Technical Notes

Far Field Predicted by Piston Theory

Earl H. Dowell*
Duke University, Durham, North Carolina 27708

DOI: 10.2514/1.J054460

Nomenclature

a_∞	=	free-stream speed of sound
c	=	airfoil chord
f	=	frequency, Hz
M	=	Mach number, which is equal to U_∞/a_∞
P, p	=	transformed amplitude of pressure, pressure or Laplace transform variable
U_∞	=	freestream velocity
W, w	=	transformed amplitude of downwash, downwash
ω	=	frequency, rad/s
ρ_∞	=	free-stream fluid density

I. Introduction

Piston theory is an unsteady aerodynamic model that is widely used in hypersonic and supersonic flows to predict the pressure on an oscillating airfoil or wing. Since the seminal paper of Ashley and Zartarian [1] (following the work of Lighthill [2]), it has indeed become an effective tool for the aeroelastician as those authors anticipated. Here the far field is considered.

Piston theory is usually thought of as a high supersonic flow theory in that it can be derived as the limit of potential flow theory as the Mach number M becomes large. However, it can also be found as the limit as the frequency of oscillation becomes large. Indeed, formally, piston theory is the limit as the frequency becomes large at all Mach numbers if one starts from classical potential flow theory. However, this limit is usually of less interest to the aeroelastician because this means that the reduced frequency, that is, the dimensional frequency times a characteristic length (e.g., airfoil or wing chord) divided by the speed of sound, must be large compared to one. Note that strictly speaking in this limit the Mach number is held fixed and indeed formally the piston theory is correct at any M if the reduced frequency is sufficiently large.

In this Note the high frequency limit is considered again because in aeroacoustics, for example, the noise contributed by the vibration of panels on an airframe, the reduced frequency may indeed be large. For example,

$$a = 343\ \mathrm{m/s}$$

$$c = 1\ \mathrm{m}$$

$$f = 1000\ \mathrm{Hz}$$

$$\frac{2\pi f c}{a} \cong 20$$

where a is the speed of sound at sea level, c is the airfoil or wing chord, and f is the frequency in hertz.

However, in noise radiation the interest is in the far field rather than in the pressure on the oscillating wing per se and this is our interest here. As it turns out, within the piston theory model, one can construct a model for all points off the airfoil or wing and thus one does not need to make a far-field approximation per se.

Also, one of the interesting outcomes of the analysis to follow is that, to the lowest order and to the next lowest order, the piston theory model is the same whether one takes the limit of large Mach number or large reduced frequency. Of course, this has been known to be true for the lowest order result for piston theory prediction for pressures on the airfoil or wing for some time [3,4]. Moreover, for the pressure on the airfoil or wing, the first-order correction is zero in both cases and the differences between the large Mach number and large frequency limits appear only in the second-order terms. This is still true for the "far-field" result, but now the first-order corrections to the classical piston theory in the far field are not zero in either limit, but fortunately they are the same, which may be important when using the theory in practice. Again for second-order and higher-order terms, there are some differences and those can be worked out using methods similar to those to be described below.

II. Analysis of the Far Field

It is suggested that the reader consult [3,4] before reading on, because the following starts from those results and generalizes them.

Now consider the relationship between pressure and downwash as derived from classical unsteady small perturbation potential flow theory, after simple harmonic motion in time has been assumed and a Laplace transform has been taken with respect to the streamwise spatial variable. It is noted that the assumption of simple harmonic motion may also be considered as taking a Fourier transform with respect to time. Also, to generalize the results from the limit of large reduced frequency, we could take a Fourier transform with respect to the streamwise spatial coordinate. The final results are the same

$$P = \rho_\infty (i\omega + \cup_\infty\ p) \frac{W}{\mu} e^{-\mu z} \tag{1}$$

Here $z > 0$ is considered. A similar analysis is applicable to $z < 0$, or symmetry (and antisymmetry) considerations may be invoked.

Also, if $z = 0$, the previous results for the pressure on the airfoil are obtained [3,4].

Now as in [4] we wish to expand μ and $1/\mu$ in terms of either an expansion in $1/M$ as M becomes large for fixed reduced frequency or in terms of $\omega c/a$ as reduced frequency becomes large for fixed M.

Consider these two limits in turn: M becomes large: $M = U_\infty/a_\infty \gg 1$.

