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# Some fixed point theorems in modular metric spaces

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**Abstract**

Mongkolkeha et al. [6] proved two fixed point theorems in modular metric spaces. Unfortunately there are many gaps in their proofs reported in [1,4,7]. Abdou and Khamsi [1] gave some corrections with the regularity assumption. In this paper, we present three fixed point theorems in modular metric spaces without assuming the regularity as was the case in Abdou and Khamsi [1].

**Keywords:** modular metric space, fixed point theorem, sequence of Banach type, sequence of Kannan type.

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## 1 Introduction

The notion of modulars on linear spaces and the corresponding theory were founded by Nakano [8] and were extensively developed by his mathematical school and many mathematicians. Due to the absence of the linear structure on the space, Chistyakov [2, 3] defined the notion of a modular on an arbitrary set and develop the systematic theory of modular metric spaces. As fixed point theory in metric spaces plays an important role in mathematics, it is interesting to study the same theory in the context of modular metric spaces. We first recall the following concept introduced by Chistyakov.

**Definition 1.1** ( [2,3]). A pair $(X, w)$ is called a *modular metric space* if $X$ is a nonempty set and $w : (0,\infty) \times X^2 \to [0,\infty]$ is a function satisfying the following properties:

(a) $w(\lambda, x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(b) $w(\lambda, x, y) = w(\lambda, y, x)$ for all $x, y \in X$ and for all $\lambda > 0$;

(c) $w(\lambda + \mu, x, z) \le w(\lambda, x, y) + w(\mu, y, z)$ for all $x, y, z \in X$ and for all $\lambda, \mu > 0$.

The function $w$ above is called a *modular* on $X$.

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**Example 1.2.** Let $(X, d)$ be a metric space. Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is nondecreasing and suppose that $w(\lambda, x, y) := d(x, y)/\varphi(\lambda)$ for all $x, y \in X$ and for all $\lambda > 0$. Then $w$ is a modular on $X$. In particular, every metric space is a modular metric space.

Observe that if $w$ is a modular on $X$, then $\lambda \mapsto w(\lambda, x, y)$ is nonincreasing. In fact, let $x, y \in X$ and $0 < \mu < \lambda$. It follows from the condition (c) of Definition 1.1 that

$$w(\lambda, x, y) \leq w(\lambda - \mu, x, x) + w(\mu, x, y) = w(\mu, x, y).$$

As studied in the linear case, we pay attention on the following subspace.

**Definition 1.3.** ( [2,3]) Let $(X, w)$ be a modular metric space and $x_0 \in X$. Define the *modular space around* $x_0$ by

$$X_w(x_0) := \left\{ x \in X : \lim_{\lambda \to \infty} w(\lambda, x, x_0) = 0 \right\}.$$

**Example 1.4.** Suppose that $(X, d)$ is a metric space and $x_0 \in X$. Let $w$ be the modular defined in Example 1.2 with respect to a nondecreasing function $\varphi$. If $\varphi$ is bounded above, then $X_w(x_0) = \{x_0\}$.

The following concepts are analogous to the one in the classical metric spaces.

**Definition 1.5** ( [2,3]). Let $(X, w)$ be a modular metric space.

- A sequence $\{x_n\}$ in $X$ is *convergent* if there exists an element $x \in X$ such that

  $$\lim_{n \to \infty} w(\lambda, x_n, x) = 0$$

  for all $\lambda > 0$. In this case, we also write $x_n \xrightarrow{w} x$ for the statement "$\lim_{n\to\infty} w(\lambda, x_n, x) = 0$ for all $\lambda > 0$".

- A sequence $\{x_n\}$ in $X$ is *Cauchy* if $\lim_{n,m\to\infty} w(\lambda, x_m, x_n) = 0$ for all $\lambda > 0$.

- A subset $C$ of $X$ is *complete* if every Cauchy sequence in $C$ is convergent and its limit is in $C$.

- A subset $C$ of $X$ is *bounded* if $\sup\{w(\lambda, x, y) : x, y \in C\} < \infty$ for all $\lambda > 0$

*Remark* 1.6. Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$. If $x_n \xrightarrow{w} x^*$ and $x_n \xrightarrow{w} x^{**}$ where $x^*, x^{**} \in X$, then $x^* = x^{**}$.

