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FIXED POINT THEOREMS IN FUZZY METRIC SPACES AND MODULAR METRIC SPACES

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Abstract. The notions of fuzzy metric spaces and of modular metric space are from different sources. Grabiec proved two fixed point theorems in fuzzy metric spaces in the sense of Kramosil and Michálek. In this paper, we introduced a generalized fuzzy metric space which does not require the left continuity as was the case in the sense of Kramosil and Michálek. Two corresponding fixed point results of Grabiec are extended. We also discuss the theorem of Vasuki and show that this result is equivalent to the first fixed point theorem of Grabiec. We also use our new extensions to recover three fixed point theorems recently proved by Martínez-Moreno et al. in modular metric spaces. Several gaps and misleading concepts appeared in the paper of Martínez-Moreno et al. are corrected.

1. Introduction

Fixed point theory is a powerful tool and has a wide application in various problems in both pure and applied science. There have been many frameworks in the study of fixed point theory. In this paper we are interested in two frameworks: fuzzy metric spaces and modular metric spaces. Let us recall the first one.

Definition 1.1. A continuous function $*:[0,1]^2 \to [0,1]$ is called a *continuous t-norm* or a *continuous triangular norm* if the following conditions are satisfied:

(T1) $*$ is associative and commutative;
(T2) $a * 1 = a$ for all $a \in [0,1]$;
(T3) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Remark 1.2. It follows from (T2) and (T3) that $a * 0 \le 1 * 0 = 0$ for all $a \in [0,1]$.

Definition 1.3. A triple $(X, M, *)$ is called a *generalized fuzzy metric space* if X is an arbitrary nonempty set, $*$ is a continuous t-norm and $M : X^2 \times [0,\infty) \to [0,1]$ is a function satisfying the following conditions:

(FM1) $M(x,y,0) = 0$ for all $x, y \in X$;
(FM2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;
(FM3) $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$ and $t > 0$;
(FM4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$ for all $x,y,z \in X$ and $s,t > 0$.

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A triple $(X, M, *)$ is called a *fuzzy metric space in the sense of Kramosil and Michálek* if it is a generalized fuzzy metric space and the following condition is satisfied:

(KM) the function $t \mapsto M(x, y, t)$ is left-continuous for all $x, y \in X$.

It is obvious that every metric space is a fuzzy metric space in the sense of Kramosil and Michálek. More precisely, if (X, d) is a metric space, then $(X, M, \min)$ is a fuzzy metric space in the sense of Kramosil and Michálek where $M(x, y, t) := t/(t + d(x, y))$ for all $x, y \in X$ and $t \geq 0$.

Grabiec [4] extended two well-known fixed point theorems of Banach and of Edelstein in fuzzy metric spaces in the sense of Kramosil and Michálek. To state and discuss these two result, we recall the following concepts in a generalized fuzzy metric space. Let us note that, in fact, these concepts are given originally in the context of fuzzy metric spaces in the sense of Kramosil and Michálek. The study of various kinds of Cauchyness and convergence in fuzzy metric spaces was recently given by Gregori et al. (see [5]).

Definition 1.4. Let $(X, M, *)$ be a generalized fuzzy metric space. We say that a sequence $\{x_n\}$ in X is

- *M-convergent* if there exists an element $x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$ (In this case, we write $x_n \xrightarrow{M} x$);
- *Cauchy* if $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$ for all $t > 0$;
- *Cauchy in the sense of Grabiec* if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and for all $p \geq 1$.

Remark 1.5. A sequence $\{x_n\}$ in a generalized fuzzy metric space $(X, M, *)$ is Cauchy in the sense of Grabiec if and only if $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

Remark 1.6. Note that

$$M\text{-convergence} \implies \text{Cauchyness} \implies \text{Cauchyness in the sense of Grabiec.}$$

None of the reverse implication holds.

Remark 1.7. Let $\{x_n\}$ be a sequence in a generalized fuzzy metric space $(X, M, *)$. If $x_n \xrightarrow{M} x$ and $x_n \xrightarrow{M} y$, then $x = y$.

The following lemma was explicitly proved by Grabiec.

