

Concept of Phonons (Quantisation of Lattice Vibration)

It is a definite and discrete unit of energy or quantum of vibrational mechanical energy. Phonons and electrons are the elementary particle (two) of excitations in solids.

It is not a fundamental particle, as it is a vibrational quantum of excitations in a crystal lattice.

So, localized travelling waves of atomic vibrations in solids, is called phonons, that are quantised with energy $\hbar\omega = h\nu$ where $\nu/2\pi = \omega$, frequency of vibration of wave.

Ex: sound wave moving through air, seismic waves travels thousand of miles to reach seismograph detector that records earthquake.

The Basic difference

Phonon: It has E.M. phase and are quantum of lattice vibrations.

Photon: It also has E.M. phase and is the quantum of light radiations.

As energy level in harmonic oscillators are quantized -
the allowed energy levels in harmonic oscillator are

$$E = (n + \frac{1}{2}) \hbar\omega \quad \text{--- (1)}$$

where n = quantum number

$$U = A e^{i(\sqrt{\omega}t - \omega t)}$$

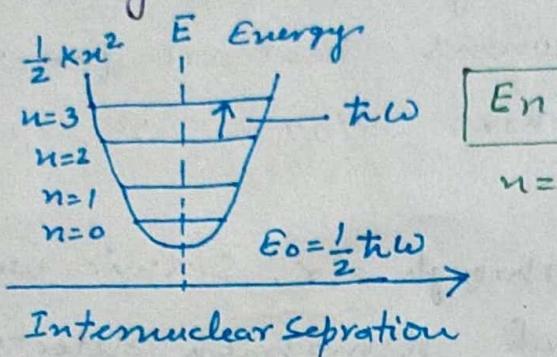
[No. of vibrational mode in crystal with frequency ω .]

Here the term $\frac{1}{2} \hbar\omega$ = zero point energy of mode.

Phonon governs the thermal properties in semiconductors & insulators. They affect the thermal, electrical, optical & other properties of bulk materials. They also have

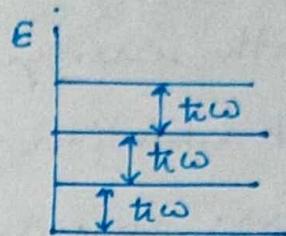
density of states ($D(PH)$) which is dimension dependent & its electronic counterpart, affects some properties of solids.

Explanation: \rightarrow Harmonic Oscillator (quantum or classical) is a particle in a potential energy well given by $V(x) = \frac{1}{2}Kx^2$. K is the force constant. It can be seen as the motion of small mass attached to a string, or a particle oscillating in a well shaped as parabola.



$$E_n = (n + \frac{1}{2}) \hbar \omega$$

$$n = 0, 1, 2, 3, 4$$



\Rightarrow Normal mode of vibrational frequency of the solid. It is necessary to quantize these normal modes, & these quantized normal modes of vibration are called Phonons.

\Rightarrow Phonons are massless quantum mechanical particle.

$\Rightarrow E_0 = \frac{1}{2} \hbar \omega$ = ground energy state of oscillator.

\Rightarrow If the system make the transition from lower to higher energy level it is always true that change in energy is integer multiple of $\hbar \omega$.

$$E_{\text{pho}} = \frac{\hbar \nu s}{\lambda} \quad \text{and} \quad \nu_{\text{pho}} = \frac{\hbar}{\lambda}$$

Concept of phonons: (Quantisation of Lattice vibrations :-)

The energy of a lattice vibration is quantised. The quantum of energy is called a "phonon". Thus, elastic waves in crystals are made up of phonons. Thermal vibrations in crystals are thermally - excited phonons.

Phonons have zero spin (like photons). Hence, they obey Bose-Einstein statistics.

The energy of each mode of vibration in an elastic wave of frequency ω is

$$\epsilon = (n + \frac{1}{2})\hbar\omega = (n + \frac{1}{2})(\frac{\hbar}{2\pi})\omega,$$

where n is the no. of phonons in the mode, $(\frac{\hbar}{2\pi})\omega$ is the quantum of elastic energy i.e. energy of phonon and $\frac{1}{2}(\frac{\hbar}{2\pi})\omega$ is the zero-point energy of the mode.

The quantisation of lattice vibration energy has an important consequence when a lattice vibration is involved in an interaction. Since the energy of a mode of vibration has to be $(n + \frac{1}{2})(\frac{\hbar}{2\pi})\omega$ with n any positive integer and since to a first order approximation, the energy changes only to adjacent allowed values, we must have

$$\Delta\epsilon = \pm (\frac{\hbar}{2\pi})\omega$$

and correspondingly

$$\Delta n = \pm 1$$

This means that the energy - change corresponds to a gain or loss of a single phonon at a time.

Experimental Evidence for phonon:-

- (1) The lattice contribution to the heat capacity of solids approaches zero as the temperature

Density of States :- It is defined as the no. of energy states per unit range. In other words density of states for electrons in band gives the no. of orbitals (states) in a given energy range. Hence the no. of filled states having energy range $E \leq E + dE$ is

$$N(E) d(E) = E(E) f(E) dE \quad \text{--- (I)}$$

From energy relation

$$E = \frac{\pi^2 k^2}{2m} \quad \text{--- (II)}$$

Q. The no. of electrons

$$N = \frac{V}{3\pi^2} k^3 \quad \text{OR } k^3 = \frac{3\pi^2 N}{V} \quad [V = L^3]$$

where E = Total energy, N = no. of electrons so from above eqn.