**Lemma 1.7** ( [5]). *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a modular metric space $(X, w)$. If $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ where $x, y \in X$, then*

$$\lim_{\mu\to\lambda^+} w(\mu, x, y) \leq \liminf_{n\to\infty} w(\lambda, x_n, y_n) \leq \limsup_{n\to\infty} w(\lambda, x_n, y_n) \leq \lim_{\mu\to\lambda^-} w(\mu, x, y)$$

*for all $\lambda > 0$.*

In 2011, Mongkolkeha et al. [6] introduced the following two kinds of mappings in modular metric spaces which are related to Banach and Kannan fixed point theorems in metric spaces.

**Definition 1.8.** Let $C$ be a subset of a modular metric space $(X, w)$. A mapping $T : C \to C$ is

- a *Banach contraction* (on $C$) if there exists $\theta \in (0, 1)$ such that

  $$w(\lambda, Tx, Ty) \leq \theta w(\lambda, x, y)$$

  for all $x, y \in C$ and for all $\lambda > 0$;

- a *strong Kannan contraction* (on $C$) if there exists $\theta \in (0,1)$ such that

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}(w(2\lambda, x, Tx) + w(2\lambda, y, Ty))$$

for all $x, y \in C$ and for all $\lambda > 0$.

Mongkolkeha et al. [6] also proved the following fixed point theorems for Banach contractions and strong Kannan contractions on $X_w$.

**Theorem 1.9** ( [6, Theorem 3.2]). *Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a Banach contraction. If $X_w$ is complete, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x \xrightarrow{w} x^*$ for all $x \in X_w$.*

**Theorem 1.10** ( [6, Theorem 3.6]). *Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a strong Kannan contraction. If $X_w$ is complete, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x \xrightarrow{w} x^*$ for all $x \in X_w$.*

Dehghan et al. [4] gave a counterexample to Theorem 1.9. Almost at the same time, Mongkolkeha et al. [7] discussed some mistakes in their results and proposed the following corresponding corrections.

**Theorem 1.11** ( [7, Theorem 2.1]). *Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a Banach contraction. If $X_w$ is complete and there exists $x_1 \in X_w$ such that $w(\lambda, x_1, Tx_1) < \infty$ for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x_1 \xrightarrow{w} x^*$.*

**Theorem 1.12** ( [7, Theorem 2.2]). *Let $(X, w)$ be a modular metric space and $X_w := X_w(x_0)$ for some $x_0 \in X$. Suppose that $T : X_w \to X_w$ is a strong Kannan contraction. If $X_w$ is complete and there exists $x_1 \in X_w$ such that $w(\lambda, x_1, Tx_1) < \infty$ for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ in $X_w$ and the sequence $T^n x_1 \xrightarrow{w} x^*$.*

After a careful reading of the proofs of the preceding two theorems, there remains several gaps in their proofs [7]. As noted in Abdou and Khamsi's paper [1], we quote the following:

> Indeed, a modular may take infinite value. This is the problem that the authors of [6] did not pay attention to. This was also pointed out in the short note [4]. In fact, the authors of [6] did try to fix this problem in another short note [7] but they used the triangle inequality in their proof knowing that $w$ does not in general satisfy the triangle inequality.

To properly establish the classical Banach contraction principle in the best possible way in modular metric spaces, Abdou and Khamsi [1] used the concept of regularity and discuss another concept of convergence to overcome this problem. Recall that a modular $w$ on $X$ is said to be *regular* if

$$x = y \iff w(\lambda, x, y) = 0 \text{ for some } \lambda > 0.$$

Together with this regularity assumption, the following concept are studied in [1].

**Definition 1.13** ( [1]). Let $(X, w)$ be a modular metric space and $X_w$ be a modular space around some point in $X$.

- A sequence $\{x_n\}$ in $X_w$ is *$w$-convergent* to $x \in X_w$ if and only if $\lim_{n\to\infty} w(\lambda, x_n, x) = 0$ for some $\lambda > 0$.
- A sequence $\{x_n\}$ in $X_w$ is *$w$-Cauchy* if $\lim_{n,m\to\infty} w(\lambda, x_m, x_n) = 0$ for some $\lambda > 0$.
- A subset $C$ of $X_w$ is *$w$-complete* if every $w$-Cauchy sequence in $C$ is convergent and its limit is in $C$.

