Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematical Modelling 33 (2009) 852–866

APPLIED MATHEMATICAL MODELLING

www.elsevier.com/locate/apm

Newton–harmonic balancing approach for accurate solutions to nonlinear cubic–quintic Duffing oscillators

S.K. Lai [a], C.W. Lim [a,*], B.S. Wu [b], C. Wang [c], Q.C. Zeng [d], X.F. He [e]

[a] *Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, China*
[b] *Department of Mechanics and Engineering Science, School of Mathematics, Jilin University, Changchun 130012, PR China*
[c] *Huazhong University of Science and Technology, Wuhan 430074, PR China*
[d] *Johnton Science and Technology Group (Hong Kong) Ltd., Kowloon, Hong Kong, China*
[e] *Dongfang Electric Autocontrol Engineering Company Ltd., Deyang, Sichuan 618201, PR China*

Received 2 March 2007; received in revised form 7 December 2007; accepted 10 December 2007
Available online 31 January 2008

Abstract

This paper presents a new approach for solving accurate approximate analytical higher-order solutions for strong nonlinear Duffing oscillators with cubic–quintic nonlinear restoring force. The system is conservative and with odd nonlinearity. The new approach couples Newton's method with harmonic balancing. Unlike the classical harmonic balance method, accurate analytical approximate solutions are possible because linearization of the governing differential equation by Newton's method is conducted prior to harmonic balancing. The approach yields simple linear algebraic equations instead of nonlinear algebraic equations without analytical solution. Using the approach, accurate higher-order approximate analytical expressions for period and periodic solution are established. These approximate solutions are valid for small as well as large amplitudes of oscillation. In addition, it is not restricted to the presence of a small parameter such as in the classical perturbation method. Illustrative examples are presented to verify accuracy and explicitness of the approximate solutions. The effect of strong quintic nonlinearity on accuracy as compared to cubic nonlinearity is also discussed.

Keywords: Newton's method; Harmonic Balance method; Duffing equation

1. Introduction

Nonlinear oscillation in engineering and applied mathematics has been a topic to intensive research for many years [1–12]. In the past, there have been many analytical and numerical methods for solving these complicated nonlinear systems. The perturbation method involving expansion over a small parameter has been the most common analytical techniques for nonlinear oscillation systems. In general, it is only useful if there exist

* Corresponding author. Tel.: +852 2788 7285; fax: +852 2788 7612.
E-mail address: bccwlim@cityu.edu.hk (C.W. Lim).

0307-904X/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.apm.2007.12.012

small parameters in the nonlinear systems where the solution can be analytically expanded into power series of the parameters. However, very frequently there exist many nonlinear problems in science and engineering where small parameters do not exist and even if such small parameters do exist, the analytical solutions given by the perturbation methods involving expansion over a small parameter have, in most cases, a small range of validity. Because the cubic–quintic Duffing oscillators described in this paper include small and large parameters, the perturbation method involving expansion over a small parameter is not suitable. The harmonic balance method [1,3,7,10] is valid even for rather large oscillation amplitudes for nonlinear oscillation systems. However, the lower-order solutions of the method are rather inaccurate while it is usually difficult to achieve higher-order analytic approximations. There are some other methods more recently for various nonlinear oscillation systems, including the incremental harmonic balance method [4,6], the modified Lindstedt–Poincaré method [5,11], the power series method [8], the homotopy analysis [9], the finite element method [12] and the iteration procedures [13,14].

The accuracy and computational effort of the analytic approximate and numerical methods highly rely on capability of the approaches and flexibility of computer programmes. Except limited circumstances where exact solutions are possible, most of these methods yield approximate or numerical solutions that hardly provide an all-encompassing understanding of the nature of systems in response to changes of parameters affecting nonlinearity. For instance, the perturbation method involving expansion over a small parameter and harmonic balance method are common but these techniques have restrictions that confine their scope of application. In many cases, exact solutions are frequently unavailable due to nonlinearity in the differential equation. Therefore, efforts have been focused on developing and improving many analytical or semi-analytical techniques for solving nonlinear differential equations. In this respect, we present in this paper a new accurate approach for accurate higher-order approximate analytical solutions of the Duffing oscillator with strong cubic and/or quintic nonlinearities.

The Duffing equation is a well-known nonlinear differential equation which is related to many practical engineering systems such as the classical nonlinear spring system with odd nonlinear restoring characteristics [1] and more recently in different physical phenomena [15]. There have been many variations of Duffing equation, for instance, the Duffing–harmonic equation [16,17] and the cubic–quintic Duffing equation. The latter, a differential equation with third and/or fifth power nonlinearity, is difficult to handle because of the presence of strong nonlinearity. Due to the presence of fifth power nonlinearity, the cubic–quintic Duffing equation inherits strong nonlinearity and thus accuracy of approximate analytical methods becomes extremely demanding. The unperturbed cubic–quintic Duffing equation can be found in the modeling of free vibration of a restrained uniform beam carrying intermediate lumped mass and undergoing large amplitude of oscillations in the unimodel Duffing type temporal problem [18,19], the nonlinear dynamics of a slender elastica [20], the generalized Pochhammer–Chree (PC) equations [21] and the compound Korteweg–de Vries (KdV) equation [22] in nonlinear wave systems, and the propagation of a short electromagnetic pulse in a nonlinear medium [23]. In these models, unlike the common Duffing model for a nonlinear spring system, the sign of the harmonic constant may be negative representing a "repulsive restoring force" while the signs of the anharmonicity constants are arbitrary representing higher-order nonlinear "attractive restoring force" or "repulsive restoring force".

The cubic–quintic Duffing model investigated here is governed by a nonlinear differential equation with all real and positive coefficients and it represents a strong nonlinear oscillator. In many cases, the nonlinear equation has been replaced by a related linear equation that approximates the original nonlinear equation closely enough to provide useful solutions. However, such linearization is not often feasible and the original nonlinear differential equation itself must be considered [7]. To complement the study of Duffing equation with harmonic balance method, an analytical approach [24] was proposed to solve the quintic Duffing oscillator. Here, first-order and second-order solutions for linearization of the governing equation with the method of harmonic balance were presented. Although this approach has been effectively applied to the conservative systems with odd nonlinearity for an inertia and static nonlinearities [25], a fractional-power nonlinearities [26], and the dispersion relation of periodic wavetrain in the nonlinear Klein–Gordon wave equation with even potential function [27], only lower-order solutions were available because higher-order solutions result in a rather complicated system of equation which, again, requires numerical techniques.

In this paper, a new coupled Newton's method with the harmonic balancing [28] has been developed. At an initial stage, the lower-order approximate analysis is presented and it is further generalized to construct higher-order analytical approximations for quantitative analysis of strongly nonlinear oscillation systems of cubic–quintic Duffing oscillators for large parameters. Newton's method is introduced here to linearize the displacement and the squared angular frequency. The lower-order analytical approximations obtained are subsequently and successively applied to construct more accurate higher-order analytical approximations through an iterative procedure. This analysis presents accurate approximate solutions valid for small as well as large amplitudes of oscillation. Contrary to most previous approaches and due to linearization prior to harmonic balancing, a system of simple and linear algebraic equations instead of complex nonlinear algebraic equations can be derived. Therefore, accurate higher-order approximate analytic expressions for the period and periodic solution can be established. The expressions for accurate approximate solutions are also applicable to large parameter regime where the conventional perturbation method involving expansion over a small parameter fails. The improved harmonic balance method presented here is to provide an effective avenue for solving the Duffing equation with strong nonlinearities which arose in any engineering applications. Several examples of cubic–quintic Duffing oscillators with different parameters are presented to illustrate the simplicity, accuracy and effectiveness of the new approach.