$$E = \frac{\pi^2}{2m} \left[\frac{3\pi^2 N}{V} \right]^{2/3} \quad \text{--- (III)}$$

$$\text{or} \quad \frac{3\pi^2 N}{V} = \left[\frac{2mE}{\pi^2} \right]^{3/2} = \left[\frac{2mE}{\pi^2 / 4\pi^2} \right]^{3/2}$$

This eqn. in terms of N

$$N = \frac{8\pi V}{3\pi h^3} (2mE)^{3/2} \quad \text{--- (IV)}$$

By differentiating eqn. (IV) w. r. to E

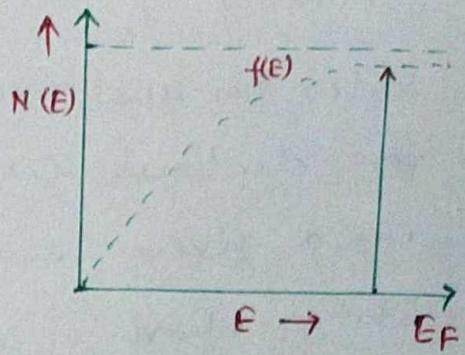
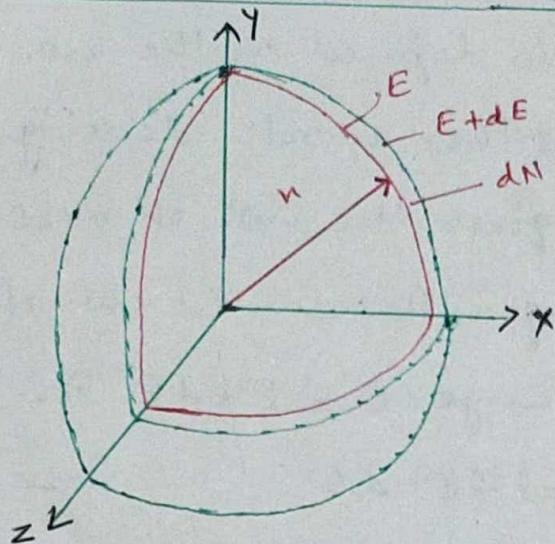
$$\frac{dN}{dE} = \frac{8\pi V}{3\pi h^3} (2m)^{3/2} \cdot \frac{3}{2} E^{1/2} = \frac{8\pi m V (2mE)^{1/2}}{h^3}$$

$$\text{or} \quad \frac{dN}{dE} = \frac{V}{2\pi^2} \left[\frac{2m}{\pi^2} \right]^{3/2} E^{1/2} \quad \left[\text{In terms of } h = \frac{h}{m} \right]$$

The quantity dN/dE is frequently referred to as density of available states $d(E)$, which on multiplication with probability of occupation $f(E)$ gives the density of states $N(E)$ as in fig.

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— Dr. Mudit Srivastava



Thus the no. of electrons whose energy lie between E & $E+dE$ is given by

$$N(E)dE = \frac{dN}{dE} f(E) dE$$

$$N(E)dE = \frac{8\pi V m}{h^3} (2mE)^{1/2} \cdot \frac{dE}{1 + e^{(E-E_F)/kT}}$$

KRONIG PENNEY MODEL :- In order to find the allowed energies of e^- in solid, The Schrodinger eqn. for e^- in a crystal lattice can be solved. Fig. shows the actual potential as seen by an e^- in crystal lattice in one dimension case. The Kronig Penney suggested a simplified model potential consisting of an infinite row of rectangular potential well separated by barrier of width b and with space periodicity a , of lattice.

- Each well represents an approximation to the potential produced by one ion.

The Solution to Schrodinger eqns. for an e^- in periodic motion lattice can be found by Bloch theorem.

According to this the sign function of free e^- travelling wave

$$\Psi(x) = e^{ikx}$$

where $k = 2\pi/\lambda$ is propagation

constant is modified by potential to be the form

$$\Psi(x) = U_k(x) e^{ikx}$$

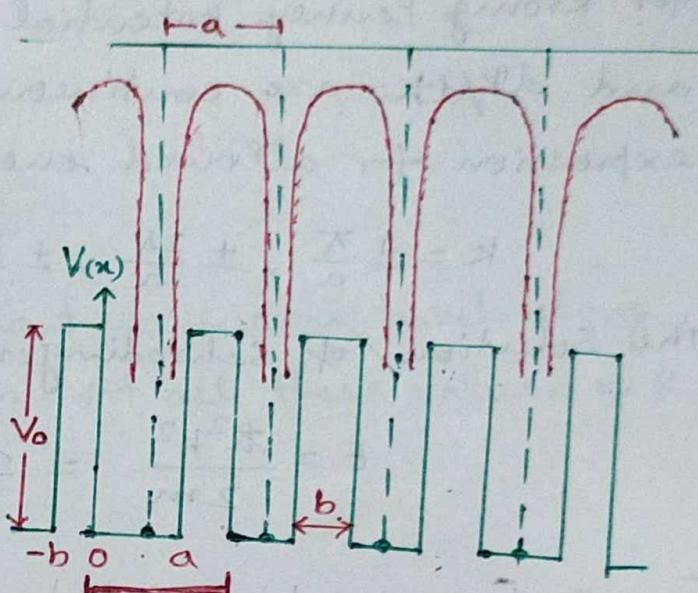
where $U_k(x)$ is modulating factor and with periodicity a

$$U_k(x) = U_k(x+a) = U_k(x+na)$$

where $n = \text{integer}$. Hence with periodicity complete wavefunction

$$\Psi(x,t) = U_k(x) e^{i(kx - \omega t)} \quad (1)$$

The exponential terms indicate a wave of wavelength $\lambda = \frac{2\pi}{k}$, which travel towards $+x$ if k is +ve and $-x$ if k is -ve