Lemma 1.8 ([4, Lemma 6]). *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a generalized fuzzy metric space $(X, M, *)$ such that $x_n \xrightarrow{M} x \in X$ and $y_n \xrightarrow{M} y \in X$. If $t \mapsto M(x, y, t)$ is continuous at t_0, then $\lim_{n\to\infty} M(x_n, y_n, t_0) = M(x, y, t_0)$.*

Using the sequential concepts above, we define the analog of completeness and compactness in this framework.

Definition 1.9. A generalized fuzzy metric space $(X, M, *)$ is

- *compact* if every sequence in X contains a M-convergent subsequence;

- *complete in the sense of Grabiec* if every Cauchy sequence in X the sense of Grabiec is M-convergent;
- *complete* if every Cauchy sequence in X is M-convergent.

Remark 1.10. Note that

$$\text{Compactness} \Longrightarrow \text{Completeness} \Longleftarrow \text{Completeness in the sense of Grabiec.}$$

None of the reverse implication holds.

We are now ready to state the following three fixed point theorems. The first two were proved by Grabiec [4] and the last one by Vasuki [10].

Theorem 1.11 ([4], Banach's type fixed point theorem). *Let $(X, M, *)$ be a fuzzy metric space in the sense of Kramosil and Michálek. Let $0 < k < 1$ and let $T : X \to X$ be a mapping such that*

$$M(Tx, Ty, kt) \geq M(x, y, t) \text{ for all } x, y \in X \text{ and for all } t > 0.$$

If $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and X is complete in the sense of Grabiec, then T has a unique fixed point.

Theorem 1.12 ([4], Edelstein's type fixed point theorem). *Let $(X, M, *)$ be a fuzzy metric space in the sense of Kramosil and Michálek and let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, t) > M(x, y, t) \text{ for all } x \neq y \in X \text{ and } t > 0.$$

If X is compact, then T has a unique fixed point.

Theorem 1.13 ([10]). *Let $(X, M, *)$ be a fuzzy metric space in the sense Kramosil and Michálek. Let $m \geq 1$ be a fixed integer and $0 < k < 1$. Suppose that $\{T_n : X \to X\}_{n=1}^{\infty}$ be a countable family of mappings such that the following property is satisfied:*

> *For any pair $i, j \geq 1$ there exists a constant $\alpha_{i,j} \in (0, k)$ such that $M(T_i^m x, T_j^m y, \alpha_{i,j} t) \geq M(x, y, t)$ for all $x, y \in X$ and for all $t > 0$.*

If $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and X is complete in the sense of Grabiec, then the family $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

In this paper, we extend two fixed point theorems of Grabiec [4] in the setting of a generalized fuzzy metric space. Using these two results, we deduce and correct three fixed point theorems of Martínez-Moreno et al. [8] in the setting of modular metric spaces. Finally, we show that there is a one-to-one correspondence between the family of our generalized fuzzy metric spaces and that of generalized modular metric spaces of Chistyakov.

2. Some notes and comments on fixed point theorems of Grabiec and of Vasuki

We now make some comments and remarks on the preceding three theorem.

2.1. **Comments on Theorems 1.11 and 1.12.** After a careful reading, we find that the proof of Theorem 1.11 does not require the condition (KM) at all. Now we call the following result Theorem 2.1.

Theorem 2.1. *Let $(X, M, *)$ be a generalized fuzzy metric space. Let $0 < k < 1$ and let $T : X \to X$ be a mapping such that*

$$M(Tx, Ty, kt) \geq M(x, y, t) \text{ for all } x, y \in X \text{ and for all } t > 0.$$

If $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and X is complete in the sense of Grabiec, then T has a unique fixed point.

For Theorem 1.12, the original proof of Grabiec makes use of the condition (KM). However, we can drop this condition and show that the conclusion remains true. The following result will be referred as Theorem 2.2.

Theorem 2.2. *Let $(X, M, *)$ be a generalized fuzzy metric space and let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, t) > M(x, y, t) \quad \text{for all } x \neq y \in X \text{ and } t > 0.$$

If X is compact, then T has a unique fixed point.

