CS-210 Homework 5

Anusheh Zohair Mustafeez Student ID: 21100072

October 24, 2018

1 Problem 4b

Collaborated with: Hajira Zaman (21100057) and Safah Barak (21100092)

Proving expression is a multiple of 3 for all 'n's
P(n): (22^n-1)%3=0

Base Step: Proving first proposition is true:
For n=0
P(0): (22^0 - 1)%3
=(1-1)%3
=0
Hence proved P(0) is true.

Inductive Hypothesis: Assuming any nth proposition is true:
P(n): (22^n-1)%3=0 is assumed to be true for any n

Inductive Step: Proving n+1 proposition is true given that P(n) is true:
P(n+1): (22^{n+1}-1)%3
=((22^n)*(22-1)%3
=((22^n)*(21+1)-1)%3
=((22^n)*21 + 22^n -1)%3
Since both (22^n)*21 and 22^n -1 are both divisible by 3, P(n+1) is a sum of two terms divisible by three hence is divisible too.
So P(n+1) is also proved to be true.
Hence concluding P(n) is true for all n.

2 Problem 9

Collaborated with: Hajira Zaman (21100057) and Safah Barak (21100092)

Proving that for any real number x > -1 and any positive integer n, $(1+x)^n \geq 1$ + n*x

Base Step: Proving first proposition is true:
P(1): $(1+x)^0 \geq 1$ + (0)*x
$1 \geq 1$
Hence proved P(0) is proved to be true.

Inductive Hypothesis: Assuming any nth proposition is true:
P(n): $(1+x)^n \geq 1$ + n*x is assumed to be true for any n.
If we add x to both sides, we get $(1+x)^n$ + x $\geq$ 1 + n*x +x

So x $\geq \frac{1+n*x+x}{(1+x)^n}$ is assumed to be true for any n.

Inductive Step: Proving n+1 proposition is true given that P(n) ia true:
P(n+1): $(1+x)^{n+1} \geq$ 1 + (n+1)*x
$((1+x)^n)$*(1+x) $\geq$ 1 + n*x +x
1+x $\geq \frac{1+n*x+x}{(1+x)^n}$
From our inductive hypothesis we know that x $\geq \frac{1+n*x+x}{(1+x)^n}$ is true and since x+1> x we can conclude 1+x $\geq \frac{1+n*x+x}{(1+x)^n}$ is also true.
So P(n+1) is also proved to be true.
Hence concluding P(n) is true for all n.

3 Problem 10

Collaborated with: Hajira Zaman (21100057) and Safah Barak (21100092)

Any amount of postage greater or equal to 18 cents can be formed using just 3 cent and 10 cent stamps.
Proving P(n) by strong induction such that P(n): 3*x + 10*y = n where x and y are non negative integers and n is any integer such that n$\geq$ 18:

Base Step: Proving first proposition is true:
P(18): 3*6 + 10*0 =18
Hence P(18) is proved to be true.

Inductive Hypothesis: Assuming P(k) is true such that 19$\leq$k$\leq$n:
P(k): 3*x + 10*y = k is assumed to be true for any k.

Inductive Step: Proving k+1 proposition is true given that P(k) is true such that 19$\leq$k$\leq$n:
To ensure P(k+1) is true for all P(k) we can divide the problem into two cases:

Case 1: Where there are at least three 3 cent stamps
In this case if P(k) can be made into P(k+1) by simply replacing the three 3 cent stamps (n = 3*3 = 9) with one 10 cent stamp (n+1=10*1=10) and the proposition would still remain true.

Case 2: Where there are at least two 10 cent stamps
In this case if P(k) can be made into P(k+1) by simply replacing the two 10 cent stamps (n = 10*2 = 20) with seven 3 cent stamps (n+1=3*7=21) and the proposition would still remain true.

This way each P(k+1) can be derived from P(k) using either case 1 or case 2 and P(k+1) is also proved to be true.
Concluding P(n) is true for all n.

4 Problem 13

Collaborated with: Hajira Zaman (21100057) and Safah Barak (21100092)

Proving P(n): $2^n <$ n! for all n $\in$ Z+, n > 3.