From Kronig Penney model in fig. each well represents an approximation to the potential produced by one ion. In the region such as $0 < x < a$, the potential is assumed to be zero, while in region $-b < x < 0$ the potential energy is taken as V_0 . Then with Schrodinger equation

$$\frac{d^2\psi}{dx^2} + \left[\frac{2m}{\hbar^2} \right] E \psi = 0 \quad [0 < x < a] \quad (2)$$

$$\frac{d^2\psi}{dx^2} + \left[\frac{2m}{\hbar^2} \right] (E - V_0) \psi = 0 \quad [-b < x < 0] \quad (3)$$

Here we will solve the Schrodinger wave equation for e^- for Kronig Penney potential under the condition that ψ and $d\psi/dx$ are continuous at boundaries. So the expression for allowed energy in terms of K

$$K = \pm \frac{\pi}{a}, \pm \frac{2\pi}{a}, \pm \frac{3\pi}{a}, \dots \quad (4)$$

The solution of Schrodinger wave eqn. for free e^- result in

$$E = \frac{\hbar^2 K^2}{2m} = \frac{\hbar^2 K^2}{8\pi^2 m} \quad (5)$$

Energy E versus Wave Vector K Diagram:

Equation (5) provide the allowed energy solution to Schrodinger equation. As the relation involve mathematical functions only certain value of K are possible. The R.H.S. of eqn.(5) is a function and can take value only +1 and -1.

Therefore L.H.S. of eqn. restricted to vary between these two limit. Hence only certain value of K is allowed and energy E is restricted to ~~allow~~ lie between these range. This concept is best in drawing the graph

between E & wave vector K. The E vs K graph is parabolic and is obtained in case of free e^- is interrupted at certain values of K. Thus energies of electrons are divided into forbidden and allowed bands.

The discontinuity can be understood by Bragg's Law

$$2a \sin\theta = n\lambda$$

where a = spacing θ = angle of incidence

$$2a = n\lambda$$

[if $\sin\theta = 90^\circ$]

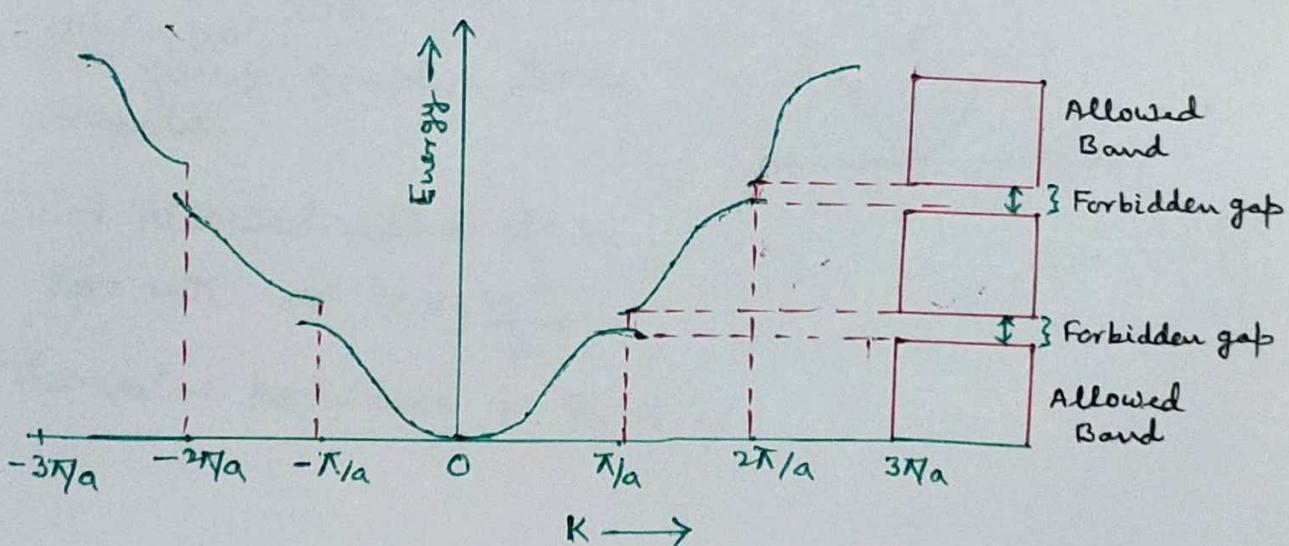
$$2a = n \cdot \frac{2\pi}{K}$$

$$a = \frac{n\pi}{K}$$

$$\text{or } K = \frac{n\pi}{a}$$

$$\text{or } K = \pm \frac{\pi}{a}, \pm \frac{2\pi}{a}, \pm \frac{3\pi}{a}$$

We put \pm sign because incident wave can travel along +x-axis as well as -x-axis. At all these values of K the gap in energy occur.



From the fig it is evident that potential energy of an e^- is maximum between the ions and minimum at the

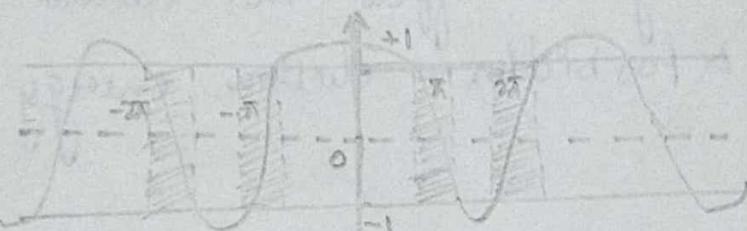
\rightarrow Multi & multi

million of tons. So we can have two different values
of energy into E_1 and E_2 for $\Delta \frac{E}{E} = \frac{1}{2}$. Then we obtain
the heat energy between E_1 & E_2 . This phenomena
makes a difference in energy (E_1, E_2) which is
known as energy gap.