As claimed in [1], the following result is the "best possible way" in modular metric spaces.

**Theorem 1.14** ( [1, Theorem 3.1])**.** *Suppose that $(X, w)$ is a modular metric space and $w$ is regular. Let $C$ be a $w$-complete subset of $X_w$ such that* $\sup\{w(1, x, y) : x, y \in C\} < \infty$*. Let $T : C \to C$ be a $w$-Banach contraction, that is, there exists a constant $\theta \in (0, 1)$ such that*

$$w(1, Tx, Ty) \leq \theta w(1, x, y)$$

*for all $x, y \in C$. Then $T$ has a unique fixed point $x^*$. Moreover, $\{T^n(x)\}$ $w$-converges to $x^*$ for each $x \in C$.*

It is our purpose to give another way to correct the results of Mongkolkeha et al. [6, 7] without the appearance of the regularity assumption.

# 2 Main Results

## 2.1 Sequences of Banach type and of Kannan type

We first introduce the following two kinds of sequences in a modular metric space which play an important role in this paper.

**Definition 2.1.** Let $(X, w)$ be a modular metric space. For a given $\lambda > 0$, we say that a sequence $\{x_n\}$ in $X$ is

- a $\lambda$-*sequence of Banach type* if there exists $\theta := \theta(\lambda) \in (0, 1)$ such that

$$w(\lambda, x_{n+1}, x_{m+1}) \leq \theta w(\lambda, x_n, x_m) \quad \text{for all } n, m \geq 1;$$

- a $\lambda$-*sequence of Kannan type* if there exists $\theta := \theta(\lambda) \in (0, 1)$ such that

$$w(\lambda, x_{n+1}, x_{m+1}) \leq \frac{\theta}{2}(w(\lambda, x_n, x_{n+1}) + w(\lambda, x_m, x_{m+1})) \quad \text{for all } n, m \geq 1.$$

We also say that $\{x_n\}$ is a *sequence of Banach type (of Kannan type, respectively)* if it is a $\lambda$-sequence of Banach type (of Kannan type, respectively) for all $\lambda > 0$.

To prove convergence of a given sequence $\{x_n\}$ in the presence of completeness of the space, the key step is to prove that $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ for all $\lambda > 0$. The following two results give a necessary and sufficient condition for the situation $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ provided that $\{x_n\}$ is a $\lambda$-sequence of Banach type and of Kannan type, respectively.

**Proposition 2.2.** *Let $(X, w)$ be a modular metric space and let $\lambda > 0$ be given. Let $\{x_n\}$ be a sequence in $X$ and set*

$$B(\lambda, n) := \sup\{w(\lambda, x_n, x_{n+k}) : k \geq 1\}.$$

*Suppose that $\{x_n\}$ is a $\lambda$-sequence of Banach type. Then the following conditions are equivalent.*

*(a) There exists an integer $n \geq 1$ such that $B(\lambda, n) < \infty$.*

*(b)* $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$.

*Proof.* (b) $\Longrightarrow$ (a) is obvious. We now prove (a) $\Longrightarrow$ (b). Without loss of generality, we assume that $B(\lambda, 1) < \infty$. For $n, k \geq 1$, we note that $w(\lambda, x_{n+1}, x_{n+1+k}) \leq \theta w(\lambda, x_n, x_{n+k}) \leq B(\lambda, n)$. In particular, $B(\lambda, n + 1) \leq \theta B(\lambda, n)$. Since $B(\lambda, 1) < \infty$, we have $\lim_{n\to\infty} B(\lambda, n) = 0$ and hence (b) is satisfied. $\square$

**Proposition 2.3.** *Let $(X, w)$ be a modular metric space and let $\lambda > 0$ be given. Let $\{x_n\}$ be a sequence in $X$ and set*

$$K(\lambda, n) := w(\lambda, x_n, x_{n+1}).$$

*Suppose that $\{x_n\}$ is a $\lambda$-sequence of Kannan type. Then the following conditions are equivalent.*

*(a) There exists $N \geq 1$ such that $K(\lambda, n) < \infty$ for all $n \geq N$.*

*(b)* $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$.

