Digital Communication Assignment

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Contents

1 Axioms

1. A and B are events such that $\Pr(A) = 0.42, \Pr(B) = 0.48$ and $\Pr(\text{A and B}) = 0.16$. Determine
   (a) $\Pr(\text{not A})$
   (b) $\Pr(\text{not B})$
   (c) $\Pr(\text{A or B})$

   Solution:

   (a) $\Pr(\text{not A})$

$$\Pr(A') = 1 - \Pr(A) \tag{1.1.1}$$
$$= 1 - 0.42 \tag{1.1.2}$$
$$= 0.58 \tag{1.1.3}$$

   (b) $\Pr(\text{not B})$

$$\Pr(B') = 1 - \Pr(B) \tag{1.1.4}$$
$$= 1 - 0.48 \tag{1.1.5}$$
$$= 0.52 \tag{1.1.6}$$

(c) Pr (A or B)

$$\Pr(\text{A+B}) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.1.7}$$
$$= 0.42 + 0.48 - 0.16 \tag{1.1.8}$$
$$= 0.74 \tag{1.1.9}$$

2. Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

   (a) mutually exclusive

   (b) independent
   Solution:

   i. In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{1.2.10}$$
$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{1.2.11}$$
$$\therefore p = \frac{1}{10} \tag{1.2.12}$$

   ii. Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.2.13}$$
$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{1.2.14}$$
$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{1.2.15}$$
$$\therefore p = \frac{1}{5} \tag{1.2.16}$$

2 Conditional Probability

1. Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.
   Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{2.1.1}$$
$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{2.1.2}$$

2. Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.
   Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{2.2.3}$$

3 Two Dice

3.1 Sum of Independent Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

3.1.1. *The Uniform distribution* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{3.3.1.0.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{3.3.1.0.2}$$
$$\implies X \in \{1, 2, \dots, 12\} \tag{3.3.1.0.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{3.3.1.0.4}$$

SOLUTION: the following python code is available for proof.

/codes/chapter3/dice_1.py

3.1.2. *Convolution:* From (3.3.1.0.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{3.3.1.0.5}$$
$$= \sum_k \Pr(X_1 = n - k | X_2 = k) \, p_{X_2}(k) \tag{3.3.1.0.6}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k)$$
$$= \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{3.3.1.0.7}$$

From (3.3.1.0.6) and (3.3.1.0.7),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{3.3.1.0.8}$$

where $*$ denotes the convolution operation. Substituting from (3.3.1.0.1) in (3.3.1.0.8),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^{6} p_{X_1}(n - k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{3.3.1.0.9}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{3.3.1.0.10}$$

From (3.3.1.0.9),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6} \sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n - 1 \le 6 \\ \frac{1}{6} \sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n - 6 \le 6 \\ 0 & n > 12 \end{cases} \tag{3.3.1.0.11}$$

Substituting from (3.3.1.0.1) in (3.3.1.0.11),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{3.3.1.0.12}$$

satisfying (3.3.1.0.4).

SOLUTION: the following python code is available for proof.

/codes/chapter3/1_2_dice_conv.py

3.1.3. *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{3.3.1.0.13}$$

From (3.3.1.0.1) and (3.3.1.0.13),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{3.3.1.0.14}$$

$$= \frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}, |z| > 1 \tag{3.3.1.0.15}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{3.3.1.0.16}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{3.3.1.0.17}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (3.3.1.0.15) and (3.3.1.0.17),

$$P_X(z) = \left\{\frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}\right\}^2 \tag{3.3.1.0.18}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{3.3.1.0.19}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{3.3.1.0.20}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{3.3.1.0.21}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{3.3.1.0.22}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{3.3.1.0.23}$$

From (3.3.1.0.13), (3.3.1.0.19) and (3.3.1.0.22)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{3.3.1.0.24}$$

which is the same as (4.4.1.1.2). Note that (4.4.1.1.2) can be obtained from (3.3.1.0.22) using contour integration as well.

SOLUTION: the following python code is available for proof.

/codes/chapter3/1_3_dice.py

3.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 3.1.1. The theoretical pmf obtained in (4.4.1.1.2) is plotted for comparison.

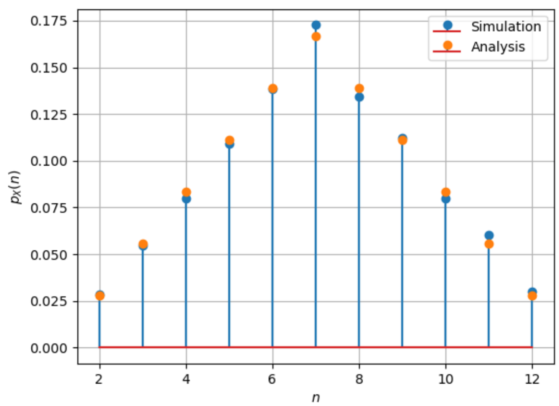

Figure 3.1.1: Plot of $p_X(n)$. Simulations are close to the analysis.

