# SOME FIXED POINT THEOREMS IN MODULAR METRIC SPACES

**MR. ANURUK NOYWISET**

**A THESIS FOR THE DEGREE OF MASTER OF SCIENCE**

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**MR. ANURUK NOYWISET**

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**Author:** Mr. Anuruk Noywiset

**Thesis Examination Committee**

| | |
|---|---|
| Assoc. Prof. Dr. Kittikorn Nakprasit | Chairperson |
| Assoc. Prof. Dr. Satit Saejung | Member |
| Asst. Prof. Dr. Chokchai Viriyapong | Member |

**Thesis Advisor:**

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . Advisor

(Assoc. Prof. Dr. Satit Saejung)

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| (Prof. Dr. Surasakdi Wongratanacheewin) | (Asst. Prof. Dr. Somkiat Srijaranai) |
| Dean, Graduate School | Dean, Faculty of Science |

อนุรักษ์ น้อยวิเศษ. 2561. **ทฤษฎีบทจุดตรึงบางบทบนปริภูมิเมตริกมอดุลาร์.** วิทยานิพนธ์ปริญญาวิทยาศาสตรมหาบัณฑิต สาขาวิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยขอนแก่น.

**อาจารย์ที่ปรึกษาวิทยานิพนธ์:** รศ. ดร. สาธิต แซ่จึง

## บทคัดย่อ

แนวคิดของทฤษฎีจุดตรึงมีความเป็นมายาวนานเช่นเดียวกับแนวคิดของปริภูมิเมตริกมอดุลาร์ Mongkolkeha และคณะได้พิสูจน์ทฤษฎีบทจุดตรึงสองบทในปริภูมิเมตริกมอดุลาร์ในปี 2011 แต่มีข้อผิดพลาดอยู่หลายแห่งและข้อผิดพลาดเหล่านี้ยังได้ถูกรายงานโดยนักวิจัยหลายท่าน ต่อมา Abdou และ Khamsi ได้ให้ข้อแก้ไขบางประการภายใต้สมมติฐาน regularity ในวิทยานิพนธ์นี้ เราจะนำเสนอทฤษฎีบทจุดตรึงสามบทในปริภูมิเมตริกมอดุลาร์โดยปราศจากสมมติฐาน regularity

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**Thesis Advisor:** Assoc. Prof. Dr. Satit Saejung

## ABSTRACT

The concept of fixed point theory has a long history as long as that of modular metric spaces. In 2011, Mongkolkeha et al. proved two fixed point theorems in modular metric spaces. Unfortunately there are many gaps in their proofs reported by many authors. Abdou and Khamsi gave some corrections with the regularity assumption. In this thesis, we present three fixed point theorems in modular metric spaces without assuming the regularity as was the case in Abdou and Khamsi.

**Goodness Portion to the Present Thesis is Dedicated for my Parents and Entire Teaching Staffs**

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# CHAPTER I

# INTRODUCTION

Let $(X, d)$ be a metric space and let $T : X \to X$ be a mapping. An element $x \in X$ is called a *fixed point* of $T$ if $x = Tx$. Many mathematicians have investigated various conditions on $T$ that guarantee the existence of a fixed point of $T$. In 1922, Banach [2] proved the following fixed point theorem which is known as Banach contraction principle:

Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping satisfying

$$\exists k \in (0, 1), \forall x, y \in X \; d(Tx, Ty) \leq kd(x, y).$$

Then $T$ has a unique fixed point.

A self-mapping $T$ on metric space $(X, d)$ is called *Kannan* if there exists $k \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$.

In 1968, Kannan [7] proved that $X$ is complete, then a Kannan mapping $T : X \to X$ has a fixed point. The notion of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [12] and were extensively developed by his mathematical school. In 2008, Chistyakov [3, 4, 5] introduced the notion of modular metric spaces generated by an $F$-modular. In 2011, Mongkolkeha, Sintunavarat and Kumam [10] proved the following two fixed point theorems related to Banach and Kannan fixed point theorems.

**Theorem 1.0.1** ([10, Theorem 3.2])**.** Let $(X, w)$ be a complete modular metric space and $T$ be a self-mapping on $X$ satisfying the inequality

$$w(\lambda, Tx, Ty) \leq \theta w(\lambda, x, y),$$

for all $x, y \in X_w$. where $\theta \in [0, 1)$. Then, $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $\{T^n x\}$ converges to $x^*$.

**Theorem 1.0.2** ([10, Theorem 3.6])**.** Let $(X, w)$ be a complete modular metric space and $T$ be a self-mapping on $X$ satisfying the inequality

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}(w(2\lambda, x, Tx) + w(2\lambda, y, Ty)),$$

for all $x, y \in X_w$. where $\theta \in (0, 1)$. Then, $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $\{T^n x\}$ converges to $x^*$.

But their proofs are not correct. In 2012, they gave some corrections in [11].

**Theorem 1.0.3** ([11, Theorem 2.1])**.** Let $(X, w)$ be a complete modular metric space and $T$ be a self-mapping on $X$ satisfying the inequality

$$w(\lambda, Tx, Ty) \leq \theta w(\lambda, x, y),$$

for all $x, y \in X_w$. where $\theta \in [0, 1)$. Suppose that there exists $x_0 \in X_w$ such that $w(\lambda, x_0, Tx_0) < \infty$ for all $\lambda > 0$. Then, $T$ has a unique fixed point $x_*$ in $X_w$ and the sequence $\{T^n x\}$ converges to $x^*$.