Base Step: Proving first proposition is true:
P(4): $2^4 <$ 4!
16 < 4*3*2*1

16 < 24
Hence P(4) is proved to be true.

Inductive Hypothesis: Assuming any nth proposition is true:
P(n): 2^n < n! is assumed to be true for any n.

Inductive Step: Proving n+1 proposition is true:
P(n+1): 2^{n+1} < (n+1)!
2*2^n < (n+1)*n!
From the IH we know that 2^n < n! is true and we can also deduce that 2< (n+1) is also true as the least value (n+1) can take is 5. Hence 2*2^n < (n+1)*n! is proved to be true.
So P(n+1) is true.
Concluding P(n) is true for all n.

5 Problem 17

Collaborated with: Hajira Zaman (21100057) and Safah Barak (21100092)

1. $a_2 = a_0 * a_1 = 1 * 2 = 2$
   $a_3 = a_1 * a_2 = 2 * 2 = 4$
   $a_4 = a_2 * a_3 = 2 * 4 = 8$
   $a_5 = a_3 * a_4 = 4 * 8 = 32$
   $a_6 = a_4 * a_5 = 8 * 32 = 256$

2. Proving that P(n): $a_n = 2^{F_n}$, where $F_0, F_1, F_2, ...$ is the Fibonacci sequence
   Fibonacci sequence:$F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for n> 1

   Base Step: Proving first proposition is true:
   P(0): $a_0 = 2^{F_0}$
   1=2^0
   1=1
   Hence P(0) is proved to be true.

   Inductive Hypothesis: Assuming P(k) is true such that 1≤k≤n
   P(k): $a_k = 2^{F_k}$ is assumed to be true for any k.

   Inductive Step: Proving k+1 proposition is true:
   P(k+1): $a_{k+1} = 2^{F_{k+1}}$
   $a_k * (a_{k-1}) = (2^{F_k})$*$(2^{F_{k-1}})$ From the inductive hypothesis we've assumed that $a_k = 2^{F_k}$ and $a_{k-1} = 2^{F_{k-1}}$ so it can be concluded that $a_k * (a_{k-1}) = (2^{F_k})$*$(2^{F_{k-1}})$ is true.
   So P(n+1) is true.
   Concluding P(n) is true for all n.

6 Problem 19c

Collaborated with: Hajira Zaman (21100057) and Safah Barak (21100092)

Obtaining closed formula of $a_n = a_{n-1} + 2 * n + 3$ with initial condition $a_0 = 4$
$a_n = a_{n-1} + 2 * n + 3$
$a_n = (a_{n-2} + 2 * (n_1) + 3) + 2 * n + 3$

$a_n = ((a_{n-3} + 2*(n-2)+3)+2*(n-1)+3)+2*n+3$
After unraveliing the recurrence formula n number of times, we'll get:
$a_n = a_0 + 2*((n)+(n-1)+(n-2)....+(n-n))+3n$
$a_n = a_0 + 2*(\frac{n*(n+1)}{2})+3n$
$a_n = a_0 + n*(n+1)+3n$
$a_n = a_0 + n^2 + n + 3n$
$a_n = a_0 + n^2 + 4n$
Since a_0=4:
$a_n = 4 + n^2 + 4n$
$a_n = (n+2)^2$

Verification using induction:
Proving P(n): $a_n = (n+2)^2$

Base Step: Proving first proposition is true:
P(0): $a_0 = (0+2)^2$ $4 = (2)^2$
4=4
Hence P(0) is proved to be true.

Inductive Hypothesis: Assuming P(k) is true for all such that 1≤k≤n:
P(k): $a_k = (k+2)^2$ is assumed to be true for any k.

Inductive Step: Proving k+1 proposition is true:
P(k+1): $a_{k+1} = k^2 + 6*k + 9$
$a_{k+1} = k^2 + 4*k + 4 + 2*k + 5$
$a_{k+1} = (k+2)^2 + 2*k + 5$ where 2*k +5 is the difference between a_{k+1} and a_k for any value of k.
Hence proved P(k+1) is true
Concluding P(n) is true for all n.