As before, the result for $1/\mu$ is as follows:

$$\frac{1}{\mu} = \frac{[p + (i\omega/U)]^{-1}}{M}\left[1 + 0\left(\frac{1}{M^2}\right)\right] \tag{2}$$

A similar result may be obtained for μ as follows:

$$\mu = Mp\left[1 + \frac{i\omega}{a_\infty}/(Mp) + 0(1/M^2)\right] \tag{3}$$

Placing Eqs. (2) and (3) in Eq. (1), one has

$$P = \rho_\infty a_\infty W e^{-Mpz} e^{-i\omega z/a_\infty} \tag{4}$$

Received 8 May 2015; accepted for publication 5 June 2015; published online 18 August 2015.

*Professor, Mechanical Engineering, P.O. Box 90300. Grade Honorary Fellow AIAA.

Inverting from the Laplace domain back to the physical domain, one has, where we have restored the harmonic time dependence explicitly:

$$p(x,z,t) = \bar{p}e^{i\omega t} = \rho_\infty a_\infty \bar{w}(x - Mz)e^{i\omega(t - z/a_\infty)} \tag{5}$$

Now thinking of the harmonic motion as really having taken a Fourier transform with respect to time, one may invert from the frequency domain to the time domain to obtain the following final result:

$$p(x,z,t) = \rho_\infty a_\infty w\left(x - Mz, t - \frac{z}{a_\infty}\right) \tag{6}$$

The physical meaning of this result is perhaps obvious but, in any event, it will be discussed later after the same results will be obtained as the limit of high reduced frequency.

Reduced frequency becomes large: $\omega(c/a_\infty) \gg 1$.

Again, we need expansion for large reduced frequency for μ and $1/\mu$. They are as follows:

$$\frac{1}{\mu} = \frac{a_\infty}{i\omega}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{2Mipa_\infty}{\omega}\right) + 0\left(\frac{\omega}{a_\infty}\right)^{-2}\right] \tag{7}$$

$$\mu = \frac{i\omega}{a_\infty}\left[1 + \frac{Mp}{i\omega/a_\infty} + 0\left(\frac{\omega}{a_\infty}\right)^{-2}\right] \tag{8}$$

Using Eqs. (7) and (8) in Eq. (1), one has the following (note that there are some cancellations in the algebra):

$$P = \rho_\infty a_\infty W e^{-[Mp + i\omega/a_\infty]z} + \mathrm{HOT} \tag{9}$$

and doing the inversions as before, one has again Eq. (6).

III. Physical Meaning of the Result and Its Use in Practice

Note that the vertical distance from the airfoil or wing, z, does not appear independently but only as a modification to the streamwise coordinate and time. In supersonic flow, the former is clearly a high Mach number approximation to Mach waves and the latter is a time delay (at any Mach number) to the propagation in the fluid of pressure waves at the speed of sound.

An interesting question is whether this result can be used at lower Mach numbers if the reduced frequency is high enough. Formally, the answer is yes, but in practice only comparisons with results from the full potential flow model will demonstrate the utility of this result at lower Mach numbers. Certainly, one would expect the piston theory result to be in better agreement with the full potential flow model the higher the Mach number.

This also raises the question of whether something like piston theory might be used as an approximation when the flow is modeled by the Euler or even the Navier–Stokes equations. Several authors have suggested that replacing the free stream values for flow density and speed of sound from experiment or computational fluid dynamics computations is a reasonable approximation and, to the extent that it is, one might be tempted to modify this "far-field" result as well.

However, a more rigorous justification for making such approximations awaits further work.

Finally, a comment about how this result might prove useful: If one is doing an aeroacoustic computation of the far field, the higher the frequency the more challenging the computations from classic potential flow theory (or some other computational model) become. So one might use piston theory as a check on the computational results at some selected high frequency and, if indeed they do agree, then the more complex computations need not be continued to yet higher frequencies.

IV. Piston Theory at All Mach Numbers

It may be of interest to recall that indeed piston theory correlates well with the full potential flow theory at all Mach numbers when the frequency is very high or equivalently (given the duality between large frequency and small times) the time is very small. For definiteness, consider an airfoil or wing that is subjected to a step change in angle of attack and the lift that is thereby generated. Setting $z = 0$ and recognizing that the total pressure differential across the lifting surface is twice that given by Eq. (6), the ratio of lift coefficient to angle of attack is given by the following equation:

$$\frac{C_L}{\alpha} = \frac{4}{M} \tag{10}$$

This well-known formula is often used for a high supersonic Mach number, but in fact piston theory is accurate for short times, that is, when ta_∞/c is much less than 1 at all M. Note in particular that Eq. (10) gives a finite result at any M not equal to zero, but becomes infinite when $M = 0$. This is the well-known singularity associated with the incompressible flow model and shows explicitly that the incompressible flow models fail at short times or large frequencies. Equation (10) may be further checked by comparison to the results of Lomax [5] for transient lift due to a step change in angle of attack using the full potential flow model. The results of Lomax [5] are presented in [3] and are perhaps most readily available there; see especially Fig. 4.11 and the associated text.