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— Dr. Madit P. Srivastava

positions of ions. So an e^- can have two different values of energies i.e. E_1 and E_2 for $K = \frac{\pi}{a}$. Hence no electron can have any energy between E_1 & E_2 . This phenomenon creates a difference in energy ($E_1 \sim E_2$) which is known as energy gap.



From the fig. shaded regions are allowed regions having values more than +1 & less than -1 are disallowed.

Now Case-I $\rightarrow V_0 \rightarrow \infty$, the curve becomes steep & allowed band become narrow — line spectrum

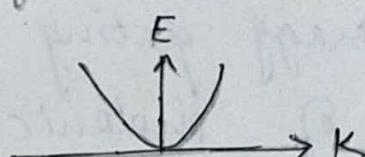
$$ka = n\pi \quad \text{or} \quad K = \frac{n\pi}{a}$$

$$2E = \frac{n^2\pi^2 h^2}{2ma^2}$$

Case-II $\rightarrow V_0 = 0$

$E = \frac{\hbar^2 K^2}{2m} \rightarrow$ the energy of free e^- will follow the parabolic path.

This gives graphs like Parabola.



Case-III \rightarrow Allowed value of K

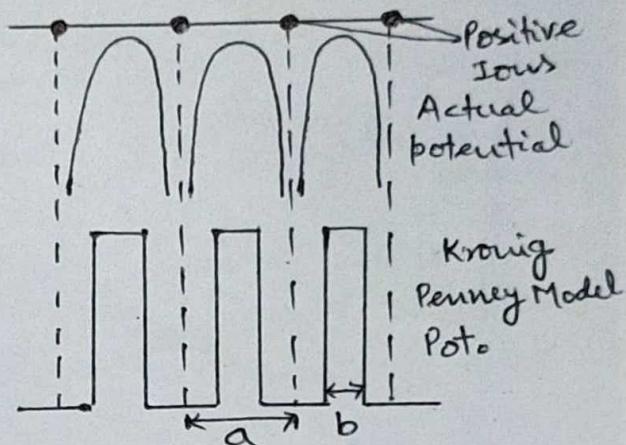
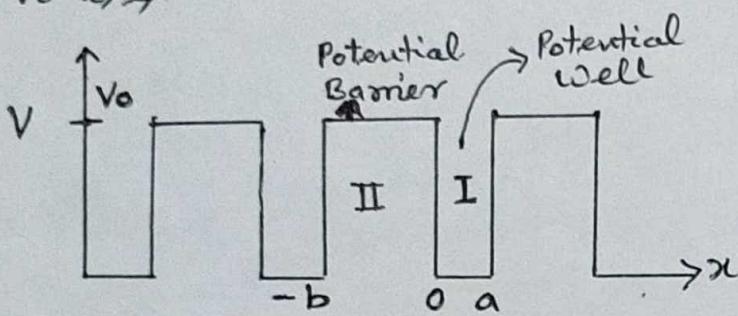
$$ka = n\pi \quad \text{or} \quad K = \pm \frac{n\pi}{a}$$

The value between +1 & -1.

Kronig-Penney Model :- (Derivation) :-

The Kronig-Penney model is a simplified model for an electron in a one dimensional periodic potential. It explains the behaviour of an electron in periodic potential. Kronig-Penney suggested a model potential consisting of an infinite row of an rectangular potential wells separated by barriers of 'b' with space periodicity 'a' which is the periodicity of lattice.

In the region $0 < x < a$, the potential energy is equal to zero and in the region $-b < x < 0$, the potential energy is V_0 i.e.



Potential Barrier \rightarrow Max. Potential $- V_0$

Potential Well \rightarrow Potential $= V = 0$

$$\begin{aligned} V(x) = V = 0 &\quad \text{for } 0 < x \leq a \\ V = V_0 &\quad \text{for } -b < x < 0 \end{aligned}$$

} — (1)

The possible states that the electrons can occupy in such a square well potential can be determined by 1D Schrodinger equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] \psi = 0 \quad \text{--- (2)}$$

Here m is the mass of electron. Eqn (2) for two region I & II takes form

$$\text{for I } \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E \psi = 0 \quad (\text{for } 0 < x < a) \quad \text{--- (3)}$$

$$\text{for II } \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} [E - V_0] \psi = 0 \quad (\text{for } -b < x < 0) \quad \text{--- (4)}$$

$$\text{Now consider } \alpha^2 = \frac{2mE}{\hbar^2} \quad \& \quad \beta^2 = \frac{2m}{\hbar^2} (V_0 - E)$$

Now eqn. (3) & (4) becomes

$$\frac{d^2\psi}{dx^2} + \alpha^2\psi = 0 \quad (\text{for } 0 < x \leq a) \quad (5)$$

$$\frac{d^2\psi}{dx^2} - \beta^2\psi = 0 \quad (\text{for } -b < x < 0) \quad (6)$$

Wave function solution for eqn. (5) & (6)

$$\psi = e^{ikx} U(x) \quad (7) \quad [\text{Given by Bloch theorem}]$$

where $U(x)$ = periodic function, ψ = wave function.

