Oscillating Airfoils at High Mach Number

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SUMMARY

A simple formula is given for the pressure distribution on an oscillating airfoil in two-dimensional flow at high Mach Number. The formula is expected to be reasonably accurate if the pressure on the surface remains within the range 0.2 to 3.5 times the main-stream pressure. To illustrate the application of the formula, some results for symmetrical airfoils performing pitching oscillations are obtained and compared with results obtained from existing theories in the case of high Mach Number.

NOTATION

x	= coordinate parallel to undisturbed stream
y	= coordinate perpendicular to undisturbed stream, measured *away* from airfoil on each side of it
y	= $Y(x, t)$ = equation of airfoil surface (upper or lower) at time t
U	= velocity of undisturbed stream
p_1, ρ_1, a_1	= pressure, density, velocity of sound, respectively, in undisturbed stream
M	= U/a_1 = Mach Number of undisturbed stream
w	= "effective piston velocity" = $(\partial Y/\partial t) + (U\partial Y \div \partial x)$
ϵ	= maximum displacement in airfoil oscillation
$\omega/2\pi$	= frequency of airfoil oscillation
c	= airfoil chord
δ	= maximum inclination of airfoil surface to the stream
p	= pressure at point on airfoil surface
γ	= ratio of specific heats (assumed constant)
y	= $Y_s(x) \pm Y_t(x, t)$ = equation of $\begin{pmatrix}\text{lower}\\ \text{upper}\end{pmatrix}$ surface of symmetrical oscillating airfoils
w_s, w_t	= values of w resulting from Y_s, Y_t, respectively
Δp	= pressure on lower surface minus pressure at same point on upper surface
$\alpha(t)$	= variable angle of attack (in pitching oscillations)
m	= nose-up moment of aerodynamic forces per unit span about axis of pitching oscillations
$m_\alpha, m_{\dot\alpha}$	= coefficients defined as $(1/\rho_1 U_1^2 c^2)\,(\partial m/\partial \alpha)_{\alpha=\dot\alpha=0}$ and $(1/\rho_1 U_1 c^3)\,(\partial m/\partial \dot\alpha)_{\alpha=\dot\alpha=0}$
τ	= thickness-chord ratio
x_0	= distance of axis of pitching oscillations behind leading edge
h	= ratio of x_0 to c

Basic formula: In motions with M large and $M[\delta + (\epsilon/c) \times (\omega c/U)] < 1$, the surfaces pressures are given to good approximation by the equation

$$\frac{p}{p_1} = 1 + \gamma\frac{w}{a_1} + \frac{\gamma(\gamma+1)}{4}\left(\frac{w}{a_1}\right)^2 + \frac{\gamma(\gamma+1)}{12}\left(\frac{w}{a_1}\right)^3$$

(1) INTRODUCTION

A THEORY OF OSCILLATING AIRFOILS in supersonic flow is needed, but a completely linearized theory has proved too inaccurate. Some second-order terms (part of an ascending series for them, in powers of the frequency parameter) have been found by Wylly[1] and by Van Dyke,[2] but their methods are complicated, and unfortunately their answers do not agree. The method of Jones and Skan[3] takes the thickness of the airfoil into account to higher order but not the amplitude of the oscillations; this method calls for an extensive program of computation in any one case. In this state of affairs it seems worth pointing out that, in the special case of high Mach Number, there is a simple theory that should give fairly accurate results (better than second-order) even when disturbances are rather large.

In two-dimensional flow past an airfoil at high Mach Number, say $M \geq 4$, the shock waves and expansion waves are set at small angles to the undisturbed flow. Two consequences of this are worth special notice.

(a) Gradients in the direction of the undisturbed flow are small compared with gradients perpendicular to it.

(b) Because velocity components parallel to shock waves or expansion waves are not changed by them, the velocity components perpendicular to the flow are large compared with the *disturbances* to components parallel to it.

Both these facts contribute to the truth of Hayes's result[4] that, to a good approximation, *any plane slab of fluid, initially perpendicular to the undisturbed flow, remains so as it is swept downstream and moves in its own plane under the laws of one-dimensional unsteady motion.*

Goldsworthy[5] showed that the relative error in this result of Hayes should be of order $1/M^2$, both causes (a) and (b) above contributing a factor $1/M$. Here it is assumed that $M\delta$ is bounded, say less than 1, where δ is the maximum inclination (in radians) of the airfoil surface to the stream. This condition, he points out, would in most practical applications need to be satisfied from considerations of drag.

