Beta Gamma Function

The Beta and Gamma functions also called Euler's Integral.

Euler's Integral is of two types

1. Euler's Integral of the first kind is known as Beta Function
2. Euler's Integral of the Second kind is known as Gamma Function

1.1 Beta Function:

The beta function was studied by **Euler and Legendre** and was given its name by **Jacques Binet.**

Beta function is denoted by $\boldsymbol{\beta(m,n)}$

$$\boldsymbol{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx} \quad ; \quad \boldsymbol{m} > 0\,,\ n > 0$$

For negative values of $m \ and \ n$, this integral does not converge.

e.g.: (I) $Integral\ \int_0^1 \sqrt{x}(1-x)^4\ dx$ is Beta function denoted by $\beta\ \left(\frac{1}{2}+1{,}4+1\right) = \beta\ \left(\frac{3}{2},5\right)$

(II) $Integral\ \int_0^1 x^{\frac{1}{3}}(1-x)^{-5}\ dx$ is not a Beta function because in this $n = -4\ \ but\ n > 0\ required$

1.2 Properties of Beta function

Property I (Symmetry Property): $\boldsymbol{\beta(m,n) = \beta(n,m)}$

Proof: By definition of Beta function $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\ dx \qquad \left\{\because\ \int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\ dx \ = \ \beta(n,m)$$

Hence $\boldsymbol{\beta(m,n) = \beta(n,m)}$

Property II: $\boldsymbol{\beta(m,n) = \beta(m,n+1) + \beta(m+1,n)}$

Proof: R.H.S. = $\beta(m,n+1) + \beta(m+1,n)$

$$= \int_0^1 x^{m-1}(1-x)^{n+1-1}\ dx \ + \int_0^1 x^{m+1-1}(1-x)^{n-1}\ dx$$

$$= \int_0^1 x^{m-1}(1-x)^n \ + x^m(1-x)^{n-1}\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[1-x+x]\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

$$= \beta(m,n) = L.H.S.$$

Property III: $\boldsymbol{\frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n}}$

Proof: $\beta(m,n+1) = \int_0^1 x^{m-1}(1-x)^n\ dx$

$$= \int_0^1 (1-x)^n x^{m-1}\ dx$$

$$= \left[(1-x)^n\left(\frac{x^{m-1+1}}{m-1+1}\right)\right]_0^1 - \int_0^1 n(1-x)^{n-1}(-1).\frac{x^{m-1+1}}{m-1+1}\ dx \qquad \{Integration\ by\ Parts\}$$

$$= (0-0) + \frac{n}{m}\int_0^1 (1-x)^{n-1}.x^m\,dx$$

$$= \frac{n}{m}\ \beta(m+1,n)$$

$$\therefore \qquad \frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m}$$

Now $\frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)+\beta(m+1,n)}{m+n}$ $\qquad \left\{\because\ \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right\}$

$\Rightarrow\ \frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n}$ $\qquad \{Using\ Property\ II\}$

Property IV: $\boldsymbol{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (Sin\,\theta)^{2m-1} (cos\,\theta)^{2n-1}\, d\theta}$

Proof: We know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$

Put $x = sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 1$ $then\ \theta = \frac{\pi}{2}$ $and\ \ when\ \ x = 0\ then\ \theta = 0$

$$\therefore \quad \beta(m,n) = \int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1}(1-sin^2\theta)^{n-1}.\ 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2}.\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\, d\theta$$

Property V: $\boldsymbol{\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$

Proof: By definition $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$

Put $x = \frac{1}{1+t}$ $so\ that\ \ dx = -\frac{1}{(1+t)^2}\, dt$

$When\ \ x = 0\ \ then\ \ t \to \infty \quad and \quad When\ \ x = 1 \quad then \quad t = 0$

$$\therefore \quad \beta(m,n) = \int_\infty^0 \left(\frac{1}{1+t}\right)^{m-1}\left(1-\frac{1}{1+t}\right)^{n-1}\left(-\frac{1}{(1+t)^2}\right) dt$$

$$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}\, dt$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$$

Also by symmetry property $\beta(m,n) = \beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

Property VI: $\boldsymbol{\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx}$

Proof: We have $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\ dx$ $\{by\ property\ V\}$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\ dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\ dx$$

$$= I_1 + I_2 \quad (say) \qquad \ldots(1)$$

Now, Consider $I_2 = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\ dx$

Put $x = \frac{1}{t}$ $so\ that\ \ dx = -\frac{1}{t^2}\, dt$

$When\ \ x = 1\ \ then\ \ t = 1 \quad and\ \ when\ \ x \to \infty \quad then\ \ t \to 0$

$$\therefore \quad I_2 = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right) dt$$

$$= -\int_1^0 \frac{1}{t^{m-1}\frac{(1+t)^{m+n}}{t^{m+n}}}\cdot\frac{1}{t^2}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\ dt$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\ dx \qquad \ldots(2)$$

From (1) and (2) , we get

$$\beta(m,n) = I_1 + I_2 = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\ dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\ dx$$

$$= \int_0^1 \left[\frac{x^{m-1}}{(1+x)^{m+n}} + \frac{x^{n-1}}{(1+x)^{m+n}}\right] dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\ dx$$

Property VII: $\boldsymbol{\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}}$, ***where m, n are positive integers.***

Proof: By definition $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$

Integration by Parts, we get

$$\beta(m,n) = \left[x^{m-1}.\frac{(1-x)^n}{n(-1)}\right]_0^1 - \int_0^1 (m-1)x^{m-2}.\left[\frac{(1-x)^n}{n(-1)}\right] dx$$

$$= (0-0) + \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\ dx$$

$$= \frac{m-1}{n}\ \beta(m-2+1, n+1)$$

$$\boldsymbol{\beta(m,n) = \frac{m-1}{n}\ \beta(m-1, n+1)} \qquad \dots(1)$$

Changing m *to* $m-1$ *and* n *to* $n+1$ *on both side of* (1), *we get*

$$\beta(m-1, n+1) = \frac{m-2}{n+1}\ \beta(m-2, n+2)$$

Continuing the above process $(m-2)$ times, we get

$\beta(m-2, n+2) = \frac{m-3}{n+2}\ \beta(m-3, n+3)$ {Changing m *to* $m-2$ *and* n *to* $n+2$ *on both side of* (1)}

$\beta(m-3, n+3) = \frac{m-4}{n+3}\ \beta(m-4, n+4)$ {Changing m *to* $m-3$ *and* n *to* $n+3$ *on both side of* (1)}

....