*Proof.* (b) $\Longrightarrow$ (a) is obvious. We now prove (a) $\Longrightarrow$ (b). Assume that Condition (a) holds. We may assume without loss of generality that $K(\lambda, n) < \infty$ for all $n \geq 1$. For $n \geq 1$, we note that

$$\begin{aligned} K(\lambda, n+1) &= w(\lambda, x_{n+1}, x_{n+2}) \\ &\leq \frac{\theta}{2}(w(\lambda, x_n, x_{n+1}) + w(\lambda, x_{n+1}, x_{n+2})) \\ &= \frac{\theta}{2}(K(\lambda, n) + K(\lambda, n+1)). \end{aligned}$$

In particular, $K(\lambda, n+1) \leq \frac{\theta}{\theta-2} K(\lambda, n)$ and hence $\lim_n K(\lambda, n) = 0$. For $n, m \geq 1$, we now have

$$\begin{aligned} w(\lambda, x_{n+1}, x_{m+1}) &\leq \frac{\theta}{2}(w(\lambda, x_n, x_{n+1}) + w(\lambda, x_m, x_{m+1})) \\ &= \frac{\theta}{2}(K(\lambda, n) + K(\lambda, m)). \end{aligned}$$

This implies that $\lim_{n,m\to\infty} w(\lambda, x_n, x_m) = 0$ and hence (b) is satisfied. $\square$

As consequences of the preceding two propositions, we immediately obtain the following theorem.

**Theorem 2.4.** *Let $\{x_n\}$ be a sequence in a modular metric space $(X, w)$. The following statements are true.*

*(1) A sequence of Banach type $\{x_n\}$ is Cauchy if and only if for each $\lambda > 0$ there exists an integer $n \geq 1$ such that $B(\lambda, n) < \infty$.*

*(2) A sequence of Kannan type $\{x_n\}$ is Cauchy if and only if for each $\lambda > 0$ there exists an integer $N$ such that $K(\lambda, n) < \infty$ for all $n \geq N$.*

## 2.2 Three fixed point theorems

In this section, we present fixed point theorems which can be viewed as a correction of the corresponding fixed point theorems proved by Mongkolkeha et al. ( [6, 7]).

### 2.2.1 Banach type mappings

We first observe the following result.

**Proposition 2.5.** *Let $C$ be a subset of a modular metric space $(X, w)$. If $T : C \to C$ is a Banach contraction, then $T$ is continuous, that is, $Tx_n \xrightarrow{w} Tx^*$ whenever $\{x_n\}$ is a sequence in $C$ such that $x_n \xrightarrow{w} x^* \in C$.*

*Proof.* Assume that $T : C \to C$ is a Banach contraction and $\{x_n\}$ is a sequence in $C$ such that $x_n \xrightarrow{w} x^* \in C$. It follows that there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \leq \theta w(\lambda, x, y)$$

for all $x, y \in C$ and for all $\lambda > 0$. To see that $Tx_n \xrightarrow{w} Tx^*$, let $\lambda > 0$ be given. Note that $w(\lambda, Tx_n, Tx^*) \leq \theta w(\lambda, x_n, x^*)$ for all $n \geq 1$ and $\lim_{n\to\infty} w(\lambda, x_n, x^*) = 0$. This implies that $\lim_{n\to\infty} w(\lambda, Tx_n, Tx^*) = 0$. Hence $Tx_n \xrightarrow{w} Tx^*$. $\square$

The following result is a Banach fixed point theorem in a modular metric space without the appearance of regularity.

