \phi(v_R)$. The Chua's diode has 3-segment piece-wise-linear odd-symmetric characteristics.

We can consider the Chua diode as a small black box with two external wires soldered across capacitance C1, as the dynamics depends on other 4 components.

Some kind of periodic behaviour can be expected because in the figure 1, the characteristics of a chua diode shown are non-linear and the graph continues symmetrically on both to behave like a passive resistance. Note this diode is the only power source.

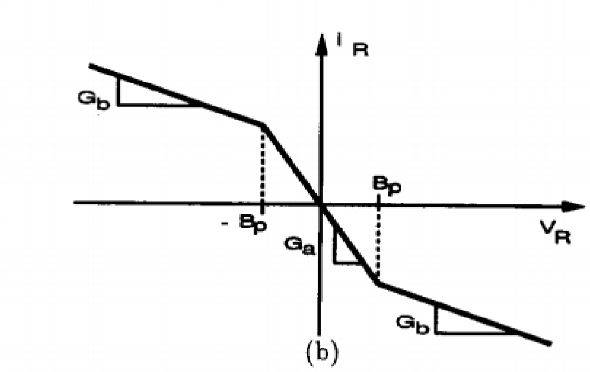
We want to locate the storage elements to get what are the state variables. Current through Inductors and Voltage through capacitors uniquely define the state variables.

5.2 System Dynamics:

$$C_1 \frac{\mathrm{d}C_1}{\mathrm{d}t} = G\left(V_{C2} - V_{C1}\right) - \phi\left(V_{C1}\right)$$

$$C_2 \frac{\mathrm{d}C_2}{\mathrm{d}t} = G\left(V_{C1} - V_{C2}\right) + i_L$$

$$L \frac{\mathrm{d}i_L}{\mathrm{d}t} = -V_{C2}$$



$$\phi\left(V_{C1}\right) = G_b V_{C1} + \frac{1}{2}\left[\left|V_{C1} + B_p\right| - \left|V_{C1} - B_p\right|\right]$$

or

,

$$\phi\left(v_R\right) = \begin{cases} m_1 V_r + m_1 - m_0 & \text{if } V_R \leq -1 \\ m_0 V_R & \text{if } -1 \leq V_R \leq 1 \\ m_1 V_r + m_1 - m_0 & \text{if } 1 \leq V_R \end{cases}$$

Here $m_0 = G_a/G, m_1 = G_b/G$ is the slope of the middle segment and m_1 the other 2 segments.

By re-scaling we obtain the following **Chua's Equations**.

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \alpha\left(y - \phi\left(x\right)\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = x - y + z$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = -\beta y - \gamma z$$

Smoothening the non-linear characteristics, we have the following function when the Chua's diode operates as an active element.

$$\phi\left(x\right)=ax^3+cx, ac<0, a\neq 0$$

These set of equations are called **Chua's Oscillator** Here, x,y,z are dimensionless state variables.

$$x=V_{C1}/B_p$$

$$y=V_{C2}/B_p$$

$$z=i_L/B_pG$$

$$\alpha=\frac{C_2}{C_1}$$

$$\beta=\frac{C_2}{LG^2}$$

$$\tau=\frac{tG}{C2}$$

$\alpha, \beta, \gamma > 0$ are the 3 dimensionless parameters. γ is 0.

5.3 Analysis

- Origin is an equilibrium point. For $a\neq 0$, the other 2 equilibrium points ($P^{\pm}$)are,

$$x=\pm\sqrt{\frac{\left(\frac{\gamma}{\gamma+\beta}-c\right)}{a}}$$

$$y=\pm\frac{\gamma}{\gamma+\beta}\sqrt{\frac{\left(\frac{\gamma}{\gamma+\beta}-c\right)}{a}}$$

$$z=\mp\frac{\beta}{\gamma+\beta}\sqrt{\frac{\left(\frac{\gamma}{\gamma+\beta}-c\right)}{a}}$$

- Let $k=1$. The Jacobian Matrices at the equilibrium points are :

$$M_0=\begin{bmatrix}-\alpha c & \alpha & 0\\ 1 & -1 & 1\\ 0 & -1\beta & -\gamma\end{bmatrix}$$

$$M_p = \begin{bmatrix} 2\alpha c - 3\alpha\gamma/(\gamma+\beta) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}$$

- The corresponding characteristic equations are as follows. We can obtain criteria for their stability.