Proof. The proof given below is modified from the original proof of Theorem 1.12 (see [4, Theorem 8]). We first note that

- $M(Tx, Ty, t) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$;
- $M(Tx, Ty, t) = M(x, y, t)$ for some $t > 0 \implies x = y$.

Let $x \in X$ and $x_n = T^n x$ for all $n \geq 1$. Since X is compact, there exist a subsequence $\{x_{n_k}\}$ and an element $y \in X$ such that $x_{n_k} \xrightarrow{M} y$. This implies that $x_{n_k+1} = Tx_{n_k} \xrightarrow{M} Ty$ and $Tx_{n_k+1} = T^2 x_{n_k} \xrightarrow{M} T^2 y$. Note that the set of discontinuous points of $t \mapsto M(y, Ty, t)$ and $t \mapsto M(Ty, T^2y, t)$ is a countable subset of $[0, \infty)$ because these two functions are nondecreasing. In particular, there exists a point $t_0 \in (0, \infty)$ such that both functions $t \mapsto M(y, Ty, t)$ and $t \mapsto M(Ty, T^2y, t)$ are continuous at t_0. Observe that

$$M(x_{n_k+1}, Tx_{n_k+1}, t_0) = M(Tx_{n_k}, T^2x_{n_k}, t_0) \geq M(x_{n_k}, Tx_{n_k}, t_0)$$

for all $k \geq 1$. It follows from Lemma 1.8 that

$$M(y, Ty, t_0) = M(Ty, T^2y, t_0).$$

Hence $y = Ty$. The uniqueness of a fixed point of T is obvious. □

2.2. **Notes on Theorem 1.13.** We now restate Theorem 1.13 in the absence of Condition (KM) and call it Theorem 2.3.

Theorem 2.3. *Let $(X, M, *)$ be a generalized fuzzy metric space. Let $m \geq 1$ be a fixed integer and $0 < k < 1$. Suppose that $\{T_n : X \to X\}_{n=1}^{\infty}$ be a countable family of mappings such that the following property is satisfied:*

For any pair $i, j \geq 1$ there exists a constant $\alpha_{i,j} \in (0, k)$ such that
$M(T_i^m x, T_j^m y, \alpha_{i,j} t) \geq M(x, y, t)$ for all $x, y \in X$ and for all $t > 0$.

If $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and X is complete in the sense of Grabiec, then the family $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Proof. We do not follow the original proof given by Vasuki [10] because the proof can be done shortly by the following observation: The contractive condition of the family $\{T_n\}_{n=1}^{\infty}$ implies that there is only one mapping in the family and hence the result follows from our Theorem 2.1. In fact, let $x \in X$ and we have $M(T_i^m x, T_1^m x, \alpha_{i,j}t) \geq M(x, x, t) = 1$ for all $t > 0$ and for all $i > 1$. This implies that $T_i^m = T_1^m$ for all $i > 1$. It follows from Theorem 2.1 that T_1^m has a unique fixed point. Suppose that $T_1^m z = z$ for some $z \in X$. Note that $T_1^m(T_1 z) = T_1(T_1^m z) = T_1 z$, that is, $T_1 z$ is a fixed point of T_1^m. Hence $z = Tz$ and the proof is finished. □

Remark 2.4. Vasuki [10] noted that Theorem 1.13 is a generalization of Theorem 1.11. However, following the preceding proof, we have

$$\text{Theorem 1.11} \iff \text{Theorem 1.13}.$$

2.3. **Completeness in the sense of Grabiec and Theorem 1.11.** In this subsection, we explain why the completeness in the sense of Grabiec rather than the "usual" one is considered in Theorem 1.11 (and Theorem 2.1). In fact, the completeness in the usual sense is not sufficient to guarantee the existence of a fixed point. The following argument given below is revised from the work of Radu [9]. Since we will discuss all fixed point theorems recently proved by Martínez-Moreno et al. in modular metric spaces (see Section 3), we now restrict ourselves to the Hamacher t-norm $\diamond$, that is,

$$a \diamond b := \begin{cases} 0 & \text{if } a = b = 0; \\ \frac{ab}{a+b-ab} & \text{otherwise.} \end{cases}$$

