Parametric forms

Circle $(x^2 + y^2 = a^2)$

$x = a\cos\theta$

$y = a\sin\theta$

Parabola $(y^2 = 4ax)$

$x = at^2$

$y = 2at$

Hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$

$x = a\sec\theta$

$y = b\tan\theta$

Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right.$

$x = a\cos\theta$

$y = b\sin\theta$

Asteriod $\left(x^{2/3} + y^{2/3} = a^{2/3}\right)$

$x = a\cos^3\theta$

$y = a\sin^3\theta$

Hyperbolic Functions

Definition

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

Hyperbolic Identities

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1-x^2}}, x \neq 0$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, |x| > 1$$

UNIT-2 Differential Equation

D.E

- Ordinary Diff. Equ^n (ODE)
  - only one I.D.V involved
    - LDE (Linear)
    - NLDE (Non-linear)
- Partial Diff. Equ^n (PDE)
  - two or more than two IDV involved

- Order: Highest derivative
- Degree: power of highest derivative

↳ $y = 5x^2 + 7x + 4$
Here $(5, 7, 4) \longrightarrow$ constants

↳ $y = c_1x^3 + c_2x^2 + c_3x + c_4$
$(c_1, c_2, c_3, c_4) \longrightarrow$ arbitary constants

* No. of arbitary constant = order of derivative

To convert into D.E
↳ Target – Remove arbitary constant

Syllabus → 2nd & 3rd order DE

Linear D.E. of nth order with constant coefficients.

If the degree of dependent variables and power of all derivatives is one is called Linear D.E.

Examples –

① $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y^4 = \sin x \rightarrow$ NLDE

② $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + y = x^5 \rightarrow$ NLDE

③ $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = x^4 \rightarrow$ NLDE

④ $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^4 \rightarrow$ LDE

⑤ $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^6 \rightarrow$ LDE

General form

2nd order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$P, Q \rightarrow$ constants

$R \rightarrow$ I.D.V. / constant

3rd order

$$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + ty = R$$

$P, Q, t \rightarrow$ constants

$R \rightarrow$ IDV / constant

Formula :

complete solution (CS) = complementary function (CF) + Particular Integral [P.I.]

CS → Dependent Variable

CF, P.I. → Independent Variable

Linear D.E. whether it is 2nd or 3rd order

LHS → C.F.

RHS → P.I.

$$\underbrace{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy}_{CF} = \underbrace{R}_{PI}$$

Method for finding complementary function (C.F)

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$D \rightarrow$ derivate $\quad D = \frac{d}{dx}$

$$Dy = \frac{dy}{dx}, \quad D^2y = \frac{d^2y}{dx^2}, \quad D^3y = \frac{d^3y}{dx^3}$$

$\frac{1}{D} \rightarrow$ Integral, $\int$

Step 1. change DE in Symbol —

$$D^2y + PDy + Qy = R$$

$$[D^2 + PD + Q]y = R$$

find CF

$$[D^2 + PD + Q]y = 0$$

let $D = m$ & $y = 1$

$m^2 + Pm + Q = 0$

$m = m_1, m_2$

<u>Case I</u> Roots – Real & Different

$m_1 \neq m_2$

$$\boxed{C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$c_1, c_2 \rightarrow$ arbitrary constant

$m_1 \neq m_2 \neq m_3$

$$\boxed{CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}}$$

$c_1, c_2, c_3 \rightarrow$ arbitrary constant

<u>Case II</u> Roots → Real & equal

$m_1 = m_2 = m$

$$\boxed{CF = [c_1 + c_2 x] e^{mx}}$$

$m_1 = m_2 = m_3 = m$

$$\boxed{CF = [c_1 + c_2 x + c_3 x^2] e^{mx}}$$

<u>Case III</u> complex Roots [Only for quadratic]

$m = \alpha \pm i\beta$

$$\boxed{CF = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]}$$

← for cubic one root can be solved & quadratic can be solved using this method

Rules to find Particular Integral (P.I.)

I. e^{ax}

II. $\sin ax$ or $\cos ax$

III. x^n $(n \to +ve)$

IV. $e^{ax} \cdot \phi(x)$ → $\sin ax$ or $\cos ax$, x^n

V. $x^n \cdot \sin ax$ or $x^n \cdot \cos ax$

$$P.I. = \frac{1}{f(D)} \times (I, II, III, IV, V)$$

$f(D) = D^2 + PD + Q$

$f(D) = D^3 + PD^2 + QD + t$

$f(D) \neq 0$

P.I. only deals with $f(D)$ only in Denominator

Target – D must be removed in $f(D)$ & bring constant.

