CSE-483: **Mobile Robotics** **Monsoon 2019**

Epipolar Geometry (Constraints, F matrix derivation and E matrix)

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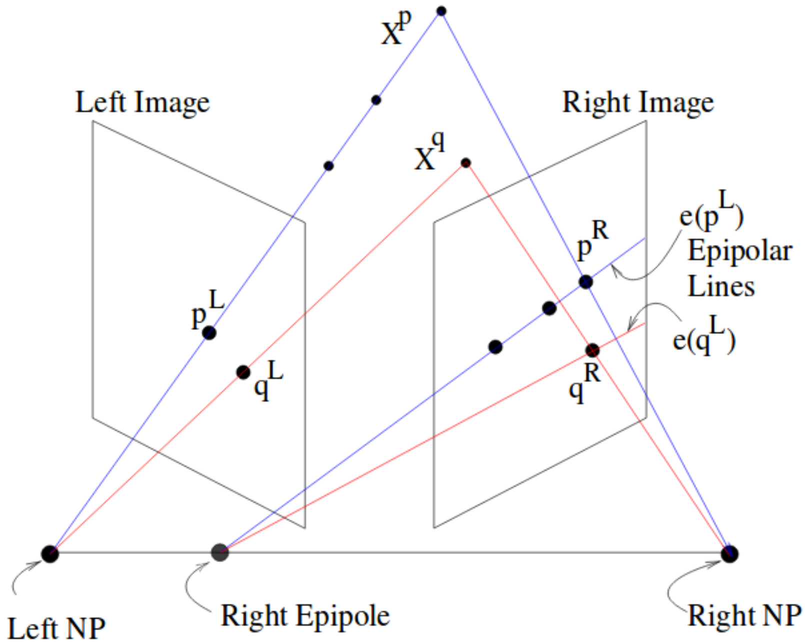
1 Introduction

Epipolar geometry is the geometry of stereo vision, i.e, the extraction of 3D information and details obtained by digital images. In the real 3D world, when two cameras view a 3D scene from two different positions, there are many geometric relations between the 3D points and the 2D images which are captured by the camera, that lead to constraints between the image points. In the following discussions, we can safely assume that the cameras can be approximated by the pinhole camera model.

2 Epipolar Geometry of a Pair of Cameras

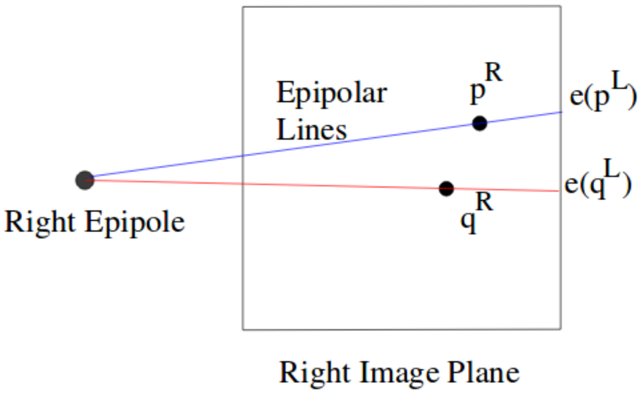


- **Epipolar Plane:** This is the plane containing the center of projection of the two cameras (or the nodal points) and the corresponding scene point. The baseline and projection lines from two cameras make the boundaries of this triangular epipolar plane. Baseline is the line segment which is connecting the two centers of projection. The intersection of this epipolar plane with the left/right image gives us the epipolar lines. Here the left nodal point is O_1 and the right nodal point is O_2.

- **Epipoles:** If the image plane is extended indefinitely, it intersects the baseline at some point, this point represents epipole for that camera. Epipoles are fixed for a given arrangement of the cameras. If enough angle of view and distance is provided, epipole represents the image of optical center of one camera in the image of other camera.

- **Epipolar lines:** The epipolar plane intersects the image plane at the epipolar line. This line passes through one of the projection points (p or q) and one of the epipoles. Thus, the epipolar lines come in corresponding pairs and if we know one projection point, say p^L, then the corresponding scene point X^p lies on the projection ray of this point. The image of this scene point on the right image plane is an epipolar line $e(p^L)$.

- **Epipolar Constraints:** Suppose q^L is the left image point for a scene point X^q, then the corresponding right image point q^R must lie on the epipolar line $e(q^L)$, and similarly, if q^R is the right image point, the left image point q^L must lie on the epipolar line $e(q^R)$.