Let $T : \mathbb{N} \to \mathbb{N}$ be defined by $Tn = n + 1$ for all $n \in \mathbb{N}$. Obviously, T is fixed point free. We define a function $M : \mathbb{N}^2 \times (0, \infty) \to [0, 1]$ such that the following properties hold:

- $(\mathbb{N}, M, \diamond)$ is a fuzzy metric space in the sense of Kramosil and Michálek;
- $(\mathbb{N}, M, \diamond)$ is complete;
- $M(Tn, Tm, \frac{1}{2}t) = M(n, m, t)$ for all $n, m \in \mathbb{N}$ and for all $t > 0$.

Let $a_n = 2^{n-1}/(1 + 2^{n-1})$ for all $n \geq 1$. It is easy to see that $\{a_n\}$ is a strictly increasing sequence in $(0, 1)$ and $\lim_n a_n = 1$. Moreover,

$$\underbrace{a_n \diamond \cdots \diamond a_n}_{2^{n-1} \text{ terms}} = a_1 = 1/2.$$

We define $G : [0, \infty) \to [0, \infty)$ by

$$G(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq 1; \\ a_1 & \text{if } 1 < t \leq 2^{2+2^{n-1}}; \\ a_{n+1} & \text{if } 2^{2n+2^{n-1}} < t \leq 2^{2n+2+2^n}. \end{cases}$$

For $n, m \in \mathbb{N}$ and $t \in [0, \infty)$, we define $M(n, n, t) := 1$ and

$$M(n, n+m, t) = M(n+m, n, t) := G(t/k^n) \diamond G(t/k^{n+1}) \diamond \cdots \diamond G(t/k^{n+m}).$$

It follows that $(\mathbb{N}, M, \diamond)$ is a fuzzy metric space in the sense of Kramosil and Michálek. Moreover, it is complete because every Cauchy sequence is eventually constant. Finally, it is clear that $M(Tn, Tm, \frac{1}{2}t) = M(n, m, t)$ for all $n, m \in \mathbb{N}$ and for all $t > 0$. It is worth mentioning that X is not complete in the sense of Grabiec.

3. Some fixed point theorems in modular metric spaces

We now recall the concept of modular metric spaces introduced by Chistyakov.

Definition 3.1 ([1, 2]). A pair (X, w) is called a *modular metric space* if X is a nonempty set and $w : (0, \infty) \times X^2 \to [0, \infty]$ is a function such that the following conditions are satisfied:

(M1) $w(\lambda, x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(M2) $w(\lambda, x, y) = w(\lambda, y, x)$ for all $x, y \in X$ and for all $\lambda > 0$;
(M3) $w(\lambda+\mu, x, z) \leq w(\lambda, x, y) + w(\mu, y, z)$ for all $x, y, z \in X$ and for all $\lambda, \mu > 0$.

The following concept of convergence, Cauchyness, and completeness are taken from [8]. It should noted that the latter two concepts are different from the ones given by Chistyakov [2, 3].

Definition 3.2 ([8]). Let (X, w) be a modular metric space.

- A sequence $\{x_n\}$ in X is *w-convergent* if there exists an element $x \in X$ such that $\lim_{n\to\infty} w(\lambda, x_n, x) = 0$ for all $\lambda > 0$. In this situation, we write $x_n \xrightarrow{w} x$.
- A sequence $\{x_n\}$ is *w-Cauchy* if $\lim_{n\to\infty} w(\lambda, x_n, x_{n+p}) = 0$ for each $\lambda > 0$ and for all $p > 1$.
- (X, w) is *w-complete* if every w-Cauchy sequence in X is w-convergent.

3.1. Every modular metric space is a generalized fuzzy metric space.

Theorem 3.3. *Let (X, w) be a modular metric space. Then $(X, M_w, \diamond)$ is a generalized fuzzy metric space where*

$$M_w(x, y, t) := \begin{cases} 0 & \text{if } w(t, x, y) = \infty \text{ or } t = 0; \\ \frac{1}{1+w(t,x,y)} & \text{if } w(t, x, y) \in [0, \infty) \end{cases}$$

and

$$a \diamond b := \begin{cases} 0 & \text{if } a = b = 0; \\ \frac{ab}{a+b-ab} & \text{otherwise.} \end{cases}$$

Moreover, if $\{x_n\}$ is a sequence in X, then the following two statements are true.