2. Problem definition, formulation and solution

A cubic–quintic Duffing oscillator of a conservative autonomous system can be described by the following second-order differential equation with cubic–quintic nonlinearities [19]

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+f(x)=0, \tag{1}$$

with initial conditions

$$x(0)=X_0, \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0)=0, \tag{2}$$

where $f(x)=\alpha x+\beta x^3+\gamma x^5$ is an odd function, and x and t are generalized dimensionless displacement and time variables while α, β and γ are positive constant parameters. The equilibrium position is located at $x=0$ and the system oscillates between symmetric bounds $[-X_0, X_0]$. The period and angular frequency of Eq. (1) are dependent on the initial amplitude of oscillation X_0. It is a simple harmonic oscillator if $\alpha\neq 0$, $\beta=0$, $\gamma=0$; a cubic Duffing oscillator if $\beta\neq 0$, $\gamma=0$; a quintic oscillator if $\beta=0$, $\gamma\neq 0$; otherwise it is a cubic–quintic oscillator if β and γ do not vanish.

By defining a new independent variable replacing the time variable, $\tau=\omega t$, Eq. (1) can be expressed in terms of angular frequency ω as:

$$\omega^2x''+f(x)=0, \tag{3}$$

and it also satisfies the initial conditions

$$x(0)=X_0, \quad x'(0)=0, \tag{4}$$

where a prime denotes differentiation with respect to τ instead of t. Because the restoring force function is an odd function of x, thus the periodic solution contains only odd multiples of τ (i.e. $x(\tau)=\sum_{i=0}^{\infty}h_i\cos[(2i+1)\tau]$).

Applying Newton's procedure, the displacement and squared angular frequency can be expressed as Eqs. (5) and (6) in which Δx_1 and $\Delta\omega_1^2$ are small increments of original displacement x_1 and squared angular frequency ω_1^2, respectively

$$x(\tau)=x_1(\tau)+\Delta x_1(\tau), \tag{5}$$

$$\omega^2=\omega_1^2+\Delta\omega_1^2. \tag{6}$$

For brevity and convenience, let $\omega^2=\Omega$, $\omega_1^2=\Omega_1$ and $\Delta\omega_1^2=\Delta\Omega_1$. Then Eq. (6) can be expressed as:

$$\Omega=\Omega_1+\Delta\Omega_1. \tag{7}$$

Substituting Eqs. (5) and (7) into Eq. (3) results in

$$(\Omega_1 + \Delta\Omega_1)(x_1'' + \Delta x_1'') + f(x_1 + \Delta x_1) = 0. \tag{8}$$

Further linearizing Eq. (8) with respect to the correction terms $\Delta x_1(\tau)$ and $\Delta\Omega_1$ yield

$$(\Omega_1 + \Delta\Omega_1)x_1'' + \Omega_1\Delta x_1'' + f(x_1) + f_x(x_1)\Delta x_1 = 0, \tag{9}$$

where $f_x(x_1) = \alpha + 3\beta x_1^2 + 5\gamma x_1^4$, the subscript x in $f_x(x_1)$ denotes differentiation with respect to x.

Let $x_1(\tau) = X_0 \cos\tau$ be an initial approximation to $x(\tau)$, it is a periodic function of τ of period 2π. Substituting $x_1(\tau) = X_0 \cos\tau$ into Eq. (9) yields

$$-(\Omega_1 + \Delta\Omega_1)X_0\cos\tau + \Omega_1\Delta x_1'' + f(X_0\cos\tau) + f_x(X_0\cos\tau)\Delta x_1 = 0, \tag{10}$$

where Δx_1 is a periodic function of τ of period 2π. We have the following Fourier series expansion

$$f(X_0\cos\tau) = \sum_{i=1}^{\infty} a_{2i-1}\cos[(2i-1)\tau], \tag{11}$$

$$f_x(X_0\cos\tau) = \frac{b_0}{2} + \sum_{i=1}^{\infty} b_{2i}\cos(2i\tau), \tag{12}$$

where the Fourier coefficients are

$$a_1 = \alpha X_0 + \frac{3\beta X_0^3}{4} + \frac{5\gamma X_0^5}{8}, \quad a_3 = \frac{4\beta X_0^3 + 5\gamma X_0^5}{16}, \tag{13a–b}$$

$$b_0 = 2\alpha + 3\beta X_0^2 + \frac{15\gamma X_0^4}{4}, \quad b_2 = \frac{X_0^2(3\beta + 5\gamma X_0^2)}{2}, \quad b_4 = \frac{5\gamma X_0^4}{8}, \quad b_6 = 0. \tag{14a–d}$$

2.1. First-order analytical approximation

For the first-order analytical approximation, we set

$$\Delta x_1(\tau) = 0, \tag{15}$$

$$\Delta\Omega_1 = 0, \tag{16}$$

and therefore

$$x(\tau) = x_1(\tau) = X_0\cos\tau. \tag{17}$$

Substituting Eqs. (11), (12), (15) and (16) into Eq. (10), expanding the resulting expression in a trigonometric series and setting the coefficient of $\cos\tau$ to zero results in the first-order analytical approximation $\sqrt{\Omega_1}$. The corresponding approximate analytical periodic solution $x_1(t)$ can then be expressed as:

$$x_1(t) = X_0\cos\left[\sqrt{\Omega_1(X_0)}t\right], \tag{18}$$

where

$$\sqrt{\Omega_1(X_0)} = \sqrt{\frac{a_1}{X_0}} = \sqrt{\alpha + \frac{3\beta X_0^2}{4} + \frac{5\gamma X_0^4}{8}}, \tag{19}$$

where angular frequency $\sqrt{\Omega_1}$ is the first-order analytical approximation.

2.2. Second-order analytical approximation

For the second analytical approximation, we set

$$\Delta x_1(\tau) = c_1(\cos\tau - \cos 3\tau). \tag{20}$$

Substituting Eqs. (11), (12) and (20) into Eq. (10), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero result in a set of simultaneous equations in terms of c_1 and $\Delta\Omega_1$

$$2a_1 + b_0c_1 - b_4c_1 - 2X_0\Delta\Omega_1 - 2X_0\Omega_1 - 2c_1\Omega_1 = 0, \tag{21}$$

$$2a_3 - b_0c_1 + b_2c_1 + b_4c_1 - b_6c_1 + 18c_1\Omega_1 = 0. \tag{22}$$

Solving Eqs. (21) and (22) simultaneously yield $\Delta\Omega_1$ and c_1

$$\Delta\Omega_1(X_0) = \frac{a_3[-2a_1 + X_0(b_0 - b_4)]}{X_0[-18a_1 + X_0(b_0 - b_2 - b_4 + b_6)]} = -\frac{3X_0^4(4\beta + 5\gamma X_0^2)^2}{16(128\alpha + 96\beta X_0^2 + 85\gamma X_0^4)}, \tag{23}$$

$$c_1(X_0) = \frac{2X_0a_3}{-18a_1 + X_0(b_0 - b_2 - b_4 + b_6)} = \frac{-4\beta X_0^3 - 5\gamma X_0^5}{128\alpha + 96\beta X_0^2 + 85\gamma X_0^4}. \tag{24}$$

The corresponding approximate analytical periodic solution $x_2(t)$ and the second-order analytical approximate frequency $\sqrt{\Omega_2}$ can be expressed in terms of α, β, γ and X_0 as

$$x_2(t) = [X_0 + c_1(X_0)]\cos\left[\sqrt{\Omega_2(X_0)}t\right] - c_1(X_0)\cos\left[3\sqrt{\Omega_2(X_0)}t\right], \tag{25}$$

and

$$\sqrt{\Omega_2(X_0)} = \sqrt{\Omega_1(X_0) + \Delta\Omega_1(X_0)}$$
$$= \sqrt{\frac{2048\alpha^2 + 1104\beta^2X_0^4 + 1860\beta\gamma X_0^6 + 775\gamma^2X_0^8 + 48\alpha X_0^2(64\beta + 55\gamma X_0^2)}{16(128\alpha + 96\beta X_0^2 + 85\gamma X_0^4)}}. \tag{26}$$

The angular frequency $\sqrt{\Omega_2}$ is a more accurate, higher-order approximation as compared to $\sqrt{\Omega_1}$.

