Objective: To learn about Fourier transforms, which are a representation of *nonperiodic* functions in terms of trigonometric functions. Also, to see the notation for some useful functions and see their Fourier transforms

   To learn useful properties of Fourier transforms, which not only enable a variety of functions to be transformed but also lead to a variety of engineering applications

   To learn some analytical tools involving Fourier transforms, including systems analysis, transfer functions and filtering, and energy relationships

Review of Fourier series formulas

Let $f(t)$ be periodic with period T_0. Then

$$\boxed{f(t) = \sum_n D_n e^{jn\omega_0 t}}$$

where $\omega_0 = 2\pi/T_0$ and

$$\boxed{D_n = \frac{1}{T_0} \int_{T_0} f(t)e^{-jn\omega_0 t}\, dt.}$$

Representation of nonperiodic functions

We have seen how we can represent periodic functions in terms of sines and cosines. This is a natural thing to contemplate (at least retrospectively). What if we want to represent *nonperiodic* functions in terms of periodic functions? This seems kind of strange at first. Why would we want to? Part of the answer comes because we know how LTI systems respond to sines and cosines. It turns out to be closely related to sinusoidal steady-state analysis of circuits and provide some useful tools for design of signal processing and communication systems.

   To represent a nonperiodic function in terms of periodic functions, one thing we will need to to use periodic functions that might have infinite period. It will also turn out to add up a whole lot of them: more than a countable infinite. Here is one approach to the problem.

   Suppose we have a function $f(t)$. We construct a periodic extension of $f(t)$ called $f_{T_0}(t)$, where $f(t)$ is repeated every T_0 seconds. Since $f_{T_0}(t)$ is periodic, it has a Fourier series

$$f_{T_0}(t) = \sum_n D_n e^{jn\omega_0 t}$$

where

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_{T_0}(t)e^{-jn\omega_0 t} dt$$

and $\omega_0 = \frac{2\pi}{T_0}$. The integral is the same (since $f(t)$ is zero outside of $-T_0, T_0$) as

$$D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} f(t)e^{-jn\omega_0 t} dt$$

Now let's define the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

Comparing these, we can see that

$$D_n = \frac{1}{T_0}F(n\omega_0)$$

So we get the F.S. coefficients by sampling the function $F(\omega)$. In the limit as $T_0 \to \infty$, we sample infinitely often.

Now the F.S. for $f_{T_0}(t)$ is

$$f_{T_0}(t) = \sum_n \frac{F(n\omega_0)}{T_0}e^{jn\omega_0 t}$$

Let $\Delta\omega = 2\pi/T_0$ and write

$$f_{T_0}(t) = \sum_n \frac{F(n\Delta\omega_0)\Delta\omega}{2\pi}e^{jn\Delta\omega t}$$

Now in the limit as $T_0 \to \infty$ we get

$$f(t) = \lim_{T_0 \to \infty} f_{T_0}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega$$

So we have been able to reconstruct our *nonperiodic* function from $F(\omega)$. These are important enough they bear repeating:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt}$$

This is called the **Fourier Transform** of $f(t)$. The inverse Fourier transform is

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega}$$

We also write

$$F(\omega) = \mathcal{F}[f(t)] \qquad f(t) = \mathcal{F}^{-1}[F(\omega)]$$

or

$$f(t) \Leftrightarrow F(\omega)$$

This is notation similar to what we used for Laplace transforms.

Some things to notice:

1. Notice that to get $f(t)$ we are doing the same kind of thing we did for F.S. except that we add up a *continuum* of frequencies.

2. Suppose that $f(t)$ is causal. In the Laplace transform, set $s = e^{j\omega}$. Then we get the Laplace transform back! But notice that the integral goes from $-\infty$ to ∞. So we explicitly don't assume that $f(t)$ is causal. We thus do not typically use the F.T. to examine things like transient response: use the Laplace transform for that.