(i) *$x_n \xrightarrow{w} x \in X$ if and only if $x_n \xrightarrow{M_w} x$;*
(ii) *$\{x_n\}$ is w-Cauchy if and only if it is M_w-Cauchy in the sense of Grabiec.*

Proof. Note that $\diamond$ is a continuous t-norm. In fact, it is a Hamacher t-norm (see [6]). It is obvious that (FM1) and (FM3) hold. Let $x, y, z \in X$ and $s, t > 0$.

(FM2)($\Rightarrow$) Assume that $M_w(x, y, t) = 1$ for all $t > 0$. Let $t > 0$. Then $w(t, x, y) < \infty$. Then $1 = M_w(x, y, t) = \frac{1}{1+w(t,x,y)}$ and hence $w(t, x, y) = 0$. So $x = y$.

(FM2)($\Leftarrow$) Assume that $x = y$. Then $w(t, x, y) = 0$ for all $t > 0$. Hence $M_w(x, y, t) = 1$ for all $t > 0$.

To show (FM4), we consider two cases.

Case 1: Either $w(t, x, y) = \infty$ or $w(s, y, z) = \infty$. In this case, either $M_w(x, y, t) = 0$ or $M_w(y, z, s) = 0$. Consequently, $M_w(x, z, t + s) \geq 0 = M_w(x, y, t) \diamond M_w(y, z, s)$.

Case 2: Both $w(t,x,y) < \infty$ and $w(s,y,z) < \infty$. In this case, it follows from (M3) that $w(t+s,x,z) < \infty$. This implies that

$$\begin{aligned} M_w(x,z,t+s) &= \frac{1}{1+w(t+s,x,z)} \\ &\geq \frac{1}{1+w(t,x,y)+w(s,y,z)} \\ &= \frac{\frac{1}{1+w(t,x,y)}\frac{1}{1+w(s,y,z)}}{\frac{1}{1+w(t,x,y)}+\frac{1}{1+w(s,y,z)}-\frac{1}{1+w(t,x,y)}\frac{1}{1+w(s,y,z)}} \\ &= M_w(x,y,t) \diamond M_w(y,z,s). \end{aligned}$$

Hence $(X, M_w, \diamond)$ is a generalized fuzzy metric space.

The statements (i) and (ii) follow from the statement: for $\varepsilon > 0$

$$w(t,x,y) < \varepsilon \iff M_w(x,y,t) > 1 - \frac{\varepsilon}{1+\varepsilon}.$$

□

In the context of modular metric space, we are interested in the following subspace.

Definition 3.4. Let (X,w) be a modular metric space and $x_0 \in X$. Define the *modular space around x_0* by

$$X_w = X_w(x_0) := \{x \in X : \lim_{\lambda\to\infty} w(\lambda, x, x_0) = 0\}.$$

Lemma 3.5. *Let (X,w) be a modular metric space and $X_w = X_w(x_0)$ for some $x_0 \in X$. Then $\lim_{t\to\infty} M_w(x,y,t) = 1$ for all $x,y \in X_w$ and for all $t > 0$.*

Proof. For $t > 0$ and $x, y \in X_w$, we have $w(t,x,y) \leq w(t/2,x,x_0) + w(t/2,y,x_0)$ and $\lim_{t\to\infty} w(t/2,x,x_0) = \lim_{t\to\infty} w(t/2,x_0,y) = 0$. It follows that $\lim_{t\to\infty} w(t,x,y) = 0$ and hence $\lim_{t\to\infty} M_w(x,y,t) = 1$. □

3.2. **Three fixed point theorems of Martínez-Moreno et al.** We first state the following three fixed point theorems given by Martínez-Moreno et al.

