(1) Let $D(N)$ denote the dearrangement of N letters.

~~No. of ways to choose correct pairings $^{N}C_N$~~

$\therefore$ Probability that atleast one letter is in the correct envelope = 1 − Probability that no letter is in the correct envelope

$$= 1 - \frac{D(N)}{N!} = 1 - \frac{N!}{N!}\sum_{k=0}^{N}\frac{(-1)^k}{k!}$$

$$= \boxed{1 - \sum_{k=0}^{N}\frac{(-1)^k}{k!}} \text{ (Ans)}.$$

For $N = 50$ (large) $\sum_{k=0}^{N}\frac{(-1)^k}{k!}$ can be approximated as $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = \frac{1}{e}$.

$\therefore$ for $N = 50$, Probability becomes =

$$\boxed{1 - \frac{1}{e}} \text{ (Ans)}.$$

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(2) Let A, B, C be the events that the gifts are in the 1st, 2nd and 3rd presents respectively.

$P(A) = P(B) = P(C) = 1/3$

Let D be the event that Host selects Present 2.

Case (1): \$1000 are in Present 1.

$P(D|A) = 1/2$ [He may choose 2 or 3]

Case ②: \$1000 are in Present 2. (Not possible) $P(D|B) = 0$

Case ③: \$1000 are in Present 3

$P(D|C) = 1$ [He can only choose 2].

$\therefore P(D) = \frac{1}{3}\cdot\frac{1}{2} + 1\cdot\frac{1}{3} = \frac{1}{6}+\frac{1}{3} = \frac{1}{2}$ (c)

$\therefore$ $P(C|D)$ = Probability to win if he switches. (c)

$= \frac{P(C\cap D)}{P(D)}$

$= \frac{P(D|C)\cdot P(C)}{P(D)} = \frac{1\cdot 1/3}{1/2} = \frac{2}{3}$

$\therefore$ Expected winnings if I switch $= \sum p_i x_i$

$= \$1000\cdot\frac{2}{3} + 0\cdot\frac{1}{3}$

$= \boxed{\$ 666.67}$

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③ (a) RHS $= P(A|B\cap C)\cdot P(B|C)$

$= \frac{P(A\cap B\cap C)}{P(B\cap C)}\cdot\frac{P(B\cap C)}{P(C)} = \frac{P(A\cap B\cap C)}{P(C)}$

$= P(A\cap B|C) =$ LHS

$\boxed{\text{TRUE}}$

(b) LHS $= P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)} = \frac{P(A\cap C)\cdot P(B)}{P(C)}$

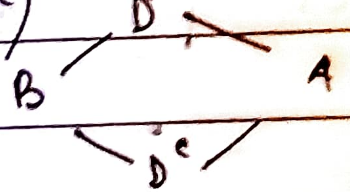
$= P(A|C)\,P(B)$

$\neq$ RHS

$\boxed{\text{FALSE}}$

(c) $P(A|B) = P(A|B\cap D) + P(A|B\cap D^c)$

$P(A|B^c) = P(A|B^c\cap D) + P(A|B^c\cap D^c)$



Now, $P(A|B\cap D) < P(A|D\cap B^c)$

$P(A|B\cap D^c) < P(A|D^c\cap B^c)$

$\therefore$ $\boxed{P(A|B^c) > P(A|B)}$ $\boxed{\text{TRUE}}$

4 (a) $E(X) = \sum x P(X=x)$.

We have $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converging and $\sum_{i=1}^{\infty} \frac{1}{n}$ diverging

Let $P(X=n) = \frac{c}{n^3}$ s.t. $\sum_{n=1}^{\infty} \frac{c}{n^3} = 1$

$\forall n \in [1,\infty)$

$$\therefore EX = \sum_{n=1}^{\infty} n \cdot \frac{c}{n^3} = c\sum_{n=1}^{\infty} \frac{1}{n^2} = c\left(\frac{\pi^2}{6}\right)$$

$$EX^2 = \sum_{n=1}^{\infty} n^2 \cdot \frac{c}{n^3} = c\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$

$\therefore$ EX converges but EX^2 diverges

$\therefore$ $\boxed{\text{Can exist}}$

(b) We have $\int_1^{\infty} \frac{1}{x}\,dx = \left[\log x\right]_1^{\infty} \to \infty$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \left[\frac{1}{x}\right]_{\infty}^{1} = 1$$

Let $f_X(x) = \frac{c}{x^3}$ $\forall x \geq 1$, $\int_1^{\infty} \frac{c}{x^3} = 1 \Rightarrow c = \frac{1}{2}$.

$$E_{x\in\Omega} X = \int_{\Omega} x f_X(x)\,dx = \int_1^{\infty} x \cdot \frac{1}{2x^3}\,dx = \frac{1}{2}$$

$$EX^2 = \int x^2 f_X(x)\,dx = \int_1^{\infty} \frac{1}{2x}\,dx \to \infty$$

$\therefore$ EX converges but EX^2 diverges.