3.1.5. The python code is available in

/codes/chapter3/1_5_dice.py

4 Random Variables

4.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

4.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

/codes/chapter4/exrand.c

/codes/chapter4/coeffs.h

4.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{4.4.1.0.1}$$

Solution: The following code plot in Fig. 4.1.1

/codes/chapter4/uni.dat

/codes/chapter4/cdf_plot.py

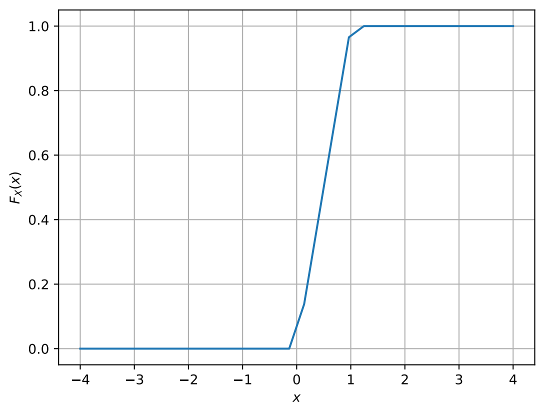

Figure 4.1.1: The CDF of U

4.1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b \end{cases} \tag{4.4.1.1.2}$$

were, a=0 and b=1.

4.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{4.4.1.1.3}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{4.4.1.1.4}$$

Write a C program to find the mean and variance of U.
Solution:

the following C program is used to find mean and variance of uniform distribution .

/codes/chapter4/uni.dat

/codes/chapter4/4_1_4_rand.c

4.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4.4.1.1.5}$$

Solution:

$$E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{4.4.1.1.6}$$

$$E\left[X^2\right] - E^2[X] = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu^2 \tag{4.4.1.1.7}$$

where μ is the mean and σ is variance of random variable. Substituting the CDF of U from (4.4.1.1.2) in (4.4.1.1.6) and (4.4.1.1.7), we get

$$E[X] = \mu = \frac{1}{2} \tag{4.4.1.1.8}$$

$$\sigma_U^2 = \frac{1}{12} \tag{4.4.1.1.9}$$

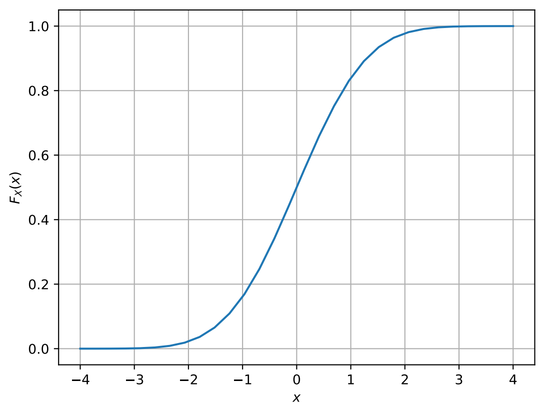

Figure 4.2.1: The CDF of X

the following python code is used to verify mean and variance of uniform distribution using (4.4.1.1.5)

Simulation results:
mean $\mu = \frac{1}{2}$
variance $\sigma^2 = \frac{1}{12}$

/codes/chapter4/4_1_5.py

4.2 Central Limit Theorem

4.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{4.4.2.0.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called Gaussian.dat
Solution: Download the following files and execute the C program.

/codes/chapter4/random_var.c

/codes/chapter4/coeffs.h

4.2.2 Load Gaussian.dat in python and plot the empirical CDF of X using the samples in Gaussian.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig.4.2.1

/codes/chapter4/cdf_plot_clt.py

Properties:

- CDF is non-decreasing function.
- Maximum value of CDF $F(+\infty) = 1$.
- Minimum value of CDF $F(-\infty) = 0$.

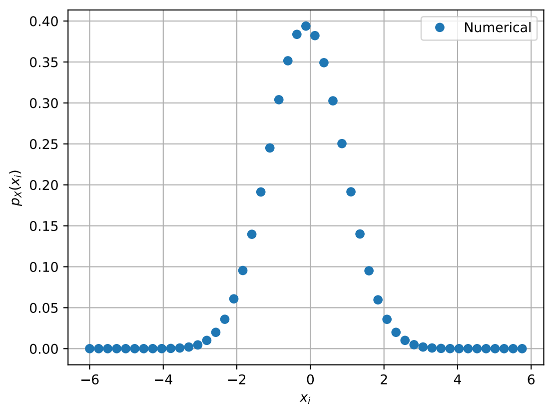

Figure 4.2.2: The PDF of X

4.2.3 Load Gaussian.dat in python and plot the empirical PDF of X using the samples in Gaussian.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{4.4.2.1.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 4.2.2 using the code below

/codes/chapter4/pdf_plot_clt.py

Properties :

- Mean,median and mode are equal.
- The curve is bell-shaped and symmetric about mean.
- Area under the curve=1.