Differentiate eqn. (7) twice we have

$$\frac{d\psi}{dx} = e^{ikx} \frac{dU(x)}{dx} + U(x) ik e^{ikx}$$

$$\frac{d^2\psi}{dx^2} = e^{ikx} \frac{d^2U}{dx^2} + \frac{dU}{dx} - ik e^{ikx} + U(x) i k^2 e^{ikx} + ik e^{ikx} \frac{dU}{dx}$$

$$\left[\frac{d^2\psi}{dx^2} = e^{ikx} \frac{d^2U}{dx^2} + \frac{dU}{dx} ik e^{ikx} - U(x) k^2 e^{ikx} + ik e^{ikx} \frac{dU}{dx} \right] \quad (8)$$

Putting eqn. (8) in eqn. (5)

$$e^{ikx} \frac{d^2U}{dx^2} + \frac{dU}{dx} ik e^{ikx} - U(x) k^2 e^{ikx} + ik e^{ikx} \frac{dU}{dx} + d^2 e^{ikx} U = 0$$

Dividing the above eqn. by e^{ikx}

$$\frac{d^2U}{dx^2} + \frac{ikdU}{dx} - k^2 U + ik \frac{dU}{dx} + \alpha^2 U = 0$$

$$\frac{d^2U}{dx^2} + 2ik \frac{dU}{dx} + (\alpha^2 - k^2) U = 0 \quad (9)$$

Now Putting eqn. (8) in eqn. (6)

$$\frac{d^2U}{dx^2} + 2ik \frac{dU}{dx} - (\beta^2 + k^2) U = 0 \quad (10)$$

Now general solution of eqn. (9) & (10)

$$U_1 = A e^{i(\alpha-k)x} + B e^{-i(\alpha+k)x} \quad (11)$$

$$U_2 = C e^{i(\beta-k)x} + D e^{-i(\beta+k)x} \quad (12)$$

Solving eqn. (11) & (12)

$$\left| \frac{\sin \alpha a}{\alpha a} + \cos \alpha a = \cos K a \right| \quad \begin{array}{l} \text{K.P. Model} \\ \text{Eqn.} \end{array}$$

where $K = \frac{2\pi}{\lambda}$ (wave vector)

$$P = \frac{mv_0 a}{t^2}$$

\Rightarrow so case-I - At $P \rightarrow \infty$

$$\frac{P \sin \alpha}{da} + \frac{\cos \alpha}{P} = \cos Kd$$

$$E = \frac{h^2 k^2}{8\pi^2 m}$$

$$t = \frac{h}{2\pi}$$

$$\frac{\sin \alpha}{da} + \frac{\cos \alpha}{P} = \frac{\cos Kd}{P} \quad [\text{Divided by } P]$$

$$\sin \alpha = 0$$

$$\alpha = n\pi \quad \text{or} \quad \alpha = n\pi/a$$

$$\text{like } d^2 = \frac{v^2 \pi^2}{a^2}$$

$$\text{We have } d^2 = \frac{2mE}{t^2}$$

$$\text{or } \frac{v^2 \pi^2}{a^2} = \frac{2mE}{t^2}$$

$$\text{or } E = \frac{v^2 \pi^2 t^2}{2ma^2}$$

\Rightarrow case-II - At $P = 0$

$$\frac{P \sin \alpha}{da} + \cos \alpha = \cos Kd$$

$$\cos \alpha = \cos Kd$$

$$d = K \quad \text{, i.e. } d^2 = K^2$$

$$\therefore d^2 = \frac{2mE}{t^2}$$

$$K^2 = \frac{2mE}{t^2}$$

$$(K = \frac{2\pi}{\lambda} \text{ wave vector})$$

$$\frac{2\pi}{\lambda} = \frac{2mE}{t^2}$$

$$E = \frac{2\pi t^2}{2m\lambda}$$

$$\text{or } E = \frac{mv^2}{2m}$$

$$\text{so } E = \frac{p^2}{2m} = \frac{1}{2}mv^2$$

(By De-Broglie concept)

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

[Proves that Energy is K.E.]

Brillouin Zones / Wigner - Seitz Primitive Cell

These are the set of points in k -space that can be reached from origin without crossing the plane. In other words it is defined as the set of points closer to the origin than to any other reciprocal lattice.

Brillouin zone is nothing but they are the allowed energy regions of e^- in momentum/reciprocal space. In this region e^- travelling will have wave like solutions. At the zone boundaries, the e^- suffers Bragg's diffraction & thus no travelling wavelike solution exist at zone boundaries. This diffraction originates the energy gap in crystals.

This is a representation of permissive value of K of e^- in one, two, three dimensions. It gives us the concept & origin of allowed & disallowed bands.

In One Dimensional :- If an e^- is moving in 1-D periodic lattice. As a result, the energy spectrum contains allowed and forbidden bands. Considering the value of K at which the discontinuities in E occur. The discontinuity occurs when the term

$$\frac{r_1 \sin \alpha}{\alpha} + \cos \alpha \text{ reaches to its maximum value}$$

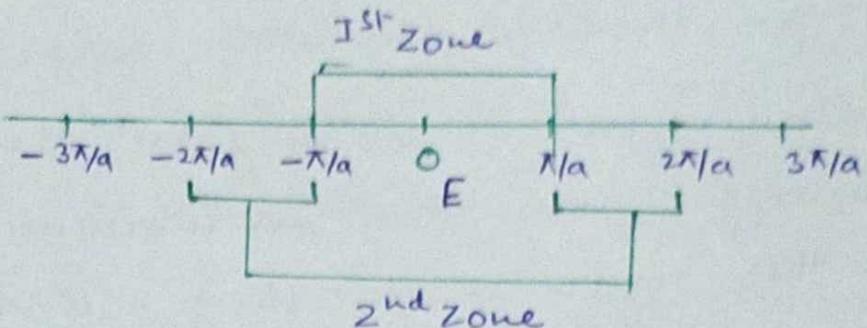
$$\text{i.e } \cos(K\alpha) = \pm 1$$

$$\text{or } K = \pm \frac{n\pi}{a} \quad \text{where } n=1, 2, 3, \dots$$

At such value of K , the e^- momentum increases and as a result its energy increases, it jumps discontinuously from the top of one allowed band to the

bottom of next. This region between the first and second value of k for which discontinuities occur is called the second Brillouin zone and so on.