$$\begin{aligned} det(\lambda I - M_0) = \lambda^3 + (1 + \gamma + \alpha\alpha c + 3\alpha\gamma/(\gamma+\beta)\lambda^2 \\ + (\gamma + \alpha c(1+\gamma)) - \alpha + \beta)\lambda \\ + (\alpha\gamma c - \alpha\gamma + \beta\alpha c) = 0 \end{aligned}$$

$$\begin{aligned} det(\lambda I - M_{p\mp}) = \lambda^3 + (1 + \gamma - 2\alpha c + \tfrac{3\alpha\gamma}{\gamma+\beta})\lambda^2 \\ +(\gamma + \beta - (2\alpha c - \tfrac{3\alpha\gamma}{\gamma+\beta})(1+\gamma) - \alpha)\lambda \\ \text{-}2\alpha\gamma c + 2\alpha\gamma - 2\beta\alpha c = 0 \end{aligned}$$

Based on the Eigen values obtained by solving for the roots of these equations, we can predict what the solution/trajectories look like when we vary the parameter G. This system is now analogous to Lorenz System, whose behaviour is determined in previous chapters.

Summary of Trajectories for different eigen values, which are possible with varying the G parameter

Case 1: All are real and negative. Then we will have a stable sink at the equilibrium point

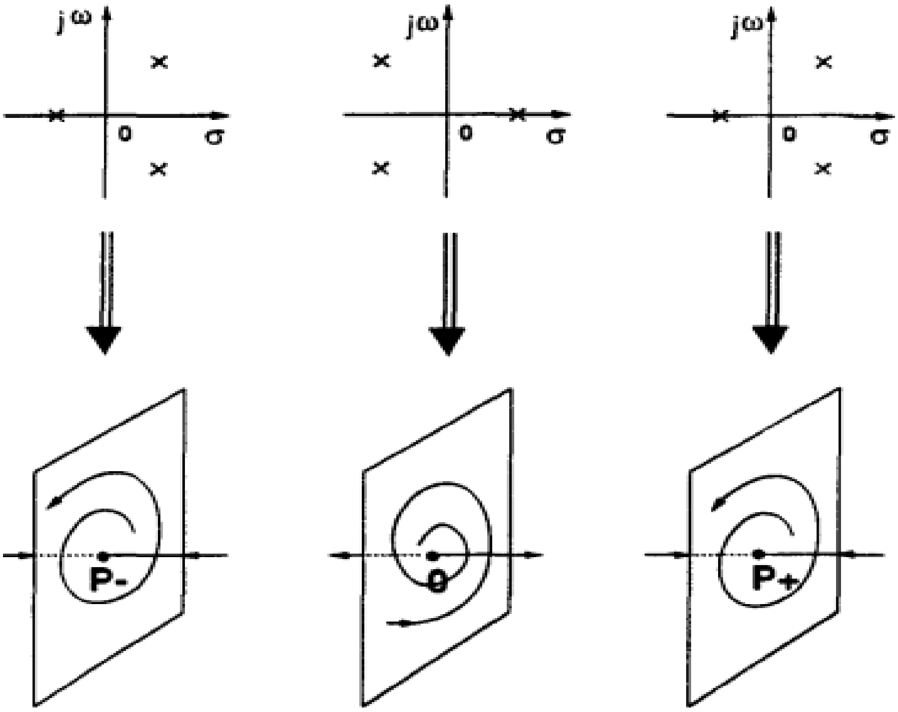
Case 2: All are real and positive. The equilibrium point is unstable.

Case 3: one is positive, other 2 are negative OR one is negative, other 2 are positive. The equilibrium point is a saddle point

case 4: One is real and negative, the other 2 are complex conjugates. We will have a spiral trajectory and it starts on the real eigen vector and converges to the equilibrium point on the complex conjugate eigen vectors' plane (real plane as the state space is real). The equilibrium point is stable.

Case 5: One is real and positive, the other 2 are complex conjugates. The trajectory starts on the real plane as an outward going spiral converging on the real eigen direction, but this direction is unstable because the eigen value is positive. Therefore, the trajectory is diverging and the equilibrium point is unstable in the real eigen vector direction.

With parameters $a = 1, c = -0.143, \alpha = 10, \beta = 16$, we can see the following flow near the equilibrium points.