2.3. Third-order analytical approximation

To construct the third-order analytical approximations, the relevant expressions must be extended. Here, $x_1(\tau)$, $\Delta x_1(\tau)$, Ω_1 and $\Delta\Omega_1$ should be replaced by $x_2(\tau)$, $\Delta x_2(\tau)$ Ω_2 and $\Delta\Omega_2$, respectively, in Eqs. (5) and (7), thus Eq. (9) becomes

$$(\Omega_2 + \Delta\Omega_2)[-X_0\cos\tau + c_1(9\cos 3\tau - \cos\tau)] + \Omega_2\Delta x_2'' + f(X_0\cos\tau + c_1(\cos\tau - \cos 3\tau))$$
$$+ f_x(X_0\cos\tau + c_1(\cos\tau - \cos 3\tau))\Delta x_2 = 0. \tag{27}$$

For the third-order analytical approximation, we set

$$\Delta x_2(\tau) = c_2(\cos\tau - \cos 3\tau) + c_3(\cos 3\tau - \cos 5\tau). \tag{28}$$

Substituting Eq. (28) into Eq. (27), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$ to zero result in c_2, c_3 and $\Delta\Omega_2$ in Eqs. (29)–(31), respectively

$$\chi_1 + \chi_2c_2 + \chi_3c_3 + \chi_4\Delta\Omega_2 = 0, \tag{29}$$

$$\chi_5 + \chi_6c_2 + \chi_7c_3 + \chi_8\Delta\Omega_2 = 0, \tag{30}$$

$$\chi_9 + \chi_{10}c_2 + \chi_{11}c_3 = 0. \tag{31}$$

Solving Eqs. (29)–(31) simultaneously yields,

$$\Delta\Omega_2(X_0) = \frac{\chi_3(-\chi_6\chi_9 + \chi_5\chi_{10}) + \chi_2(\chi_7\chi_9 - \chi_5\chi_{11}) + \chi_1(-\chi_7\chi_{10} + \chi_6\chi_{11})}{\chi_8(-\chi_3\chi_{10} + \chi_2\chi_{11}) + \chi_4(\chi_7\chi_{10} - \chi_6\chi_{11})}, \tag{32}$$

$$c_2(X_0) = \frac{\chi_8(\chi_3\chi_9 - \chi_1\chi_{11}) + \chi_4(-\chi_7\chi_9 + \chi_5\chi_{11})}{\chi_8(-\chi_3\chi_{10} + \chi_2\chi_{11}) + \chi_4(\chi_7\chi_{10} - \chi_6\chi_{11})}, \tag{33}$$

and

$$c_3(X_0) = \frac{\chi_8(-\chi_2\chi_9 + \chi_1\chi_{10}) + \chi_4(\chi_6\chi_9 - \chi_5\chi_{10})}{\chi_8(-\chi_3\chi_{10} + \chi_2\chi_{11}) + \chi_4(\chi_7\chi_{10} - \chi_6\chi_{11})}, \tag{34}$$

where the variables χ_i ($i = 1 - 11$) are presented in Appendix A.

The corresponding approximate analytical periodic solution $x_3(t)$ is

$$x_3(t) = [X_0 + c_1(X_0) + c_2(X_0)]\cos\left[\sqrt{\Omega_3(X_0)}t\right] + [c_3(X_0) - c_2(X_0) - c_1(X_0)]\cos\left[3\sqrt{\Omega_3(X_0)}t\right] - c_3(X_0)\cos\left[5\sqrt{\Omega_3(X_0)}t\right]. \tag{35}$$

The third-order analytical approximate frequency $\sqrt{\Omega_3}$ can be expressed in terms of α, β, γ and X_0 as

$$\begin{aligned}\sqrt{\Omega_3(X_0)} &= \sqrt{\Omega_2(X_0) + \Delta\Omega_2(X_0)} \\ &= \left[\frac{2048\alpha^2 + 1104\beta^2X_0^4 + 1860\beta\gamma X_0^6 + 775\gamma^2X_0^8 + 48\alpha X_0^2(64\beta + 55\gamma X_0^2)}{16(128\alpha + 96\beta X_0^2 + 85\gamma X_0^4)}\right. \\ &\quad \left. + \frac{\chi_3(-\chi_6\chi_9 + \chi_5\chi_{10}) + \chi_2(\chi_7\chi_9 - \chi_5\chi_{11}) + \chi_1(-\chi_7\chi_{10} + \chi_6\chi_{11})}{\chi_8(-\chi_3\chi_{10} + \chi_2\chi_{11}) + \chi_4(\chi_7\chi_{10} - \chi_6\chi_{11})}\right]^{\frac{1}{2}}.\end{aligned} \tag{36}$$

Here $\sqrt{\Omega_3}$ is a more accurate higher-order approximation as compared to $\sqrt{\Omega_2}$.

It should be clear how the procedure works for constructing further analytical approximate solutions. For example, for the kth order analytical approximation, we may set

$$\Delta x_{k-1}(\tau) = \sum_{j=1}^{k-1} d_j\{\cos[(2j-1)\tau] - \cos[(2j+1)\tau]\}. \tag{37}$$

3. Illustrative examples of cubic–quintic Duffing oscillators

A few examples of the cubic–quintic Duffing oscillators are presented and discussed in this section. In these examples, comparison of the higher-order analytical approximate frequency $\sqrt{\Omega}$ with the exact solutions ω_e is shown to illustrate and verify the accuracy of this approximate analytical approach. The various approximate solutions are presented in Figs. 1–4 for the cubic–quintic Duffing equation. The superscript "1/2" of Ω in these figures denotes square root. The corresponding numerical results of angular frequencies are shown in Tables 1–4. These figures and tables correspond to small as well as large amplitudes of oscillation for different parameters of α, β and γ. The three analytical approximate frequencies in this paper are denoted as $\sqrt{\Omega_1}$, $\sqrt{\Omega_2}$ and $\sqrt{\Omega_3}$, respectively.

For reference, the exact frequency ω_e is obtained by direct integration of governing nonlinear differential Eq. (1) of the dynamical system. Imposing the initial conditions, the solution is

$$\omega_e(X_0) = \frac{\pi k_1}{2\int_0^{\frac{\pi}{2}}(1 + k_2\sin^2 t + k_3\sin^4 t)^{-\frac{1}{2}}\mathrm{d}t}, \tag{38}$$

where k_1, k_2 and k_3 are constants. Derivation of the exact solution and expression of constants are presented in Appendix B.

The approach presented herein is suitable for small as well as large amplitudes of oscillation. The various limiting approximations for $\gamma X_0^4 \to \infty$ can be derived. Based on Eqs. (19), (26), (36) and (38), the various approximations compared with respect to the exact solution are

$$\lim_{\gamma X_0^4\to\infty}\frac{\sqrt{\Omega_1(X_0)}}{\omega_e(X_0)} = \lim_{\gamma X_0^4\to\infty}\frac{T_e(X_0)}{T_1(X_0)} = 1.05856, \tag{39}$$

$$\lim_{\gamma X_0^4\to\infty}\frac{\sqrt{\Omega_2(X_0)}}{\omega_e(X_0)} = \lim_{\gamma X_0^4\to\infty}\frac{T_e(X_0)}{T_2(X_0)} = 1.01078, \tag{40}$$

$$\lim_{\gamma X_0^4 \to \infty} \frac{\sqrt{\Omega_3(X_0)}}{\omega_e(X_0)} = \lim_{\gamma X_0^4 \to \infty} \frac{T_e(X_0)}{T_3(X_0)} = 1.0023. \tag{41}$$

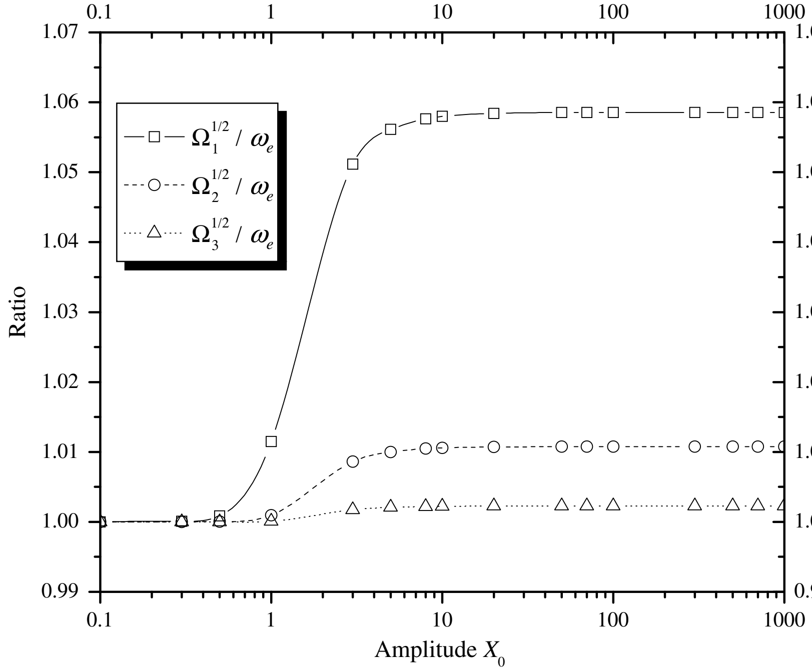

Fig. 1. Comparison of approximate frequencies with exact frequency for the cubic–quintic Duffing oscillator for $\alpha = \beta = \gamma = 1$.