   As will be mentioned below, however, the F.T. cannot be used for unstable signals, since the integral does not exist in this case. The F.T. is not usually used to examine stability, as the L.T. can.

3. Notice again that in the forward transform, a negative is used in the exponent.

4. The frequency variable in the F.T. is ω, which is frequency in radians/sec. If we make the change of variable $\omega = 2\pi f$ and find the transform in terms of f we get the following transform pair:

$$F(2\pi f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$$

This is sometimes written simply as $F(f)$ (which, strictly speaking, is not correct notation, so you have to be careful). In this notation, we would have

$$\boxed{F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\, dt.}$$

The inverse transform is

$$f(t) = \int_{-\infty}^{\infty} F(2\pi f)e^{j2\pi ft}d\omega$$

This has the convenience of not having the factor of $1/2\pi$ in front of it. This is usually written (again, by an abuse of notation)

$$\boxed{f(t) = \int_{-\infty}^{\infty} F(f)e^{j2\pi ft}df}$$

As we did for F.S., we can plot the magnitude and phase of the F.T. We can write

$$F(\omega) = |F(\omega)|e^{j\angle F(\omega)}$$

Notice that if $f(t)$ is real, then $F(-\omega) = \overline{F(\omega)}$ (they are complex conjugates) so that

$$|F(-\omega)| = |F(\omega)|$$
$$\angle F(-\omega) = -\angle F(\omega)$$

or, in other words, if $f(t)$ is a real function, then the magnitude function is even and the phase function is odd.

Example 1 Find the F.T. of $f(t) = e^{-at}u(t)$

$$F(\omega) = \frac{1}{a + j\omega}$$

if $a > 0$ Does this look familiar? The magnitude and phase are

$$|F(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \qquad \angle F(\omega) = -\tan^{-1}(\omega/a)$$

Plot and discuss. What kind of filter is this? □

Some useful functions

Some functions come up often enough that it is nice to give them names.

The unit gate or **unit pulse** is defined by

$$\text{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

Some interesting variations: Plot $\text{rect}(x/2)$, $\text{rect}((x-1)/2)$, $\text{rect}((x-a)/T)$. Note that the **total** width is T and the function is centered around a.

The unit triangle function, also known as the hat function (or the chapeau function), is defined by

$$\Delta(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ 1 - 2|x| & |x| < \frac{1}{2} \end{cases}$$

Plot. Also look at $\Delta(x/T)$. Notice that the **total** width of the function is T.

The sinc function we have already met in the context of the F.S. of a square wave. To reiterate,

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

Observe the following:

1. $\text{sinc}(x)$ is even.
2. $\text{sinc}(x)$ is zero for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$.
3. $\text{sinc}(0) = 1$
4. $\text{sinc}(x)$ decays as $1/x$.

The signum function is defined by

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

F.T. of some simple functions

Get some students down for this part!

Example 2 Find the F.T. of $f(t) = \text{rect}(t/T)$.

$$\text{rect}(t/\tau) \Leftrightarrow \tau \, \text{sinc}(\omega\tau/2)$$

Plot this function. The zeros occur at $\pm 2n\pi/T$. Also plot as a function of f. The zeros occur at $\pm 1/T$. Notice that the longer the function, the narrower the sinc, and vice versa. What is the bandwidth of the signal? What is the "null-to-null" bandwidth? □

Example 3 Find the F.T. of $\delta(t)$.

$$\delta(t) \Leftrightarrow 1$$

Notice that every frequency has the same response! This is what would be termed a *white* spectrum: all components equal. □

Example 4 Find the inverse F.T. of $\delta(\omega)$. Use the inverse F.T. formula.

$$\frac{1}{2\pi} \Leftrightarrow \delta(\omega)$$

Multiplying through,

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

So a constant signal only has a response at $\omega = 0$. Does this make sense? □

Example 5 Find the inverse F.T. of $\delta(\omega - \omega_0)$ where ω_0 is a fixed frequency.