For 1-D



The first two Brillouin zone in 1-D lattice

In 2-Dimension :- If the e^- is moving in 2-D square lattice, then let us consider

$$\lambda = \frac{h}{P} = \frac{h}{\sqrt{2mE}}$$

Low energy e^- will have longer wavelength & without being diffracted they travel freely through a crystal.

On the other hand, the more energetic e^- (near to the fermi level $E \approx E_F$) has complexable wavelength they suffer Bragg's diffraction Law

$$2a \sin \theta = n\lambda \quad \text{where } n=1, 2, 3, \dots$$

where θ = angle of incidence of beam of e^-

Replacing λ by k

$$k = \frac{2\pi}{\lambda}$$

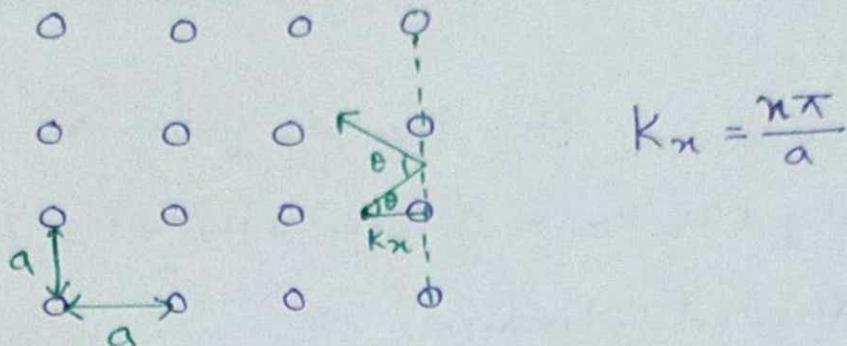
from $n\lambda = 2a \sin \theta$ and $(k) = \frac{2\pi}{\lambda}$ we get

$$(k) = \frac{n\pi}{a \sin \theta} = K$$

K depends upon angle of incidence θ .

Let us take the two dimensional lattice. For vertical rows - K in n -direction

$$k_{2n} = \frac{n\pi}{a} \Rightarrow k_n = \frac{n\pi}{a} \quad ; \quad k_y = \frac{\pi n}{a} \quad (\text{for horizontal rows})$$



Hence for first zone $n = \pm 1$

$$k_n = \pm \frac{\pi}{a} \quad \& \quad k_y = \pm \frac{\pi}{a} \quad (\text{For } n_1 = \pm 1, n_2 = \pm 0, n_3 = 0 \\ n_2 = \pm 1)$$