**Theorem 1.0.4** ([11, Theorem 2.2])**.** Let $(X, w)$ be a complete modular metric space and $T$ be a self-mapping on $X$ satisfying the inequality

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}(w(2\lambda, x, Tx) + w(2\lambda, y, Ty)),$$

for all $x, y \in X_w$. where $\theta \in (0, 1)$. Suppose that there exists $x_0 \in X_w$ such that $w(\lambda, x_0, Tx_0) < \infty$ for all $\lambda > 0$. Then, $T$ has a unique fixed point $x_*$ in $X_w$ and the sequence $\{T^n x\}$ converges to $x^*$.

Unfortunately, the second version are not correct again. Abdou and Khamsi [1] used the concept of regularity and discuss another concept of convergence to overcome this problem.

**Theorem 1.0.5** ([1, Theorem 3.1])**.** Suppose that $(X, w)$ is a modular metric space and $w$ is regular. Let $C$ be a $w$-complete subset of $X_w$ such that $\delta_w(C) := \sup\{w(1, x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be a $w$-Banach contraction, that is, there exists a constant $\theta \in (0, 1)$ such that

$$w(1, Tx, Ty) \leq \theta w(1, x, y)$$

for all $x, y \in C$. Then $T$ has a unique fixed point $x^*$. $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

In this thesis, we will correct Theorem 1.0.3 and Theorem 1.0.4 in another way. Namely, we do not assume the regularity as was the case in Abdou and Khamsi. We give a brief outline of how we intend to proceed and what each chapter contains. This thesis is organized as follows.

Chapter II mainly presents some basic concepts, definitions, properties and fixed point theorems and modular metric spaces which are important for Chapter IV. In Chapter III, we discuss the process of the research and the thesis. Chapter IV consists of two sections. In the first section, we introduce the two kinds of sequences in a modular metric space and give a necessary and sufficient condition for such a sequence to be a Cauchy sequence. In the last section, we present three fixed point theorems in modular metric spaces. In Chapter V, we summarize the results of this thesis.

# CHAPTER II

# LITERATURE REVIEWS

In this chapter, we recall some basic concepts, definitions, properties and theorems of modular metric spaces that will be used in Chapter IV.

## 2.1 Modular metric spaces

**Definition 2.1.1** ([4, 5]). A pair $(X, w)$ is called a *modular metric space* if $X$ is a nonempty set and $w : (0, \infty) \times X^2 \to [0, \infty]$ is a function satisfying the following properties:

(a) $w(\lambda, x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(b) $w(\lambda, x, y) = w(\lambda, y, x)$ for all $x, y \in X$ and for all $\lambda > 0$;

(c) $w(\lambda + \mu, x, z) \leq w(\lambda, x, y) + w(\mu, y, z)$ for all $x, y, z \in X$ and for all $\lambda, \mu > 0$.

The function $w$ above is called a *modular* on $X$.

**Example 2.1.2.** The following indexed objects $w$ are simple examples of modulars on a set $X$. Let $\lambda > 0$ and $x, y \in X$. We have:

- $w^a(\lambda, x, y) = \infty$ if $x \neq y$ and $w^a(\lambda, x, y) = 0$ if $x = y$.

Let $(X, d)$ be a metric space. Then we also have:

- $w^b(\lambda, x, y) = \infty$ if $\lambda \leq d(x, y)$ and $w^b(\lambda, x, y) = 0$ if $\lambda > d(x, y)$;
- $w^c(\lambda, x, y) = \infty$ if $\lambda < d(x, y)$ and $w^c(\lambda, x, y) = 0$ if $\lambda \geq d(x, y)$.

**Example 2.1.3.** Let $(X, d)$ be a metric space. Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is nondecreasing and suppose that $w(\lambda, x, y) := d(x, y)/\varphi(\lambda)$ for all $x, y \in X$ and for all $\lambda > 0$. Then $w$ is a modular on $X$. In particular, every metric space is a modular metric space.

Observe that if $w$ is a modular on $X$, then $\lambda \mapsto w(\lambda, x, y)$ is nonincreasing. In fact, let

$x, y \in X$ and $0 < \mu < \lambda$. It follows from the condition (c) of Definition 2.1.1 that

$$w(\lambda, x, y) \leq w(\lambda - \mu, x, x) + w(\mu, x, y) = w(\mu, x, y).$$

As studied in the linear case, we pay attention on the following subspace.

**Definition 2.1.4.** ([4, 5]) Let $(X, w)$ be a modular metric space and $x_0 \in X$. Define the *modular space around* $x_0$ by

$$X_w(x_0) := \left\{ x \in X : \lim_{\lambda \to \infty} w(\lambda, x, x_0) = 0 \right\}.$$

**Example 2.1.5.** Let $x_0 \in X$. For modulars $w = w^a, w^b, w^c$ from Example 2.1.2 we have, respectively:

- $X_{w^a}(x_0) = \{x_0\}$;
- $X_{w^b}(x_0) = X_{w^c}(x_0) = X$.