Applications: Simulation of Brain Dynamics Evolution of natural languages Music Image processing Neural networks Dynamic associative memories, etc

References:

https://drive.google.com/file/d/1UYWFTOmCLB-BBjv4CuJ1BCodsnf8VtQC/view?usp=sharing

https://people.eecs.berkeley.edu/ chua/papers/Chua92.pdf

Chapter 6

Image Encryption Algorithm : Lorenz System

6.1 Introduction

In this section , we discuss about a new image encryption algorithm has been introduced which includes the Lorenz system in order to produce chaotic data and uses it to mask the images for secure encryption. The increase of communication networks and digital data has to lead to the development of data security and encryption. The information can be in the form of images, text, video, or other sound formats. There are existing algorithms like Data Encryption Standard (DES), Advanced Encryption Standard (AES), International Data Encryption Algorithm (IDEA), and the Rivest-Shamir-Adleman cryptosystem (RSA) , but these are not suitable for encryption methods for image/video format.

6.2 Proposed Algorithm

The following steps assume that you have an image of size - M X N , with 3 channels (Red , Blue , Green) , whose intensity values have a range between 0 - 255 .

- Like other Lorenz System , we initialise the initial values (x_0, y_0, z_0) and system parameters (a,b,c) which will comprise of our secret key .As time iterations are realised , we get a pseudo-random sequence and to remove transient effect we ignore first n (i.e n=3400) values .

- The chaotic sequences of x,y,z are used for rotating pixel values of the RGB channels of the plain image respectively. So, the pixel values are converted to 8 bit binary data for pixel value transformation . Processing of double values of chaotic sequences are prepossessed as follows:

$$k(i,j) = mod(abs(k(i,j)) - floor(abs(k(i,j))x10^{15}, 256))$$

Here , k can either be x/y/z chaotic sequence , and floor term gives us the value of k(i,j) to the nearest integer less than or equal to k(i,j).

We also compute l(i,j) term , for every k(i,j) which shows the sum of digits of k(i,j) , then we define :

m = mod(l(i,j),2)

n= mod(l(i,j),9)

$$p_n(i,j) = (p_0(i,j), m, n)$$

Here, p_n is the transformed color value , and we have the following relationship with m , n :

$$m = \begin{cases} RotateLeft, & \text{if } m = 1 \\ RotateRight, & \text{otherwise } m = 0 \end{cases} \tag{6.1}$$

Here, n denotes the number of steps need for rotation of these bits in the desired direction .

- Initial column vectors have been generated from the chaotic system for red, green and blue color maps of the size $Mx1$. Then, all elements of the first columns of RGB components of image have been made XOR with the elements of initial column vector mutually. XOR results were adjusted to first column of the image. Later on, the first column was made XOR with the second one and adjusted to the second column and all columns were made XOR in similar manner as the figure :

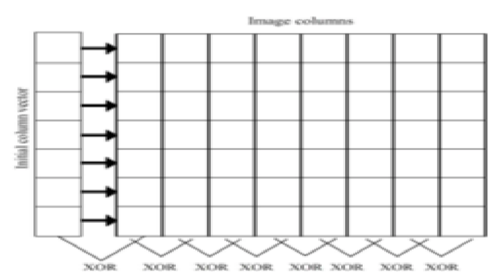


Figure 2. Xor operations applied to the columns

- For the pixel shuffling process, two chaotic sequences (sizes M and N) have been used. Their values are, of course, unrepeatable in the range $0 - M$ and $0 - N$, respectively. These two sequences have been used for the creation of MxN size matrix, which leads to new pixel positions. Thus, all the color values of red, green and blue channels are transformed randomly to new positions.
- In this step, the pixel value transformation is made one more time as in Step 2. All of the pixel values of the plain image have been rotated randomly by using new chaotic sequences.
- Row xor operation is the same as in step 3. Firstly, the initial row vectors have been generated from the chaotic system for RGB color maps with the sizes $1xN$. Then, XOR has been applied to all elements of the first row of image with the initial row vector's elements mutually. After that, XOR has been applied to the first row with the second row and adjusted to second row and all rows have been made XOR respectively as :

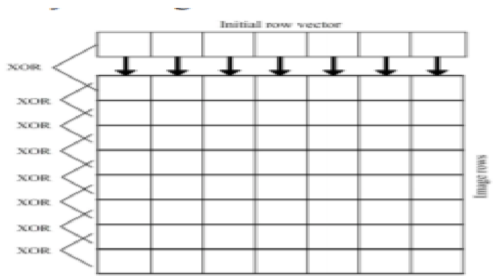


Figure 3. Xor operations applied to the rows

- In this step, pixel shuffling has been applied one more time for the diffusion and confusion processes as in Step 4. As a result, the last form of the encrypted image is obtained.