$$n_1 k_n + n_2 k_y = \frac{\pi}{a} (n_1^2 + n_2^2)$$

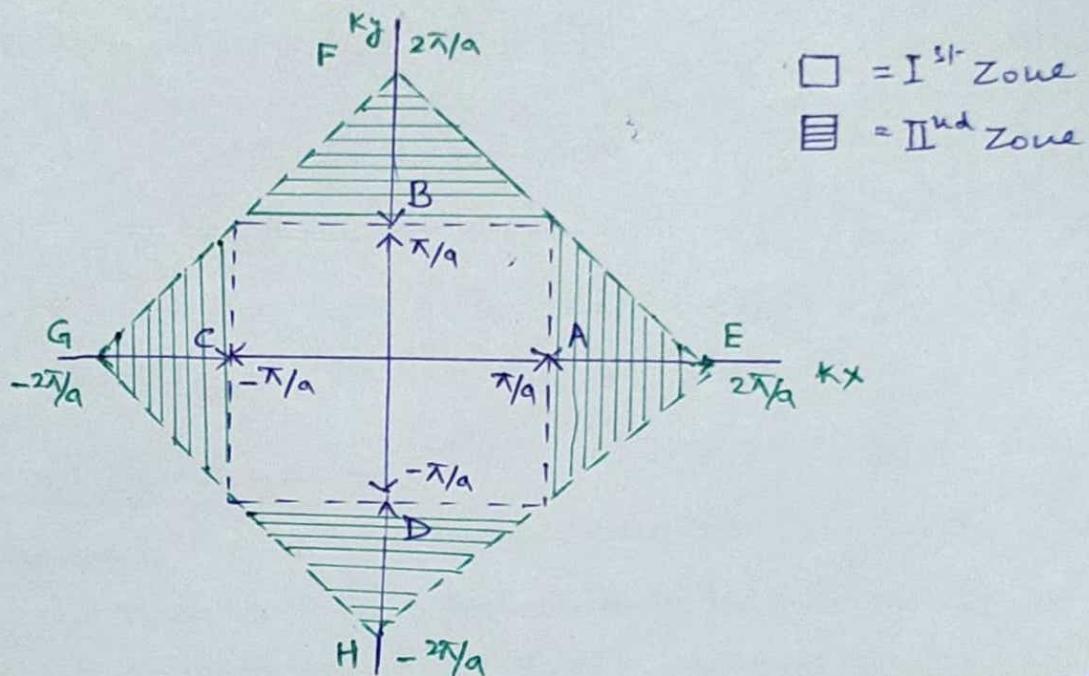


Fig. First three Brillouin zones for a two dimensional lattice

Now to get the sketch we use integers values

$$n_1 = +1 ; n_2 = +1 \rightarrow k_n + k_y = 2\pi/a \quad (1)$$

$$n_1 = -1 ; n_2 = +1 \rightarrow -k_n + k_y = 2\pi/a \quad (2)$$

$$n_1 = +1 ; n_2 = -1 \rightarrow k_n - k_y = 2\pi/a \quad (3)$$

$$n_1 = -1 ; n_2 = -1 \rightarrow -k_n - k_y = 2\pi/a \quad (4)$$

for $\Delta \mu$ & $\Delta \mu_{\text{eff}}$ and $\Delta \mu_{\text{eff}}^2 + \frac{\Delta \mu}{2} (\omega_1^2 + \omega_2^2)$

The above equation is then solved at first by the help of trial fitting through E, F, G, H and shown in fig. 1. Finally the third value is taking values with its given as 0, 81, 81.

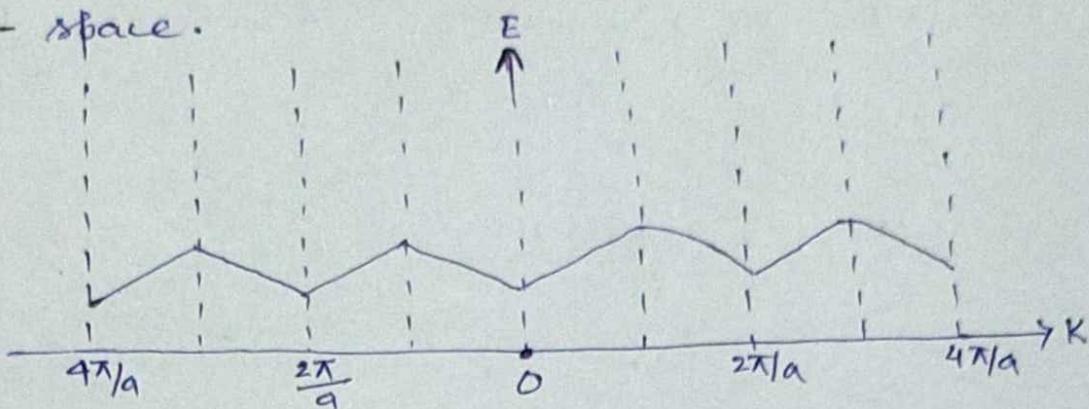
For these determined the eqn is given as

$$\Delta \mu_{\text{eff}}^2 + \Delta \mu_{\text{eff}} = \frac{\Delta \mu}{2} (\omega_1^2 + \omega_2^2 + \omega_3^2)$$

from the complicated theories.

① Periodic Zone Scheme :-

The periodic representation of allowed energy values corresponding to each allowed band, which is obtained by the periodic repetition of the region of $-\pi/a < K < \pi/a$ through out the whole K -space.

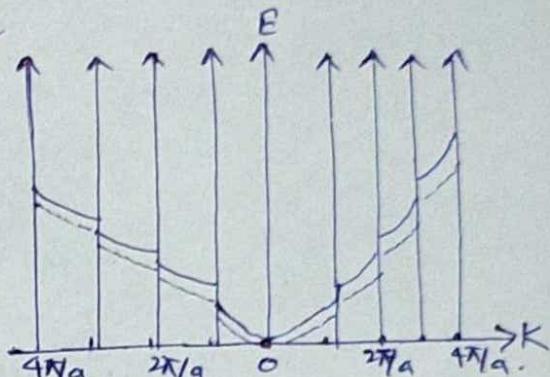


② Extended Zone Scheme -

In this scheme different bands are drawn in different zones. In K -space discontinuity is obtained at $K = \pm \frac{n\pi}{a}$ where $n = \pm 1, \pm 2, \dots$

For $K = -\pi/a$ to $\pi/a \rightarrow 1^{\text{st}}$ Brillouin zone

for $K = -\pi/a$ to $-\frac{2\pi}{a}$ & $\frac{2\pi}{a}$ to $\pi/a \rightarrow 2^{\text{nd}}$ Brillouin zone

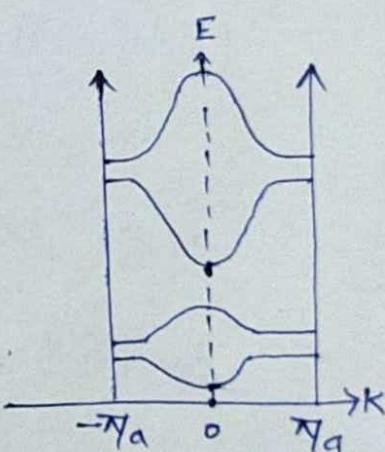


This continuity in the curves is due to Bragg's law of reflection at the edge of allowed bands i.e. $K = \pm \frac{n\pi}{a}$