....

$\beta(2, m+n-2) = \frac{1}{m+n-2}\ \beta(1, m+n-1)$ {Changing m *to* 2 *and* n *to* $m+n-2$ *on both side of* (1)}

Using above equations in equation (1) , we get

$$\beta(m,n) = \frac{(m-1)(m-2)(m-3)\dots\dots\dots\dots 1}{n(n+1)(n+2)\dots\dots\dots\dots(m+n-2)}\ \beta(1, m+n-1)$$

Multiply and divide by $[1.2.3 \dots \dots \dots \dots (n-1)]$ We get

$$= \frac{(m-1)![1.2.3\dots\dots\dots(n-1)]}{[1.2.3\dots\dots\dots(n-1)]\ n(n+1)(n+2)\dots\dots\dots\dots(m+n-2)}\ \beta(1, m+n-1)$$

$$= \frac{(m-1)!(n-1)!}{(m+n-2)!}\ \int_0^1 x^{1-1}(1-x)^{m+n-1-1}dx$$

$$= \frac{(m-1)!(n-1)!}{(m+n-2)!}\ \int_0^1 (1-x)^{m+n-2}dx$$

$$= \frac{(m-1)!(n-1)!}{(m+n-2)!}\ \left[\frac{(1-x)^{m+n-1}}{(m+n-1)(-1)}\right]_0^1$$

$$= \frac{(m-1)!(n-1)!}{(m+n-2)!}\left\{\left(0 + \frac{1}{m+n-1}\right)\right\}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-2)!(m+n-1)}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Hence $\boldsymbol{\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}}$ **, where** $\boldsymbol{m\ and\ n\ are}$ **positive integers.**

Another Proof of Property III with the help of another property.

(I) $\boldsymbol{\beta(m+1,n) = \beta(m,n)\,.\frac{m}{m+n}}$

Proof: We know that $\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$ $\dots(1)$

$$\therefore \qquad \beta(m+1, n) = \frac{m!(n-1)!}{(m+n)!}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-1)!} * \frac{m}{m+n}$$

$$= \beta(m,n).\frac{m}{m+n} \qquad \{using\ (1)\}$$

(II) $\boldsymbol{\beta(m,n+1) = \beta(m,n)\,.\frac{n}{m+n}}$

Proof: We know that $\beta(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$ $\quad\ldots(2)$

$$\therefore \quad \beta(m,n+1)=\frac{(m-1)!n!}{(m+n)!}$$

$$=\frac{(m-1)!(n-1)!}{(m+n-1)!}*\frac{n}{m+n}$$

$$=\beta(m,n).\frac{n}{m+n} \qquad \{using\ (2)\}$$

Examples

Example: 1 Express $\int_0^1 x^5(1-x^2)^3\ dx$ as Beta function and hence evaluate it.

Solution: Let $I=\int_0^1 x^5(1-x^2)^3\ dx$ ...(1)

Step (i) Put $x^2=t\ so\ that\ \ 2xdx=dt \ \Rightarrow\ dx=\frac{dt}{2t^{\frac{1}{2}}}$

Step (ii) $When\ \ x=0\ \ then\ t=0\ and\ \ when\ x=1\ then\ t=1$

Step (iii) Substitute the values of step (i) &(ii) in equation no. (1) , we get

$$=\int_0^1 t^{\frac{5}{2}}(1-t)^3\ \frac{dt}{2t^{\frac{1}{2}}} \ \Rightarrow\ \frac{1}{2}\int_0^1 t^2(1-t)^3\ dt$$

Step (iv) $\int_0^1 x^5(1-x^2)^3\ dx \ =\frac{1}{2}\int_0^1 t^2(1-t)^3\ dt$

$$=\frac{1}{2}\ \beta(2+1,3+1)$$

$$=\frac{1}{2}\ \beta(3,4) \qquad \left\{using\ \beta(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}\right\}$$

$$=\frac{1}{2}*\frac{(3-1)!(4-1)!}{(3+4-1)!}$$

$$=\frac{1}{2}*\frac{(2)!3!}{(6)!} \ \Rightarrow\ \frac{1}{2}*\frac{1.2\ (3!)}{6.5.4.3!} \ \Rightarrow\ \frac{1}{6.5.4} \ \Rightarrow\ \frac{1}{120} \quad \text{Ans.}$$

Example: 2 Express $\int_0^1 x^m(1-x^n)^p\ dx$ in terms of Beta function and hence evaluate $\int_0^1 x^5(1-x^2)^3\ dx$

Solution: Let $I=\int_0^1 x^m(1-x^n)^p\ dx$ ...(1)

Step (i) Put $x^n=t\ so\ that\ \ nx^{n-1}dx=dt \ \Rightarrow\ dx=\frac{1}{n}t^{\frac{1}{n}-1}dt$

Step (ii) $When\ \ x=0\ \ then\ t=0\ and\ \ when\ x=1\ then\ t=1$

Step (iii) Substitute the values of step (i) &(ii) in equation no. (1) , we get

$$=\int_0^1 \left(t^{\frac{1}{n}}\right)^m\ (1-t)^p\ .\frac{1}{n}t^{\frac{1}{n}-1}dt \ \Rightarrow\ \frac{1}{\mathrm{n}}\int_0^1 t^{\frac{m}{n}+\frac{1}{n}-1}(1-t)^p\ dt$$

Step (iv) $\int_0^1 x^m(1-x^n)^p\ dx \ =\frac{1}{\mathrm{n}}\int_0^1 t^{\frac{m}{n}+\frac{1}{n}-1}(1-t)^p\ dt$

$$=\frac{1}{n}\ \beta\left(\frac{m+1}{n},p+1\right) \qquad \ldots(2)$$

Step (v) comparing $\int_0^1 x^5(1-x^2)^3\ dx$ with $\int_0^1 x^m(1-x^n)^p\ dx$, $we\ have$

$$m=5\ ,n=2\ and\ p=3$$

$\therefore\ \ from\ (2),we\ get$

$$\int_0^1 x^5(1-x^2)^3\ dx\ =\frac{1}{2}\ \beta\left(\frac{5+1}{2},3+1\right)$$

$$=\frac{1}{2}\ \beta(3,4) \qquad \left\{using\ \beta(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}\right\}$$

$$=\frac{1}{2}*\frac{(3-1)!(4-1)!}{(3+4-1)!}$$

$$=\frac{1}{2}*\frac{(2)!3!}{(6)!} \ \Rightarrow\ \frac{1}{2}*\frac{1.2\ (3!)}{6.5.4.3!} \ \Rightarrow\ \frac{1}{6.5.4} \ \Rightarrow\ \frac{1}{120} \quad \text{Ans.}$$