The decryption sequence is just the inverse process , and there is lossless reconstruction of the entire image . Now , after experiments we have found the optimal values of system and initial condition , as :

x_0= 1.100000000000000 , a = 10.000000000000000

y_0= 1.300000000000000 , b= 2.666666666666667

z_0= 1.500000000000000 , c= 28.000000000000000

6.3 Evaluation Metrics

- **Secret Key Sensitivity Test**

  In order to test the proposed image encryption scheme, the secret key analysis is important. In general , the key space should be more than 2^{128} to provide a sufficient security in order to avoid brute-force attacks. In the algorithm, the initial conditions of the chaotic system have been used as the secret keys with the precision 10^{-15} The key space of the proposed system is 10^{45}. We test with 2 initial value variation - x_0 = 1.100000000000000 and x = 1.100000000000001, with a minimal difference of $10^(- 15$, and we can observe how the values digress , by computing the L2 mean loss metric (Decrypted Images) :

  TABLE II. MSE RESULTS OF RED, GREEN AND BLUE COMPONENTS OF THE DECRYPTED BABOON AND LENA IMAGES WITH THE WRONG INITIAL VALUES.

	Lena Image	Baboon Image
Red	5437.00	5460.33
Green	5452.60	5459.36
Blue	5444.41	5457.73

  The decrypted images seem to vary significantly between these two values of x .

- **Histogram Analysis**: The image histogram determines the distribution of each color intensity level of the individual pixels of an image. An ideal histogram for an encrypted image should be flat, implying the uniform distribution on the image range and the complete randomness. In fact, the flatness of a histogram could better prevent the leakage of the information by the statistical attacks.

- **Correlation Analysis** : In this section, the horizontal, vertical and diagonal correlation coefficients of the 3000 ciphered pixels and regular image have been calculated. Here x and y are color value of two adjacent pixels. Cov(x,y), D(x) and E(x) denote the covariance, variance and the mean. We can observe the there is high

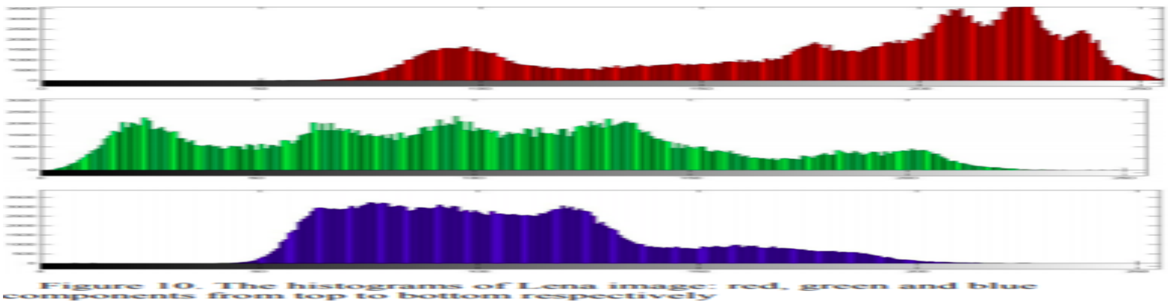


Figure 6.1: Regular Image Histogram

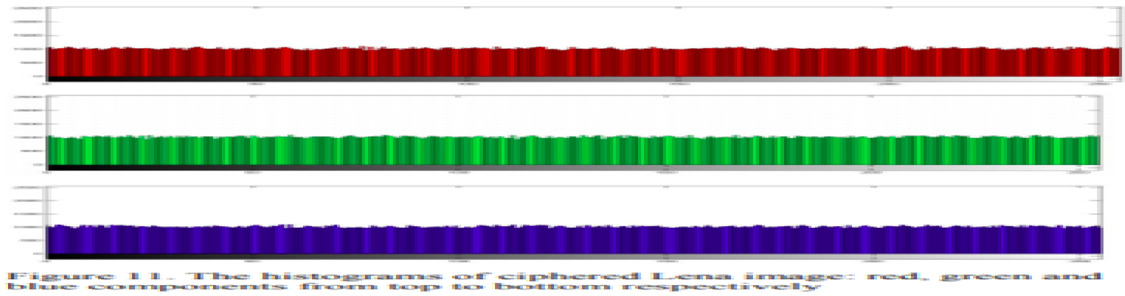


Figure 6.2: Encryption Image Histogram

correlation in original case ,and almost none in other case .