Now the result in italics must remain true even if the whole flow is unsteady—for example, if the airfoil oscillates. This is so evident physically from (a) and (b) above† that its mathematical deduction from the equations of motion, an easy extension of Goldsworthy's work, is here omitted.

Now as the slab of fluid moves downstream with approximately the velocity U of the undisturbed flow, the position of the piece of solid wall bounding it will move normal to the stream with a velocity

$$w = (\partial Y/\partial t) + U(\partial Y/\partial x) \tag{1}$$

where $y = Y(x, t)$ is the equation of the upper or lower surface of the airfoil, with y measured away from the airfoil in each case.

Accordingly, the problem of determining the forces on the oscillating airfoil at high Mach Number reduces, as in the steady case, to the one-dimensional flow problem of finding the pressure on a piston moved, with a given dependence of velocity w on time, into otherwise undisturbed fluid.

This, by comparison, is a not too complicated problem. It becomes particularly simple if a condition like Goldsworthy's is imposed (as indeed it must if the statement in italics is to be at all accurate)—namely, that the magnitude $|w|$ of the piston velocity, Eq. (1), never exceeds the speed of sound in the undisturbed fluid. In an oscillation with maximum displacement ϵ and frequency $\omega/2\pi$, this condition can be written

$$\epsilon\omega + U\delta < a_1 \tag{2}$$

where δ, as before, is the maximum inclination of the airfoil surface to the stream. This condition could be rewritten

$$M[\delta + (\epsilon/c)\,(\omega c/U)] < 1 \tag{3}$$

where $M = U/a_1$ is the Mach Number and c is the chord, so that $\omega c/U$ is the frequency parameter. Evidently, condition (3) might well be satisfied in practical problems. (The theory would still have value as a rough approximation even if the left-hand side somewhat exceeded 1.) As will be seen below, it permits compression up to three and one-half times and rarefaction down to one-fifth of the undisturbed pressure.

The advantage of imposing this condition is that the pressure on the piston then depends, with fair accuracy, only on its instantaneous velocity. The pressure distribution is in fact a simple function of the piston velocity w, given by Eq. (1).

When there is no shock wave, indeed, this is true *without* any restriction on amplitude, and the pressure is given by the "simple wave" condition

$$\frac{p}{p_1} = \left(1 + \frac{\gamma - 1}{2}\frac{w}{a_1}\right)^{2\gamma/(\gamma-1)} \tag{4}$$

where γ is the ratio of the specific heats (assumed constant). When there is an initial shock wave (because the piston moves initially with finite velocity into the fluid), Eq. (4) can be improved by use of the "shock-expansion" theory, which takes into account the exact pressure change at the shock [which is less than Eq. (4)] and assumes a simple wave (i.e., neglects lateral entropy *gradients*) behind the shock. But this theory gives pressures depending not only on the instantaneous value of w/a_1 but also on the value that w/a_1 had when the shock wave was produced.

TABLE 1

w/a_1	Simple Wave	Shock Expansion	Quadratic	Cubic
1	3.583	3.473	3.240	3.520
0.5	1.949	1.916	1.910	1.945
0	1.000	1.000	1.000	1.000
−0.5	0.478	0.488	0.510	0.475
−1	0.210	0.220	0.440	0.160

In Table 1 is shown the dependence on w/a_1 (for $\gamma = 1.4$) of (a) the simple wave expression, Eq. (4), for p/p_1; (b) the value of p/p_1 on "shock-expansion" theory, taking the shock at its maximum permitted strength, corresponding to $w/a_1 = 1$; (c) the "Busemann" quadratic approximation—namely, the first three terms of the binomial expansion of Eq. (4); (d) the cubic approximation (first four terms of the expansion)—namely,

$$\frac{p}{p_1} = 1 + \gamma\frac{w}{a_1} + \frac{\gamma(\gamma+1)}{4}\left(\frac{w}{a_1}\right)^2 + \frac{\gamma(\gamma+1)}{12}\left(\frac{w}{a_1}\right)^3 \tag{5}$$

Since for a given value of w/a_1 the value of p/p_1 may vary continuously between the extremes of the "simple wave" and "shock-expansion" values (as the value of w/a_1 at the shock varies from 0 to 1), any approximate value that always lies reasonably close to both is acceptable. The cubic approximation, for example, which is always within 0.06 p_1 of both values (not a great error where such large pressure differences are occurring), even if w/a_1 is as high as 1, is in many ways to be preferred even to the exact simple wave value, which is in error by 0.11 p_1 immediately behind the shock wave. Since, also, the cubic is reasonably simple, it will here be adopted. (Note that cubic terms in the expansion of pressure ratio in a simple wave have often been discarded on the grounds that their coefficient is not the same as in the expansion of the pressure ratio through a shock; however, the coefficient in the latter case is $\gamma(\gamma + 1)^2/32$, which differs from that in Eq. (5) by only 10 per cent for $\gamma = 1.4$, so that the terms are well worth retaining.)