Example: 3 **Express $\int_0^2 x^4(16-x^4)^{-\frac{1}{4}}\, dx$ in terms of Beta function.**

Solution: Let $I=\int_0^2 x^4(16-x^4)^{-\frac{1}{4}}\, dx$ ...(1)

Step (i) Put $x^4=16t \Rightarrow x=2t^{\frac{1}{4}}$ $so\ that$ $dx=\frac{1}{2}.t^{-\frac{3}{4}}dt$

Step (ii) $When\ x=0\ then\ t=0\ and\ when\ x=2\ then\ t=1$

Step (iii) Substitute the values of step (i) &(ii) in equation no. (1) , we get

$$I=\int_0^1 16t(16-16t)^{-\frac{1}{4}}.\frac{1}{2}.t^{-\frac{3}{4}}dt$$

$$=\int_0^1 16t(16-16t)^{-\frac{1}{4}}.\frac{1}{2}.t^{-\frac{3}{4}}dt$$

$$=4\int_0^1 t^{\frac{1}{4}}(1-1t)^{-\frac{1}{4}}dt = 4\beta\left(\frac{1}{4}+1,-\frac{1}{4}+1\right) = 4\beta\left(\frac{5}{4},\frac{3}{4}\right)$$ Ans.

Example: 4 **Using the property $\beta(m,n)=\beta(n,m)$, evaluate $\int_0^1 x^2(1-x)^{\frac{1}{2}}dx$**

Solution : $\int_0^1 x^2(1-x)^{\frac{1}{2}}dx=\beta\left(2+1,\frac{1}{2}+1\right)=\beta\left(3,\frac{3}{2}\right)=\beta\left(\frac{3}{2},3\right)$ $\{\because \beta(m,n)=\beta(n,m)\}$

$$=\int_0^1 x^{\frac{3}{2}-1}(1-x)^{3-1}dx = \int_0^1 x^{\frac{1}{2}}(1-x)^2dx$$

$$=\int_0^1 x^{\frac{1}{2}}(1+x^2-2x)dx = \int_0^1 x^{\frac{1}{2}}+x^{\frac{5}{2}}-2x^{\frac{3}{2}}\ dx$$

$$=\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{7}{2}}}{\frac{7}{2}}-2\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = \left[\left(\frac{2}{3}+\frac{2}{7}-\frac{4}{5}\right)-(0+0-0)\right] = \frac{16}{105}$$ Ans.

Example: 5 **Show that $\int_0^1 x^{m-1}(a-x)^{n-1}dx=a^{m+n-1}\beta(m,n)$**

Solution: Put $x=at$ $so\ that$ $dx=adt$

When $x=0$ $then\ t=0\ and\ when\ x=1\ then\ t=1$ in L.H.S. , we get

L.H.S. $=\int_0^1 x^{m-1}(a-x)^{n-1}dx=\int_0^1(at)^{m-1}(a-at)^{n-1}.adt$

$$=a^{m+n-1}\int_0^1 t^{m-1}(1-t)^{n-1}.dt = a^{m+n-1}\beta(m,n) \quad = \boldsymbol{R.H.S}$$

Example: 6 **Prove that $\int_0^\infty \frac{x^2}{(1+x)^6}\, dx=\frac{1}{30}$**

Solution: we know that $\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

Here $m-1=2 \Rightarrow m=3\ and\ m+n=6 \Rightarrow 3+n=6 \Rightarrow n=3$

$$\int_0^\infty \frac{x^2}{(1+x)^6}\, dx = \beta(3,3) = \frac{(3-1)!(3-1)!}{(3+3-1)!}$$

$$\frac{2!2!}{5!}=\frac{1}{30}$$

Example: 7 **Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx=\frac{1}{(a+b)^m a^n}\beta(m,n)$**

Solution: $Let\ \ I=\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx$

Put $\frac{x}{a+bx}=t \Rightarrow x=\frac{at}{1-bt}$ $so\ that$ $dx=\frac{a}{(1-bt)^2}dt$

$When\ x=0\ then\ t=0\ and\ when\ x=1\ then\ t=\frac{1}{a+b}$

$$\therefore \int_0^{\frac{1}{a+b}}\left(\frac{\left(\frac{at}{1-bt}\right)^{m-1}\left(1-\frac{at}{1-bt}\right)^{n-1}}{\left(\frac{a}{1-bt}\right)^{m+n}}\right).\frac{a}{(1-bt)^2}dt$$

$$=\int_0^{\frac{1}{a+b}}\frac{1}{a^n}t^{m-1}[1-(a+b)t]^{n-1}\, dt$$

Now Put $(a+b)t = u$ $\quad so\ that \quad dt = \frac{du}{a+b}$

$When\ \ t = 0\ \ then\ \ u = 0\ \ and\ \ when\ \ t = \frac{1}{a+b}\ \ \ then\ \ u = 1$

$$\therefore \quad \int_0^1 \frac{1}{a^n} \frac{u^{m-1}}{(a+b)^{m-1}} (1-u)^{n-1} . \frac{du}{a+b}$$

$$= \frac{1}{a^n (a+b)^m} \int_0^1 u^{m-1} (1-u)^{n-1} dx$$

$$= \frac{1}{(a+b)^m a^n}\ \beta(m,n) \quad = \text{R.H.S}$$

Example: 8 Show that $\boldsymbol{\beta(m,m) = 2^{1-2m}\beta\left(m,\frac{1}{2}\right)}$

Solution: By definition of Beta function $\quad \beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta \qquad \ldots(1)$

$Put\ n = m\ in\ above\ equation\ \ i.e\ equ.\ (1)\ ,we\ get$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2m-1}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}\, d\theta \quad \Rightarrow \quad 2\int_0^{\frac{\pi}{2}} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1}\, d\theta$$