Theorem 3.6 ([8, Theorem 8]). *Let (X,w) be a modular metric space and $k \in (0,1)$. Let $X_w = X_w(x_0)$ for some $x_0 \in X$. Let $T : X_w \to X_w$ be a mapping such that*

$$w(k\lambda, Tx, Ty) \leq w(\lambda, x, y) \text{ for all } x,y \in X_w \text{ and for all } \lambda > 0.$$

If X is w-complete, then T has a unique fixed point.

Theorem 3.7 ([8, Theorem 9]). *Let (X,w) be a modular metric space, let $k \in (0,1)$, and let $m \geq 1$ be a fixed integer. Let $X_w = X_w(x_0)$ for some $x_0 \in X$. Suppose that $\{T_n : X_w \to X_w\}_{n=1}^{\infty}$ is a countable family of mappings such that the following property is satisfied:*

For any pair $i,j \geq 1$ there exists a constant $\alpha_{i,j} \in (0,k)$ such that

$$w(\alpha_{i,j}\lambda, T_i^m x, T_j^m y) \leq w(\lambda, x, y)$$

for all $x,y \in X_w$ and for all $\lambda > 0$.

If X_w is w-complete, then the family $\{T_n\}$ has a unique common fixed point.

Theorem 3.8 ([8, Theorem 12]). *Let (X, w) be a modular metric space and let $T : X \to X$ be a mapping satisfying*

$$w(\lambda, Tx, Ty) < w(\lambda, x, y) \text{ for all } \lambda > 0 \text{ and for all } x, y \in X_w \text{ with } x \neq y.$$

*If X is w-compact, then T has a unique fixed point. Recall that X is w-*compact *(see [8, Definition 6]) if every sequence in $X_w = X_w(x_0)$ (for some $x_0 \in X$) contains a w-convergent subsequence.*

First we note that the proof lines of Theorems 3.6, 3.7, and 3.8 are exactly the same as those of Theorems 1.11, 1.13, and 1.12, respectively. Unfortunately, there are no references in their paper citing [4] and [10]. We now discuss each theorem above and we present various remarks on them.

3.3. **Comments on Theorem 3.6.** The proof of Theorem 3.6 given there [8] is *not* correct due to the assumptions that X is complete and $T : X_w \to X_w$. In fact, their proof is correct if we assume that X_w is complete. But we do not know that the completeness of X implies that of X_w. So we present the following two corrections of Theorem 3.6.

Theorem 3.9. *Let (X, w) be a modular metric space and $k \in (0,1)$. Let $X_w = X_w(x_0)$ for some $x_0 \in X$. Let $T : X_w \to X_w$ be a mapping such that*

$$w(k\lambda, Tx, Ty) \leq w(\lambda, x, y) \text{ for all } x, y \in X_w \text{ and for all } \lambda > 0.$$

If X_w is w-complete, then T has a unique fixed point.

Theorem 3.10. *Let (X, w) be a modular metric space and $k \in (0,1)$. Let $T : X \to X$ be a mapping such that*

$$w(k\lambda, Tx, Ty) \leq w(\lambda, x, y) \text{ for all } x, y \in X \text{ and for all } \lambda > 0.$$

If X is w-complete and $\lim_{\lambda\to\infty} w(\lambda, x, y) = 0$ for all $x, y \in X$, then T has a unique fixed point.

Proof of Theorem 3.10 via Theorem 3.3. It follows from Theorem 3.3 that $(X, M_w, \diamond)$ is a generalized fuzzy metric space and it is complete the sense of Grabiec. Finally, we prove that

$$M_w(Tx, Ty, kt) \geq M_w(x, y, t)$$

for all $x, y \in X$ and for all $t > 0$. To this end, let $x, y \in X$ and $t > 0$. If $w(t, x, y) = \infty$, then $M_w(x, y, t) = 0$, then the inequality above holds. If $w(t, x, y) < \infty$, then $w(kt, Tx, Ty) < \infty$ and hence

$$M_w(Tx, Ty, kt) = \frac{1}{1 + w(kt, Tx, Ty)} \geq \frac{1}{1 + w(t, x, y)} = M_w(x, y, t).$$

Consequently, Theorem 3.9 follows from Theorem 2.1. □

Proof of Theorem 3.9 via Theorem 2.1. The proof is very similar to the proof above but in this situation we need Lemma 3.5. We leave the proof for the reader to verify. □

Before moving to the next subsection, we recall the concept of Cauchyness and completeness in the usual sense.