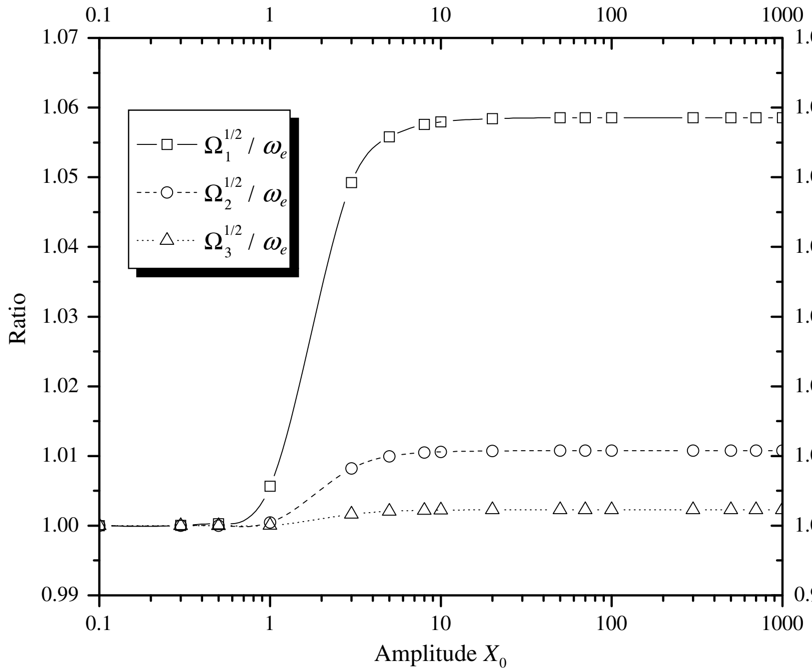

Fig. 2. Comparison of approximate frequencies with exact frequency for the cubic–quintic Duffing oscillator for $\alpha = 2$ and $\beta = \gamma = 1$.

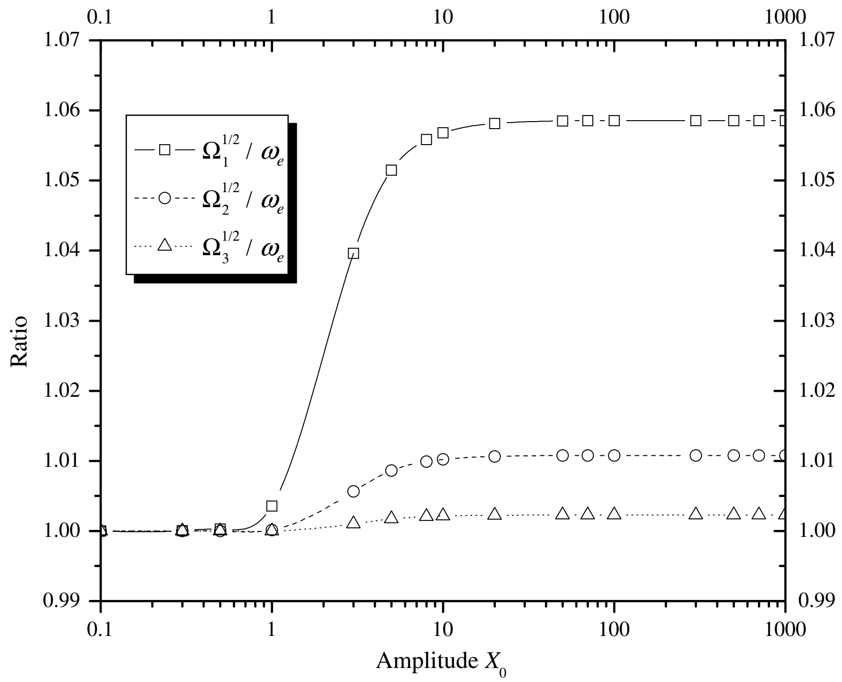

Fig. 3. Comparison of approximate frequencies with exact frequency for the cubic–quintic Duffing oscillator for $\alpha = 5$, $\beta = 3$ and $\gamma = 1$.

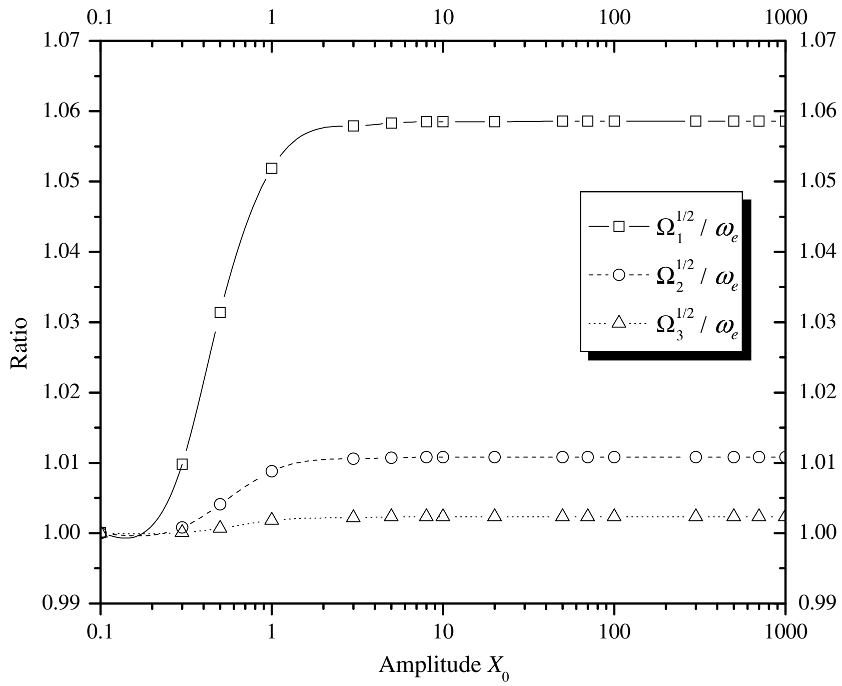

Fig. 4. Comparison of approximate frequencies with exact frequency for the cubic–quintic Duffing oscillator for $\alpha = 1$, $\beta = 10$ and $\gamma = 100$.