$$\frac{1}{2\pi} e^{j\omega_0 t} \Leftrightarrow \delta(\omega - \omega_0)$$

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - \omega_0)$$

□

Example 6 Find the F.T. of $\cos(\omega_0 t)$

$$\cos(\omega_0 t) \Leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

Plot and discuss. □

Example 7 Find the F.T. of the sgn function. This function is difficult: the upper and lower limits of integration cannot be handled directly. There are actually several approaches to the problem; I even wrote a paper describing a different approach. But we will go with the approach in the book, as it is probably the easiest. Note that we can write

$$\text{sgn}(t) = \lim_{\alpha \to 0}[e^{-\alpha t}u(t) - e^{\alpha t}u(-t)].$$

Plot and discuss. Then we can write

$$\mathcal{F}[\text{sgn}(t)] = \lim_{\alpha \to 0}[\mathcal{F}[e^{-\alpha t}u(t)] - \mathcal{F}[e^{\alpha t}u(t)] = \lim_{\alpha \to 0}\left[\frac{1}{a + j\omega} - \frac{1}{a - j\omega}\right] = \frac{2}{j\omega}$$

There are issues the math guys would have to stew about: can we really interchange the limits? But we will just believe this. □

Example 8 The real reason why the last example is useful is for this one: Find the F.T. of $u(t)$. It is not a simple matter of changing s to $j\omega$ in the Laplace transform, because of region of convergence issues. (This function does not satisfy the Dirichlet conditions.) So we have to be a bit tricky. Note that

$$u(t) = \frac{1}{2}[1 + \text{sgn}(t)]$$

Using linearity,

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}.$$

□

System response using Fourier transforms

Let $f(t)$ be the input to an LTIC system with transfer function $H(s)$ and output $y(t)$. The response to the everlasting signal $f(t) = e^{j\omega t}$ is $y(t) = H(\omega)e^{j\omega t}$. Thus $f(t) = e^{jn\Delta\omega t} \to H(n\Delta\omega)e^{jn\Delta\omega t}$. Now let the input be

$$\frac{1}{2\pi}\lim_{\Delta\omega \to 0}\sum_n F(n\Delta\omega)e^{jn\Delta\omega t}.$$

By the definition of the FT, this is $f(t)$. By linearity, the output is

$$y(t) = \frac{1}{2\pi}\lim_{\Delta\omega \to 0}\sum_n F(n\Delta\omega)H(n\Delta\omega)e^{jn\Delta\omega t}.$$

That is,

$$y(t) = \frac{1}{2\pi}\int_\infty^\infty F(\omega)H(\omega)e^{j\omega t}\,dt = \frac{1}{2\pi}\int_\infty^\infty Y(\omega)e^{j\omega t}\,dt$$

so we can say

$$Y(\omega) = F(\omega)H(\omega).$$

In a sense, this is the whole point of Fourier analysis!

Properties of Fourier Transforms

We saw earlier a variety of properties associated with the Laplace transform: linearity, time shift, convolution, differentiation, and integration. These opened up a variety of applications. There are similar the properties of F.T. which are also very useful.

Linearity The F.T. is linear:

$$\mathcal{F}[af(t) + bg(t)] = a\mathcal{F}[f(t)] + b\mathcal{F}[g(t)]$$

This follows from the linearity of the integral.

Time/Frequency Duality The duality property is one that is not shared by the Laplace transform. While slightly confusing perhaps at first, it essentially doubles the size of our F.T. table. The duality property follows from the similarity of the forward and inverse F.T. It states that if

$$f(t) \Leftrightarrow F(\omega)$$

then

$$F(t) \Leftrightarrow 2\pi f(-\omega)$$

where the function on the left is the function of time and the function on the right is the function of frequency.