Definition 3.11 ([8]). Let (X, w) be a modular metric space.

- A sequence $\{x_n\}$ is *Cauchy* if $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ for each $\lambda > 0$.
- (X, w) is *complete* if every Cauchy sequence in X is w-convergent.

It obvious that

- Cauchyness $\implies$ w-Cauchyness;
- w-completeness $\implies$ Completeness.

It is natural to ask about Theorem 3.9 (and Theorem 3.10) in the presence of completeness in the usual sense above. As mentioned in Subsection 2.3, such results are impossible. In other word, we cannot weaken w-completeness to completeness in Theorems 3.9 and 3.10.

3.4. **Comments on Theorem 3.7.** The proof of Theorem 3.7 is correct. But as mentioned in Subsection 2.2, it follows from the assumption imposed in the family of mappings that there exists actually only one mapping in the family. Hence Theorem 3.7 is equivalent to our Theorem 3.9.

3.5. **Comments on Theorem 3.8.** There are several issues about Theorem 3.8. First, the definition of compactness (given as Definition 3.11 there) is just the compactness of X_w. The corrected definition of the compactness is given in Theorem 3.12. Secondly, the statement of this theorem is inconsistency, for example, the domain of the mapping is X but the contractivity is given for pair $x, y \in X_w$. Finally, the proof of the theorem is *not* correct because the right continuity of the mappings $t \mapsto w(t, y, Ty)$ and $t \mapsto w(t, Ty, T^2y)$ in the last line of the proof are used without assuming in the statement of the theorem. Following the proof of our Theorem 3.9 and our Theorem 2.2, we have the following correction of Theorem 3.8.

Theorem 3.12. *Let (X, w) be a modular metric space and let $T : X \to X$ be a mapping satisfying*

$$w(\lambda, Tx, Ty) < w(\lambda, x, y) \text{ for all } \lambda > 0 \text{ and for all } x, y \in X \text{ with } x \neq y.$$

If X is compact, then T has a unique fixed point. Recall that X is compact *if every sequence in X contains a w-convergent subsequence.*

Proof of Theorem 3.12 via Theorem 2.2. It follows from Theorem 3.3 that $(X, M_w, \diamond)$ is a generalized fuzzy metric space. Since X is w-compact, it follows that X is M_w-compact. Finally, we prove that

$$M_w(Tx, Ty, t) > M_w(x, y, t)$$

for all $x, y \in X$ with $x \neq y$ and for all $t > 0$. To this end, let $x, y \in X$ be such that $x \neq y$ and $t > 0$. If $w(t, x, y) = \infty$, then $w(t, Tx, Ty) < \infty$. Hence

$$M_w(Tx, Ty, t) = \frac{1}{1 + w(t, Tx, Ty)} > 0 = M_w(x, y, t).$$

If $w(t, x, y) < \infty$, then $w(t, Tx, Ty) < \infty$ and hence

$$M_w(Tx, Ty, t) = \frac{1}{1 + w(t, Tx, Ty)} > \frac{1}{1 + w(t, x, y)} = M_w(x, y, t).$$

Consequently, Theorem 3.12 follows from our Theorem 2.2. ∎

3.6. Generalized modular metric spaces and generalized fuzzy metric spaces are the same. Recently, Chistyakov [1] generalized modular metric spaces by introducing the concept of F-operation on $\overline{\mathbb{R}}_+ := [0, \infty]$.

Definition 3.13 ([1]). A continuous function $F : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is said to be an *F-operation* if the following conditions are satisfied:

(F1) F is associative and commutative;
(F2) $F(s, 0) = s$ for all $s \in \overline{\mathbb{R}}_+$;
(F3) $F(s, t) \leq F(w, z)$ whenever $s \leq w$ and $t \leq z$ for all $s, t, w, z \in \overline{\mathbb{R}}_+$.