**Example 2.1.6.** Suppose that $(X, d)$ is a metric space and $x_0 \in X$. Let $w$ be the modular defined in Example 2.1.3 with respect to a nondecreasing function $\varphi$.

- If $\varphi$ is bounded above, then $X_w(x_0) = \{x_0\}$.
- If $\varphi(\lambda) \to \infty$ as $\lambda \to \infty$, then $X_w(x_0) = X$.

The following concepts are analogous to the one in the classical metric spaces.

**Definition 2.1.7** ([4, 5])**.** Let $(X, w)$ be a modular metric space.

- A sequence $\{x_n\}$ in $X$ is *convergent* if there exists an element $x \in X$ such that

  $$\lim_{n \to \infty} w(\lambda, x_n, x) = 0$$

  for all $\lambda > 0$. In this case, we also write $x_n \xrightarrow{w} x$ for the statement "$\lim_{n \to \infty} w(\lambda, x_n, x) = 0$ for all $\lambda > 0$".
- A sequence $\{x_n\}$ in $X$ is *Cauchy* if $\lim_{n,m \to \infty} w(\lambda, x_m, x_n) = 0$ for all $\lambda > 0$.
- A subset $C$ of $X$ is *complete* if every Cauchy sequence in $C$ is convergent and its limit is in $C$.
- A subset $C$ of $X$ is *bounded* if $\sup\{w(\lambda, x, y) : x, y \in C\} < \infty$ for all $\lambda > 0$.

**Remark 2.1.8.** Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$. If $x_n \xrightarrow{w} x^*$ and $x_n \xrightarrow{w} x^{**}$ where $x^*, x^{**} \in X$, then $x^* = x^{**}$.

*Proof.* Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$. Suppose that $x_n \xrightarrow{w} x^*$ and $x_n \xrightarrow{w} x^{**}$ where $x^*, x^{**} \in X$. For $\lambda > 0$ and $n \in \mathbb{N}$, we have

$$w(\lambda, x^*, x^{**}) \leq w(\lambda/2, x_n, x^*) + w(\lambda/2, x_n, x^{**}).$$

Since $x_n \xrightarrow{w} x^*$ and $x_n \xrightarrow{w} x^{**}$, we have $w(\lambda, x^*, x^{**}) = 0$. Hence $w(\lambda, x^*, x^{**}) = 0$ for all $\lambda > 0$ and so $x^* = x^{**}$. $\square$

**Remark 2.1.9.** If $\{x_n\}$ is a convergent sequence in a modular metric space $(X, w)$, then it is a Cauchy sequence.

*Proof.* Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$ that converges to the limit $x^* \in X$. Let $\varepsilon > 0$ and $\lambda > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$,

$$w(\lambda/2, x_n, x^*) < \varepsilon/2.$$

For $m > n > N$, we have

$$w(\lambda, x_m, x_n) \leq w(\lambda/2, x_m, x^*) + w(\lambda/2, x_n, x^*) < \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. $\square$

**Lemma 2.1.10** ([4])**.** Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a modular metric space $(X, w)$. If $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ where $x, y \in X$, then

$$\lim_{\mu \to \lambda^+} w(\mu, x, y) \leq \liminf_{n \to \infty} w(\lambda, x_n, y_n) \leq \limsup_{n \to \infty} w(\lambda, x_n, y_n) \leq \lim_{\mu \to \lambda^-} w(\mu, x, y)$$

for all $\lambda > 0$.

*Proof.* Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a modular metric space $(X, w)$ such that $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ where $x, y \in X$. For $\lambda > 0$ and $0 < \varepsilon < \lambda$,

$$w(\lambda, x_n, y_n) \leq w(\varepsilon/2, x_n, x) + w(\lambda - \varepsilon, x, y) + w(\varepsilon/2, y_n, y).$$

Note that $\lim_{n\to\infty} w(\varepsilon/2, x_n, x) = \lim_{n\to\infty} w(\varepsilon/2, y_n, y) = 0$. Thus

$$\limsup_{n \to \infty} w(\lambda, x_n, y_n) \leq w(\lambda - \varepsilon, x, y).$$

Note that since $\lambda \mapsto w(\lambda, x, y)$ is nonincreasing, $\lim_{\mu\to\lambda^-} w(\mu, x, y)$ and $\lim_{\mu\to\lambda^+} w(\mu, x, y)$ exist. Hence $\limsup_{n\to\infty} w(\lambda, x_n, y_n) \le \lim_{\mu\to\lambda^-} w(\mu, x, y)$. On the other hand,

$$w(\lambda + \varepsilon, x, y) \le w(\varepsilon/2, x, x_n) + w(\lambda, x_n, y_n) + w(\varepsilon/2, y_n, y).$$

Note that $\lim_{n\to\infty} w(\varepsilon/2, x_n, x) = \lim_{n\to\infty} w(\varepsilon/2, y_n, y) = 0$. Thus

$$w(\lambda + \varepsilon, x, y) \le \liminf_{n\to\infty} w(\lambda, x_n, y_n).$$

Since $\lim_{\mu\to\lambda^+} w(\mu, x, y)$ exists, $\lim_{\mu\to\lambda^+} w(\mu, x, y) \le \liminf_{n\to\infty} w(\lambda, x_n, y_n)$. $\square$

## 2.2 Some fixed point theorems in modular metric spaces

In 2011, Mongkolkeha et al. [10] introduced the following two kinds of mappings in modular metric spaces which are related to Banach and Kannan fixed point theorems in metric spaces.