Taking O_1 as origin,

$O_1 = [I_{3*3}|O_{3*1}]$

And $O_2 = [R_{12}|t_{12}]$

The image point p can be expressed as:

$\lambda_1 p = K[I|O]X_{4*1}$

$\lambda_1 K^{-1} p = X_1$

Let $K^{-1}p = x_1$ be normalized image coordinates

$K^{-1}p$ is the direction vector of the ray from the center O_1 to the 3D point X

$\lambda_2 q = K[R_{21}|t_{21}]X_{4*1}$

Ray from O_2 to q can be written as:

$\lambda_2 K^{-1} q = X_2$

$\lambda_2 K^{-1} q = [R_{21}|t_{21}]X_{4*1}$

Now, $K^{-1}q = x_2$ (normalized image coordinates)

So, $\lambda_2 x_2 = [R_{21}|t_{21}]X_{4*1}$

Since, O_1, O_2 and X forms a triangular plane (epipolar plane)

So, $O_1X.(O_1O_2 * O_2X) = 0$

We know that, $O_1X = \lambda_1(x_1)^T$, $O_1O_2 = t_{12}$ and $O_2X = R_{12}\lambda_2x_2$

substituting in (1)

$\lambda_1 x_1^T(t_{12} * R_{12}\lambda_2x_2) = 0$

$\lambda_1\lambda_2 x_1^T(t_{12} * R_{12}x_2) = 0$

$$Now, a_* = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

represents the matrix that expresses the cross product of a with any other vector.

So the above equation can be written as

$x_1^T[t_{12}]_*R_{12}x_2 = 0$

And this equation is called the *Epipolar constraint*

and $E_{12} = [t_{12}]_*R_{12}$

is called the *Essential matrix (E matrix)* that relates 2^{nd} image to the 1^{st} image.

Now, $x_1^T E_{12}x_2 = 0$ and similarly $x_2^T E_{21}x_1 = 0$

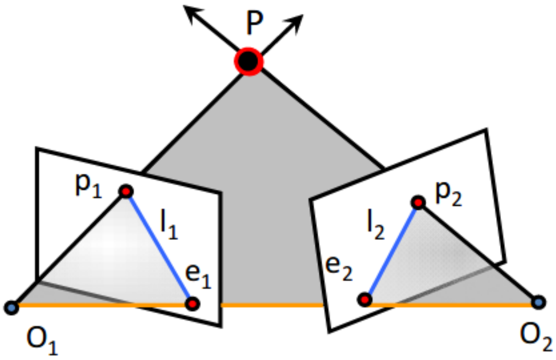
3 E matrix

The *Essential matrix* is a 3 x 3 matrix *E* which relates the corresponding points in stereo images, given the rotation and translation, assuming that the camera satisfies the pinhole camera model.

The following is the algebraic representation of epipolar geometry for the camera calibration relating to the E matrix:

E = Rt_*

Here R is the 3 x 3 rotation matrix and t_* is the matrix that expresses the cross product of *t*, the 3D translational vector, with any other vector.



Properties of E matrix:

- Ep_2 is the epipolar line associated with p_2 ($l_1 = Ep_2$)
- $E^T p_1$ is the epipolar line associated with p_1 ($l_2 = E^T p_1$)
- E is a singular matrix of rank 2
- E$e_2 = 0$ and $E^T e_1 = 0$
- E is a 3 x 3 matrix having 5 degrees of freedom.

4 F matrix

The *Fundamental matrix, F* is a 3 x 3 matrix which relates corresponding points in stereo images. In epipolar geometry with homogeneous image coordinates y and y', of corresponding points in a stereo image pair, **Fy** describes an epipolar line on which the corresponding point y' on the other image must lie.

So for all pairs of corresponding points y and y', the following equation holds

$y'^T F y = 0$

From the *Epipolar Constraint* equation, we have

$x_1^T [t_{12}]_* R_{12} x_2 = 0$

Now, $x_1 = K^{-1} p$ and $x_2 = K^{-1} q$

Replacing the values of x_1 and x_2 in above equation

$K^{-1} p^T [t_{12}]_* R_{12} K^{-1} q = 0$

$p^T K^{-T} [t_{12}]_* R_{12} K^{-1} q = 0$

$p^T F q = 0$

So, $F = K^{-T}[t_{12}]_* R_{12} K^{-1}$ is the **Fundamental Matrix**

Properties of F matrix:

- Fp_2 is the epipolar line associated with p_2 ($l_1 = Fp_2$)
- $F^T p_1$ is the epipolar line associated with p_1 ($l_2 = F^T p_1$)
- F is a singular matrix of rank 2
- F$e_2 = 0$ and $F^T e_1 = 0$
- F matrix is a more general form of the E matrix.
- F is a 3 x 3 matrix having 7 degrees of freedom.

5 Conclusion

In this document, we discussed about the basics of *Epipolar Geometry* and by defining some of the important terms associated with it like, *Epipolar planes, Epipolar lines and Epipoles.* We also discussed about the *Epipolar constraints* and its relation with the *E matrix* and the *F matrix.* This also includes the properties and the derivation of the *Essential Matrix* and the *Fundamental Matrix.*

6 References

[1] About Epipolar geometry

[2] Lecture slides from moodle

[3] Lectures from YouTube