Finally, then, the pressure at any point on an oscillating airfoil will be taken as given by the cubic (5) in w/a_1, where w is defined in terms of the equation $y = Y(x, t)$ of the airfoil surface by means of Eq. (1).*

It is unnecessary for the author to make extensive calculations here on the basis of the postulated pressure distribution, since it is a simple matter for anyone concerned in technical applications to make use of it to obtain any particular results he may need on the pres-

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† Note that it is of little importance to such a slab of fluid, while it is being deformed by the wall, whether or not earlier slabs have been subjected to the same deformations.

* It should be noted that a formula of somewhat similar character was suggested by Jones[8] as an empirical approximation for general M. The present work indicates that Jones's formula is right for large M, and so gives it some added support.

sure forces acting on oscillating airfoils under the conditions here treated. Some specimen results will, however, be displayed and compared with existing theories.

(2) Symmetrical Airfoils

Supersonic airfoils are usually symmetrical, so that at zero angle of attack the equation of both surfaces [with y measured away from the airfoil in each case, as in Section (1)] is the same, say $y = Y_s(x)$. Changes in y due to oscillations, whether pitching or bending, will have opposite sign on both sides, say $Y_t(x, t)$ on the lower surface and $-Y_t(x, t)$ on the upper surface. The full equation of both surfaces is then

$$y = Y_s(x) \pm Y_t(x, t) \quad \begin{Bmatrix} \text{lower surface} \\ \text{upper surface} \end{Bmatrix} \tag{6}$$

Hence, by Eq. (1),

$$w = UY_s'(x) \pm \left(\frac{\partial Y_t}{\partial t} + U\frac{\partial Y_t}{\partial x}\right) = w_s(x) \pm w_t(x, t) \tag{7}$$

Now in many problems one would wish to know not the pressure distribution, Eq. (5), on each surface but only the load distribution, obtained by subtracting the pressure distribution on the upper surface from that on the lower surface. This is

$$\begin{aligned}\frac{\Delta p}{p_1} &= 1 + \gamma\frac{w_s + w_t}{a_1} + \frac{\gamma(\gamma+1)}{4}\left(\frac{w_s + w_t}{a_1}\right)^2 + \frac{\gamma(\gamma+1)}{12}\left(\frac{w_s + w_t}{a_1}\right)^3 - \\ &\qquad 1 - \gamma\frac{w_s - w_t}{a_1} - \frac{\gamma(\gamma+1)}{4}\left(\frac{w_s - w_t}{a_1}\right)^2 - \frac{\gamma(\gamma+1)}{12}\left(\frac{w_s - w_t}{a_1}\right)^3 \\ &= \left[2\gamma + \gamma(\gamma+1)\frac{w_s}{a_1} + \frac{\gamma(\gamma+1)}{2}\left(\frac{w_s}{a_1}\right)^2\right]\frac{w_t}{a_1} + \frac{1}{6}\gamma(\gamma+1)\left(\frac{w_t}{a_1}\right)^3\end{aligned} \tag{8}$$

As would be expected, there are no terms in the square of the oscillatory part w_t of w. The main effect of nonlinearity is to change the coefficient of w_t/a_1 by a factor depending on Mach Number and on the local slope of the airfoil contour.