**Definition 2.2.1.** Let $C$ be a subset of a modular metric space $(X, w)$. A mapping $T : C \to C$ is

- a *Banach contraction* (on $C$) if there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \le \theta w(\lambda, x, y)$$

  for all $x, y \in C$ and for all $\lambda > 0$;

- a *strong Kannan contraction* (on $C$) if there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \le \frac{\theta}{2}(w(2\lambda, x, Tx) + w(2\lambda, y, Ty))$$

  for all $x, y \in C$ and for all $\lambda > 0$.

Mongkolkeha et al. [10] also proved the following fixed point theorems for Banach contractions and strong Kannan contractions on $X_w$.

**Theorem 2.2.2** ([10, Theorem 3.2])**.** Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a Banach contraction. If $X_w$ is complete, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x \xrightarrow{w} x^*$ for all $x \in X_w$.

**Theorem 2.2.3** ([10, Theorem 3.6])**.** Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a strong Kannan contraction. If $X_w$ is complete, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x \xrightarrow{w} x^*$ for all $x \in X_w$.

Dehghan et al. [6] gave a counterexample to Theorem 2.2.2. Almost at the same time, Mongkolkeha et al. [11] discussed some mistakes in their results and proposed the following corresponding corrections.

**Theorem 2.2.4** ([11, Theorem 2.1])**.** Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a Banach contraction. If $X_w$ is complete and there exists $x_1 \in X_w$ such that $w(\lambda, x_1, Tx_1) < \infty$ for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x_1 \xrightarrow{w} x^*$.

**Theorem 2.2.5** ([11, Theorem 2.2])**.** Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a strong Kannan contraction. If $X_w$ is complete and there exists $x_1 \in X_w$ such that $w(\lambda, x_1, Tx_1) < \infty$ for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x_1 \xrightarrow{w} x^*$.

After a careful reading of the proofs of the preceding two theorems, there remain several gaps in their proofs [11]. As noted in Abdou and Khamsi's paper [1], we quote the following:

> Indeed, a modular may take infinite value. This is the problem that the authors of [10] did not pay attention to. This was also pointed out in the short note [6]. In fact, the authors of [10] did try to fix this problem in another short note [11] but they used the triangle inequality in their proof knowing that $w$ does not in general satisfy the triangle inequality.

To properly establish the classical Banach contraction principle in the best possible way in modular metric spaces, Abdou and Khamsi [1] used the concept of regularity and discuss another concept of convergence to overcome this problem. Recall that a modular $w$ on $X$ is said to be *regular* if

$$x = y \iff w(\lambda, x, y) = 0 \text{ for some } \lambda > 0.$$

Together with this regularity assumption, the following concepts studied in [1].

**Definition 2.2.6** ([1])**.** Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$.

- A sequence $\{x_n\}$ in $X_w$ is *w-convergent* to $x \in X_w$ if $\lim_{n\to\infty} w(\lambda, x_n, x) = 0$ for some $\lambda > 0$.
- A sequence $\{x_n\}$ in $X_w$ is *w-Cauchy* if $\lim_{n,m\to\infty} w(\lambda, x_m, x_n) = 0$ for some $\lambda > 0$.
- A subset $C$ of $X_w$ is *w-complete* if every $w$-Cauchy sequence in $C$ is convergent to some point in $C$.

As claimed in [1], the following result is the "best possible way" in modular metric spaces.

**Theorem 2.2.7** ([1, Theorem 3.1])**.** Suppose that $(X, w)$ is a modular metric space and $w$ is regular. Let $C$ be a $w$-complete subset of $X_w$ such that $\delta_w(C) := \sup\{w(1, x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be a $w$-Banach contraction, that is, there exists a constant $\theta \in (0, 1)$ such that

$$w(1, Tx, Ty) \leq \theta w(1, x, y)$$

for all $x, y \in C$. Then $T$ has a unique fixed point $x^*$. $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

**Remark 2.2.8.** Let $(X, w)$ be a modular metric space.

- If $\{x_n\}$ is a sequence on $X$ such that $x_n \xrightarrow{w} x^* \in X$, then $\{x_n\}$ is $w$-convergent to $x^*$.
- If $\{x_n\}$ is a Cauchy sequence in $X$, then $\{x_n\}$ is a $w$-Cauchy sequence in $X$.

# CHAPTER III

# RESEARCH METHODOLOGY

After a careful reading of two papers of Mongkolkeha et al. [10, 11], we start our work by defining two new concepts of certain sequences generated from Banach and Kannan contractions. We find a characterization of the Cauchyness of theses two sequences in modular metric spaces without the regularity as was the case in the work of Abdou and Khamsi [1]. Using such characterizations, we study fixed point theorems for Banach and Kannan contractions in a complete modular metric space.

# CHAPTER IV

# RESULTS AND DISCUSSIONS

In this chapter, we present the main results of this thesis.

## 4.1 Sequences of Banach type and of Kannan type

We first introduce the following two kinds of sequences in a modular metric space which play an important role in this thesis.