Proof

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x)e^{jxt}dx$$

so

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(x)e^{-jxt}dx = \mathcal{F}[F(x)]$$

But observe that the integral on the right is the F.T. of $F(x)$. Now replace $t \to \omega$ to see the result. □

Example 9 We have seen that

$$\mathrm{rect}(t/T) \Leftrightarrow T\,\mathrm{sinc}(\omega T/2)$$

In this case, $f(t) = \mathrm{rect}(t/T)$ and $F(\omega) = T\,\mathrm{sinc}(\omega T/2)$. Then $F(t) = T\,\mathrm{sinc}(tT/2)$ and $f(\omega) = \mathrm{rect}(\omega/T)$. By duality,

$$T\,\mathrm{sinc}(tT/2) \Leftrightarrow 2\pi\,\mathrm{rect}(-\omega/T) = 2\pi\,\mathrm{rect}(\omega/T).$$

□

Example 10 We have already seen another example of the duality property in the F.T. pairs

$$\delta(t) \Leftrightarrow 1$$

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

□

In terms of the alternative notation $F(f)$ (in Hz), we have that if

$$g(t) \Leftrightarrow G(f)$$

then

$$G(t) \Leftrightarrow g(-f)$$

Scaling Property We have seen this for Laplace transforms: If

$$f(t) \Leftrightarrow F(\omega)$$

then

$$f(at) \Leftrightarrow \frac{1}{|a|} F(\omega/a)$$

Proof By student... □

Discuss in terms of time and bandwidth: when a signal is expanded in time, it is compressed in frequency, and vice versa. We cannot be simultaneously short in time and short in frequency. This is the basis for the famed **Heisenberg uncertainty principle**.

Time-Shift Property If

$$f(t) \Leftrightarrow F(\omega)$$

then

$$f(t - t_0) \Leftrightarrow F(\omega)e^{-j\omega t_0}$$

In other words, a shift in time corresponds to a change in phase in the F.T.

Proof By student... □

What is interesting is that higher frequencies experience a greater phase shift than lower frequencies. A plot of the phase is linear in frequency.

Example 11 Find the F.T. of $f(t) = e^{-|t-t_0|}$. From the table we see that

$$\mathcal{F}[e^{-|t|}] = \frac{2a}{a^2 + \omega^2}$$

So

$$\mathcal{F}[e^{-|t-t_0|}] = \frac{2a}{a^2 + \omega^2} e^{-j\omega t_0}$$

□

Frequency-shift Property This innocuous-looking property forms a basis for every radio and TV transmitter in the world! It simply states that if

$$f(t) \Leftrightarrow F(\omega)$$

then

$$f(t)e^{j\omega_0 t} \Leftrightarrow F(\omega - \omega_0)$$

Note that this is a dual of the time-shift property.

Proof By student... □

Show a block diagram of what happens. Since $e^{j\omega_0 t}$ is a complex signal, this is really only a mathematical description. But we can multiply by $\cos(\omega_0 t)$. Using linearity we get

$$f(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}[F(\omega - \omega_0) + F(\omega + \omega_0)]$$

What we get out is two images in the frequency domain, at positive and negative frequencies.

As an application, suppose that $f(t)$ is some information-bearing signal, say, the signal from a microphone. Plot. Now multiply by $\cos(\omega_0 t)$ and plot the

result, and show the effect in frequency. This is what KSL does! This kind of modulation is called **amplitude modulation**.

Observe that we can modulate signals onto a variety of different frequencies. This makes it possible to have several radio (or TV) stations. The higher frequencies also propagate further through the air than the baseband frequencies.

Convolution Property If

$$f_1(t) \Leftrightarrow F_1(\omega) \quad \text{and} \quad f_2(t) \Leftrightarrow F_2(\omega)$$

then

$$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega)F_2(\omega)$$

(where $*$ is convolution) and

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

Proof By student...   □

Example 12 Show that

$$\int_{-\infty}^{t} f(x)dx \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

Notice that

$$f(t) * u(t) = \int_{-\infty}^{t} f(x)dx$$

Then using $\mathcal{F}[u(t)] = 1/j\omega + \pi\delta(\omega)$ the result follows.   □