It will be sufficient as an example to consider a pitching oscillation, say one about the axis $x = x_0$, $y = 0$, with variable angle of attack $\alpha(t)$. Then

$$Y_t(x, t) = \alpha(t)\,(x - x_0), \qquad w_t = \dot\alpha(x - x_0) + U\alpha \tag{9}$$

Substituting also for w_s in Eq. (8), it becomes

$$\frac{\Delta p}{p_1} = \left[2\gamma + \gamma(\gamma+1)MY_s'(x) + \frac{\gamma(\gamma+1)}{2}M^2Y_s'^2(x)\right]\left[\frac{\dot\alpha(x - x_0)}{a_1} + M\alpha\right] + \frac{\gamma(\gamma+1)}{6}\left[\frac{\dot\alpha(x - x_0)}{a_1} + M\alpha\right]^3 \tag{10}$$

The terms in (10) which are linear in α and $\dot\alpha$ are especially important for determining the aerodynamic stiffness and damping, respectively, of the oscillation. The stiffness is the coefficient of α, and the damping is that of $\dot\alpha$, in the nose-down moment,

$$-m = \int_0^c (x - x_0)\Delta p\, dx \tag{11}$$

of the load distribution about the axis $x = x_0$. (Here, c is the chord.)

Thus, by Eq. (10), the aerodynamic stiffness coefficient is

$$-m_\alpha = \frac{1}{\rho_1 U_1^2 c^2}\left(-\frac{\partial m}{\partial \alpha}\right)_{\alpha=\dot\alpha=0} = \frac{1}{Mc^2}\int_0^c (x - x_0)\left[2 + (\gamma+1)MY_s'(x) + \frac{\gamma+1}{2}M^2Y_s'^2(x)\right]dx \tag{12}$$

and the aerodynamic damping coefficient is

$$-m_{\dot\alpha} = \frac{1}{\rho_1 U_1 c^3}\left(-\frac{\partial m}{\partial \dot\alpha}\right)_{\alpha=\dot\alpha=0} = \frac{1}{Mc^3}\int_0^c (x - x_0)^2\left[2 + (\gamma+1)MY_s'(x) + \frac{\gamma+1}{2}M^2Y_s'^2(x)\right]dx \tag{13}$$

The latter is essentially positive—that is, there is always damping at high Mach Number. (This is obvious from the point of view of this paper, the damping being essentially the radiation damping of a piston in air.)

To fix the ideas still further, consider a biconvex airfoil of thickness-chord ratio τ, given by the equation

$$Y_s(x) = 2\tau[x - (x^2/c)] \tag{14}$$

(Note that, by the restriction (3), we must have $\tau < (2M)^{-1}$ even for small oscillations; somewhat less latitude can be allowed if the oscillations are large.) Then writing $x_0/c = h$ for the position of the axis of pitching oscillations relative to the chord, Eqs. (12) and (13) become

$$-m_\alpha = \frac{1}{M}\left\{1 - \frac{1}{3}(\gamma+1)M\tau + \frac{1}{3}(\gamma+1)M^2\tau^2 - h\left[2 + \frac{2}{3}(\gamma+1)M^2\tau^2\right]\right\} \tag{15}$$

$$-m_{\dot\alpha} = \frac{1}{M}\left\{\frac{2}{3} - \frac{1}{3}(\gamma+1)M\tau + \frac{4}{15}(\gamma+1)M^2\tau^2 - 2h\left[1 - \frac{1}{3}(\gamma+1)M\tau + \frac{1}{3}(\gamma+1)M^2\tau^2\right] + h^2\left[2 + \frac{2}{3}(\gamma+1)M^2\tau^2\right]\right\} \tag{16}$$

The value of h for which Eq. (15) vanishes, corresponding to the aerodynamic center, is tabulated against $M\tau$ for $\gamma = 1.4$ in Table 2. This table also gives the value of $-m_{\dot\alpha}$ in the special case ($h = 0.5$) of oscillations about an axis at mid-chord, when

$$-m_{\dot\alpha} = \frac{1}{M}\left(\frac{1}{6} + \frac{1}{10}(\gamma+1)M^2\tau^2\right) \tag{17}$$

Now the expression given by Wylly[1] for this last quantity is, in our notation,

$$-m_{\dot\alpha} = \frac{M^2 - 2}{6(M^2 - 1)^{3/2}} + \frac{\tau}{6(M^2 - 1)^2}\left[4M^6 + (2\gamma - 16)M^4 + (\gamma + 19)M^2 - 4\right] \tag{18}$$

This gives, as $M \to \infty$ for fixed $M\tau$,

$$-m_{\dot\alpha} \sim (1/M)\,[(1/6) + (2/3)M\tau] \tag{19}$$

Now Wylly does not retain terms of high enough order to give a term in $M^2\tau^2$ in $-Mm_{\dot\alpha}$, but the coefficient of $M\tau$ ought to be correct. However, it is seen from Eq. (17) that, in this case of $h = 0.5$, the coefficient of $M\tau$ in $-Mm_{\dot\alpha}$ for large M should be zero.