③ Reduced Zone Scheme -

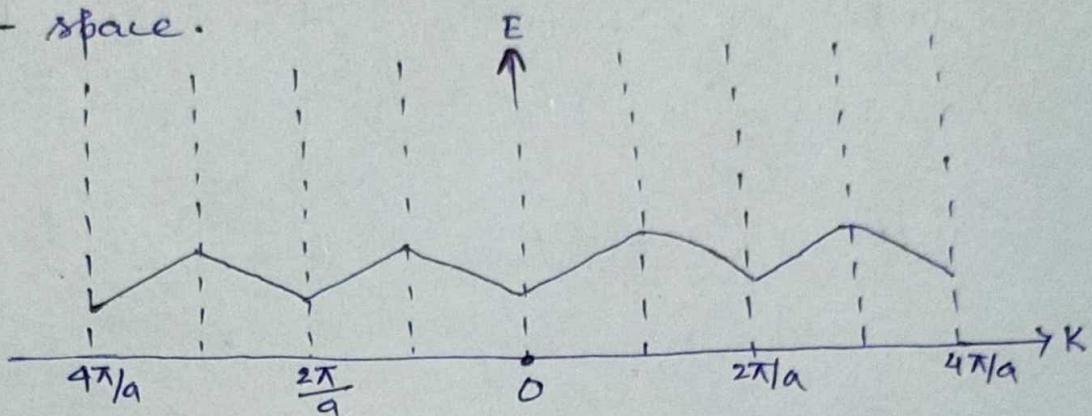
→ In this scheme, the 1st Brillouin zone is shown since the E-K diagram is periodic. It is sufficient to restrict to 1st Brillouin zone in reduced zone scheme.

→ If we know the energy value of 1st zone next K then we know everywhere because energy eigen values are periodic.



① Periodic Zone Scheme :-

The periodic representation of allowed energy values corresponding to each allowed band, which is obtained by the periodic repetition of the region of $-\pi/a < K < \pi/a$ throughout the whole K -space.

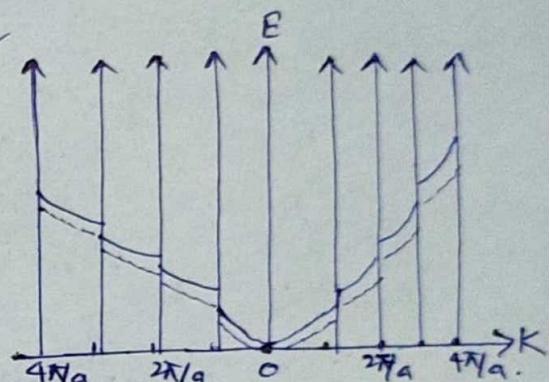


② Extended Zone Scheme -

In this scheme different bands are drawn in different zones. In K -space discontinuity is obtained at $K = \pm \frac{n\pi}{a}$ where $n = \pm 1, \pm 2, \dots$

In $K = -\pi/a$ to π/a \rightarrow 1st Brillouin zone

for $K = -\pi/a$ to $-\frac{2\pi}{a}$ & $\frac{2\pi}{a}$ to $+\frac{2\pi}{a}$ \rightarrow 2nd Brillouin zone

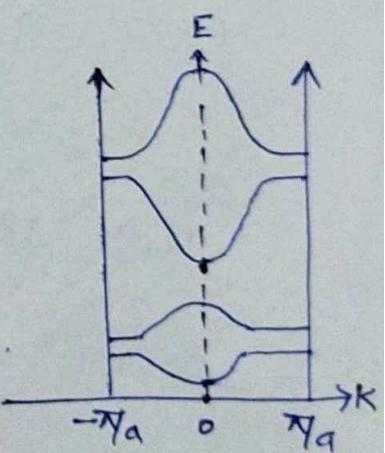


This discontinuity in the curves is due to Bragg's law of reflection at the edge of allowed bands i.e. $K = \pm \frac{n\pi}{a}$

③ Reduced Zone Scheme -

→ In this scheme, the 1st Brillouin zone is shown since the E-K diagram is periodic. It is sufficient to restrict to 1st Brillouin zone in reduced zone scheme.

→ If we know the energy value of 1st zone next K then we know everywhere because energy eigenvalues are periodic.



$$D(E) = \frac{dn(E)}{dE} \quad \text{--- (1)}$$

where $n(E)$ = no. of photons in phase space.

$dn(E)$ = no. of photons in indefinite size of phase space.

$$dn(E) = \frac{d^3n \cdot d^3p}{h^3} \quad \text{--- (2)}$$

$$\therefore n(E) = \int \frac{d^3n \cdot d^3p}{h^3}$$

$$= \frac{1}{h^3} \int d^3n \int d^3p \quad \text{where } \int d^3r = \text{Volume} \& \int d^3p = \text{energy}$$

$$= \frac{V}{h^3} \int p^2 dp \sin \alpha d\theta d\phi$$

$$= \frac{V}{h^3} \int p^2 dp (4\pi)$$

$$= \frac{4\pi V}{h^3} \int p^2 dp$$

$$= \frac{4\pi V}{h^3 c^3} \int E^2 dE$$

$$= \frac{4\pi V}{h^3 c^3} \cdot \frac{E^3}{3}$$

$$\begin{cases} E = pc \\ \frac{E}{c} = p, \frac{dE}{c} = dp \end{cases}$$

$$n(E) = \frac{4\pi V E^2}{3 h^3 c^3} \quad \text{--- (3)}$$

Photons have the +ve polarization states

$$\therefore n(E) = 2 \cdot \frac{4\pi V E^2}{3 h^3 c^3} = \frac{8\pi V E^3}{3 h^3 c^3} \quad \text{--- (4)}$$

$$D(E) = \frac{8\pi V}{3 h^3 c^3} \cdot 3 E^2 = \frac{8\pi V E^2}{h^2 c^3} \quad \text{--- (5)}$$

In form of frequency $E = h\nu$

$$\therefore D(\nu) = \frac{8\pi V \nu^2}{h c^3} \quad \text{in the required Eqn.}$$