Time Differentiation If

$$f(t) \Leftrightarrow F(\omega)$$

then

$$\frac{df}{dt} \Leftrightarrow j\omega F(\omega)$$

Proof By student...   □

Time Integration We saw above that

$$\int_{-\infty}^{t} f(x)dx \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

If f is zero mean (i.e. $F(0) = 0$) then

$$\int_{-\infty}^{t} f(x)dx \Leftrightarrow \frac{F(\omega)}{j\omega}$$

Example 13 Use the time differentiation property to find the F.T. of $f(t) = \Delta(t/T)$. Note that

$$\frac{d^2 f}{dt^2} = \frac{2}{T}[\delta(t + T/2) - 2\delta(t) + \delta(t - T/2)].$$

Then the differentiation property states that

$$\frac{d^2 f}{dt^2} \Leftrightarrow -\omega^2 F(\omega)$$

Using $\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}$ we get

$$-\omega^2 F(\omega) = \frac{2}{T}[e^{j\omega T/2} - 2 + e^{-j\omega T/2}] = \frac{4}{T}[\cos(\omega T/2) - 1] = -\frac{8}{T}\sin^2(\omega T/4)$$

So we get

$$F(\omega) = \frac{8}{\omega^2 T}\sin^2(\omega T/4) = T/2\operatorname{sinc}^2(\omega T/4)$$

$\square$

Example 14 Find the F.T. of $f(t) = |t|$. This is hard to do by other techniques. But notice that

$$\frac{df}{dt} = \operatorname{sgn}(t).$$

So

$$j\omega F(\omega) = \mathcal{F}[\operatorname{sgn}(t)] = \frac{2}{j\omega}$$

$$F(\omega) = \frac{-2}{\omega^2}$$

$\square$

System Analysis using Fourier Transforms

We have seen that we can use Laplace transforms for analysis of LTIC systems. We can do similar things with Fourier transforms, except that we will be able to obtain only steady-state responses, not the transient response. For the differential equation

$$Q(D)y(t) = P(D)f(t)$$

we can use the time-differentiation property of the F.T.:

$$D^k y(t) \Leftrightarrow (j\omega)^k Y(\omega)$$

so that we can write

$$Q(j\omega)Y(\omega) = P(j\omega)F(\omega)$$

or

$$Y(\omega) = H(\omega)F(\omega),$$

where

$$H(\omega) = \frac{P(j\omega)}{Q(j\omega)}.$$

The function $H(\omega)$ is the transfer function of the system, and is related to the transfer function we saw for Laplace transforms.

Analysis with the F.T. is not generally preferable to analysis with the L.T., because the system must always be asymptotically stable and we have to lug around $j\omega$, instead of the more compact s. However, for systems or signals that are non-causal and Fourier transformable, the analysis may be easier using F.T.

Example 15 Let

$$H(s) = \frac{1}{s + 2}$$

and let $f(t) = u(t)$. Find the zero-state response $y(t)$. We have

$$H(\omega) = \frac{1}{j\omega + 2}$$

and

$$F(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

(Notice that the F.T. is more complicated than the L.T.) Then

$$
\begin{aligned}
Y(\omega) &= \frac{1}{j\omega + 2}(\pi\delta(\omega) + \frac{1}{j\omega}) \\
&= \frac{\pi\delta(\omega)}{2} + \frac{1}{j\omega(j\omega + 2)} \\
&= \frac{1}{2}\left\{\pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{j\omega + 2}\right\}
\end{aligned}
$$

The inverse F.T. is

$$y(t) = \frac{1}{2}(1 - e^{-2t})u(t)$$

□

Example 16 For the same system, find the response when

$$f(t) = e^t u(-t)$$

Note that this is *non-causal*, and hence could not be analyzed using the (single-sided) Laplace transform. We have (from the table)

$$F(\omega) = \frac{-1}{j\omega - 1}$$

$$Y(\omega) = H(\omega)F(\omega) = \frac{-1}{(j\omega - 1)(j\omega + 2)}$$

Doing the PFE, we obtain

$$y(t) = \frac{1}{3}[e^t u(-t) + e^{-2t}u(t)]$$

□

Signal distortion during transmission

Fourier transforms form a natural way of looking at the frequency response of filters, since they examine the response due to a a trigonometric exponential. We have already seen how to characterize a filter by its frequency response, using the Laplace transform. We can do the same thing (somewhat more naturally) using Fourier transforms.