The theory of Section (1), if no slip has been made (and it is very simple), must give correctly the asymptotic form as $M \to \infty$ of any term of second order in the disturbances, since there is no difference between the simple wave and shock-expansion theories to this order. (In contrast, the third-order terms, which have been retained, should merely improve the accuracy, without being mathematically exact.) Therefore, the conclusion seems inescapable that some error has crept into Wylly's calculations.

Van Dyke's expression[3] for the pressure on the upper surface may be written, in our notation, as

$$\frac{p}{p_1} = 1 + \frac{\gamma M}{(M^2 - 1)^{1/2}}\left(\frac{w_s - w_t}{a_1}\right) + \frac{(1/4)\gamma(\gamma+1)M^4 - \gamma M^2 + \gamma}{(M^2 - 1)^2}\left(\frac{w_s^2 - 2w_sw_t}{a_1^2}\right) + \frac{\gamma\dot\alpha}{a_1}\left\{\frac{M}{(M^2 - 1)^{3/2}}x + \frac{M^2[2 + (\gamma - 1)M^2]}{(M^2 - 1)^3}\left[MY_s(x) + \frac{1}{2}xMY_s'(x)\right]\right\} \tag{20}$$

To obtain this result, he neglects w_t^2 and all the third-order terms and takes the limit as the frequency parameter tends to zero. For large M the last term in Eq. (20) vanishes, and the rest becomes identical with the basic formula of this paper (which holds independently of frequency parameter) if, in it, all second-order terms except w_s^2, but no third-order terms, are included. This constitutes a valuable check on Van Dyke's work.

The greatest Mach Number for which Jones and Skan[5] have carried out their calculations is $M = 2$, at which poor agreement with the present theory would be expected. However, their results do agree with it in predicting that the dependence of $-m_{\dot\alpha}$ on the thickness-chord ratio τ for oscillations of a biconvex airfoil about a mid-chord axis becomes negligible (for moderate τ) at $M = 2$. Indeed, their actual values of $-m_{\dot\alpha}$ (from 0.062 to 0.072 as the frequency parameter varies from 0 to 0.75), do not differ enormously from Eq. (17), which has the value 0.084 (independent of frequency parameter). Since the Jones and Skan values for $-m_{\dot\alpha}$ are expected to have good accuracy, this result at a moderate Mach Number augurs well for the present

Table 2

Distance of Aerodynamic Center from Leading Edge as Fraction of Chord, and Aerodynamic Damping Coefficient for Pitching Oscillations About a Mid-Chord Axis, as Functions of the Product of Mach Number and Thickness-Chord Ratio for a Biconvex Airfoil at High Mach Number

$M\tau$	0	0.1	0.2	0.3	0.4	0.5
$[h]_{m_\alpha=0}$	0.500	0.460	0.422	0.388	0.358	0.333
$[-m_{\dot\alpha}]_{h=0.5}$	$\frac{0.167}{M}$	$\frac{0.169}{M}$	$\frac{0.176}{M}$	$\frac{0.188}{M}$	$\frac{0.205}{M}$	$\frac{0.227}{M}$

theory at the rather high Mach Numbers for which it was designed to work.

One may note, in conclusion, that of the cubic terms in α and $\dot{\alpha}$ in Eq. (10), not so far discussed, the one that will predominate when the frequency parameter is less than 1 is

$$(1/6)\gamma(\gamma + 1)M^3\alpha^3 \tag{21}$$

This uniform load distribution along the airfoil constitutes a nonlinearity in the stiffness for a pitching oscillation, which will help to stabilize it when the axis is ahead of the mid-chord position (but may render possible some subharmonic resonance) and to destabilize it for axes to the rear of it.

References

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The Upwash Correction for an Oscillating Wing in a Wind Tunnel

(*Concluded from page 386*)

$$\int_0^\infty f(q)\, dq/(q \pm k)$$

If μ appears in the numerator and $(q + k)$ in the denominator, then, as $\mu \to 0$, the integral vanishes.

If μ appears in the numerator and $q - k$ in the denominator, then, as $\mu \to 0$, the integral vanishes everywhere except in a small region about $q = k$; here $f(q)$ may be replaced by $f(k)$ and brought outside the integral, provided $f(q)$ is continuous. The resulting integral may be evaluated, and then, as $\mu \to 0$ the result is $\pi f(k)$. This is a special case of Fourier's integral theorem.