**Definition 4.1.1.** Let $(X, w)$ be a modular metric space. For a given $\lambda > 0$, we say that a sequence $\{x_n\}$ in $X$ is

- a $\lambda$-*sequence of Banach type* if there exists $\theta := \theta(\lambda) \in (0, 1)$ such that

$$w(\lambda, x_{n+1}, x_{m+1}) \leq \theta w(\lambda, x_n, x_m) \quad \text{for all } n, m \geq 1;$$

- a $\lambda$-*sequence of Kannan type* if there exists $\theta := \theta(\lambda) \in (0, 1)$ such that

$$w(\lambda, x_{n+1}, x_{m+1}) \leq \frac{\theta}{2}(w(\lambda, x_n, x_{n+1}) + w(\lambda, x_m, x_{m+1})) \quad \text{for all } n, m \geq 1.$$

We also say that $\{x_n\}$ is a *sequence of Banach type (of Kannan type, respectively)* if it is a $\lambda$-sequence of Banach type (of Kannan type, respectively) for all $\lambda > 0$.

To prove convergence of a given sequence $\{x_n\}$ in the presence of completeness of the space, the key step is to prove that $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ for all $\lambda > 0$. The following two results give a necessary and sufficient condition for the situation $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ provided that $\{x_n\}$ is a $\lambda$-sequence of Banach type and of Kannan type, respectively.

**Proposition 4.1.2.** Let $(X, w)$ be a modular metric space and let $\lambda > 0$ be given. Let $\{x_n\}$ be a sequence in $X$ and set

$$B(\lambda, n) := \sup\{w(\lambda, x_n, x_{n+k}) : k \geq 1\}.$$

Suppose that $\{x_n\}$ is a $\lambda$-sequence of Banach type. Then the following conditions are equivalent.

(a) There exists an integer $n \geq 1$ such that $B(\lambda, n) < \infty$.

(b) $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$.

*Proof.* (b) $\Longrightarrow$ (a) is obvious. We now prove (a) $\Longrightarrow$ (b).
Without loss of generality, we assume that $B(\lambda, 1) < \infty$. For $n, k \geq 1$, we note that $w(\lambda, x_{n+1}, x_{n+1+k}) \leq \theta w(\lambda, x_n, x_{n+k}) \leq B(\lambda, n)$. In particular, $B(\lambda, n+1) \leq \theta B(\lambda, n)$. Since $B(\lambda, 1) < \infty$, we have $\lim_{n\to\infty} B(\lambda, n) = 0$ and hence (b) is satisfied. □

**Proposition 4.1.3.** Let $(X, w)$ be a modular metric space and let $\lambda > 0$ be given. Let $\{x_n\}$ be a sequence in $X$ and set

$$K(\lambda, n) := w(\lambda, x_n, x_{n+1}).$$

Suppose that $\{x_n\}$ is a $\lambda$-sequence of Kannan type. Then the following conditions are equivalent.

(a) There exists $N \geq 1$ such that $K(\lambda, n) < \infty$ for all $n \geq N$.

(b) $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$.

*Proof.* (b) $\Longrightarrow$ (a) is obvious. We now prove (a) $\Longrightarrow$ (b). Assume that Condition (a) holds. We may assume without loss of generality that $K(\lambda, n) < \infty$ for all $n \geq 1$. For $n \geq 1$, we note that

$$\begin{aligned} K(\lambda, n+1) &= w(\lambda, x_{n+1}, x_{n+2}) \\ &\leq \frac{\theta}{2}(w(\lambda, x_n, x_{n+1}) + w(\lambda, x_{n+1}, x_{n+2})) \\ &= \frac{\theta}{2}(K(\lambda, n) + K(\lambda, n+1)). \end{aligned}$$

In particular, $K(\lambda, n+1) \leq \frac{\theta}{2-\theta} K(\lambda, n)$ and hence $\lim_{n\to\infty} K(\lambda, n) = 0$. For $n, m \geq 1$, we now have

$$\begin{aligned} w(\lambda, x_{n+1}, x_{m+1}) &\leq \frac{\theta}{2}(w(\lambda, x_n, x_{n+1}) + w(\lambda, x_m, x_{m+1})) \\ &= \frac{\theta}{2}(K(\lambda, n) + K(\lambda, m)). \end{aligned}$$

This implies that $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ and hence (b) is satisfied. □

As consequences of the preceding two propositions, we immediately obtain the following theorem.

**Theorem 4.1.4.** Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$. The following statements are true.

(1) A sequence of Banach type $\{x_n\}$ is Cauchy if and only if for each $\lambda > 0$ there exists an integer $n \geq 1$ such that $B(\lambda, n) < \infty$.

(2) A sequence of Kannan type $\{x_n\}$ is Cauchy if and only if for each $\lambda > 0$ there exists an integer $N$ such that $K(\lambda, n) < \infty$ for all $n \geq N$.

**Example 4.1.5.** Let $(X, d)$ be a metric space and define modular

$$w(\lambda, x, y) := d(x, y)$$

for all $x, y \in X$ and for all $\lambda > 0$. If $\{x_n\}$ is a sequence of Banach type then $\{x_n\}$ is Cauchy.

In general, a sequence of Banach type may be not Cauchy.

