HOME WORK 6

1)
a)

1) (a) In a simple bipartite graph G (planar), each face has at least four edges, since each cycle must have even length. each edge is shared by 2 faces. So we have $2m \geq 4f$.

euler formal $n - m + f = 2$

$$n - m + \frac{2m}{4} \leq 2$$

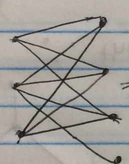
$$4n - 4m + 2m \leq 4 \times 2$$

$$4n - 2m \leq 8$$

$$-2m \leq -4n + 8$$

$$m \leq \frac{4n - 8}{2} \Rightarrow m \leq 2n - 4$$

(i) $G(U, V, E)$ where $|U| = 3$; $|V| = 4$ and $|E| = 10$



$\Rightarrow m = 10$

$m \leq 2n - 4$

$\leq 2 \times 7 - 4 \Rightarrow 10$

$m \leq 10$ Satisfies but graph is not planar.

$\therefore$ $m \leq 2n - 4$ is necessary but not sufficient condition.

1)

b) 7.8

1

(b) Euler's formula: $V - E + F = 2$

If there are no traingles then each face is enclosed by at least 4 edges. So $\frac{4F}{2} < E$

$\Rightarrow F < E/2$. From this we get, $V > E/2 + 2$. $\Rightarrow$ ($\because V - E + \frac{E}{2} \geq 2$) — ①

If all vertices have order ≥ 4 then $\frac{4V}{2} < E$ i.e. $\Rightarrow$

$V < E/2$. From this we get $0 \not> 2$. (substitute $V < E/2$ in ① $E/2 > E/2 + 2$)

So, we cannot have no traingles and at the same time all vertices with degree ≥ 4

7.9)

Graph with n- vertices

m- edges

f- faces

each faces bounded by ate least K edges

$\Rightarrow$ degree of each face $\geq K$

$$\sum deg(f) \leq 2m \text{ — ① \& } \sum deg(f) \geq Kf \text{ — ②}$$

euler formula:

$$n - m + f = 2 \text{ — (i)}$$

from ① & ② $Kf \leq 2m \Rightarrow f \leq \frac{2m}{K} \rightarrow$ ③

substitue ③ in (i)

$$n - m + \frac{2m}{K} \geq 2$$

$$nK - mK + 2m \geq 2K$$

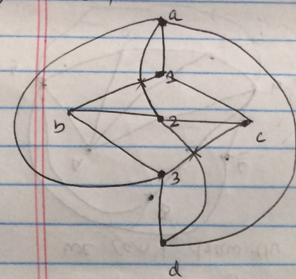
$$-m(K-2) \geq 2K - nK$$

$$m(K-2) \leq nK - 2K$$

$$m(K-2) \leq K(n-2)$$

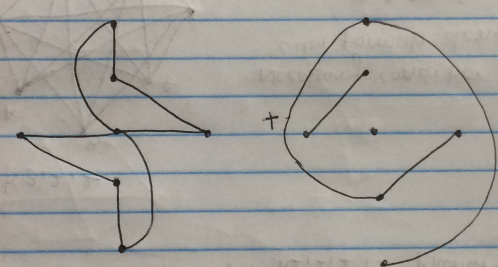
2)

2) $K_{3,4}$



crossing number = 2

Thickness

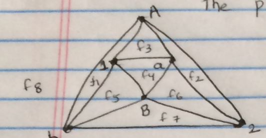


Layers required to draw without crossing is 2.

∴ thickness = 2

3) $K_{2,2,2}$ is planar but $K_{3,2,2}$ is not a planar graph

② $K_{2,2,2}$



The planar embedding of graph $K_{2,2,2}$ is:

Necessary condition: euler formula satisfied.

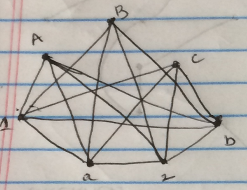
$n - m + f = 2$

$6 - 12 + 8 = 2$

More over, graph doesn't have any $K_{3,3}$ & K_5 as subgraph.