$\therefore$ $\boxed{\text{Can exist}}$

4(c) e^{-x} is a convex function.

By Jensen's inequality : $g(E(X)) \leq E(g(X))$

$\Rightarrow \quad e^{-E(X)} \leq E(e^{-X}) \qquad E(X) = 1$

$\Rightarrow \quad e^{-1} \leq E(e^{-X})$

$\Rightarrow \quad E(e^{-X}) \geq \frac{1}{e} \geq 1/3$

$\therefore$ $\boxed{\text{It cannot exist}}$

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5) Let X be the random variable of the maximum price. $F(x) = P(X \leq x) = \frac{x}{N}$

For n draws, $F(x) = P(X \leq x) = \left(\frac{x}{N}\right)^n$.

$PMF = P[X=x] = F(x) - F(x-1)$.

$$= \left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n$$

$$\therefore E(M) = \sum x_i P(x_i)$$

$$= \sum_{x=1}^{N} x\left[\left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n\right]$$

$$= \boxed{\sum_{x=1}^{N} \frac{x^{n+1}}{N^n} - \sum_{x=1}^{N} x\left(\frac{x-1}{N}\right)^n}$$

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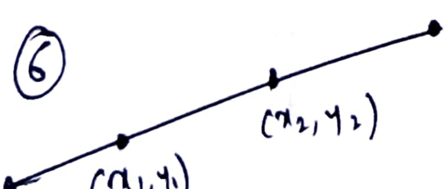
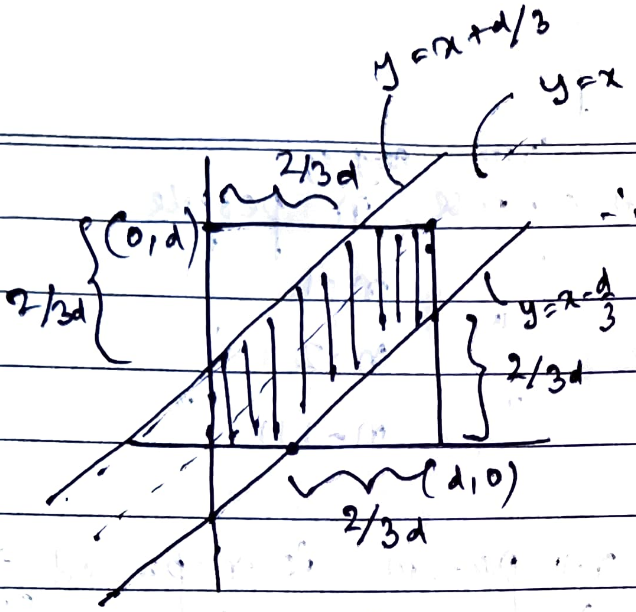
(6)



Let X and Y be the two points. $0 \leq X \leq d$ and $0 \leq Y \leq d$

We can take X and Y to be the coordinate axes



$\therefore$ (X, Y) can be any pt. on the square

We want $|X - Y| < d/3$

$\Rightarrow -\frac{d}{3} < X - Y < \frac{d}{3}$

$\Rightarrow Y < X + \frac{d}{3}$ and $Y > X - \frac{d}{3}$

$$P(|X-Y| < d/3) = \frac{\text{Area of shaded region}}{\text{Area of square}}$$

$$= \frac{1 - \text{Area of } 2\,\Delta\text{'s}}{\text{Area of square}} = \frac{1 - 2 \times \frac{1}{2} \times \frac{2}{3} \times \frac{2}{3} d^2}{d^2}$$

$$= \frac{1 - \frac{4}{9} d^2}{d^2} \quad \boxed{\cancel{\frac{9 - 4d^2}{9d^2}}}$$

$$= \boxed{\frac{5}{9}}$$

7) (a) Without loss of generality let 1st person is the originator. He chooses the 1st listner in ${}^{n+1}C_1 = (n+1)$ ways

Probability that in one single conversation originator is not involved $= \frac{n-1}{n}$

For remaining $r-1$ conversations, Probability

$$= \boxed{\left(\frac{n-1}{n}\right)^{r-1}}$$

(b) 1st person can choose from ~~any of~~ n people.
2nd " " " " n-1 "
3rd " " " " n-2 "
ith " " " " n-i+1 "