$put\ \ 2\theta = \emptyset \quad so\ that\ \ d\theta = \frac{d\emptyset}{2}$

$When\ \ \theta = 0\ \ then\ \ \emptyset = 0\ \ \ and\ \ when\ \ \theta = \frac{\pi}{2}\ \ then\ \ \ \emptyset = \pi$

$$\therefore \quad \beta(m,n) = \frac{2}{2^{2m-1}} \int_0^{\pi} (\sin\emptyset)^{2m-1} \frac{d\emptyset}{2}$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin\emptyset)^{2m-1}\ d\emptyset$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin\emptyset)^{2m-1}\ d\emptyset \qquad \left\{\because \quad \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx\ \ if\ \ f(2a-x) = f(x)\right\}$$

$$= \frac{1}{2^{2m-1}} \left[2\int_0^{\frac{\pi}{2}} (\sin\emptyset)^{2m-1} . (\cos\emptyset)^{2.\frac{1}{2}-1}\ d\emptyset\right]$$

$$= 2^{1-2m}\beta\left(m,\frac{1}{2}\right)$$

$$= R.H.S \qquad\qquad \{using\ (1)\ \}$$

Excercise

Q: 1 Express the following integrals in terms of Beta function and then evaluate

(i) $\int_0^1 x^7(1-x^4)^9\ dx$ Ans. $\frac{1}{4}\beta(2,10)\ ;\ \frac{1}{440}$

(ii) $\int_0^1 x^5(1-x^3)^3\ dx$ Ans. $\frac{1}{3}\beta(2,4)\ ;\ \frac{1}{60}$

(iii) $\int_0^1 y(1-\sqrt{y})\ dy$ Ans. $2\beta(4,2)\ ;\ \frac{1}{10}$

Q: 2 Express the following integrals in terms of Beta function

(i) $\int_0^3 x^3(27-x^3)^{-\frac{1}{3}}\ dx$ Ans. $9\beta\left(\frac{4}{3},\frac{2}{3}\right)$

(ii) $\int_0^2 x^3(8-x^3)^{-\frac{1}{3}}\ dx$ Ans. $\frac{8}{3}\beta\left(\frac{4}{3},\frac{2}{3}\right)$

(iii) $\int_0^2 \sqrt{t}(8-t^2)^{-\frac{1}{4}}\ d\square$ Ans. $\beta\left(\frac{3}{4},\frac{3}{4}\right)$

(iv) $\int_0^1 x^{m-1}(1-x^2)^{n-1}\ dx$ Ans. $\frac{1}{2}\beta\left(\frac{m}{2},n\right)$

(v) $\int_0^1 x^2(1-x^3)^{\frac{3}{2}}\ dx$ Ans. $\frac{1}{3}\beta\left(1,\frac{5}{2}\right)$

Q: 3 Using the property $\beta(m,n)=\beta(n,m)$, evaluate $\int_0^1 x^3(1-x)^{\frac{4}{3}}dx$ Ans. $\frac{243}{7280}$

Q: 4 Show that $\int_0^p x^m(p^q-x^q)^n dx=\frac{p^{qn+m+1}}{q}\beta\left(n+1,\frac{m+1}{q}\right)$ where $p>0, q>0, m>-1\ , n>-1$

$\left\{Hint:\ \ Put\ \ \frac{x^q}{p^q}=t\ \ i.e\ \ x=pt^{\frac{1}{q}}\right\}$

Q: 5 Prove that $\int_0^\infty \frac{x^3}{(1+x)^7}\ dx=\frac{1}{60}$

Q: 6 Express $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\ dx$ in terms of Beta function, $where\ \ a,b,m,n\ are\ all\ positive$

$\left\{Hint:\ \ Put\ \ bx=at\ \ i.e\ \ x=\frac{a}{b}t\right\}$ Ans. $\frac{1}{a^n b^m}\beta(m,n)$

Q: 7 Prove that $\frac{\beta(m+1,n)}{\beta(m,n)}=\frac{m}{m+n}$

Q: 8 Show that:

(i) $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\ dx=2\beta(m,n)$

(ii) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\ dx=0$

(iii) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\ dx=0$

Q: 9 Prove that $\int_{-1}^\infty \frac{x+1}{(x+2)^6}\ dx=\frac{1}{20}$ $\left\{Hint:\ \ Put\ \frac{x+1}{x+2}=t\ \ i.e\ \ x=\frac{2t-1}{1-t}\right\}$

Q: 10 Show that $\int_0^1 x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}(1+2x)^{-1}dx\ =\ \frac{1}{(9)^{\frac{1}{3}}}\beta\left(\frac{2}{3},\frac{1}{3}\right)$ by substituting $\frac{x}{1-x}=\frac{az}{1-z}$, where a is constant suitably selected. $\{Hint:\ \ Taking\ \ 1-3a=0\}$

Q: 11 Prove that $\int_0^{\frac{\pi}{2}}(\sin\theta)^m(\cos\theta)^n\,d\theta=\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$, $where\ \ p>-1\ \ and\ \ q>-1$

Deduce that $\int_0^2 x^4(8-x^3)^{-\frac{1}{3}}dx=\frac{16}{3}\ \beta\left(\frac{5}{3},\frac{2}{3}\right)$

Q: 12 Prove that $\int_0^{\frac{\pi}{2}}(\sin\theta)^3(\cos\theta)^{\frac{5}{2}}d\theta=\frac{8}{77}$

Q: 13 Prove that $\int_0^{\frac{\pi}{2}}(x-a)^m(x-b)^n\,dx=\ (b-a)^{m+n+1}\beta(m+1,n+1)$

Gamma Function

Definition:
Gamma function is denoted by $\Gamma(n)$ & is defined as $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$, where $n > 0$

e.g. (i) $\int_0^\infty x^3 e^{-x}\, dx = \Gamma(3+1) = \Gamma(4)$

(ii) $\int_0^\infty x^{-4} e^{-x}\, dx$ is not a Gamma function $\because \quad n-1 = -4 \quad i.e \quad n = -3 < 0$

Theorem: (Recurrence Relation) $\boldsymbol{\Gamma(n+1) = n\Gamma(n)}$

Proof: By definition of Gamma function $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$

$= \left(\frac{x^n e^{-x}}{-1}\right)_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} dx$ $\quad\{Integration\ by\ parts\}$

$= n\int_0^\infty x^{n-1} e^{-x} dx$ $\quad\left\{\because \left(\frac{x^n e^{-x}}{-1}\right)_0^\infty = (0-0)\right\}$

$n\Gamma(n)$ $\quad\{by\ definition\ of\ Gamma\ function\}$

$\boldsymbol{Cor.\ If\ n\ is\ a\ positive\ integer\ , then\ \Gamma(n+1) = n!}$

Proof: We know that $\Gamma(n+1) = n\Gamma(n)$ ... (1)

Replace $n\ by\ n-1\ , we\ get$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

Substitute the value of $\Gamma(n)$ in equation no. (1) we get

$$\Gamma(n+1) = n(n-1)\Gamma(n-1)$$

Continue this process, we get

$$\Gamma(n+1) = n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots 2.1.\Gamma(1) \quad \ldots (2)$$

Now $\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\, dx = \int_0^\infty e^{-x}\, dx = \left(\frac{e^{-x}}{-1}\right)_0^\infty = (0+1) = 1$

Substitute the value of $\Gamma(1)$ in equation (2) , we get

$$\Gamma(n+1) = n!$$

Note: $\Gamma(0) = \infty \quad and \quad \Gamma(-n) = -\infty \quad if \quad n > 0$

Relationship between Beta and Gamma functions

Theorem: Prove that $\boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$, $\boldsymbol{m} > 0$ & $n > 0$.

