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Sol3

a) Defining the Markov chains for the cat and mouse independently.

Cat's Movement

States $\begin{cases} S_1 \rightarrow \text{Cat in Room 1.} \\ S_2 \rightarrow \text{Cat in Room 2} \end{cases}$

Transn matrix P_c for the cat's

$$P_C = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

Let stationary distribution be $\pi_c = [\pi_1, \pi_2]$

we solve $\pi_c P_c = \pi_c$ and $\pi_1 + \pi_2 = 1$.

$$\pi_1 = 0.2\pi_1 + 0.8(1 - \pi_1)$$

$$\underline{\pi_1 = 0.5}, \ \underline{\pi_2 = 0.5}$$

so, stationary medium for cat's

$$\pi_c = [0.5, 0.5]$$

Mouse Movement

Moves from Room 1 to 2 with prob 0.3
Moves from Room 2 to 1 with prob 0.6

So transition matrix P_M:

$$P_M = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

Let $\pi_M = [\mu_1, \mu_2]$

Solve. $\pi_M P_M = \pi_M$, $\mu_1 + \mu_2 = 1$.

$$\mu_1 = 0.7\mu_1 + 0.6\mu_2$$

$$\boxed{\mu_1 = \frac{2}{3}, \mu_1 = \frac{1}{3}}$$

So, the stationary distribution for the mouse is

$$\pi_M = \left[\frac{2}{3}, \frac{1}{3}\right]$$

b) Is Z_n a Markov chain?

Define

- $Z_n = (C_n, M_n)$ where
- C_n = state (room) of the cat at time n

There are 4 possible states for Z_n;

1) Room 1, Room 1

2) Room 1, Room 2

3) Room 2, Room 1

4) Room 2, Room 2

Since the cat and the mouse move independently, the joint process $Z_n = (C_n, M_n)$ depends only on the previous state $Z_{n-1} = (C_{n-1}, M_{n-1})$ and not on earlier values.

Thus, yes the process $\{Z_n\}$ is a Markov chain on the state space of 4 pairs.

Q16

a) State space:

→ Each state is a permutation g of the 26 letters. (so there are $26!$ states)

→ At each step, we randomly select two different positions $i, j \in \{1, \dots 26\}$ and swap the entries at these positions in the current permutation.

Transition probability from g to h in one step.

Let's define

- $P(g \to h)$ → probability of transitioning from permutation g to h.
- If h can be obtained from g by swapping

→ $^{26}C_2 = 325$ possible swaps.

→ So, if h differs from g → $P(g \to h) = \frac{1}{32}$

If $h = g$, $\boxed{P(g \to g) = 0}$

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Stationary distribution

This Markov chain is:

- Irreducible -> Any permutation can be reached from any other by a sequence of swaps.

- Aperiodic: Any state can be revisited at different state

- Symmetric
$$P(g \to h) = P(h \to g)$$

So, uniform distribution over all 26! permutation.

(b) We now bias the Markov chain using a score function $s(g) > 0$, which measures how likely a permutation is.

From state g:

1) Propose a new state h by swapping two entries (same as part a)

2) • Probability 1 if $s(h) \geq s(g)$

• Probability $\frac{s(h)}{s(g)}$ if $s(h) < s(g)$.

This is standard Metropolis - Hastings rule

Reversibility of the Markov chain

We define the transition $q(g \to h)$ in the Metropolis - Hastings algorithm as

$$q(g \to h) = \begin{cases} \frac{1}{325} \cdot \min\left(1, \frac{s(h)}{s(g)}\right) & \text{if } h \text{ is reachable from } g \text{ by 1 swap} \\ 0 & \text{otherwise} \end{cases}$$

Similarly the reversible transition is

$$q(h \to g) = \frac{1}{325} \cdot \min\left(1, \frac{s(g)}{s(h)}\right)$$

To show reversibility we verify

$$s(g) \cdot q(g \to h) = s(h)\, q(h \to g)$$

$$s(g)\,(q(g \to h)) = \frac{s(g)}{325} \cdot \min\left(1, \frac{s(h)}{s(h)}\right)$$

$$= \begin{cases} \frac{s(h)}{325} & \text{if } s(h) < s(g) \\ \frac{s(g)}{} & \text{if } s(h) \geq s(g) \end{cases}$$

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Similarly

$$s(h) \cdot q(h \to g) = \frac{s(h)}{325} \cdot \min\left(1, \frac{s(g)}{s(h)}\right)$$

$$= \begin{cases} \frac{s(g)}{325} & \text{if } s(h) > s(g) \\ \frac{s(h)}{325}, & \text{if } s(g) \geq s(h) \end{cases}$$

A markov chain that satisfies detailed balance has stationary distribution.

$$\pi(g) \propto s(g)$$

So the chain converges to a distribution where the probability of being in state g. is $\propto$ to its score.