7 Problem 21

Collaborated with: None

1. $a_0 = 0$
   $a_1 = 1$
   $a_2 = 2$
   a_n=n
   So total cars made in n months will be the sum of all cars made in that month and the previous months
   P(n): T_n= n + (n-1) + (n-2)+...+(n-n) where ((n-1)+(n-2)+....+(n-n)) is T_{n-1} and ((n-2)+(n+3)....+(n-n)) is T_{n-2} and so on.
   So recurrence relation of P(n) is $T_n = n + T_{n-1}$

2. Towards the end of the first year, n =12 so
   $T_{12} = n + T_{11}$ = 12 + 11 + 10 + 9 ......+1+ 0= 78

3. Obtaining formula using recurrence relation:
   $T_n = n + T_{n-1}$
   $T_n = n + ((n-1) + T_{n-2})$
   $T_n = n + ((n-1) + ((n-2) + T_{n-3}))$
   After unraveliing the recurrence formula n number of times, we'll get:

$T_n = n + (n-1) + (n-2) + ... + (n-n)$
$T_n = \frac{n*(n+1)}{2}$
$T_n = \frac{n^n+n}{2}$

8 Problem 24

Collaborated with: None

1. The market shares of A and B entering into a new year are dependent on their market shares in the previous year. According to the data given:
   Market share retained by A when going into next year: 0.7 *a_{n-1}
   Market share retained by B when going into next year: 0.6 * $(1 - a_{n-1})$
   Market share gained by A from B: 0.4 * $(1 - a_{n-1})$
   Market share gained by B from A: 0.3 * a_{n-1}
   Total Market share of A in year n: $a_n = 0.7$ *a_{n-1} + 0.4 * $(1 - a_{n-1})$
   Total Market share of B in year B: $1 - a_n = 0.6$ *(1 - a_{n-1}) + 0.3 * (a_{n-1})
   We can use any of the above two closed formulas for our purpose.
   Using A's market share as reference: $a_n = 0.7$ *a_{n-1} + 0.4 * $(1 - a_{n-1})$
   $a_n = 0.7$ *a_{n-1} + 0.4 * $(1 - a_{n-1})$
   $a_n = 0.7$ *a_{n-1} + 0.4 - 0.4*a_{n-1}
   $a_n = 0.3$*a_{n-1} + 0.4

2. Obtaining closed formula from recurrence relation:
   $a_n = 0.3$*a_{n-1} + 0.4
   $a_n = 0.3$*(0.3*a_{n-2} + 0.4) + 0.4
   $a_n = 0.3$*(0.3*(0.3*a_{n-3} + 0.4) + 0.4) + 0.4
   After unraveliing the recurrence formula n number of times, we'll get:
   $a_n = 0.3^n * a_0$ + 0.4*$((0.3^{n-1})$+(0.3^{n-2})+(0.3^{n-3})+...+$(0.3^{n-n}))$
   $a_n = 0.3^n * a_0$ + 0.4*$\sum_{i=0}^{n-1} 0.3^i$ where 0.4*$\sum_{i=0}^{n-1} 0.3^i$ is the sum of a geometric progression so:
   $a_n = 0.3^n * a_0$ + 0.4*$\frac{0.3^n-1}{0.3-1}$
   $a_n = 0.3^n * a_0$ + $\frac{0.4*0.3^n-0.4}{-0.7}$
   $a_n = 0.3^n * a_0$ + $\frac{0.4*0.3^n-0.4}{-0.7}$
   $a_n = 0.3^n * a_0$ - $\frac{4}{7}$*(0.3^n) + $\frac{4}{7}$
   $a_n = 0.3^n * (a_0$ - $\frac{4}{7})$ + $\frac{4}{7}$

3. $a_n = 0.3^n * (a_0$ - $\frac{4}{7})$ + $\frac{4}{7}$
   When n $\to \infty$, $0.3^n \to 0$
   So $a_n \to \frac{4}{7}$
   So in the long run, A's market share with become more or less constant at $\frac{4}{7}$ and become independent of its initial market share $a_0.a_n = 0.3^n * (a_0$ - $\frac{4}{7})$ + $\frac{4}{7}$
   $a_n = 0.3^n * (a_0$ - $\frac{4}{7})$ + $\frac{4}{7}$