4.2.4 Find the mean and variance of X by writing a C program.
Solution:

/codes/chapter4/random_var_.c

/codes/chapter4/coeffs.h

4.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{4.4.2.2.3}$$

repeat the above exercise theoretically.
Solution: Mean$(\mu) = E[X]$

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \tag{4.4.2.2.4}$$

$$= 0 \tag{4.4.2.2.5}$$

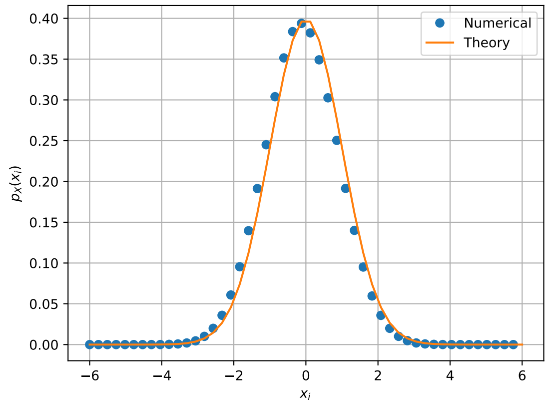

Figure 4.2.3: The PDF of $p(x)$

$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \quad (even function) \tag{4.4.2.2.6}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{4.4.2.2.7}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2u} e^{-u} du \quad \left(Let \frac{x^2}{2} = u\right) \tag{4.4.2.2.8}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u} u^{\frac{3}{2}-1} du \tag{4.4.2.2.9}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{4.4.2.2.10}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \tag{4.4.2.2.11}$$

$$= 1 \tag{4.4.2.2.12}$$

where

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{4.4.2.2.13}$$

Thus, the variance is

$$\sigma^2 = E\left(X\right)^2 - E^2\left(X\right) = 1 \tag{4.4.2.2.14}$$

theoretical and numerical plots as shown in fig.4.2.3

/codes/chapter4/pdf_plot_clt.py

4.3 From Uniform to Other

4.3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{4.4.3.0.1}$$

and plot its CDF.
Solution: The CDF plot of V is in fig.4.3.1

/codes/chapter4/uniform_var_.c

/codes/chapter4/coeffs.h

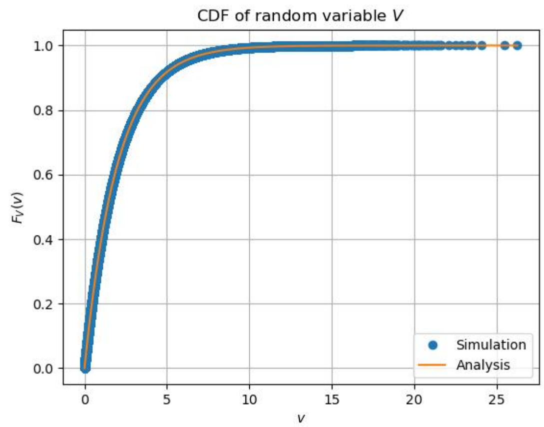

Figure 4.3.1: The CDF of V

/codes/chapter4/uni.dat

/codes/chapter4/cdf_plot_uni.py

4.3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$
\begin{aligned}
F_V(x) &= \Pr(V \le x) \\
&= \Pr(-2\ln(1-U) \le x) \\
&= \Pr\left((1-U) \le e^{\frac{-x}{2}}\right) \\
F_U(x) &= \Pr\left(U \le 1 - e^{\frac{-x}{2}}\right)
\end{aligned}
$$

/codes/chapter4/cdf_plot_uni.py

4.4 Triangular Distribution

4.4.1 Generate

$$T = U_1 + U_2 \tag{4.4.4.0.1}$$

Solution:

/codes/chapter4/tri.c

/codes/chapter4/coeffs.h

/codes/chapter4/uni1.dat

/codes/chapter4/uni2.dat

4.4.2 Find the CDF of T.

Solution: For following CDF python code plot is as shown in Fig.4.4.1

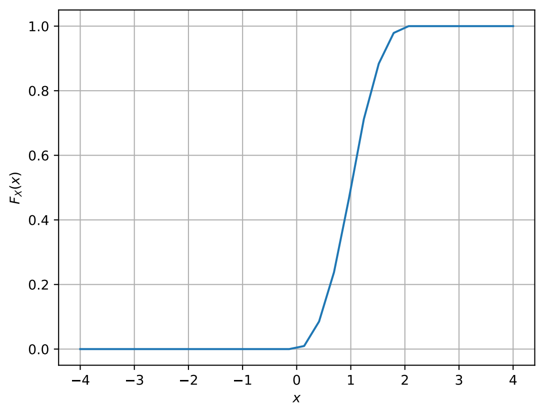

Figure 4.4.1: The CDF of T

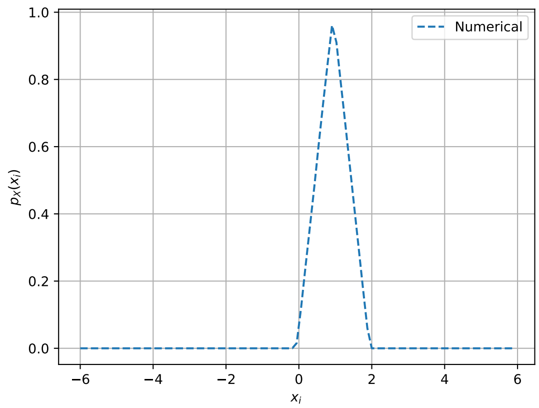

Figure 4.4.2: The CDF of T

/codes/chapter4/T.dat

/codes/chapter4/cdf_plot_tri.py

4.4.3 Find the PDF of T.
Solution: : The following PDF python code is to plot Fig.4.4.2

/codes/chapter4/T.dat

/codes/chapter4/pdf_plot_tri.py

4.4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: CDF $F_U(x)$ is

$$F_T(x) = \begin{cases} 0 & x \le a \\ \frac{(x-a)^2}{(b-a)(c-a)} & a < x \le c \\ 1 - \frac{(b-x)^2}{(b-a)(c-a)} & c < x \le b \\ 1 & x > b \end{cases} \tag{4.4.4.2.2}$$

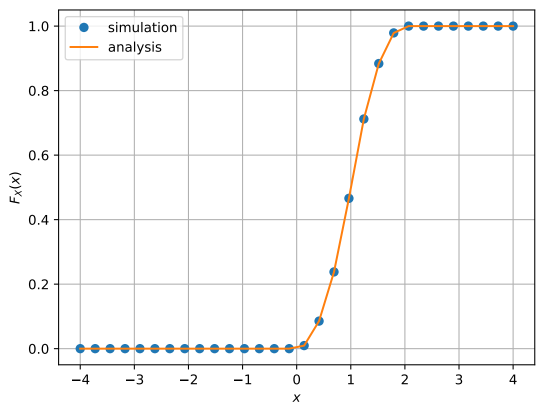

Figure 4.4.3: The CDF of T

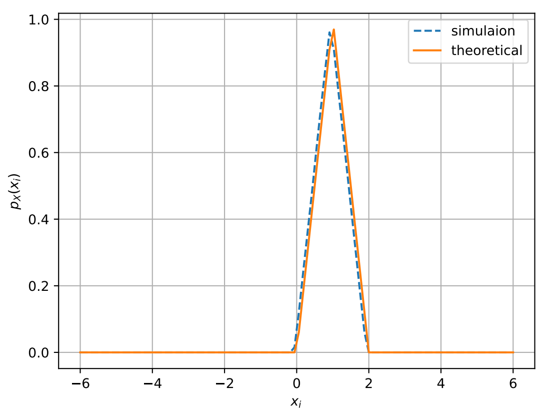

Figure 4.4.4: The PDF of T

PDF $p_T(x)$

$$p_T(x) = \frac{d}{dx} F_T(x) \tag{4.4.4.2.3}$$

$$p_T(x) = \begin{cases} 0 & x \leq a \\ \frac{2(x-a)}{(b-a)(c-a)} & a < x \leq c \\ \frac{2(b-x)}{(b-a)(c-a)} & c < x \leq b \\ 0 & x > b \end{cases} \tag{4.4.4.2.4}$$

4.4.5 Verify your results through a plot.
Solution: Compare theoretical and simulation results of CDF and PDF of triangular distribution using following python codes as given below and plots as shown in 4.4.3,4.4.4.

/codes/chapter4/cdf_plot_tri2.py

/codes/chapter4/pdf_plot_tri2.py

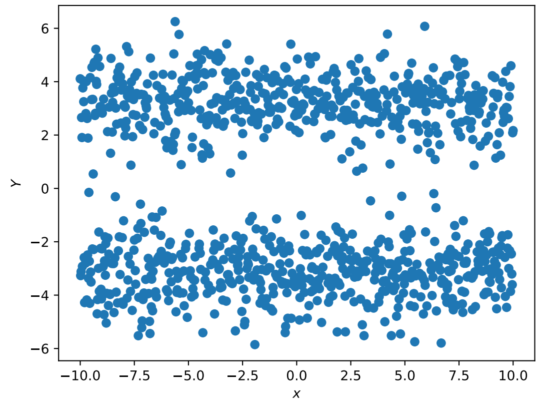

Figure 5.1.1: The scatter plot of Y

5 Maximum Likelyhood Detection:BPSK

5.1 Maximum Likelihood

5.1.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:

/codes/chapter4/5_1_1.py

5.1.2 Generate

$$Y = AX + N, \tag{5.5.1.0.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

/codes/chapter5/5_1_2.py

5.1.3 Plot Y using a scatter plot.
Solution: Code for plot of Y using scatter plot is given below and it's plot is in fig.5.1.1

/codes/chapter5/5_1_3.py

5.1.4 Guess how to estimate X from Y.
Solution: Given two signals are represented by two signals
'1' for X=1 and
'0' for X=-1. according to decision rule $P(Y > y)$

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{5.5.1.1.2}$$

5.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.5.1.1.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.5.1.1.4}$$