Proof: By definition of Gamma function $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$ ... (1)

Put $t = x^2 \quad so\ that \quad dt = 2xdx$

$When\ t = 0\ then\ x = 0\ and\ when\ t \to \infty\ then\ x \to \infty$

$\therefore \quad \Gamma(n) = \int_0^\infty x^{2n-2} e^{-x^2} .2xdx$

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2} .dx \quad \ldots (2)$$

Similarly $\Gamma(m) = 2\int_0^\infty y^{2m-1} e^{-y^2} .dy$

Now $\Gamma(m)\Gamma(n) = 2\int_0^\infty y^{2m-1} e^{-y^2} dy.\ 2\int_0^\infty x^{2n-1} e^{-x^2} .dx$

$$= 4\int_0^\infty \int_0^\infty y^{2m-1} e^{-y^2} . x^{2n-1} e^{-x^2}\ dydx$$

$$= 4\int_0^\infty \int_0^\infty x^{2n-1} y^{2m-1} e^{-(x^2+y^2)}\ dydx \quad \ldots (3)$$

Change to Polar coordinates by substitution

$x = r\cos\theta \quad and \quad y = r\sin\theta \qquad so\ that \quad dxdy = rdrd\theta.$

From equ. (3) Show that region entirely in first quadrant

$\Rightarrow \quad r \ varies\ from\ 0\ to\ \infty$ and $\quad \theta \ varies\ from\ 0\ to\ \frac{\pi}{2}$ so equ. (3) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} cos^{2n-1}\theta\ sin^{2m-1}\theta\ drd\theta$$

$$= \left[2\int_0^{\frac{\pi}{2}} cos^{2n-1}\theta\ sin^{2m-1}\theta\ d\theta\right].\left[2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right]$$

$$= \beta(m,n)\ \Gamma(m+n)$$

$$\left\{\because \begin{array}{c} 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr = \Gamma(m+n)\ \ using\ equ.\ (2) \\ 2\int_0^{\frac{\pi}{2}} cos^{2n-1}\theta\ sin^{2m-1}\theta\ d\theta = \beta(m,n)\ \ by\ definition\ of\ Beta\ function \end{array}\right\}$$

Hence $\quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m+n})}$

Cor. $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Proof: We know $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m+n})}$

Taking $m = \frac{1}{2}\ \ and\ \ n = \frac{1}{2}\ \ ,\ we\ have$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \beta\left(\frac{1}{2},\frac{1}{2}\right) \qquad \{\because\ \Gamma(1) = 1\}$$

$$= \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\ dx = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\ dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}\ dx$$

Put $x = sin^2\theta \quad so\ that \quad dx = 2\sin\theta\cos\theta\ d\theta$

$When\ \ x = 0\ \ then\ \ \theta = 0\ \ and\ \ when\ \ x = 1\ \ then\ \ \theta = \frac{\pi}{2}$

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\sqrt{1-sin^2\theta}} .2\sin\theta\cos\theta\ d\theta = \int_0^{\frac{\pi}{2}} 2\ d\theta = 2[\theta]_0^{\frac{\pi}{2}} = 2.\frac{\pi}{2}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Theorem: Legendre's Duplication formula

Prove that $\boldsymbol{\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\ \Gamma(2m)}$

Proof: By definition of Beta function in polar form $\beta(m,n) = 2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$

Taking $n = m\ \ ,we\ get$

$$\beta(m,m) = 2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-1}(\cos\theta)^{2m-1}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left(\frac{2\sin\theta\ \cos\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}(\sin 2\theta)^{2m-1}\,d\square \qquad \ldots(1)$$

$Put\ \ 2\theta = \varphi \quad so\ that\ \ 2d\theta = d\varphi$

$When\ \ \theta = 0\ \ then\ \ \varphi = 0\ \ and\ \ \theta = \frac{\pi}{2}\ \ then\ \ \varphi = \pi$

$$\therefore \quad (1)\ becomes \quad \beta(m,m) = \frac{2}{2^{2m-1}}\int_0^{\pi}(\sin\varphi)^{2m-1}\frac{d\varphi}{2}$$

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(\mathrm{m+m})} = \frac{1}{2^{2m-1}}\int_0^{\pi}(\sin\varphi)^{2m-1}\,d\varphi \qquad \left\{\because\ \ \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m+n})}\right\}$$

$$\frac{[\Gamma(m)]^2}{\Gamma(\mathrm{m+m})} = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}(\sin\varphi)^{2m-1}\,d\varphi \qquad \left\{\because\ \ \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx\ \ if\ \ f(2a-x) = f(x)\right\}$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin\varphi)^{2m-1} cos^0\varphi \, d\varphi$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin\varphi)^{2m-1} cos^{2.\frac{1}{2}-1}\varphi \, d\varphi$$

$$= \frac{1}{2^{2m-1}} \; \beta\left(m, \frac{1}{2}\right)$$

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \; \beta\left(m, \frac{1}{2}\right)$$

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \; \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right\}$$

$$\frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \; \frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left\{\because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right\}$$

$$\Gamma(m)\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \; \Gamma(2m)$$

Examples

Example: 1 Evaluate the following integral

(i) $\int_0^\infty x^4 e^{-x} dx$

(ii) $\int_0^\infty \sqrt{x} e^{-x^3} dx$

(iii) $\int_0^\infty x^{2n-1} e^{-x^2} dx$

(iv) $\int_0^\infty 3^{-4x^2} dx$

(v) $\int_0^\infty \frac{x^a}{a^x} dx \quad , \; a > 1$

(vi) $\int_0^1 \sqrt{\frac{1-x}{x}} \; dx$

(vii) $\int_0^1 x(1-\sqrt{x}) \, dx$

Solution: (i) $\int_0^\infty x^4 e^{-x} dx = \int_0^\infty x^{5-1} e^{-x} dx \qquad \{\because \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx\}$