**Example 4.1.6.** Let $X = \mathbb{R}$. Define

$$w(\lambda, x, y) := \begin{cases} \infty & \text{if } \lambda \leq |x - y|; \\ 0 & \text{if } \lambda > |x - y| \end{cases}$$

for all $x, y \in X$ and for all $\lambda > 0$. Let $\{x_n\} := \{1, 2, 3, \dots\}$. It follows that

- $\{x_n\}$ is a sequence of Banach type;
- $\{x_n\}$ is not Cauchy.

## 4.2 Three fixed point theorems

In this section, we present fixed point theorems which can be viewed as a correction of the corresponding fixed point theorems proved by Mongkolkeha et al. ([10, 11]).

### 4.2.1 Banach type mappings

We first observe the following result.

**Proposition 4.2.1.** Let $C$ be a subset of a modular metric space $(X, w)$. If $T : C \to C$ is a Banach contraction, then $T$ is continuous, that is, $Tx_n \xrightarrow{w} Tx^*$ whenever $\{x_n\}$ is a sequence in $C$ such that $x_n \xrightarrow{w} x^* \in C$.

*Proof.* Assume that $T : C \to C$ is a Banach contraction and $\{x_n\}$ is a sequence in $C$ such that $x_n \xrightarrow{w} x^* \in C$. It follows that there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \leq \theta w(\lambda, x, y)$$

for all $x, y \in C$ and for all $\lambda > 0$. To see that $Tx_n \xrightarrow{w} Tx^*$, let $\lambda > 0$ be given. Note that $w(\lambda, Tx_n, Tx^*) \leq \theta w(\lambda, x_n, x^*)$ for all $n \geq 1$ and $\lim_{n\to\infty} w(\lambda, x_n, x^*) = 0$. This implies that $\lim_{n\to\infty} w(\lambda, Tx_n, Tx^*) = 0$. Hence $Tx_n \xrightarrow{w} Tx^*$. □

The following result is a Banach fixed point theorem in a modular metric space without the appearance of regularity.

**Theorem 4.2.2.** Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a Banach contraction. Then the following statements are true.

(1) If there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded, then $T$ has a fixed point $x^*$ and $T^n x_1 \xrightarrow{w} x^*$.

(2) If there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded and $w(\lambda, x, y) < \infty$ for all $x, y \in C$ and for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

*Proof.* (1) Suppose that there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded. Define $x_{n+1} := Tx_n$ for all $n \geq 1$. It follows that $\{x_n\}$ is a sequence of Banach type. It follows from the boundedness of $\{T^n x_1\}$ that $B(\lambda, 1) := \sup\{w(\lambda, x_1, x_k) : k \geq 2\} < \infty$ for all $\lambda > 0$. Hence, by Theorem 4.1.4, $\{x_n\}$ is a Cauchy sequence and hence $x_n \xrightarrow{w} x^*$ for some $x^* \in C$. It follows from Proposition 4.2.1 that $T$ is continuous and hence $x_{n+1} = Tx_n \xrightarrow{w} Tx^*$. In particular, $x^* = Tx^*$. The proof is finished.

(2) Assume that $C$ is bounded. It follows from (1) that $T$ has a fixed point $x^*$ for some $x^* \in C$ and $T^n x_1 \xrightarrow{w} x^*$. To see the latter statement, let $x \in C$ and $\lambda > 0$. It follows

that

$$w(\lambda, T^n x, T^n x_1) \leq \theta^n w(\lambda, x, x_1)$$

for all $n \geq 1$. Since $w(\lambda, x, x_1) < \infty$, we have $\lim_{n\to\infty} w(\lambda, T^n x, T^n x_1) = 0$. In particular,

$$w(2\lambda, T^n x, x^*) \leq w(\lambda, T^n x, T^n x_1) + w(\lambda, T^n x_1, x^*).$$

Since $\lim_{n\to\infty} w(\lambda, T^n x, T^n x_1) = \lim_{n\to\infty} w(\lambda, T^n x_1, x^*) = 0$, $\lim_{n\to\infty} w(2\lambda, T^n x, x^*) = 0$. This implies that $T^n x \xrightarrow{w} x^*$. This completes the proof. $\square$

**Remark 4.2.3.** The set $C$ mentioned in the preceding theorem is a subset of $X$. This setting is different from the one of Mangkolkeha et al. [10, 11] and from the one of Abdou and Khmasi [1]. In fact, they considered only a subset of $X_w := X_w(x_0)$ for some $x_0 \in X$. In Example 2.1.6, we find that in some setting the space $X_w$ is just a singleton and hence the Banach contraction on $X_w$ becomes a constant mapping. This shows the importance of our setting.

### 4.2.2 Kannan type mappings

We now discuss the result in the context of Kannan's setting.

**Definition 4.2.4.** Let $C$ be a subset of a modular metric space $(X, w)$. A mapping $T : C \to C$ is called a *Kannan contraction (on C)* if there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}(w(\lambda, x, Tx) + w(\lambda, y, Ty))$$

for all $x, y \in C$ and for all $\lambda > 0$.