Solution: From above problem solution (5.5.1.1.2)

$$
\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) \\
&= \Pr\left(Y < 0 | X = 1\right) \\
&= \Pr\left(AX + N < 0 | X = 1\right) \\
&= \Pr\left(A + N < 0\right) \\
&= \Pr\left(N < -A\right)
\end{aligned}
$$

$$
\begin{aligned}
P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) \\
&= \Pr\left(Y > 0 | X = -1\right) \\
&= \Pr\left(AX + N > 0 | X = -1\right) \\
&= \Pr\left(-A + N > 0\right) \\
&= \Pr\left(N > A\right)
\end{aligned}
$$

where,$N \sim \mathcal{N}(0, 1)$

$$\therefore \Pr\left(N > A\right) = \Pr\left(N < -A\right) \tag{5.5.1.1.5}$$

$$P_{e|0} = P_{e|1} \tag{5.5.1.1.6}$$

5.1.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = \Pr\left(X = 1\right) P_{e|1} + \Pr\left(X = -1\right) P_{e|0} \tag{5.5.1.1.7}$$

given X is equiprobable

$$P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0} \tag{5.5.1.1.8}$$

Substituting from (5.5.1.1.6)

$$P_e = \Pr\left(N > A\right) \tag{5.5.1.1.9}$$

Given a random varible $X \sim \mathcal{N}(0, 1)$ the Q-function is defined as

$$Q(x) = \Pr\left(X > x\right) \tag{5.5.1.1.10}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \tag{5.5.1.1.11}$$

Using the Q-function, P_e is rewritten as

$$P_e = Q(A) \tag{5.5.1.1.12}$$

5.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: theoretical and simulation results are shown in 5.1.2

```
/codes/chapter5/5_1_7.py
```

5.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: From 5.5.1.1.2 the decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{5.5.1.2.13}$$

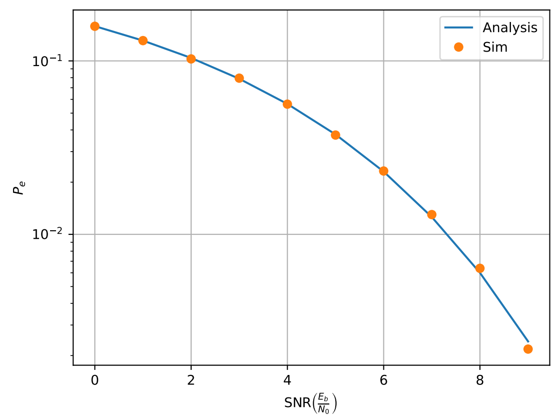

Figure 5.1.2: P_e of X wrt SNR(A)

$$
\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) \\
&= \Pr\left(Y < \delta | X = 1\right) \\
&= \Pr\left(AX + N < \delta | X = 1\right) \\
&= \Pr\left(A + N < \delta\right) \\
&= \Pr\left(N < -A + \delta\right) \\
&= \Pr\left(N > A - \delta\right) \\
&= Q(A - \delta)
\end{aligned}
$$

$$
\begin{aligned}
P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) \\
&= \Pr\left(Y > \delta | X = -1\right) \\
&= \Pr\left(N > A + \delta\right) \\
&= Q(A + \delta)
\end{aligned}
$$

Using 5.5.1.1.7 P_e is given by

$$
P_e = \frac{1}{2}Q(A+\delta) + \frac{1}{2}Q(A-\delta) \tag{5.5.1.2.14}
$$

Using the integral for Q-function from 5.5.1.1.11,

$$
P_e = k(\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right)\, du + \int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right)\, du) \tag{5.5.1.2.15}
$$

where k is a constant
Differentiating 5.5.1.2.15. wrt δ (using Leibniz's rule) and equating to 0, we get

$$
\begin{aligned}
\exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) &= 0 \\
\frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)} &= 1 \\
\exp\left(-\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) &= 1 \\
\exp\left(-2A\delta\right) &= 1
\end{aligned}
$$

Taking log on both sides

$$-2A\delta = 0$$
$$\implies \delta = 0$$

P_e is maximum for $\delta = 0$

5.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.5.1.2.16}$$

Solution: Given X is not equiprobable, P_e is given by,

$$P_e = (1-p)P_{e|1} + pP_{e|0} \tag{5.5.1.2.17}$$
$$= (1-p)Q(A+\delta) + pQ(A-\delta) \tag{5.5.1.2.18}$$

Using the integral for Q-function from 5.5.1.1.11

$$P_e = k((1-p)\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + p\int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \tag{5.5.1.2.19}$$

where k is a constant.
differentiate P_e wrt δ and equate to zero,

$$(1-p)\exp\left(-\frac{(A+\delta)^2}{2}\right) - p\exp\left(-\frac{(A-\delta)^2}{2}\right) = 0$$
$$\frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)} = \frac{p}{(1-p)}$$
$$\exp\left(-\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = \frac{p}{(1-p)}$$
$$\exp(-2A\delta) = \frac{p}{(1-p)}$$

Taking log on both sides

$$\delta = \frac{1}{2A}\log\left(\frac{1}{p} - 1\right)$$

6 Transformation of random variables

6.1 Gaussian to Other

6.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.6.1.0.1}$$

Solution: The plots of CDF and PDF of V as shown in fig.6.1.1 , 6.1.2 respectively.