The relative error of third-order analytical approximation compared with the exact solution is less than 0.23% in the limit as $\gamma X_0^4 \to \infty$. Hence, the method proposed is suitable for solving Eq. (1) with any large parameter γ

Table 1
Percentage errors for comparison of approximate frequencies with exact frequency for $\alpha = \beta = \gamma = 1$

X_0	$\sqrt{\Omega_1}$ (% error)	$\sqrt{\Omega_2}$ (% error)	$\sqrt{\Omega_3}$ (% error)	ω_e
0.1	1.00377 (0)	1.00377 (0)	1.00377 (0)	1.00377
0.3	1.03565 (0.01)	1.03554 (0)	1.03554 (0)	1.03554
0.5	1.10750 (0.09)	1.10658 (0)	1.10655 (0)	1.10654
1	1.54110 (1.15)	1.52507 (0.10)	1.52375 (0.01)	1.52359
3	7.64035 (5.11)	7.33114 (0.86)	7.28115 (0.17)	7.26863
5	20.2577 (5.61)	19.3735 (1.00)	19.2215 (0.21)	19.1815
8	51.0784 (5.76)	48.8010 (1.05)	48.4011 (0.22)	48.2946
10	79.5362 (5.80)	75.9738 (1.06)	75.3454 (0.22)	75.1774
20	316.703 (5.84)	302.435 (1.07)	299.903 (0.23)	299.223
50	1976.90 (5.85)	1887.69 (1.08)	1871.84 (0.23)	1867.57
70	3874.26 (5.86)	3699.42 (1.08)	3668.33 (0.23)	3659.98
100	7906.17 (5.86)	7549.34 (1.08)	7485.89 (0.23)	7468.83
300	71151.72 (5.86)	67940.22 (1.08)	67369.12 (0.23)	67215.57
500	197642.83 (5.86)	188721.99 (1.08)	187135.59 (0.23)	186709.04
700	387379.49 (5.86)	369894.64 (1.08)	366785.29 (0.23)	365949.25
1000	790569.89 (5.86)	754886.52 (1.08)	748540.91 (0.23)	746834.69

Table 2
Percentage errors for comparison of approximate frequencies with exact frequency for $\alpha = 2$ and $\beta = \gamma = 1$

X_0	$\sqrt{\Omega_1}$ (% error)	$\sqrt{\Omega_2}$ (% error)	$\sqrt{\Omega_3}$ (% error)	ω_e
0.1	1.41688 (0)	1.41688 (0)	1.41688 (0)	1.41688
0.3	1.43964 (0.003)	1.43960 (0)	1.43960 (0)	1.43960
0.5	1.49217 (0.03)	1.49179 (0)	1.49178 (0)	1.49177
1	1.83712 (0.56)	1.82763 (0.04)	1.82689 (0)	1.82682
3	7.70552 (4.92)	7.40403 (0.82)	7.35578 (0.16)	7.34386
5	20.2824 (5.58)	19.4014 (0.99)	19.2501 (0.21)	19.2104
8	51.0882 (5.76)	48.8120 (1.05)	48.4125 (0.22)	48.3061
10	79.5424 (5.80)	75.9809 (1.06)	75.3527 (0.22)	75.1848
20	316.705 (5.84)	302.437 (1.07)	299.905 (0.23)	299.225
50	1976.90 (5.85)	1887.70 (1.08)	1871.84 (0.23)	1867.57
70	3874.26 (5.85)	3699.42 (1.08)	3668.33 (0.23)	3659.98
100	7906.17 (5.86)	7549.34 (1.08)	7485.89 (0.23)	7468.83
300	71151.72 (5.86)	67940.22 (1.08)	67369.12 (0.23)	67215.57
500	197642.83 (5.86)	188721.99 (1.08)	187135.59 (0.23)	186709.04
700	387379.49 (5.86)	369894.64 (1.08)	366785.29 (0.23)	365949.25
1000	790569.89 (5.86)	754886.52 (1.08)	748540.91 (0.23)	746834.69

Table 3
Percentage errors for comparison of approximate frequencies with exact frequency for $\alpha = 5$, $\beta = 3$ and $\gamma = 1$

X_0	$\sqrt{\Omega_1}$ (% error)	$\sqrt{\Omega_2}$ (% error)	$\sqrt{\Omega_3}$ (% error)	ω_e
0.1	2.24111 (0)	2.24111 (0)	2.24111 (0)	2.24111
0.3	2.28201 (0.004)	2.28193 (0)	2.28193 (0)	2.28193
0.5	2.36676 (0.03)	2.36616 (0)	2.36615 (0)	2.36615
1	2.80624 (0.36)	2.79670 (0.02)	2.79630 (0)	2.79627
3	8.71063 (3.96)	8.42601 (0.56)	8.38730 (0.10)	8.37877
5	21.2574 (5.15)	20.3911 (0.86)	20.2514 (0.17)	20.2164
8	52.0481 (5.58)	49.7844 (0.99)	49.3969 (0.21)	49.2955
10	80.4984 (5.68)	76.9487 (1.02)	76.3326 (0.21)	76.1698
20	317.655 (5.81)	303.399 (1.06)	300.879 (0.22)	300.204
50	1977.85 (5.85)	1888.65 (1.08)	1872.81 (0.23)	1868.55
70	3875.21 (5.85)	3700.38 (1.08)	3669.31 (0.23)	3660.95
100	7907.12 (5.85)	7550.30 (1.08)	7486.86 (0.23)	7469.81
300	71152.67 (5.86)	67941.18 (1.08)	67370.09 (0.23)	67216.54
500	197643.78 (5.86)	188722.95 (1.08)	187136.56 (0.23)	186710.01
700	387380.44 (5.86)	369895.60 (1.08)	366786.26 (0.23)	365950.22
1000	790570.84 (5.86)	754887.48 (1.08)	748541.88 (0.23)	746835.66

Table 4
Percentage errors for comparison of approximate frequencies with exact frequency for $\alpha = 1$, $\beta = 10$ and $\gamma = 100$

X_0	$\sqrt{\Omega_1}$ (% error)	$\sqrt{\Omega_2}$ (% error)	$\sqrt{\Omega_3}$ (% error)	ω_e
0.1	1.03983 (0.01)	1.03970 (0)	1.03970 (0)	1.03970
0.3	1.47691 (0.98)	1.46373 (0.08)	1.46271 (0.01)	1.46259
0.5	2.60408 (3.14)	2.53505 (0.41)	2.52642 (0.07)	2.52469
1	8.42615 (5.19)	8.08069 (0.88)	8.02429 (0.18)	8.01005
3	71.6310 (5.79)	68.4255 (1.06)	67.8606 (0.22)	67.7097
5	198.119 (5.83)	189.203 (1.07)	187.623 (0.23)	187.199
8	506.440 (5.85)	483.608 (1.08)	479.552 (0.23)	478.463
10	791.044 (5.85)	755.366 (1.08)	749.027 (0.23)	747.323
20	3162.75 (5.85)	3020.02 (1.08)	2994.65 (0.23)	2987.83
50	19764.71 (5.86)	18872.63 (1.08)	18714.00 (0.23)	18671.34
70	38738.38 (5.86)	36989.90 (1.08)	36678.97 (0.23)	36595.36
100	79057.42 (5.86)	75489.08 (1.08)	74854.53 (0.23)	74683.91
300	711512.95 (5.86)	679397.92 (1.08)	673686.86 (0.23)	672151.27
500	1976424.01 (5.86)	1887215.59 (1.08)	1871351.54 (0.23)	1867085.99
700	3873790.61 (5.86)	3698942.10 (1.08)	3667848.54 (0.23)	3659488.07
1000	7905694.62 (5.86)	7548860.93 (1.08)	7485404.69 (0.23)	7468342.49

and any large amplitude of oscillation X_0. The corresponding period T_i from the various approximations $(i = 1, 2, 3)$ and exact solution $(i = e)$ is obtained from $T = 2\pi/\omega$.