The output of a signal with transfer function $H(\omega)$ is

$$Y(\omega) = H(\omega)F(\omega)$$

Let us express this in polar form:

$$|Y(\omega)|e^{j\angle Y(\omega)} = |F(\omega)||H(\omega)|e^{j(\angle F(\omega) + \angle H(\omega))}$$

The magnitude response of the output is therefore

$$|Y(\omega)| = |F(\omega)||H(\omega)|,$$

so the magnitude response is the input magnitude response "shaped" by the system magnitude response. In a word, the output is *filtered*. We can also consider the phase response:

$$\angle Y(\omega) = \angle F(\omega) + \angle H(\omega)$$

The phase angle of the output is the phase angle of the input plus whatever phase the system contributes.

There are applications in which we would like the system to not change the input signal as it passes through the system, other than to delay the signal and possibly uniformly attenuating the whole signal. For example, the signals that pass through the air to our radios: we would hope that they would arrive at the radio in the same form as they left the transmitter, except delayed by the propagation through the air and attenuated. A system which does not distort a signal (other than delaying it and possibly changing the overall gain) is said to be a *distortionless system*. The output of a distortionless system may be written as

$$y(t) = kf(t - t_d)$$

where k is the overall gain and t_d is the delay introduced by the system. In terms of Fourier transforms, we have

$$Y(\omega) = kF(\omega)e^{-j\omega t_d}$$

So the transfer function of a distortionless system is

$$H(\omega) = ke^{-j\omega t_d}$$

Observe that the magnitude does not change with frequency, and the phase is a linear function of ω:

$$|H(\omega)| = k$$
$$\angle H(\omega) = -\omega t_d$$

In practice, distortionless transmission is never exactly obtained over all frequencies. Over a localized region of frequency, however, the magnitude response may be flat enough, and the phase response may be linear enough, so the distortion is acceptable. For any system, including those with nonlinear phase, we can compute the time delay of a signal as it passes through a system by

$$t_d(\omega) = -\frac{d}{d\omega}\angle H(\omega)$$

Example 17 Examine the distortion of the system with frequency response

$$H(\omega) = \frac{1}{a + j\omega}$$

The magnitude response is

$$|H(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

Plot the magnitude response. Clearly there is magnitude distortion. The phase is

$$\angle H(\omega) = -\tan^{-1}\frac{\omega}{a}$$

Then

$$t_d(\omega) = -\frac{d}{d\omega}\angle H(\omega) = \frac{1}{a(1 + (\omega/a)^2)}$$

Plot. Note that the delay is a function of frequency. □

Ideal and Practical Filters

The ideal low-pass filter allows all signals below some cutoff of W rad/sec to pass through undistorted, while completely cutting off all other frequencies. This is sometimes said to be the ideal "brick wall" filter. If $f(t)$ is strictly bandlimited to W rad/sec, then the output is simply delayed:

$$y(t) = f(t - t_d)$$

The ideal filter has

$$|H(\omega)| = \text{rect}(\omega/2W) \qquad \angle H(\omega) = e^{-j\omega t_d}$$

The impulse response of the filter is then

$$h(t) = \mathcal{F}^{-1}\left[\text{rect}(\omega/(2W))e^{-j\omega t_d}\right] = \frac{W}{\pi}\text{sinc}(W(t - t_d))$$

Plot $h(t)$. Observe that it is *noncausal*, and hence physically nonrealizable. We cannot build an ideal lowpass filter, or an ideal bandpass either, for that matter. We could get pretty close, but creating a system with a large t_d, so that most of the sinc function would be included. One approach to determine the effect of this truncation of the impulse response

$$\hat{h}(t) = h(t)u(t)$$

is to use the properties of F.T.

$$\hat{H}(\omega) = \frac{1}{2\pi}H(\omega) * U(\omega) = \frac{1}{2\pi}H(\omega) * \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

The effect is to smear the response by the convolution.