∴ $K_{2,2,2}$ is planar.

$K_{3,2,2}$



Necessary condition for a graph to be planar.
euler formula derivation

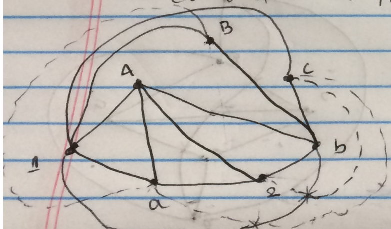
$m \le 3n-6$

$16 \le 3\times(7)-6$

$16 \nleq 15$ ∴ graph is not planar.

$m = 16$
$n = 7$

we can't draw the planar embedding of the $K_{3,2,2}$



∴ $K_{3,2,2}$ is not planar.

any planar graph $K_{r,s,t}$ is planar if:

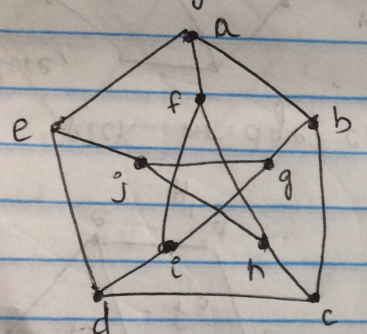
case 1
(i) if $r \ge 3$ & $s+t \ge 3$ then the graph is not planar, because in this case it contains $K_{3,3}$

case 2
(ii) if $r \le 2$ or $s+t \le 2$ we can the planar embedding of the graph so graph is planar.

$K_{2,2,2}$ falls in the second case
$K_{3,2,2}$ falls under first case.

4) Peterson graph is not planar. Since, any planar graph satisfies m<=3n-6
In Peterson graph n=10, m=15→ 15<=3*10-6➔ 15<=24 which is wrong. It doesn't satisfy the formula. Therefore, graph is not planer.

Peterson graph:



Peterson graph contains K_5 subdivision as a subgraph.

By Kuratowski theorem any graph is planar iff it doesnt contain K_5, $K_{3,3}$ or a subdivi[sion] of K_5 or $K_{3,3}$ as a subgraph.

∴ peterson graph is not planar.

5)

(5) A Graph to be planar, it has to satisfy that $m \le 3n-6$ where m & n are edges, vertices respectively.

The maximum no of edges any graph can have $= \frac{n(n-1)}{2}$

If the G contains m edges then $\bar{G}$ contains $\frac{n(n-1)}{2} - m$ edges

If G is planar, then

$$m \le 3n-6$$

If G^c is also planar,

$$\frac{n(n-1)}{2} - m \le 3n-6$$

$$G + G^c \Rightarrow \cancel{m} + \frac{n(n-1)}{2} - \cancel{m} \le 6n-12$$

$$\frac{n(n-1)}{2} \le 6n-12$$

$$n^2 - n - 12n \le -24$$

$$n^2 - 11n + 24 \le 0$$

$$n^2 - 8n - 3n + 24 \le 0$$

$$n(n-8) - 3(n-8) \le 0$$

$$n \le 8$$

So, there always ∃ a plan graph and its compliment is also planar if no. of nodes ≤ 8.

For $n \le 8$

we can have edges at most $\frac{n(n-1)}{2}$ in a graph G, with nodes 'n'.

consider, $n = 8$

Suppose that G is a graph on 8 vertices such that G and $\bar{G}$ are both planar.

Then, $m(G) \le 3 \times 8 - 6 = 18$ and also, $\bar{m}(G) \le 3 \times 8 - 6 = 18$

that implies, $m(G) + m(\bar{G}) \le 36$, $m(K_8) = 28$

So, we can draw planar embeding of K_8 where G & $\bar{G}$ both are planar.

Suppose that G is a planar graph on 11 vertices such that G and $\bar{G}$ are both planar. Then $m(G) \leq 3 \times 11 - 6 = 27$ and also $m(\bar{G}) \leq 3 \times 11 - 6 = 27$

Thus, $m(G) + m(\bar{G}) \leq 54$.