V. Expansion in $1/M$ to Next Highest Order

If one carries out the expansion to the next highest order, then the results are as follows:

$$\mu = Mp\left\{1 - (1/2)(1/M^2) + \frac{i\omega}{a_\infty}\frac{1}{Mp}\right\} \tag{11}$$

and

$$\frac{1}{\mu} = \left[\frac{p + (i\omega/U_\infty)}{M}\right]^{-1}\left[1 + \frac{1}{2}\frac{p^2}{[p + (i\omega/U_\infty)]^2}\frac{1}{M^2}\right] \tag{12}$$

Now note that, to second order,

$$\beta \equiv (M^2 - 1)^{1/2} = \left[1 - \frac{1/2}{M^2}\right]M \tag{13}$$

Thus, to second order the results may alternatively be written as follows:

$$P = \rho_\infty a_\infty\left[1 + \frac{1}{2}\frac{p^2}{[p + (i\omega/U_\infty)]^2}\frac{1}{M^2}\right]e^{(-p\beta - i\omega/a_\infty)z} \tag{14}$$

With the further assumption that $\omega/U_\infty p \ll 1$, Eq. (14) may be written in a particularly attractive form as follows:

$$P = \rho_\infty a_\infty\left[\left(\frac{1}{\beta}\right) + \frac{i\omega}{a_\infty p}\right]We^{(-p\beta - i\omega/a_\infty)z} \tag{15}$$

Recognizing the essential role that β plays in the full supersonic potential flow theory, one is led to speculate that the above is the preferred form of the second-order model and that it might indeed be more accurate than one might usually expect a strictly second-order theory should be. The inversion of Eq. (14) or (15) is straightforward. Note that $\omega/U_\infty p = (\omega/a_\infty p)(1/M)$, and thus, for fixed ω, this term will always be small (and strictly of higher order) as $M \to \infty$. However, for aeroacoustics ω may be large and so we prefer to retain the terms in Eqs. (14) and (15) as shown.

Finally then, the inversion of Eq. (15) is as follows:

$$p = \rho_\infty a_\infty \left[\frac{w}{\beta}(x - \beta z, t - z/a_\infty) + \int_0^{x-\beta z} \frac{\dot{w}}{a_\infty}(\xi, t - z/a_\infty)\,\mathrm{d}\xi \right] \tag{16}$$

VI. Conclusions

Piston theory has been extended to predict the far field as well as the near field to higher orders and it has been noted that, aside from its intrinsic interest, this extension may have some utility in aeroacoustics.

References

[1] Ashley, H., and Zartarian, G., "Piston Theory: A New Aerodynamic Tool for the Aeroelastician," *Journal of the Aeronautical Sciences*, Vol. 23, No. 12, 1956, pp. 1109–1118. doi:10.2514/8.3740

[2] Lighthill, M. J., "Oscillating Airfoils at High Mach Number," *Journal of the Aeronautical Sciences*, Vol. 20, No. 6, 1953, pp. 402–406. doi:10.2514/8.2657

[3] Dowell, E. H., Clark, R. L., Cox, D., Curtiss, H. C., Jr., Edwards, J. W., Hall, K. C., Peters, D. A., Scanlan, R. H., Simiu, E., Siso, F., Strganac, T. W., and Tang, D., *A Modern Course in Aeroelasticity*, 5th ed., Springer, Switzerland, 2015, pp. 176–187, 230–235.

[4] Dowell, E. H., and Bliss, D. B., "New Look at Piston Unsteady Supersonic Potential Flow Aerodynamics and Piston Theory," *AIAA Journal*, Vol. 51, No. 9, 2013, pp. 2278–2281. doi:10.2514/1.J052088

[5] Lomax, H., "Indicial Aerodynamics," Part II, *AGARD Manual on Aeroelasticity*, NATO, Cedex, France, 1960, Chap. 6.

C. Cesnik
Associate Editor