/codes/chapter6/6_1_cdf.py

/codes/chapter6/6_1_pdf.py

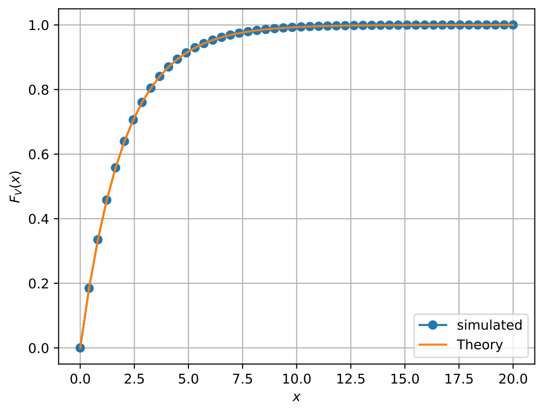

Figure 6.1.1: The CDF of V

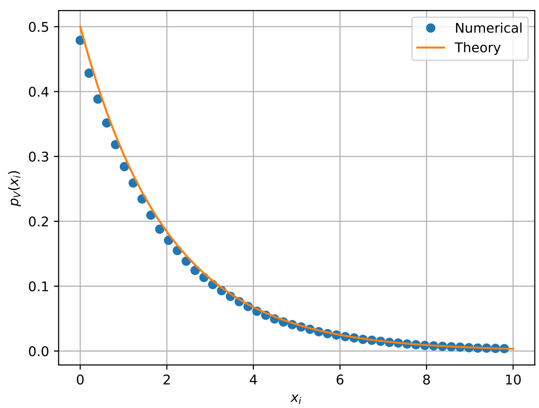

Figure 6.1.2: The PDF of V

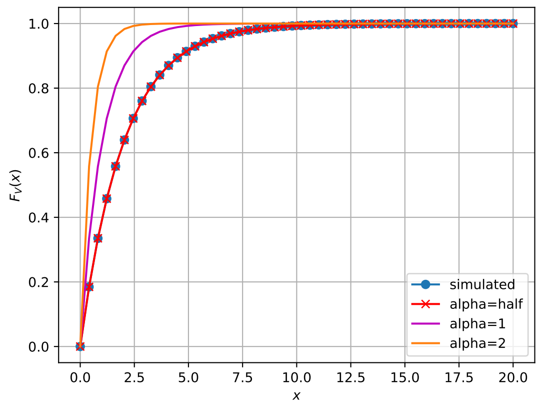

Figure 6.1.3: The CDF of V for different alpha

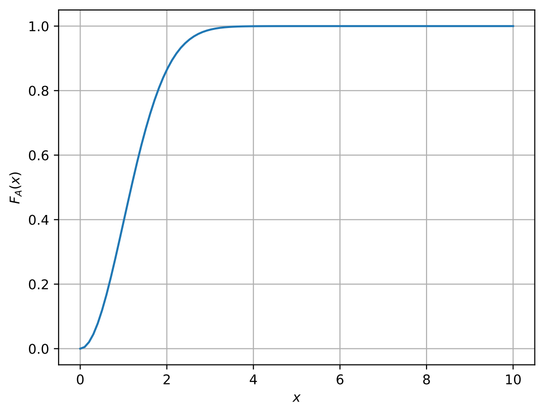

Figure 6.1.4: The CDF of A

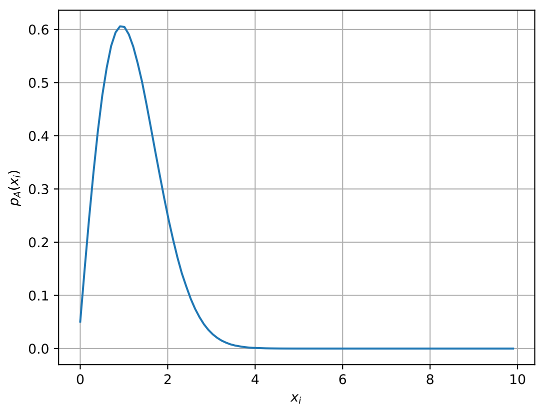

Figure 6.1.5: The PDF of A

6.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.6.1.2.2}$$

find α.
Solution:

/codes/chapter6/6_2_cdf.py

from 6.1.3 $alpha$= 0.5

6.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.6.1.3.3}$$

Solution: CDF,PDF plots are as shown in 6.1.4,6.1.5

/codes/chapter6/6_3_cdf.py

/codes/chapter6/6_3_pdf.py

6.2 Conditional Probability

6.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{6.6.2.0.1}$$

for

$$Y = AX + N, \tag{6.6.2.0.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

6.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution: The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{6.6.2.0.3}$$