$\Gamma(5) = 4! = 24 \qquad \{\because \quad \Gamma(n+1) = n!\,\}$

(ii) $\boldsymbol{Let} \;\; \mathrm{I} = \int_0^\infty \sqrt{x} e^{-x^3} dx$

Put $x^3 = t \quad \Longrightarrow \quad x = t^{\frac{1}{3}} \quad so\; that \;\; dx = \frac{1}{3} t^{-\frac{2}{3}} dt$

$When \;\; x = 0 \;\; then \;\; t = 0 \;\; and \;\; x \to \infty \;\; then \;\; t \to \infty$

$\therefore \quad I = \int_0^\infty t^{\frac{1}{6}} e^{-t} \frac{1}{3} t^{-\frac{2}{3}} dt = \frac{1}{3} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$

$= \frac{1}{3} \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt \qquad \{\because \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx\}$

$= \frac{1}{3} \, \Gamma\left(\frac{1}{2}\right) = \frac{1}{3} \sqrt{\pi} \qquad \left\{\because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right\}$

(iii) $\boldsymbol{Let} \quad \boldsymbol{I} = \int_0^\infty x^{2n-1} e^{-x^2} dx$

Put $x^2 = t \quad \Longrightarrow \quad x = t^{\frac{1}{2}} \quad so\; that \;\; dx = \frac{1}{2} t^{-\frac{1}{2}} dt$

$When \;\; x = 0 \;\; then \;\; t = 0 \;\; and \;\; x \to \infty \;\; then \;\; t \to \infty$

$\therefore \quad I = \int_0^\infty t^{\frac{2n-1}{2}} e^{-t} \frac{1}{2} t^{-\frac{1}{2}} dt \quad = \quad \frac{1}{2} \int_0^\infty t^{n-1} e^{-t} dt$

$= \frac{1}{2} \, \Gamma(n) \qquad \{\because \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx\}$

(iv) $\boldsymbol{Let} \;\; \boldsymbol{I} = \int_0^\infty 3^{-4x^2} dx$

$= \int_0^\infty e^{-4x^2 log\, 3} dx$

Put $4x^2 log\, 3 = t \quad \Longrightarrow \quad x = \left(\frac{t}{4 \log 3}\right)^{\frac{1}{2}} \quad so\; that \;\; dx = \frac{1}{2} \left(\frac{1}{4 \log 3}\right)^{\frac{1}{2}} t^{-\frac{1}{2}} dt$

$When \;\; x = 0 \;\; then \;\; t = 0 \;\; and \;\; x \to \infty \;\; then \;\; t \to \infty$

$\therefore \quad I = \int_0^\infty e^{-t} . \frac{1}{2} \left(\frac{1}{4 \log 3}\right)^{\frac{1}{2}} t^{-\frac{1}{2}} dt \quad = \quad \frac{1}{2} \left(\frac{1}{4 \log 3}\right)^{\frac{1}{2}} \int_0^\infty e^{-t} . t^{\frac{1}{2}-1} dt$

$= \frac{1}{4} \left(\frac{1}{\log 3}\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \qquad \{\because \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx\}$

$= \frac{1}{4\sqrt{\log 3}} \sqrt{\pi} \qquad \left\{\because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right\}$

(v) $\boldsymbol{Let \quad I = \int_0^\infty \frac{x^a}{a^x} dx}$

$$= \int_0^\infty x^a . a^{-x} dx = \int_0^\infty x^a . e^{-x\log a} dx$$

Put $x\log a = t \implies x = \frac{t}{\log a}$ $\quad$ *so that* $dx = \frac{dt}{\log a}$

When $x = 0$ *then* $t = 0$ *and* $x \to \infty$ *then* $t \to \infty$

$$\therefore \quad I = \int_0^\infty \left(\frac{t}{\log a}\right)^a . e^{-t} \frac{dt}{\log a} = \left(\frac{1}{\log a}\right)^{a+1} \int_0^\infty t^a . e^{-t} dt$$

$$= \left(\frac{1}{\log a}\right)^{a+1} \int_0^\infty t^{a+1-1} . e^{-t} dt \qquad \left\{\because \quad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx\right\}$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

(vi) $\boldsymbol{Let \quad I = \int_0^1 \sqrt{\frac{1-x}{x}}\, dx}$

$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}\, dx \qquad \left\{\because \quad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx\right\}$$

$$= \beta\left(\frac{1}{2},\frac{3}{2}\right) \qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m+n})}\right\}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)} \qquad \left\{\because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \;\;\&\;\; \Gamma(n+1) = n\Gamma(n)\right\}$$

$$= \frac{\sqrt{\pi}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \qquad \{\because \quad \Gamma(2) = 1!\,\}$$

$$= \frac{\pi}{2}$$

(vii) $\boldsymbol{Let \quad I = \int_0^1 x(1-\sqrt{x})\, dx}$

Put $\sqrt{x} = t \implies x = t^2$ $\quad$ *so that* $dx = 2t\, dt$

When $x = 0$ *then* $t = 0$ *and* $x = 1$ *then* $t = 1$

$$\therefore \quad I = \int_0^1 x(1-\sqrt{x})\, dx = \int_0^1 t^2(1-t)2t dt$$

$$= 2\int_0^1 t^3(1-t) dt \qquad \left\{\because \quad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx\right\}$$

$$= 2.\beta(4,2)$$

$$= 2.\frac{\Gamma(4)\Gamma(2)}{\Gamma(4+2)} \qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m+n})}\right\}$$

$$= 2.\frac{3!\,.1!}{5!} \qquad \{\because \quad \Gamma(\mathrm{n}+1) = n!\}$$

$$= \frac{1}{10}$$

Example: 2 Express $\int_0^1 x^m(1-x^p)^n\, dx$ in terms of Gamma function, $\boldsymbol{where \quad m,n,p} > 0.$

Hence evaluate: $\int_0^1 x^7(1-x^4)^9\, dx$

Solution: Let $\boldsymbol{I = \int_0^1 x^m(1-x^p)^n\, dx}$

Put $x^p = t \implies x = t^{\frac{1}{p}}$ $\quad$ *so that* $dx = \frac{1}{p} t^{\frac{1}{p}-1} dt$

When $x = 0$ *then* $t = 0$ *and* $x = 1$ *then* $t = 1$

$$\therefore \quad I = \int_0^1 t^{\frac{m}{p}}(1-t)^n \; \frac{1}{p} t^{\frac{1}{p}-1} dt$$

$$= \frac{1}{p}\int_0^1 t^{\frac{m-p+1}{p}}(1-t)^n\, dt$$

$$= \frac{1}{p}\int_0^1 t^{\frac{m+1}{p}-1}(1-t)^{n+1-1}\, dt$$

$= \frac{1}{p}\ \beta\left(\frac{m+1}{p}, n+1\right)$ $\qquad \left\{\because \quad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx\right\}$