Fig. 1 shows the frequency ratio and amplitude relationship of the cubic–quintic Duffing oscillator governed by $\frac{d^2x}{dt^2} + x + x^3 + x^5 = 0$. As observed, the second- and the third-order analytical approximations are in satisfactory to excellent agreement with the exact solution for $X_0 \in [0.1, 1000]$. The relative errors of the first-order, second-order and third-order analytical approximations as compared to the exact solution are 5.86%, 1.08% and 0.23%, respectively for $X_0 = 1000$. Particularly, the error of $\sqrt{\Omega_1}$ is relatively large for

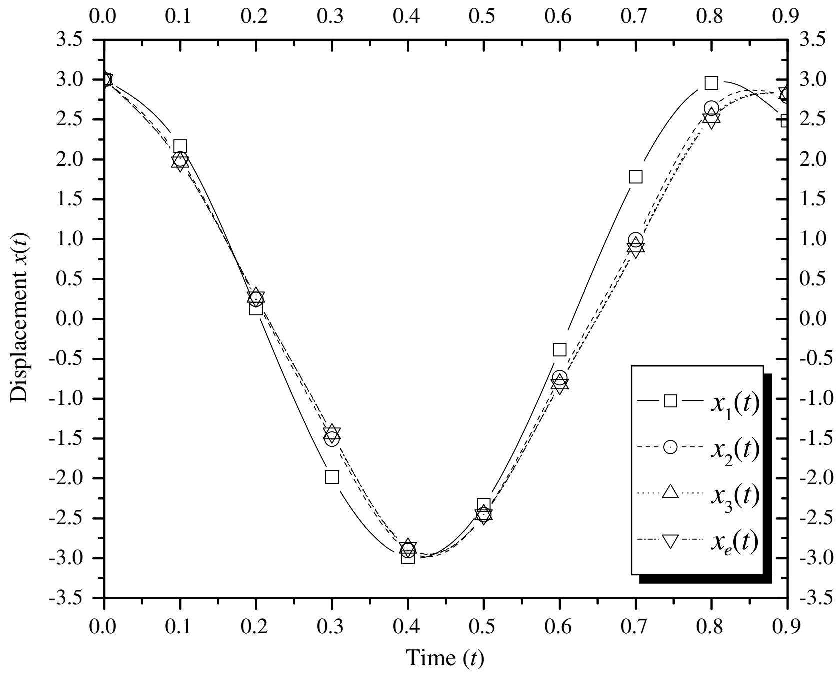

Fig. 5. Comparison of approximate periodic solutions with exact solution for the cubic–quintic Duffing oscillator for $\alpha = \beta = \gamma = 1$ and $X_0 = 3$.

the whole range of $X_0 \in [0.1, 1000]$ and therefore the first-order solution is relatively inaccurate. However, $\sqrt{\Omega_2}$ is very accurate and $\sqrt{\Omega_3}$ is excellent. From Tables 1–4, it is clearly shown that for different combination of

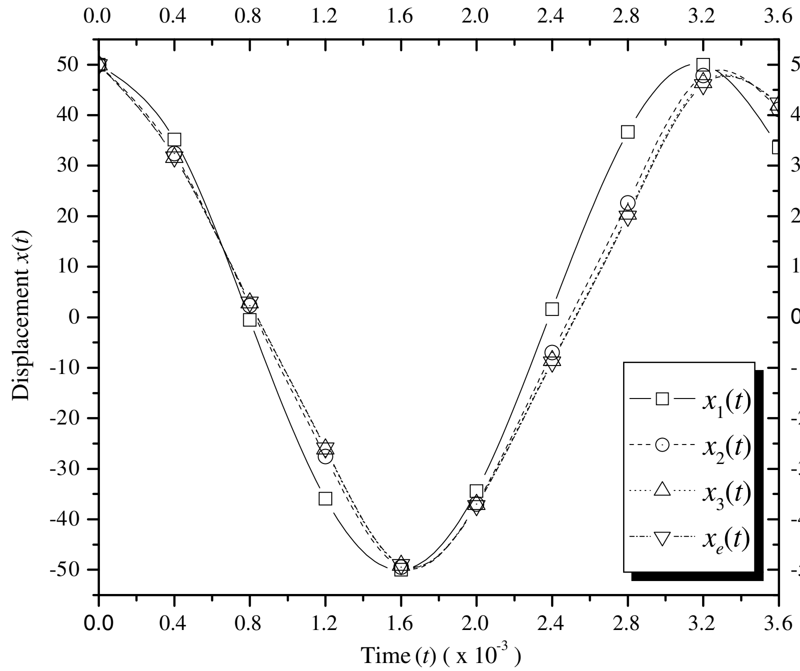

Fig. 6. Comparison of approximate periodic solutions with exact solution for the cubic–quintic Duffing oscillator for $\alpha = 2$, $\beta = \gamma = 1$ and $X_0 = 50$.

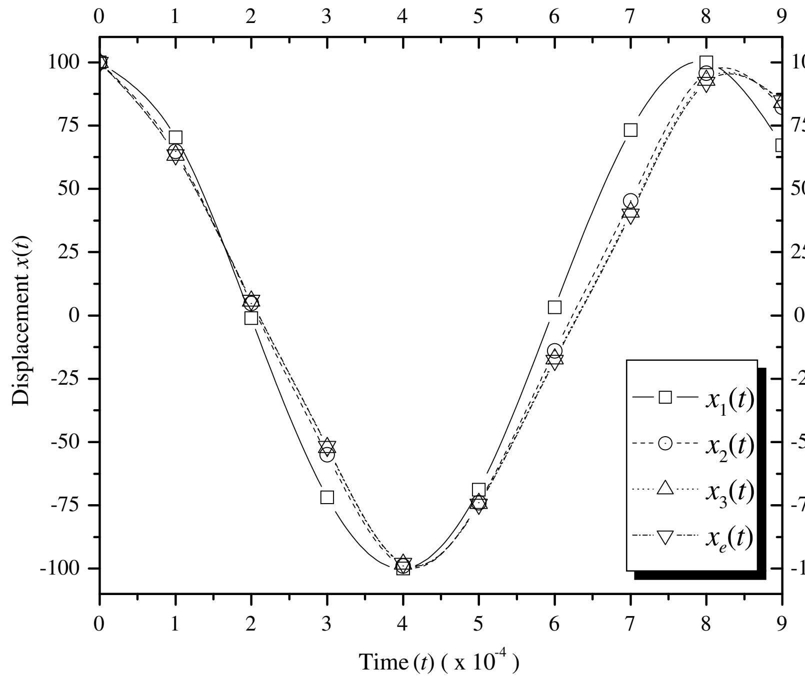

Fig. 7. Comparison of approximate periodic solutions with exact solution for the cubic–quintic Duffing oscillator for $\alpha = 5$, $\beta = 3$, $\gamma = 1$ and $X_0 = 100$.

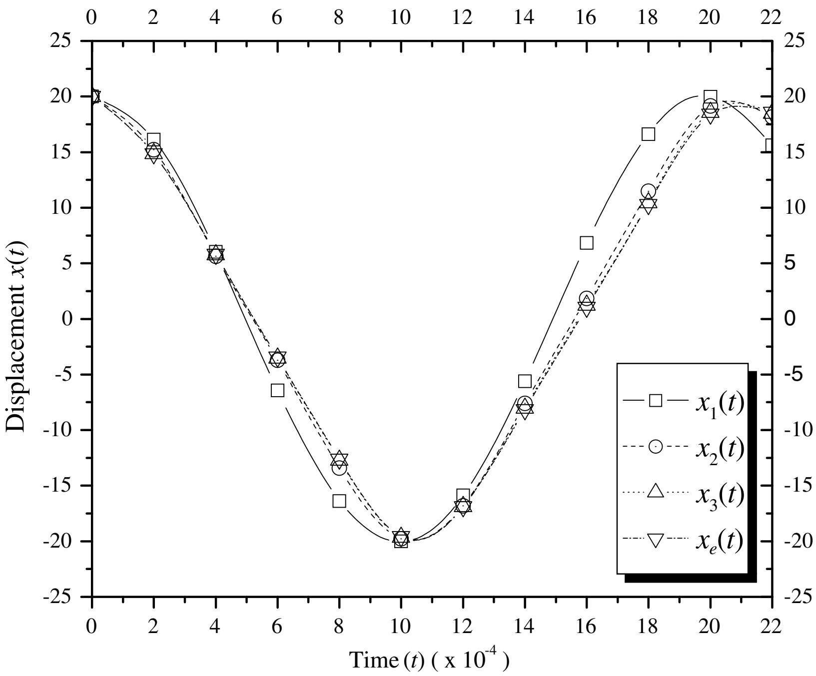

Fig. 8. Comparison of approximate periodic solutions with exact solution for the cubic–quintic Duffing oscillator for $\alpha = 1$, $\beta = 10$, $\gamma = 100$ and $X_0 = 20$.

parameters α, β, γ and X_0, the relative errors for $\sqrt{\Omega_1}$, $\sqrt{\Omega_2}$ and $\sqrt{\Omega_3}$ remain stable up to an amplitude as large as $X_0 = 1000$ and even for $X_0 \to \infty$.