For $X = 1$,

$$Y = A + N \tag{6.6.2.0.4}$$

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{6.6.2.0.5}$$

$$= \Pr\left(Y < 0|X = 1\right) \tag{6.6.2.0.6}$$

$$= \Pr\left(A < -N\right) \tag{6.6.2.0.7}$$

$$= F_A(-N) \tag{6.6.2.0.8}$$

$$= \int_{-\infty}^{-N} f_A(x)dx \tag{6.6.2.0.9}$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & otherwise \end{cases} \tag{6.6.2.0.10}$$

If $N > 0, f_A(x) = 0$. Then,

$$P_e = 0 \tag{6.6.2.0.11}$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x)dx \tag{6.6.2.0.12}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{-N} f_A(x)dx \tag{6.6.2.0.13}$$

$$= \int_{0}^{-N} \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx \tag{6.6.2.0.14}$$

$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \tag{6.6.2.0.15}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{6.6.2.0.16}$$

6.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{6.6.2.0.17}$$

Find $P_e = E\left[P_e(N)\right]$.
Solution: Since $N \sim \mathcal{N}(0,1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{6.6.2.0.18}$$

And from (6.6.2.0.16)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{6.6.2.0.19}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \tag{6.6.2.0.20}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{6.6.2.0.21}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{6.6.2.0.22}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$$

$$- \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{6.6.2.0.23}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{6.6.2.0.24}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{6.6.2.0.25}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{6.6.2.0.26}$$

$$\gamma = 2\sigma^2 \tag{6.6.2.0.27}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{6.6.2.0.28}$$

6.2.4 Plot P_e in problems 6.2.0 and 6.2.0 on the same graph w.r.t γ. Comment.
Solution: P_e is plotted w.r.t γ in 6.2.1 using the code below.

/codes/chapter6/6_2_4_pe.py

7 Bivariate Random Variables:FSK

7.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{7.4.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.4.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{7.4.3}$$

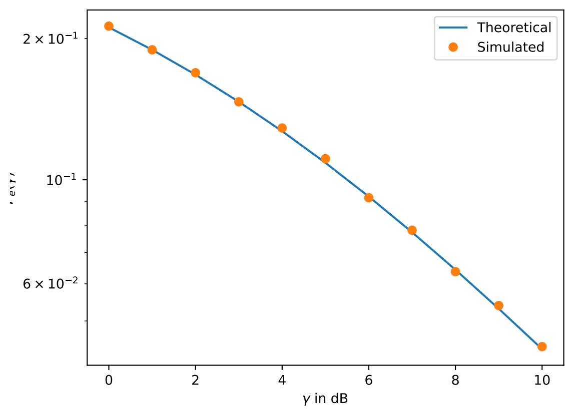

Figure 6.2.1: The P_e wrt γ

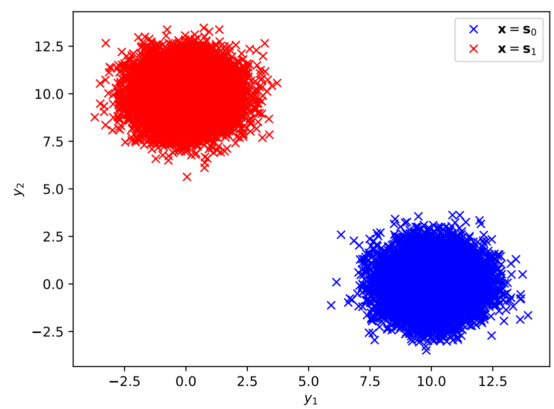

Figure 7.1.1: Y scatter plot

7.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{7.7.1.0.4}$$

on the same graph using a scatter plot.
Solution: The following python code plots the scatter plot when $\mathbf{x} = \mathbf{s}_0$ and $\mathbf{x} = \mathbf{s}_1$ in Fig. 7.1.1

```
/codes/chapter7/7_1_1.py
```

7.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution: The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\mathbf{\Sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right\} \tag{7.7.1.1.5}$$

where μ is the mean vector, $\mathbf{\Sigma} = E\left[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right]$ is the covariance matrix and $|\mathbf{\Sigma}|$ is the determinant of $\mathbf{\Sigma}$. For a bivariate gaussian distribution,