However, $m(G) + m(\bar{G}) = m(K_{11}) = 55$

This contradiction shows that our assumption must be false and no such graph can exist.

6)

a)

the cycle graph C_K

nodes: K

edges: K

no. of faces for any $C_K = 2$

euler's formula: $n - m + f = 2$

$K - K + 2 = 2$

b)

b)

board graph B_K

nodes $= (K+1)(K+1)$

faces $= K^2 + 1$

edges $= 2K(K+1)$

euler's formula: $n - m + f = 2$

$(K+1)(K+1) - 2K(K+1) + K^2 + 1 = 2$

$\cancel{K^2} + 2K + 1 - \cancel{2K^2} - 2K + \cancel{K^2} + 1 = 2$

$+ 2 = 2$; verified.

c)

$K_{2,m}$

$k = m$

no. of nodes $(n) = K + 2$

edges $(m) = 2K$

$f = K - 1$

euler's formula: $n - m + f = 2$

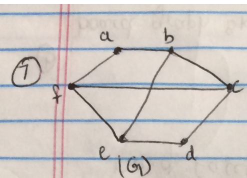
$K + 2 - 2K + K = 2$

$2 + 2K - 2K = 2$

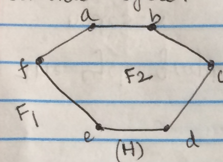
$2 = 2$; verifi

7)

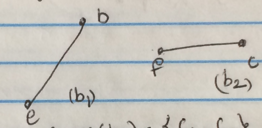
(7)



consider cycle:



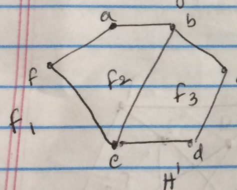
Bridges of (G) with respective to (H).



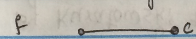
faces(b1) = {f1, f2}   faces(b2) = {f1, f2}

Pick any bridge and ~~find~~ path in it and add to graph (H)

suppose pick {be} and add to face ~~two~~ (2) since it can be embeded in any face



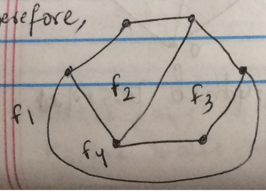
Remaining bridges



faces(b) = {f1}

pick bridge fc add to H', it can be add only in face1,

therefore,

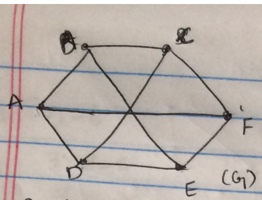


Now, there no bridges are present to add, we end up with planar embedding of (G).

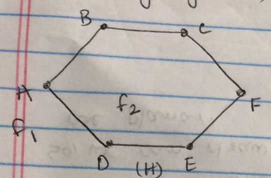
∴ G is planar.

After adding an edge BE to the graph

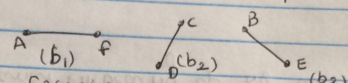
(b)



find any cycle (H) in G



bridges of G with respective to (H)



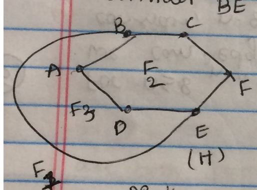
faces(b_1) = $\{f_1, f_2\}$

faces(b_2) = $\{f_1, f_2\}$

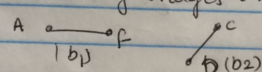
faces(b_3) = $\{f_1, f_2\}$

Pick any edge | bridge add to H.

consider BE and add to f_1, since it can be added in any of the two faces.

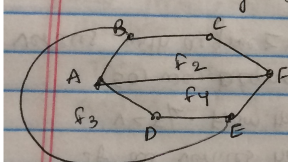


Remaining bridges are.

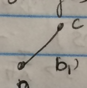


faces(b_1) = $\{f_2\}$; faces(b_2) = $\{f_2\}$

Pick any bridge add to 'H'. pick 'AF' only feasible face is F_2 so add to f_2 face.



Remaining bridges are



faces b_2 = { }

Now, we can't embeded bridge in any of the faces without crossing.

∴ Graph is not planar.