To further illustrate and verify the accuracy for this new approximate analytical approach, a comparison of the time history periodic response of the oscillation derived from various approximations is presented in Figs. 5–8. These figures correspond to four different amplitudes of oscillation for $X_0 = 3$ for $\frac{d^2x}{dt^2} + x + x^3 + x^5 = 0$, $X_0 = 50$ for $\frac{d^2x}{dt^2} + 2x + x^3 + x^5 = 0$, $X_0 = 100$ for $\frac{d^2x}{dt^2} + 5x + 3x^3 + x^5 = 0$ and $X_0 = 20$ for $\frac{d^2x}{dt^2} + x + 10x^3 + 100x^5 = 0$. In all figures, the first-order periodic solution $x_1(t)$ is generally acceptable as compared to the exact solution $x_e(t)$ while the second-order and third-order periodic solutions, $x_2(t)$ and $x_3(t)$, are in good and excellent comparison with the exact solution, respectively, for different amplitudes of oscillation. In general, the periodic solution $x(t)$ depends on initial condition X_0. Corresponding expressions for the analytical approximations $x_1(t)$, $x_2(t)$ and $x_3(t)$ can be obtained from Eqs. (18), (25) and (35), respectively. In Fig. 5 for a typical cubic–quintic Duffing oscillators, curve $x_3(t)$ virtually coalesces with curve $x_e(t)$, curve $x_2(t)$ starts to have observable deviation from $x_e(t)$ as time goes on while curve x_1 (t) starts to deviate from $x_e(t)$ from the beginning at $t = 0$. Similar time history characteristics can be observed for other cubic–quintic Duffing oscillators in Figs. 6–8. In general, deviations of solutions are expected to increase as time progresses but the third-order solution maintains excellent accuracy for the period shown.

The proposed linearized harmonic balance method with Newton's method for higher-order analytical approximation has illustrated noticeable improvement as compared with lower-order analytical approximation. It is also observed that the linearized harmonic balance method is effective for solving strong nonlinear oscillator with cubic and/or quintic nonlinearity and it has clear advantage over the classical perturbation method which is restricted by the presence of a small parameter in the governing differential equation.

4. Conclusions

This paper has proposed a new method of linearized harmonic balance with Newton's method for solving accurate analytical approximations to strong nonlinear oscillations with cubic and quintic terms in the restoring force, the so-called cubic–quintic Duffing oscillators. The proposed approach provides excellent higher-order analytical approximation comparing to exact solution. The most important advantages of this method as compared to the previous methods are its simplicity, flexibility in application, and avoidance of complicated numerical integration because it yields a system of simple, algebraic, linear simultaneous equations depending only on initial conditions and without numerical integration. Unlike the classical perturbation method involving expansion over a small parameter, these approximate analytical frequencies are valid for small as well as large amplitudes of oscillation as it is not restricted to the presence of a small parameter. Furthermore, the method does not require a known initial condition at the outset which is a condition for most numerical methods. As in many cases the initial conditions may not be given, this analytical approximation method is more preferred for analyzing the solution of those nonlinear differential equations. Several illustrative examples of cubic–quintic Duffing oscillators have been presented to verify the capability and accuracy of the approach. The results not only conclude that the second-order and third-order analytical approximations provide excellent agreement with respect to the exact solutions, but also show that quintic nonlinearity is strong and thus has marked effect on the accuracy of solutions. In addition, the method proposed allows one to estimate Fourier amplitudes of unperturbed motion, that is worth for examining resonance phenomena under periodic or quasiperiodic forcing.

Acknowledgement

The work described in this paper was supported by a Grant from Research Grants Council of the Hong Kong Special Administrative Region (Project No. 9041145).

Appendix A

The variables for $\chi_i (i = 1 - 11)$ in Eqs. (29)–(34) and (36) are presented as follows:

$$\chi_1 = \frac{5\gamma\widetilde{A}^5}{4} - \frac{25\gamma\widetilde{A}^4 c_1}{8} + \frac{3\widetilde{A}^3(\beta + 5\gamma c_1^2)}{2} - \frac{3\widetilde{A}^2(2\beta c_1 + 5\gamma c_1^3)}{4} - 2\Omega_2\widetilde{A} + \widetilde{A}\left(2\alpha + 3\beta c_1^2 + \frac{15\gamma c_1^4}{4}\right), \tag{A.1}$$

$$\chi_2 = \frac{1}{8}[25\gamma\widetilde{A}^4 + 20\gamma\widetilde{A}^3 c_1 + 6\widetilde{A}^2(4\beta + 15\gamma c_1^2) + 12\widetilde{A}(2\beta c_1 + 5\gamma c_1^3) + 2(8\alpha + 12\beta c_1^2 + 15\gamma c_1^4 - 8\Omega_2)], \tag{A.2}$$

$$\chi_3 = \frac{1}{2}[3\beta\widetilde{A}^2 + 5\gamma\widetilde{A}^4 - 10\gamma\widetilde{A}^3 c_1 - 3\beta c_1^2 - 5\gamma c_1^4 - 3\widetilde{A}(2\beta c_1 + 5\gamma c_1^3)], \tag{A.3}$$

$$\chi_4 = -2\widetilde{A}, \tag{A.4}$$

$$\chi_5 = \frac{1}{8}[5\gamma\widetilde{A}^5 - 30\gamma\widetilde{A}^4 c_1 + \widetilde{A}^3(4\beta + 30\gamma c_1^2) - 12\widetilde{A}^2(2\beta c_1 + 5\gamma c_1^3) - 2(8\alpha c_1 + 6\beta c_1^3 + 5\gamma c_1^5)] + 18 c_1\Omega_2, \tag{A.5}$$

$$\chi_6 = \frac{1}{8}\{-5\gamma\widetilde{A}^4 - 60\gamma\widetilde{A}^3 c_1 - 6\widetilde{A}^2(2\beta + 15\gamma c_1^2) - 24\widetilde{A}(2\beta c_1 + 5\gamma c_1^3) - 2[18\beta c_1^2 + 25\gamma c_1^4 + 8(\alpha - 9\Omega_2)]\}, \tag{A.6}$$

$$\chi_7 = \frac{1}{4}[5\gamma\widetilde{A}^4 + 18\beta c_1^2 + 25\gamma c_1^4 + \widetilde{A}^2(6\beta + 45\gamma c_1^2) + 4\widetilde{A}(3\beta c_1 + 10\gamma c_1^3) + 8(\alpha - 9\Omega_2)], \tag{A.7}$$

$$\chi_8 = 18 c_1, \tag{A.8}$$

$$\chi_9 = \frac{\widetilde{A}}{8}[\gamma\widetilde{A}^4 - 20\gamma\widetilde{A}^3 c_1 + 30\gamma\widetilde{A}^2 c_1^2 + 4c_1^2(3\beta + 5\gamma c_1^2) - 6\widetilde{A}(2\beta c_1 + 5\gamma c_1^3)], \tag{A.9}$$

$$\chi_{10} = \frac{1}{8}(-12\beta\widetilde{A}^2 - 15\gamma\widetilde{A}^4 - 20\gamma\widetilde{A}^3 c_1 + 12\beta c_1^2 + 20\gamma\widetilde{A}c_1^3 + 20\gamma c_1^4) \tag{A.10}$$

and

$$\chi_{11} = \frac{1}{4}(-8\alpha - 6\beta\widetilde{A}^2 - 5\gamma\widetilde{A}^4 - 12\beta\widetilde{A}c_1 - 10\gamma\widetilde{A}^3 c_1 - 12\beta c_1^2 - 15\gamma\widetilde{A}^2 c_1^2 - 30\gamma\widetilde{A}c_1^3 - 15\gamma c_1^4 + 200\Omega_2), \tag{A.11}$$

where

$$\widetilde{A} = X_0 + c_1 = \frac{4X_0(32\alpha + 23\beta X_0^2 + 20\gamma X_0^4)}{128\alpha + 96\beta X_0^2 + 85\gamma X_0^4}. \tag{A.12}$$

Ω_2 in Eqs. (A.1)–(A.11) are obtained from Eq. (26).