$= \frac{1}{p}\ \frac{\Gamma\left(\frac{m+1}{p}\right)\Gamma(n+1)}{\Gamma\left(\frac{m+1}{p}+n+1\right)}$ $\qquad \ldots(1)$ $\qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right\}$

Comparing $\int_0^1 x^7(1-x^4)^9\ dx$ with $\int_0^1 x^m(1-x^p)^n\ dx$, we have

$\boldsymbol{m = 7\ , p = 4\ , n = 9}$

Putting $m = 7\ , p = 4\ , n = 9$ in equ. (1) , we get

$$\int_0^1 x^7(1-x^4)^9\ dx = \frac{1}{4}\ \frac{\Gamma\left(\frac{7+1}{4}\right)\Gamma(9+1)}{\Gamma\left(\frac{7+1}{4}+9+1\right)}$$

$$= \frac{1}{4}\ \frac{\Gamma(2)\Gamma(10)}{\Gamma(12)} \qquad \{\because \quad \Gamma(n+1) = n!\}$$

$$= \frac{1}{4}\ \frac{1!\,9!}{11!} = \frac{1}{4}\ \frac{1}{11.10} = \frac{1}{440}$$

Example: 3 Evaluate $\int_0^\infty \frac{x^3}{(1+x)^7}\ dx$

Solution: We Know that $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\ dx = \beta(m,n)$

Putting $m = 4$ *and* $n = 3$, we get

$$\therefore \qquad \int_0^\infty \frac{x^3}{(1+x)^7}\ dx = \beta(4,3) = \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} \qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right\}$$

$$= \frac{3!2!}{6!} = \frac{2}{6.5.4} = \frac{1}{60} \qquad \{\because \quad \Gamma(n+1) = n!\}$$

Example: 4 show that $\boldsymbol{\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1}\ dx}$

Solution: By definition $\Gamma(n) = \int_0^\infty y^{n-1}e^{-y}\ dy \qquad ;\ n > 0$

Put $x = e^{-y}$ *i.e* $y = \log\left(\frac{1}{x}\right)$ *so that* $dy = -\frac{1}{x}\ dx$

When $y = 0$ *then* $x = 1$ *and when* $y = \infty$ *then* $x = 0$

$$\therefore \qquad \Gamma(n) = \int_1^0 \left(\log\frac{1}{x}\right)^{n-1} x.\left(-\frac{1}{x}\ dx\right)$$

$$= \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx \quad = \quad \text{R.H.S.}$$

Example: 5 Prove that $\boldsymbol{\int_0^{\frac{\pi}{2}} (\sin\theta)^m(\cos\theta)^n\ d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}+\frac{n+1}{2}\right)}}$, ***where*** $\boldsymbol{m} > -1$ *and* $n > -1$

Solution: ***Let*** $\boldsymbol{I = \int_0^{\frac{\pi}{2}} (\sin\theta)^m(\cos\theta)^n\ d\theta}$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{m-1}(\cos\theta)^{n-1}\sin\theta\ \cos\theta\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{m-1}{2}}(\cos^2\theta)^{\frac{n-1}{2}}.\,2\sin\theta\ \cos\theta\ d\theta$$

Put $\sin^2\theta = t$ *so that* $2\sin\theta\ \cos\theta\, d\theta = dt$

When $\theta = 0$ *then* $t = 0$ *and* $\theta = \frac{\pi}{2}$ *then* $t = 1$

$$= \frac{1}{2}\int_0^1 t^{\frac{m-1}{2}}(1-t)^{\frac{n-1}{2}}.\,dt$$

$$= \frac{1}{2}\ \beta\left(\frac{m-1}{2}+1, \frac{n-1}{2}+1\right) = \frac{1}{2}\ \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2}\ \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}+\frac{n+1}{2}\right)} \qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right\}$$

Hence $\boldsymbol{\int_0^{\frac{\pi}{2}} (\sin\theta)^m(\cos\theta)^n\ d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}+\frac{n+1}{2}\right)}}$

Example: 6 **Evaluate** $\int_0^{\frac{\pi}{2}} (\sin\theta)^3 (\cos\theta)^{\frac{5}{2}}\, d\theta$

Solution: **We know that** $\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta = \frac{1}{2}\, \beta(m,n)$

Putting $2m-1 = 3$ *and* $2n-1 = \frac{5}{2}$ *i.e* $m = 2$ *and* $n = \frac{7}{4}$

$$\therefore \int_0^{\frac{\pi}{2}} (\sin\theta)^3 (\cos\theta)^{\frac{5}{2}}\, d\theta = \frac{1}{2}\, \beta\left(2, \frac{7}{4}\right) \qquad \left\{\because \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m+n})}\right\}$$

$$= \frac{1}{2}\, \frac{\Gamma(2)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(2+\frac{7}{4}\right)} = \frac{1}{2}\, \frac{1!\, \Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{15}{4}\right)}$$

$$= \frac{1}{2}\, \frac{\Gamma\left(\frac{7}{4}\right)}{\frac{11}{4}\Gamma\left(\frac{11}{4}\right)} = \frac{1}{2}\, \frac{\Gamma\left(\frac{7}{4}\right)}{\frac{11}{4}.\frac{7}{4}\Gamma\left(\frac{7}{4}\right)} \qquad \{\because \quad \Gamma(n+1) = n\Gamma(n)\}$$

$$= \frac{1}{2}.\frac{16}{77} = \frac{8}{77}$$

Example: 7 **Show that** $\int_0^1 \frac{1}{\sqrt{1-x^4}}\, dx = \frac{1}{4\sqrt{2\pi}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2$

Solution: $\boldsymbol{Let\ \ I = \int_0^1 \frac{1}{\sqrt{1-x^4}}\, dx}$

Put $x^2 = \sin\theta$ *i.e* $x = \sqrt{\sin\theta}$ *so that* $dx = \frac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta$