$\therefore$ Probability that no person is repeated till r^{th} conversation

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{n^r}$$

$$= \boxed{\frac{(n-1)(n-2)\cdots(n-r+1)}{n^{r-1}}}$$

For N persons:

(a) Probability that originator is not invoked in a single conversation =

$$\frac{{}^{n-1}C_N}{{}^{n}C_N} = \frac{n-N}{n}$$

For r conversations, $P = \boxed{\left(\frac{n-N}{n}\right)^{r-1}}$

(b) Probability will be

$$= \frac{{}^{n}C_N \cdot {}^{n-N}C_N \cdot {}^{n-2N}C_N \cdots {}^{n-(r-1)N}C_N}{({}^{n}C_N)^r}$$

$$= \boxed{\frac{(n-N)! \cdot [(n-N)!]^{r-1}}{(n!)^{r-1} \cdot (n-(r+1)N)!}}$$

(8) $P(\cap A_i^c) = \cancel{\prod_{i=1}} \prod_i P(A_i^c) = \prod_i (1-P(A_i))$

$\leq \prod_i (e^{-P(A_i)})$ $\left[\because e^{-x} \geq 1-x \quad 0<x<1\right]$

$= e^{-P(A_1)-P(A_2)-\cdots-P(A_n)}$

$\therefore$ LHS = RHS (Proved)

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(9) Let $f(x)$ and $g(x)$ be 2 distribution functions

$h(x) = f(x) * g(x)$

$= \int_{-\infty}^{\infty} f(x-\alpha)\, g(\alpha)\, d\alpha$

$f(x) \geq 0$ and $g(x) \geq 0 \quad \forall x \in (-\infty, \infty)$

$\therefore \boxed{h(x) \geq 0 \quad \forall x \in (-\infty, \infty)}$ — (1)

$\int_{-\infty}^{\infty} h(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-\alpha)\, g(\alpha)\, d\alpha \right) dx$

$= \int_{-\infty}^{\infty} \left(g(\alpha) \int_{-\infty}^{\infty} f(x-\alpha)\, dx \right) d\alpha$

$= \int_{-\infty}^{\infty} g(\alpha) \left(\int_{-\infty}^{\infty} f(x)\, dx \right) d\alpha$

$= \int_{-\infty}^{\infty} g(\alpha)\, d\alpha = 1$

$\therefore \boxed{\int_{-\infty}^{\infty} h(x) = 1}$ Hence $h(x)$ is a valid p.d.f.

(10) $$X(\omega) = \int_0^\infty \mathbb{I}_{[0, X(\omega))}\, dx.$$

$$E(X) = E\left(\int_0^\infty \mathbb{I}_{[0, X(\omega))}\, dx\right).$$

$$= \int_0^\infty E\left(\mathbb{I}_{[0, X(\omega))}\right) dx$$

Now $E\left(\mathbb{I}_{[0, X(\omega))}\right) = P(x < X(\omega))$

$$= P(X(\omega) \geq x)$$

$$= 1 - F(x)$$

$$\therefore \boxed{E(X) = \int_0^\infty (1 - F(x))\, dx} \quad \text{(Proved)}$$

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(11) (i) $$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E e^{uX} = \int_{-\infty}^{\infty} e^{ux} \cdot \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ux - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2}(2\sigma^2 ux - x^2 - \mu^2 + 2x\mu)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 + \mu^2 - 2x(\mu + u\sigma^2) - u^2\sigma^4 - 2\mu u\sigma^2)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x - (\mu + u\sigma^2))^2} \cdot e^{\frac{(u^2\sigma^2 + 2\mu u)}{2}} dx$$

$$= e^{\mu u + \frac{1}{2}u^2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+u\sigma^2))^2}{2\sigma^2}} dx$$

$$= e^{\mu u + \frac{1}{2}u^2\sigma^2} \cdot 1$$

$$= \boxed{e^{\mu u + \frac{1}{2}u^2\sigma^2}}$$

(ii) $$E[\phi(x)] = \int_{-\infty}^{\infty} e^{ux} \cdot f(x)\, dx$$

$$= e^{\mu u + \frac{1}{2}u^2\sigma^2}$$

$$\phi(Ex) = e^{uEx} = e^{u\mu}$$

Clearly $e^{\mu u} \cdot e^{\frac{1}{2}u^2\sigma^2} > e^{u\mu}$

$\therefore$ $\boxed{E(\phi(x)) > \phi(Ex)}$ (Verified)

Teacher's Signature.......................