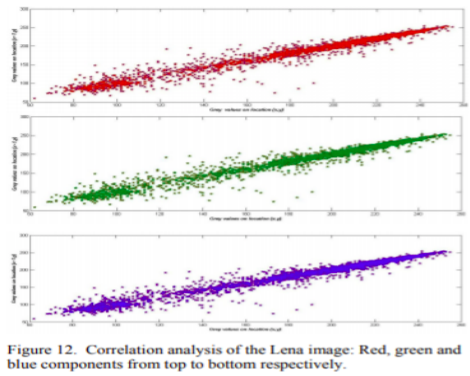


Figure 6.3: Regular Image Correlation Analysis

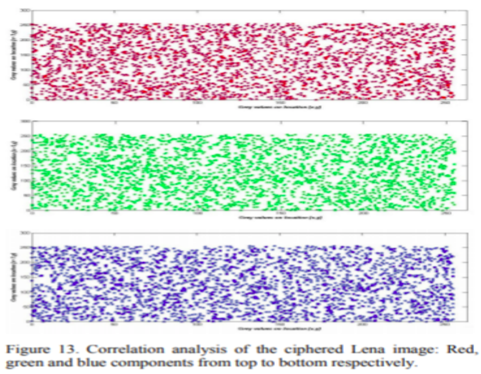


Figure 6.4: Ciphered Image Correlation Analysis

- **Information Entropy Analysis (H)** : The information entropy measures the strength of the cryptosystem and gives the equally probable gray levels for a good image encryption algorithm by applying it to the encrypted image. In this sense, entropy is given by :

$$H = \sum_{i=0}^{2^N-1} \text{p(mi)}*\log(1/\text{p(mi)})$$

Here N is the number of bits of the pixel value (m) and p(mi) represents the probability of m. If there are 256 state of m and they have the same probability, H(m) will be theoretically equal to 8. The smaller entropy value means greater degree of predictability, which threatens the encryption system security.

	Original lena	Ciphered lena	Original baboon	Ciphered baboon
R	7.2531	7.9992	7.7379	7.9993
G	7.5940	7.9993	7.4608	7.9994
B	6.9684	7.9994	7.7681	7.9993

6.4 Visual Examples

Here, the paper takes examples of 2 images of 512 x 512 dimensions , and performs encryption analysis on it .

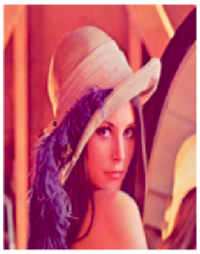


Figure 4. Plain Lena Image

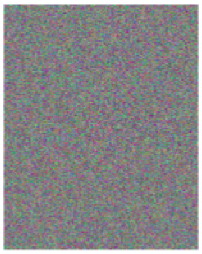


Figure 5. Ciphered Lena Image

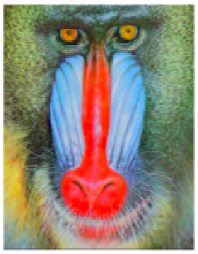


Figure 6. Plain Baboon Image

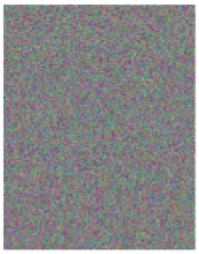


Figure 7. Ciphered Baboon Image

6.5 Conclusion

A new image encryption algorithm based on the Lorenz chaotic system has been introduced. The main features of the chaotic systems, namely the unpredictability and the sensitivity to the initial conditions, have contributed at the results of the ciphered images.

Chapter 7

References

- https://www.researchgate.net/publication/313951756_A_new_image_encryption_algorithm_based_on_lorenz_system
- http://www2.me.rochester.edu/courses/ME406/webexamp6/loreq.pdf
- https://www.vibrantcleanenergy.com/wp-content/uploads/2017/07/Dynamics-of-the-Lorenz-E pdf
- https://www2.physics.ox.ac.uk/sites/default/files/profiles/read/lect6-43147.pdf
- https://www.colby.edu/mathstats/wp-content/uploads/sites/81/2017/08/2017-Manning-Thesi pdf
- https://web.math.ucsb.edu/~jhateley/paper/lorenz.pdf
- https://www.sciencedirect.com/topics/mathematics/lorenz-equation
- http://www.cmp.caltech.edu/~mcc/Chaos_Course/Lesson1/Introduction.html
- https://www.researchgate.net/publication/342734248_CHAOS_THEORY_AND_ITS_APPLICATIONS_IN_OUR_REAL_LIFE
- https://www.tutorialspoint.com/control_systems/control_systems_stability_analysis.htm#:~:text=Routh%20Array%20Method,the%20control%20system%20is%20unstable
- https://nptel.ac.in/courses/108/105/108105054/