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}\right] \tag{7.7.1.1.6}$$

where

$$\mu = \begin{pmatrix} \mu_{\mathbf{x}} \\ \mu_{\mathbf{y}} \end{pmatrix}, \mathbf{\Sigma} = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \tag{7.7.1.1.7}$$

$$\rho = \frac{E\left[\left(x-\mu_x\right)\left(y-\mu_y\right)\right]}{\sigma_x\sigma_y}. \tag{7.7.1.1.8}$$

$$\mathbf{y}|s_0 = \begin{pmatrix} A+n_1 \\ n_2 \end{pmatrix} \tag{7.7.1.1.9}$$

$$\mathbf{y}|s_1 = \begin{pmatrix} n_1 \\ A+n_2 \end{pmatrix} \tag{7.7.1.1.10}$$

Substituting these values in (7.7.1.1.6),

$$p\left(\mathbf{y}|s_0\right) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_1^2}} \exp\left[-\frac{1}{2\left(1-\rho_1^2\right)} \times \left\{\frac{\left(y_1-A\right)^2}{\sigma_{y_1}^2} + \frac{\left(y_2\right)^2}{\sigma_{y_2}^2} - \frac{2\rho_1\left(y_1-A\right)\left(y_2\right)}{\sigma_{y_1}\sigma_{y_2}}\right\}\right] \tag{7.7.1.1.11}$$

$$p\left(\mathbf{y}|s_1\right) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_2^2}} \exp\left[-\frac{1}{2\left(1-\rho_2^2\right)} \times \left\{\frac{\left(y_1\right)^2}{\sigma_{y_1}^2} + \frac{\left(y_2-A\right)^2}{\sigma_{y_2}^2} - \frac{2\rho_2\left(y_1\right)\left(y_2-A\right)}{\sigma_{y_1}\sigma_{y_2}}\right\}\right] \tag{7.7.1.1.12}$$

where,

$$\begin{aligned} \rho_1 &= E[(y_1-A)(y_2)] = E[n_1n_2] = 0, \\ \rho_2 &= E[(y_1)(y_2-A)] = E[n_1n_2] = 0, \\ \sigma_{y_1} &= \sigma_{y_2} = 1 \end{aligned} \tag{7.7.1.1.13}$$

For equiprobably symbols, the MAP criterion is defined as

$$p\left(\vec{y}|s_0\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(\vec{y}|s_1\right) \tag{7.7.1.1.14}$$

Using (7.7.1.1.11) and (7.7.1.1.12) and substituting the values from (7.7.1.1.13), we get

$$\left(y_1-A\right)^2 + y_2^2 \underset{s_0}{\overset{s_1}{\gtrless}} y_1^2 + \left(y_2-A\right)^2 \tag{7.7.1.1.15}$$

On simplifying, we get the decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{7.7.1.1.16}$$

7.1.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0\right) \tag{7.7.1.1.17}$$

with respect to the SNR from 0 to 10 dB.
Solution: code available below and plot as shown in 7.1.2

/codes/chapter7/7_1_3.py

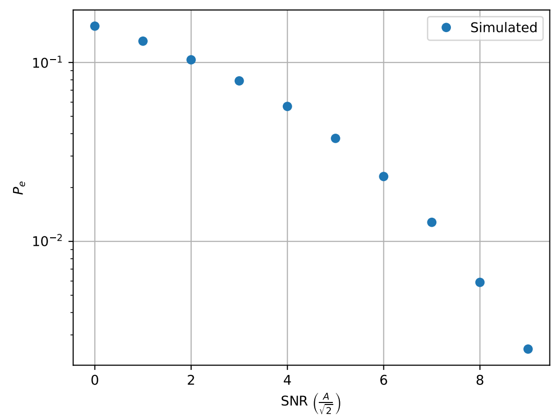

Figure 7.1.2: Pe vs SNR

7.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Solution:

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{7.7.1.2.18}$$

Given that $\mathbf{s}_0$ was transmitted, the received signal is

$$\mathbf{y}|\mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{7.7.1.2.19}$$

From (7.7.1.1.16), the probability of error is given by

$$P_e = \Pr\left(y_1 < y_2 | \mathbf{s}_0\right) = \Pr\left(A + n_1 < n_2\right) \tag{7.7.1.2.20}$$

$$= \Pr\left(n_2 - n_1 > A\right) \tag{7.7.1.2.21}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, 2)$. Thus,

$$P_e = \Pr\left(\sqrt{2}w > A\right) \tag{7.7.1.2.22}$$

$$\Pr\left(w > \frac{A}{\sqrt{2}}\right) \tag{7.7.1.2.23}$$

$$\Rightarrow P_e = Q\left(\frac{A}{\sqrt{2}}\right) \tag{7.7.1.2.24}$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the P_e curve in Fig. (7.1.3).

/codes/chapter7/7_1_4.py

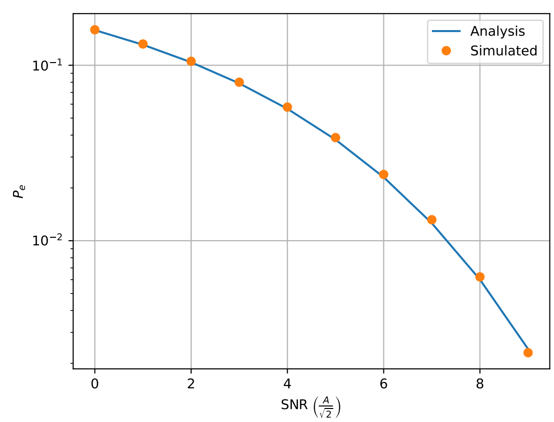

Figure 7.1.3: Pe vs SNR on theory and simulation based