**Remark 4.2.5.** Every strong Kannan contraction is a Kannan contraction. The converse is not true.

**Example 4.2.6.** Let $X = [0, 1]$. Define $w(\lambda, x, y) = |x - y|/\lambda$ for all $x, y \in X$ and for all $\lambda > 0$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{5}, & \text{if } x \in [0, \frac{1}{2}) \\ \frac{x}{4}, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then $T$ is a Kannan contraction and $T$ is not a strong Kannan contraction. To see this, let $\lambda > 0$ and $\theta = 2/3$.

- If $x, y \in [0, \frac{1}{2})$, then

$$w(\lambda, Tx, Ty) = \frac{|x-y|}{5\lambda} \leq \frac{1}{3}\Big(\frac{|x-\frac{x}{5}|}{\lambda} + \frac{|y-\frac{y}{5}|}{\lambda}\Big) = \frac{\theta}{2}\big(w(\lambda, x, Tx) + w(\lambda, y, Ty)\big).$$

- If $x, y \in [\frac{1}{2}, 1]$, then

$$w(\lambda, Tx, Ty) = \frac{|x-y|}{4\lambda} \leq \frac{1}{3}\Big(\frac{|x-\frac{x}{4}|}{\lambda} + \frac{|y-\frac{y}{4}|}{\lambda}\Big) = \frac{\theta}{2}\big(w(\lambda, x, Tx) + w(\lambda, y, Ty)\big).$$

- If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then

$$w(\lambda, Tx, Ty) = \frac{|\frac{x}{5}-\frac{y}{4}|}{\lambda} \leq \frac{1}{3}\Big(\frac{|x-\frac{x}{5}|}{\lambda} + \frac{|y-\frac{y}{4}|}{\lambda}\Big) = \frac{\theta}{2}\big(w(\lambda, x, Tx) + w(\lambda, y, Ty)\big).$$

Thus $T$ is a Kannan contraction. Moreover, we show that if

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}\big(w(\lambda, x, Tx) + w(\lambda, y, Ty)\big)$$

for all $x, y \in X$ and for all $\lambda > 0$, then $\theta \geq \frac{2}{3}$. To see this, let $x = 0$, $y = 1$ and $\lambda = 1$, then $w(\lambda, x, Tx) = 0$, $w(\lambda, y, Ty) = 3/4$ and $w(\lambda, Tx, Ty) = 1/4$. Hence $\theta \geq \frac{2}{3}$. Finally, we show that $T$ is not a strong Kannan contraction. Suppose that there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}\big(w(2\lambda, x, Tx) + w(2\lambda, y, Ty)\big)$$

for all $x, y \in X$ and for all $\lambda > 0$. Note that $w(2\lambda, x, Tx) = \frac{1}{2}w(\lambda, x, Tx)$ and $w(2\lambda, y, Ty) = \frac{1}{2}w(\lambda, y, Ty)$. This implies that

$$w(\lambda, Tx, Ty) \leq \frac{\theta/2}{2}\big(w(\lambda, x, Tx) + w(\lambda, y, Ty)\big)$$

for all $x, y \in X$ and for all $\lambda > 0$. Then $\theta/2 \geq \frac{2}{3}$, that is, $\theta \geq \frac{4}{3}$ which is a contradiction.

It is worth mentioning that the proof of Theorem 2.2.5 ([10, 11]) contains several gaps as appeared in that of Banach fixed point theorem. So we present the following two corrections.

**Theorem 4.2.7.** Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a Kannan contraction. Suppose that the following conditions hold:

(K1) For each $x \in C$, $w(\lambda, x, Tx) < \infty$ for all $\lambda > 0$.

(K2) For each $x \in C$, the function $\lambda \mapsto w(\lambda, x, Tx)$ is right-continuous.

Then $T$ has a unique fixed point and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

*Proof.* Let $x_1 \in C$ and we define $x_{n+1} := Tx_n$ for all $n \geq 1$. It follows from Definition 4.2.4 that $\{x_n\}$ is a sequence of Kannan type in $C$. Moreover, it follows from Condition (K1) and Theorem 4.1.4 that $\{x_n\}$ is a Cauchy sequence. In particular, $\lim_{n\to\infty} w(\lambda, x_n, x_{n+1}) = 0$ for all $\lambda > 0$ and $x_n \xrightarrow{w} x^*$ for some $x^* \in C$. Now we show that $x^*$ is a fixed point of $T$. To see this, let $\lambda > 0$ be given and we consider the following

$$\begin{aligned} w(\lambda, Tx^*, x_{n+1}) &= w(\lambda, Tx^*, Tx_n) \\ &\leq \frac{\theta}{2}(w(\lambda, x^*, Tx^*) + w(\lambda, x_n, Tx_n)) \\ &= \frac{\theta}{2}(w(\lambda, x^*, Tx^*) + w(\lambda, x_n, x_{n+1})). \end{aligned}$$

Since $\lim_{n\to\infty} w(\lambda, x_n, x_{n+1}) = 0$, we have

$$\limsup_{n\to\infty} w(\lambda, Tx^*, x_{n+1}) \leq \frac{\theta}{2} w(\lambda, x^*, Tx^*).$$

On the other hand, it follows from Condition (K2) and Lemma 2.1.10 that

$$\liminf_{n\to\infty} w(\lambda, Tx^*, x_{n+1}) \geq \lim_{\varepsilon\to\lambda^+} w(\varepsilon, x^*, Tx^*) = w(\lambda, x^*, Tx^*).$$

Since $w(\lambda, x^*, Tx^*) < \infty$, we have $w(\lambda, x^*, Tx^*) = 0$. Hence $x^* = Tx^*$.