**Theorem 2.6.** *Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a Banach contraction. Then the following statements are true.*

*(1) If there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded, then $T$ has a fixed point $x^*$ and $T^n x_1 \xrightarrow{w} x^*$.*

*(2) If there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded and $w(\lambda, x, y) < \infty$ for all $x, y \in C$ and for all $\lambda > 0$, then $T$ has a unique fixed point $x^*$ and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.*

*Proof.* (1) Suppose that there exists $x_1 \in C$ such that $\{T^n x_1\}$ is bounded. Define $x_{n+1} := Tx_n$ for all $n \geq 1$. It follows that $\{x_n\}$ is a sequence of Banach type. It follows from the boundedness of $\{T^n x_1\}$ that $B(\lambda, 1) := \sup\{w(\lambda, x_1, x_k) : k \geq 2\} < \infty$ for all $\lambda > 0$. Hence, by Theorem 2.4, $\{x_n\}$ is a Cauchy sequence and hence $x_n \xrightarrow{w} x^*$ for some $x^* \in C$. It follows from Proposition 2.5 that $T$ is continuous and hence $x_{n+1} = Tx_n \xrightarrow{w} Tx^*$. In particular, $x^* = Tx^*$. The proof is finished.

(2) Assume that $C$ is bounded. It follows from (1) that $T$ has a fixed point $x^*$ for some $x^* \in C$ and $T^n x_1 \xrightarrow{w} x^*$. To see the latter statement, let $x \in C$ and $\lambda > 0$. It follows that

$$w(\lambda, T^n x, T^n x_1) \leq \theta^n w(\lambda, x, x_1)$$

for all $n \geq 1$. Since $w(\lambda, x, x_1) < \infty$, we have $\lim_n w(\lambda, T^n x, T^n x_1) = 0$. In particular,

$$w(2\lambda, T^n x, x^*) \leq w(\lambda, T^n x, T^n x_1) + w(\lambda, T^n x_1, x^*).$$

Since $\lim_{n\to\infty} w(\lambda, T^n x, T^n x_1) = \lim_{n\to\infty} w(\lambda, T^n x_1, x^*) = 0$, $\lim_{n\to\infty} w(2\lambda, T^n x, x^*) = 0$. This implies that $T^n x \xrightarrow{w} x^*$. This completes the proof. □

*Remark* 2.7. The set $C$ mentioned in the preceding theorem is a subset of $X$. This setting is different from the one of Mangkolkeha et al. [6,7] and from the one of Abdou and Khmasi [1]. In fact, they considered only a subset of $X_w := X_w(x_0)$ for some $x_0 \in X$. In Example 1.4, we find that in some setting the space $X_w$ is just a singleton and hence the Banach contraction on $X_w$ becomes a constant mapping. This shows the importance of our setting.

## 2.3 Kannan type mappings

We now discuss the result in the context of Kannan's setting.

**Definition 2.8.** Let $C$ be a subset of a modular metric space $(X, w)$. A mapping $T : C \to C$ is called a *Kannan contraction (on $C$)* if there exists $\theta \in (0, 1)$ such that

$$w(\lambda, Tx, Ty) \leq \frac{\theta}{2}(w(\lambda, x, Tx) + w(\lambda, y, Ty))$$

for all $x, y \in C$ and for all $\lambda > 0$.

*Remark* 2.9. Every strong Kannan contraction is a Kannan contraction. We do not know whether the converse is true. However, the following two fixed point theorems suggest that the converse may not be true.

It is worth mentioning that the proof of Theorem 1.12 ( [6, 7]) contains several gaps as appeared in that of Banach fixed point theorem. So we present the following two corrections.

**Theorem 2.10.** *Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a Kannan contraction. Suppose that the following conditions hold:*

*(K1) For each $x \in C$, $w(\lambda, x, Tx) < \infty$ for all $\lambda > 0$.*

*(K2) For each $x \in C$, the function $\lambda \mapsto w(\lambda, x, Tx)$ is right-continuous.*

*Then $T$ has a unique fixed point and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.*

*Proof.* Let $x_1 \in C$ and we define $x_{n+1} := Tx_n$ for all $n \geq 1$. It follows from Definition 2.8 that $\{x_n\}$ is a sequence of Kannan type in $C$. Moreover, it follows from Condition (K1) and Theorem 2.4 that $\{x_n\}$ is a Cauchy sequence. In particular, $\lim_n w(\lambda, x_n, x_{n+1}) = 0$ for all $\lambda > 0$ and $x_n \xrightarrow{w} x^*$ for some $x^* \in C$. Now we show that $x^*$ is a fixed point of $T$. To see this, let $\lambda > 0$ be given and we consider the following

$$
\begin{aligned}
w(\lambda, Tx^*, x_{n+1}) &= w(\lambda, Tx^*, Tx_n) \\
&\leq \frac{\theta}{2}(w(\lambda, x^*, Tx^*) + w(\lambda, x_n, Tx_n)) \\
&= \frac{\theta}{2}(w(\lambda, x^*, Tx^*) + w(\lambda, x_n, x_{n+1})).
\end{aligned}
$$