Appendix B

The exact solution of the dynamical system can be obtained by integrating the governing differential equation in Eq. (1) and imposing the initial conditions in Eq. (2) as follows. Eq. (1) can be expressed as

$$\frac{1}{2}(x')^2 + \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 + \frac{\gamma}{6}x^6 = C \quad \forall t, \tag{B.1}$$

where C is a constant. Imposing the initial conditions in Eq. (2) yields

$$C = \frac{\alpha}{2}X_0^2 + \frac{\beta}{4}X_0^4 + \frac{\gamma}{6}X_0^6. \tag{B.2}$$

Equating Eqs. (B.1) and (B.2) yields

$$\frac{1}{2}(x')^2 + \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 + \frac{\gamma}{6}x^6 = \frac{\alpha}{2}X_0^2 + \frac{\beta}{4}X_0^4 + \frac{\gamma}{6}X_0^6 \tag{B.3}$$

or equivalently

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{\alpha(X_0^2 - x^2) + \frac{\beta}{2}(X_0^4 - x^4) + \frac{\gamma}{3}(X_0^6 - x^6)}}. \tag{B.4}$$

Integrating Eq. (B.4), the period of oscillation T_e is

$$T_\mathrm{e}(X_0) = 4\int_0^{X_0} \frac{\mathrm{d}x}{\sqrt{\alpha(X_0^2 - x^2) + \frac{\beta}{2}(X_0^4 - x^4) + \frac{\gamma}{3}(X_0^6 - x^6)}}. \tag{B.5}$$

Substituting $x = X_0 \sin t$ into Eq. (B.5) and integrating yield

$$T_\mathrm{e}(X_0) = 4\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{\sqrt{\alpha + \frac{\beta X_0^2}{2}(1 + \sin^2 t) + \frac{\gamma X_0^4}{3}(1 + \sin^2 t + \sin^4 t)}}, \tag{B.6}$$

or

$$T_\mathrm{e}(X_0) = \frac{4}{k_1}\int_0^{\frac{\pi}{2}} (1 + k_2 \sin^2 t + k_3 \sin^4 t)^{-\frac{1}{2}}\mathrm{d}t, \tag{B.7}$$

where

$$k_1 = \sqrt{\alpha + \frac{\beta X_0^2}{2} + \frac{\gamma X_0^4}{3}}, \tag{B.8}$$

$$k_2 = \frac{3\beta X_0^2 + 2\gamma X_0^4}{6\alpha + 3\beta X_0^2 + 2\gamma X_0^4}, \tag{B.9}$$

$$k_3 = \frac{2\gamma X_0^4}{6\alpha + 3\beta X_0^2 + 2\gamma X_0^4}. \tag{B.10}$$

The exact frequency ω_e is also a function of X_0 and can be obtained from the period of the oscillation as:

$$\omega_\mathrm{e}(X_0) = \frac{\pi k_1}{2\int_0^{\frac{\pi}{2}} (1 + k_2 \sin^2 t + k_3 \sin^4 t)^{-\frac{1}{2}}\mathrm{d}t}. \tag{B.11}$$

References

[1] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, Wiley, New York, 1979.
[2] A.H. Nayfeh, Problems in Perturbation, Wiley, New York, 1985.
[3] P. Hagedorn, Non-linear Oscillations, Clarendon, Oxford, 1988, translated by Wolfram Stadler.
[4] Y.K. Cheung, S.H. Chen, S.L. Lau, Application of the incremental harmonic balance method to cubic non-linearity systems, J. Sound Vib. 140 (1990) 273–286.
[5] Y.K. Cheung, S.H. Chen, S.L. Lau, A modified Lindstedt–Poincaré method for certain strongly non-linear oscillators, Int. J. Non-Linear Mech. 26 (1991) 367–378.
[6] A.Y.T. Leung, S.K. Chui, Non-linear vibration of coupled Duffing oscillators by an improved incremental harmonic balance method, J. Sound Vib. 181 (1995) 619–633.
[7] R.E. Mickens, Oscillations in Planar Dynamic Systems, Word Scientific, Singapore, 1996.
[8] M.I. Qaisi, A power series approach for the study of periodic motion, J. Sound Vib. 196 (1996) 401–406.
[9] S.J. Liao, A.T. Chwang, Application of homotopy analysis method in nonlinear oscillations, J. Appl. Mech.-Trans. ASME 65 (1998) 914–922.
[10] H.P.W. Gottlieb, Harmonic balance approach to periodic solutions of non-linear Jerk equations, J. Sound Vib. 271 (2004) 671–683.
[11] P. Amore, A. Aranda, Improved Lindstedt–Poincaré method for the solution of nonlinear problems, J. Sound Vib. 283 (2005) 1115–1136.
[12] Z.G. Xiong, H. Hu, Simplified continuous finite element method for a class of nonlinear oscillating equations, J. Sound Vib. 287 (2005) 367–373.
[13] H. Hu, Solutions of nonlinear oscillators with fractional powers by an iteration procedure, J. Sound Vib. 294 (2006) 608–614.
[14] H. Hu, Solutions of a quadratic nonlinear oscillator: iteration procedure, J. Sound Vib. 298 (2006) 1159–1165.
[15] F. Xie, X. Gao, Exact travelling wave solutions for a class of nonlinear partial differential equations, Chaos Soliton Fract 19 (2004) 1113–1117.
[16] C.W. Lim, B.S. Wu, A new analytical approach to the Duffing–harmonic oscillator, Phys. Lett. A 311 (2003) 365–373.
[17] H. Hu, Solutions of the Duffing–harmonic oscillator by an iteration procedure, J. Sound Vib. 298 (2006) 446–452.
[18] H. Wagner, Large-amplitude free vibrations of a beam, J. Appl. Mech.-Trans. ASME 32 (1965) 887–892.
[19] M.N. Hamdan, N.H. Shabaneh, On the large amplitude free vibrations of a restrained uniform beam carrying an intermediate lumped mass, J. Sound Vib. 199 (1997) 711–736.
[20] S. Lenci, G. Menditto, A.M. Tarantino, Homoclinic and heteroclinic bifurcations in the non-linear dynamics of a beam resting on an elastic substrate, Int. J. Non-Linear Mech. 34 (1999) 615–632.
[21] Z. Yan, A new sine–Gordon equation expansion algorithm to investigate some special nonlinear differential equations, Chaos Soliton Fract 23 (2005) 767–775.
[22] D.J. Huang, H.Q. Zhang, Link between travelling waves and first order nonlinear ordinary differential equation with a sixth-degree nonlinear term, Chaos Soliton Fract 29 (2006) 928–941.
[23] A.I. Maĭmistov, Propagation of an ultimately short electromagnetic pulse in a nonlinear medium described by the fifth-order Duffing model, Opt. Spectrosc. 94 (2003) 251–257.
[24] B.S. Wu, P.S. Li, A method for obtaining approximate analytic periods for a class of nonlinear oscillators, Meccanica 36 (2001) 167–176.
[25] B.S. Wu, C.W. Lim, Y.F. Ma, Analytical approximation to large-amplitude oscillation of a non-linear conservative system, Int. J. Non-Linear Mech. 38 (2003) 1037–1043.
[26] C.W. Lim, B.S. Wu, Accurate higher-order approximations to frequencies of nonlinear oscillators with fractional powers, J. Sound Vib. 281 (2005) 1157–1162.
[27] C.W. Lim, B.S. Wu, L.H. He, A new approximate analytical approach for dispersion relation of the nonlinear Klein–Gordon equation, Chaos 11 (2001) 843–848.
[28] B.S. Wu, W.P. Sun, C.W. Lim, An analytical approximate technique for a class of strongly nonlinear oscillators, Int. J. Non-Linear Mech. 41 (2006) 766–774.