Recall that $F : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is *continuous* if whenever $\{s_n\}$ and $\{t_n\}$ are two sequences in $\overline{\mathbb{R}}_+$ such that $\lim_{n\to\infty} s_n = s \in \overline{\mathbb{R}}_+$ and $\lim_{n\to\infty} t_n = t \in \overline{\mathbb{R}}_+$ it follows that $\lim_{n\to\infty} F(s_n, t_n) = F(s, t)$. Note that it is easy to see that $F(s, t) = \infty$ if either $s = \infty$ or $t = \infty$.

The following are examples of F-operations ([1]).

Example 3.14 ([1]). Let $s, t \in \overline{\mathbb{R}}_+$.

(a) $F_p(s, t) := (s^p + t^p)^{1/p}$ where $p > 0$. In particular, $F_1(s, t) = s + t$.
(b) $F_\infty(s, t) := \max\{s, t\}$.
(c) $F_{\exp}(s, t) := \frac{1}{p} \ln(e^{ps} + e^{pt} - 1)$ where $p > 0$.
(d) $F_{\log}(s, t) := s + t + pst$ where $p > 0$.

Definition 3.15 ([1]). A triple (X, w, F) is called a *generalized modular metric space* if X is a nonempty set, $w : (0, \infty) \times X^2 \to [0, \infty]$ is a function, and $F : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is an F-operation such that the following conditions are satisfied:

(M1') $w(\lambda, x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(M2') $w(\lambda, x, y) = w(\lambda, y, x)$ for all $x, y \in X$ and for all $\lambda > 0$;
(M3') $w(\lambda + \mu, x, z) \leq F(w(\lambda, x, y), w(\mu, y, z))$ for all $x, y, z \in X$ and for all $\lambda, \mu > 0$.

Remark 3.16. If $F = F_1$, that is, $F(s, t) = s + t$, then a generalized modular metric space (X, w, F) is nothing but a modular metric space in Definition 3.1.

We end this paper by showing that there is a one-to-one correspondence between the family of generalized fuzzy metric spaces and that of generalized modular metric spaces. The proofs of the following three theorems are straight forward so they are omitted.

Theorem 3.17. *Let (X, w, F) be a generalized modular metric space. Define $M_w : X \times X \times [0, \infty) \to [0, 1]$ and $*_F : [0, 1] \times [0, 1] \to [0, 1]$ by*

$$M_w(x, y, t) := \begin{cases} 0 & \text{if } w(t, x, y) = \infty \text{ or } t = 0; \\ \frac{1}{1 + w(t, x, y)} & \text{if } w(t, x, y) \in [0, \infty); \end{cases}$$

and

$$a *_F b := \begin{cases} 0 & \text{if } \min\{a, b\} = 0 \text{ or } F(\frac{1}{a} - 1, \frac{1}{b} - 1) = \infty; \\ \frac{1}{1 + F(\frac{1}{a} - 1, \frac{1}{b} - 1)} & \text{otherwise.} \end{cases}$$

Then $(X, M_w, *_F)$ is a generalized fuzzy metric space.

Theorem 3.18. *Let $(X, M, *)$ be a generalized fuzzy metric space. Define w_M : $(0, \infty) \times X \times X \to [0, \infty]$ and $F_* : [0, \infty] \times [0, \infty] \to [0, \infty]$ by*

$$w_M(\lambda, x, y) := \begin{cases} \infty & \text{if } M(x, y, \lambda) = 0; \\ \frac{1}{M(x,y,\lambda)} - 1 & \text{if } M(x, y, \lambda) \neq 0; \end{cases}$$

and

$$F_*(s, t) := \begin{cases} \infty & \text{if } \max\{s, t\} = \infty \text{ or } \frac{1}{1+s} * \frac{1}{1+t} = 0; \\ \frac{1}{\frac{1}{1+s} * \frac{1}{1+t}} - 1 & \text{otherwise.} \end{cases}$$

Then (X, w_M, F_*) is a generalized modular metric space.

Theorem 3.19. *Let X, w, F, M and $*$ be the same as Theorems 3.17 and 3.18. Then the following statements are true.*

- $w_{(M_w)} = w$ *and* $F_{(*_F)} = F$;
- $M_{(w_M)} = M$ *and* $*_{(F_*)} = *$.

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