When $x = 0$ *then* $\theta = 0$ *and when* $x = 1$ *then* $\theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta\, \cos^0\theta\, d\theta \qquad \left\{\because \quad \int_0^{\frac{\pi}{2}} (\sin\theta)^m (\cos\theta)^n\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{\mathrm{m+1}}{2}+\frac{\mathrm{n+1}}{2}\right)}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2}\, \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}\right] = \frac{1}{4}\, \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\, \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2 \sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\, \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2 \sqrt{\pi}}{\sqrt{2}\,\pi} \qquad \left\{\because \quad \Gamma(\mathrm{m})\,\Gamma(1-\mathrm{m}) = \frac{\pi}{\sin m\pi}\right\}$$

$$= \frac{1}{4\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Exercise

Q:1 Evaluate the following integral

(i) $\int_0^\infty x^5 e^{-x} dx$ $\{Ans.: 120\}$

(ii) $\int_0^\infty x^3 e^{-x} dx$ $\{Ans.: 6\}$

(iii) $\int_0^\infty x^6 e^{-2x} dx$ $\left\{Ans.: \frac{45}{8}\right\}$

(iv) $\int_0^\infty x^5 e^{-x^2} dx$ $\{Ans.: 1\}$

(v) $\int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}} dx$ $\left\{Ans.: \frac{3}{2}\sqrt{\pi}\right\}$

(vi) $\int_0^\infty e^{-x^3} dx$ $\left\{Ans.: \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\right\}$

(vii) $\int_0^1 x^4(1-x)^3 \; dx$ $\left\{Ans.: \frac{1}{280}\right\}$

(viii) $\int_0^1 x^2(1-x)^3 \; dx$ $\left\{Ans.: \frac{1}{60}\right\}$

(ix) $\int_0^1 x^4(1-x)^5 \; dx$ $\left\{Ans.: \frac{1}{1260}\right\}$

(x) $\int_0^1 x^5(1-x^3)^{10} \; dx$ $\left\{Ans.: \frac{1}{396}\right\}$

(xi) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$ $\{Ans.: \pi\}$

(xii) $\int_0^\infty a^{-bx^2} \; dx$ $\left\{Ans.: \frac{\sqrt{\pi}}{2\sqrt{b\log a}}\right\}$

(xiii) $\int_0^1 x^5 \left[\log\left(\frac{1}{x}\right)\right]^3 dx$ $\left\{Hint: put\ x = e^{-\frac{t}{5}}\right\}$ $\left\{Ans.: \frac{6}{625}\right\}$

Q: 2 Prove that $\int_0^\infty x^n e^{-a^2x^2} dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$, $n > -1$. Hence evaluate $\int_{-\infty}^\infty e^{-a^2x^2} dx$

Q: 3 Prove that $\int_0^\infty x^n e^{-ax^m} dx = \frac{1}{ma^{\frac{n+1}{m}}}\Gamma\left(\frac{n+1}{m}\right)$, *where* m, n & a *are positive constants.*

Q: 4 Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\ \Gamma\left(\frac{1}{n}\right)}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$ $\{Hint:\ put\ x^n = t\}$

Q: 5 Express $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$ in terms of Beta and Gamma functions, where $m, n, a, b > 0$

$\{Hint:\ Put\ bx = at\}$

Q: 6 Show that $\Gamma(n) = k^n \int_0^\infty x^{n-1} e^{-kx} dx$ $\{Hint:\ Put\ kx = t\}$

Q: 7 Show that

(i) $\int_0^\infty \frac{x^2}{(1+x)^6} dx = \frac{1}{30}$ $\{Hint:\ By\ def.\ of\ Beta\ function\}$

(ii) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}} dx = \frac{1}{5005}$

Q: 8 Show that $\int_0^\infty \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{1}{a^n(1+a)^m}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ $\left\{Hint:\ Put\ \frac{x}{a+x} = \frac{t}{a+1}\right\}$

Q: 9 Prove that (i) $\int_0^{\frac{\pi}{2}} (\sin\theta)^7 \, d\theta = \frac{16}{35}$

(ii) $\int_0^{\frac{\pi}{2}} (\sin\theta)^p \, d\theta \int_0^{\frac{\pi}{2}} (\sin\theta)^{p+1} \, d\theta = \frac{\pi}{2(p+1)}$

(iii) $\int_0^{\frac{\pi}{2}} (\sin\theta)^5 \, d\theta = \frac{8}{15}$

(iv) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} \, d\theta = \frac{\pi}{\sqrt{2}}$

(v) $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}} \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx = \pi$

(vi) $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta} \, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

Q: 10 Show that (i) $\int_0^\infty \frac{x}{1+x^6}\, dx = \frac{\pi}{3\sqrt{3}}$ $\{Hint: \; Put\; x^3 = tan\,\theta\}$

(ii) $\int_0^\infty \frac{x^2}{1+x^4}\, dx = \frac{\pi}{2\sqrt{2}}$ $\{Hint: \; Put\; x^2 = tan\,\theta\}$

(iii) $\int_0^\infty \frac{1}{1+x^4}\, dx = \frac{\pi}{2\sqrt{2}}$ $\{Hint: \; Put\; x^2 = tan\,\theta\}$

Q: 11 Show that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx \int_0^\infty \frac{1}{\sqrt{1+x^4}}\, dx = \frac{\pi}{2\sqrt{2}}$

Q: 12 Show that $2^n\,\Gamma\left(n+\frac{1}{2}\right) = 1.3.5 \ldots\ldots\ldots (2n-1)\sqrt{\pi}$, $where\; n\; is + ve\; integer$

Q: 13 Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$

Q: 14 Using gamma function, Prove that $\int_0^a x^4\sqrt{a^2-x^2}\, dx = \frac{\pi\, a^6}{32}$

Summary:

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$	$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\, d\theta$
$\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\ dx$	$\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$
$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\ dx$	$\beta(m,n) = \frac{m-1}{n}\ \beta(m-1,n+1)$
$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\ dx$	$\beta(m,n) = \beta(n,m)$
$\beta(m+1,n) = \beta(m,n)\,.\frac{m}{m+n}$	$\beta(m,n) = \beta(m,n+1) + \beta(m+1,n)$
$\beta(m,n+1) = \beta(m,\square)\,.\frac{n}{m+n}$	$\int_0^{\frac{\pi}{2}} (\sin\theta)^m(\cos\theta)^n\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{\mathrm{m}+1}{2}+\frac{\mathrm{n}+1}{2}\right)}$
$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\, dx$, where $n > 0$	$\Gamma(n+1) = n\Gamma(n)$
$\Gamma(0) = \infty \quad and \quad \Gamma(-n) = -\infty\ \ if\ \ n > 0$	$\Gamma(n+1) = n!$
Legendre's Duplication formula $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\ \Gamma(2m)$	$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(\mathrm{m}+\mathrm{n})}$
	$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$