Finally, we prove the uniqueness. Suppose that $y^* = Ty^*$ for some $y^* \in C$. Let $\lambda > 0$ and we consider

$$w(\lambda, x^*, y^*) = w(\lambda, Tx^*, Ty^*) \leq \frac{\theta}{2}(w(\lambda, x^*, Tx^*) + w(\lambda, y^*, Ty^*)) = 0.$$

It follows that $w(\lambda, x^*, y^*) = 0$ and hence $x^* = y^*$. $\square$

For strong Kannan contractions, we do not need the right continuity of $\lambda \mapsto w(\lambda, x, Tx)$.

**Theorem 4.2.8.** Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a strong Kannan contraction. Suppose that the following condition holds:

(K1) For each $x \in C$, $w(\lambda, x, Tx) < \infty$ for all $\lambda > 0$.

Then $T$ has a unique fixed point and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

*Proof.* Let $x_1 \in C$ and we define $x_{n+1} = Tx_n$ for all $n \geq 1$. It follows from Definition 4.2.4 that $\{x_n\}$ is a sequence of Kannan type in $C$. Moreover, it follows from Condition (K1) and Theorem 4.1.4 that $\{x_n\}$ is a Cauchy sequence. In particular, $\lim_{n\to\infty} w(\lambda, x_n, x_{n+1}) = 0$ for all $\lambda > 0$ and $x_n \xrightarrow{w} x^*$ for some $x^* \in C$. Now we show that $x^*$ is a fixed point of $T$. To see this, let $\lambda > 0$ be given and we consider the following

$$\begin{aligned}
&w(2\lambda, x^*, Tx^*)\\
&\leq w(\lambda, x^*, x_{n+1}) + w(\lambda, x_{n+1}, Tx^*)\\
&= w(\lambda, x^*, x_{n+1}) + w(\lambda, Tx_n, Tx^*)\\
&\leq w(\lambda, x^*, x_{n+1}) + \frac{\theta}{2}(w(2\lambda, x_n, Tx_n) + w(2\lambda, x^*, Tx^*))\\
&= w(\lambda, x^*, x_{n+1}) + \frac{\theta}{2}(w(2\lambda, x_n, x_{n+1}) + w(2\lambda, x^*, Tx^*)).
\end{aligned}$$

Note that $\lim_{n\to\infty} w(\lambda, x^*, x_{n+1}) = \lim_{n\to\infty} w(2\lambda, x_n, x_{n+1}) = 0$ and $w(2\lambda, x^*, Tx^*) < \infty$. It follows that $w(2\lambda, x^*, Tx^*) = 0$. Hence $x^* = Tx^*$. The uniqueness can be proved in the same way as Theorem 4.2.7, so it is omitted. $\square$

# CHAPTER V

# CONCLUSIONS

In this chapter, we summarize the results of our study.

1. Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$. The following statements are true.

- A sequence of Banach type $\{x_n\}$ is Cauchy if and only if for each $\lambda > 0$ there exists an integer $n \geq 1$ such that $B(\lambda, n) < \infty$, where $B(\lambda, n) := \sup\{w(\lambda, x_n, x_{n+k}) : k \geq 1\}$.
- A sequence of Kannan type $\{x_n\}$ is Cauchy if and only if for each $\lambda > 0$ there exists an integer $N \geq 1$ such that $K(\lambda, n) < \infty$ for all $n \geq N$, where $K(\lambda, n) := w(\lambda, x_n, x_{n+1})$.

2. Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a Banach contraction. Then the following statements are true.

- If there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded, then $T$ has a fixed point $x^*$ and $T^n x_1 \xrightarrow{w} x^*$.
- If there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded and $w(\lambda, x, y) < \infty$ for all $x, y \in C$ and for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

3. Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a Kannan contraction. Suppose that the following conditions hold:

- For each $x \in C$, $w(\lambda, x, Tx) < \infty$ for all $\lambda > 0$.
- For each $x \in C$, the function $\lambda \mapsto w(\lambda, x, Tx)$ is right-continuous.

Then $T$ has a unique fixed point and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

4. Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a strong Kannan contraction. Suppose that the following condition holds:

- For each $x \in C$, $w(\lambda, x, Tx) < \infty$ for all $\lambda > 0$.

Then $T$ has a unique fixed point and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.

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# VITAE

| | |
|---|---|
| Name: | Mr. Anuruk Noywiset |
| Date of birth: | November 12th, 1991 |
| Place of birth: | Bangkok, Thailand |
| Email: | anurukn@kkumail.com |

Education:

| | |
|---|---|
| 2003-2008 | Mattayom, Pathumpittayakom School, Ubon Rachatani, Thailand. |
| 2009-2012 | Bachelor Degree of Engineering (Computer Engineering), Khon Kaen University, Thailand. |
| 2013-2017 | Enrolled in Master of Science Program in Mathematics, Khon Kaen University. |
| Scholarship: | The Research Capability Enhancement Program Through Graduate Student Scholarship, Faculty of Science, Khon Kaen University. |