Since $\lim_{n\to\infty} w(\lambda, x_n, x_{n+1}) = 0$, we have

$$
\limsup_{n\to\infty} w(\lambda, Tx^*, x_{n+1}) \leq \frac{\theta}{2} w(\lambda, x^*, Tx^*).
$$

On the other hand, it follows from Condition (K2) and Lemma 1.7 that

$$
\liminf_{n\to\infty} w(\lambda, Tx^*, x_{n+1}) \geq \lim_{\varepsilon\to\lambda^+} w(\varepsilon, x^*, Tx^*) = w(\lambda, x^*, Tx^*).
$$

Since $w(\lambda, x^*, Tx^*) < \infty$, we have $w(\lambda, x^*, Tx^*) = 0$. Hence $x^* = Tx^*$.

Finally, we prove the uniqueness. Suppose that $y^* = Ty^*$ for some $y^* \in C$. Let $\lambda > 0$ and we consider

$$
w(\lambda, x^*, y^*) = w(\lambda, Tx^*, Ty^*) \leq \frac{\theta}{2}(w(\lambda, x^*, Tx^*) + w(\lambda, y^*, Ty^*)) = 0.
$$

It follows that $w(\lambda, x^*, y^*) = 0$ and hence $x^* = y^*$. □

For strong Kannan contractions, we do not need the right continuity of $\lambda \mapsto w(\lambda, x, Tx)$.

**Theorem 2.11.** *Let $C$ be a complete subset of a modular metric space $(X, w)$. Let $T : C \to C$ be a strong Kannan contraction. Suppose that the following condition holds:*

*(K1) For each $x \in C$, $w(\lambda, x, Tx) < \infty$ for all $\lambda > 0$.*

*Then $T$ has a unique fixed point and $T^n x \xrightarrow{w} x^*$ for all $x \in C$.*

*Proof.* Let $x_1 \in C$ and we define $x_{n+1} = Tx_n$ for all $n \geq 1$. It follows from Definition 2.8 that $\{x_n\}$ is a sequence of Kannan type in $C$. Moreover, it follows from Condition (K1) and Theorem 2.4 that $\{x_n\}$ is a Cauchy sequence. In particular, $\lim_n w(\lambda, x_n, x_{n+1}) = 0$ for all $\lambda > 0$ and $x_n \xrightarrow{w} x^*$ for some $x^* \in C$. Now we show that $x^*$ is a fixed point of $T$. To see this, let $\lambda > 0$ be given and we consider the following

$$
\begin{aligned}
& w(2\lambda, x^*, Tx^*) \\
&\leq w(\lambda, x^*, x_{n+1}) + w(\lambda, x_{n+1}, Tx^*) \\
&= w(\lambda, x^*, x_{n+1}) + w(\lambda, Tx_n, Tx^*) \\
&\leq w(\lambda, x^*, x_{n+1}) + \frac{\theta}{2}(w(2\lambda, x_n, Tx_n) + w(2\lambda, x^*, Tx^*)) \\
&= w(\lambda, x^*, x_{n+1}) + \frac{\theta}{2}(w(2\lambda, x_n, x_{n+1}) + w(2\lambda, x^*, Tx^*)).
\end{aligned}
$$

Note that $\lim_{n\to\infty} w(\lambda, x^*, x_{n+1}) = \lim_{n\to\infty} w(2\lambda, x_n, x_{n+1}) = 0$ and $w(2\lambda, x^*, Tx^*) < \infty$. It follows that $w(2\lambda, x^*, Tx^*) = 0$. Hence $x^* = Tx^*$. The uniqueness can be proved in the same way as Theorem 2.10, so it is omitted. □

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