Signal Energy and Parseval's formula

If we assume for definiteness a unit load (a 1 Ω resistor), then if $f(t)$ is the voltage across the resistor or the current through the resistor, then the energy dissipated is

$$E_f = \int_{-\infty}^{\infty} f^2(t)dt$$

If this quantity is finite, then $f(t)$ is said to be an **energy signal**. Not every signal that we consider analytically is an energy signal: for example, $f(t) = \cos(\omega_o t)$ is not an energy signal.

Substituting in for the $f(t)$ in terms of the inverse F.T.,

$$E_f = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\int_{-\infty}^{\infty} f(t)e^{j\omega t}dtd\omega$$

So we can write

$$E_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F(-\omega)d\omega$$

For a real $f(t)$, the F.T. satisfies $F(\omega) = F^*(-\omega)$ (they are conjugates), so that

$$E_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

This is **Parseval's theorem** for Fourier Transforms. It can be written using inner product notation as

$$\langle f(t), f(t) \rangle = \frac{1}{2\pi} \langle F(\omega), F(\omega) \rangle.$$

It can also be generalized for products of different functions as

$$\langle f(t), g(t) \rangle = \frac{1}{2\pi} \langle F(\omega), G(\omega) \rangle.$$

We can think of an increment of energy lying in an increment of bandwidth:

$$\Delta E_f = \frac{1}{2\pi} |F(\omega)|^2 \Delta\omega$$

To get the total energy, add up all of the pieces. The function $|F(\omega)|^2$ is therefore referred to as the **energy spectral density** of the function: it tells where the energy is distributed in frequency.

The energy which is passed by a bandpass filter with cutoff frequencies ω_1 and ω_2 is

$$E_f(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

Example 18 Find the energy in the signal $f(t) = e^{-at} u(t)$, and find the bandwidth W such that 95% of the energy is contained in frequencies below W.

$$E_f = \int f^2(t) dt = \int_0^\infty e^{-2at} dt = \frac{1}{2a}$$

The F.T. is

$$F(\omega) = \frac{1}{j\omega + a}$$

We therefore have the identity

$$E_f = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{2a}.$$

This can actually be verified by integrating:

$$E_f = \frac{1}{\pi} \int_0^\infty \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{\pi a} \tan^{-1} \frac{\omega}{a} \Big|_0^\infty = \frac{1}{2a}$$

To find the 95% containment bandwidth, we solve

$$\frac{0.95}{2a} = \frac{1}{\pi} \int_0^W \frac{d\omega}{\omega^2 + a^2}$$

for W. then

$$\frac{0.95}{2a} = \frac{1}{\pi a} \tan^{-1} \frac{\omega}{a} \Big|_0^W.$$

We get

$$\frac{0.95}{2} = \tan^{-1} \frac{W}{a}$$

so $W = 12.706$ rad. $\qquad \square$

Some observations:

1. We can use the Parseval's theorem to integrate some messy functions by simply applying the theorem. For example, we can compute

$$I = \int_0^\infty \frac{\sin^2(x)}{x^2} dx$$

   by writing

$$I = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin^2(x)}{x^2} dx = \frac{1}{4\pi} \int_{-\infty}^\infty (\pi \operatorname{rect}(\omega/2))^2 d\omega = \frac{\pi}{4} \int_{-1}^1 1 d\omega = \frac{\pi}{2}$$

   The previous example also demonstrated how we can sometimes do easier integrals in one domain than the other.

2. We can also use the idea of the last example to introduce the idea of an **essential bandwidth**: include enough bandwidth that most of the energy of the signal can get through. Often the 95% or 99% level is sufficient.