Case I e^{ax}

eg: $PI = \frac{1}{f(D)} \cdot e^{ax}$

$= \frac{1}{D^3+3D^2+3D+1} \cdot e^{ax}$

[put $D = a = -1$]

$= \frac{1}{(-1)^3+3(-1)^2+3(-1)+1} \cdot e^{-x}$

$= \frac{1}{0} e^{-x}$

$\because f(D) = f(a) = 0$

[$\therefore$ $N^r \to$ multiply by independent variable
$D^r \to f'(D)$]

$\hookrightarrow P.I = \frac{x}{(3D^2+6D+3)} \cdot e^{-x}$

$= \frac{x}{3(-1)^2+6(-1)+3} \cdot e^{-x}$

$= \frac{x}{0} e^{-x}$

$\because f'(D) = f'(a) = 0$

$\hookrightarrow PI = \frac{x^2}{f''(D)} \cdot e^{ax}$

$= \frac{x^2}{6D+6} e^{-x} = \frac{x^2}{6(-1)+6} \cdot e^{-x}$

$= \frac{x^2}{0} e^{-x}$

$\hookrightarrow PI = \frac{x^3}{f'''(D)} \cdot e^{ax} = \frac{x^3}{6} \cdot e^{-x}$

Case II: $\cos ax$ or $\sin ax$

$$PI = \frac{1}{f(D)} \cdot \sin ax$$

↳ put the value of a in D^2 only.

$D^3 = D.D^2$; $D^4 = D^2 . D^2$

$D^2 = (a)^2$

$= -a^2$ (put −ve sign)

Example: $\frac{d^2y}{dx^2} + 4y = \cos 2x$

$D = \frac{d}{dx}$

$D^2y + 4y = \cos 2x$

$(D^2+4)y = \cos 2x$

find CF.

$(D^2+4)y = 0$

let $D = m$ & $y = 1$

~~D^2+4~~

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$

$m = 0 \pm 2i$

$m = \alpha \pm i\beta$

$\alpha = 0$

$\beta = 2$

$CF = c_1 \cos 2x + c_2 \sin 2x$

find P.I.

$$P.I. = \frac{1}{f(D)} \cdot \cos 2x$$

$$= \frac{1}{D^2+4} \cos 2x$$

$$= \frac{1}{-4+4} \cos 2x = \frac{1}{0} \cos 2x$$

$f(D) = f(-a^2) = 0$

$$PI = \frac{x}{f'(D)} \cos 2x = \frac{x}{2D} \cos 2x$$

$$= \frac{xD}{2D^2} \cos 2x = \frac{xD}{2(-4)} \cos 2x$$

$$= \frac{x}{-8} D \cos 2x = \frac{x}{-8} \frac{d}{dx} (\cos 2x)$$

$$PI = \frac{+\cancel{2}\sin 2x \cdot x}{+\cancel{8}4} = \frac{x}{4} \cdot \sin 2x$$

★★ $PI = \frac{1}{D^3-3D^2+4D-2}(\cos x + e^{2x})$

$PI = I_1 + I_2$

solve I_1 & I_2 separately.

★ when P.I. = 5 (any constant)

$= 5e^{0x}$

Case III x^n

Ques: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$

D^2+2D+1

$\boxed{m = -1, -1}$

for P.I.

↳ focus on D ki small power or constant in $f(D)$.

↳ Take common to make 1 in expression.

1. $PI = \frac{1}{D^2+2D+1} \cdot x$

$= \frac{1}{1}\left[\frac{1}{D^2+2D+1}\right]x$

$= \left[\frac{1}{1+2D+D^2}\right]x$

Bracket mein likho

$= \left[\frac{1}{1+(2D+D^2)}\right]x$

fraction khatam

$= \left[1+(2D+D^2)\right]^{-1} \cdot x$

2. $PI = \frac{1}{D^2-2D+4} \cdot x^2$

$= \frac{1}{4}\left[\frac{1}{\frac{D^2}{4} - \frac{2D}{4} + 1}\right]x^2$

changing order

$= \frac{1}{4}\left[\frac{1}{1-\frac{D}{2}+\frac{D^2}{4}}\right]x^2$

$= \frac{1}{4}\left[\frac{1}{1+\left(-\frac{D}{2}+\frac{D^2}{4}\right)}\right]x^2$

$= \frac{1}{4}\left[1+\left(-\frac{D}{2}+\frac{D^2}{4}\right)\right]^{-1}x^2$

Binomial series

$$(1+y)^{-1} = 1-y+y^2-y^3+y^4 \ldots$$

$$(1-y)^{-1} = 1+y+y^2+y^3+y^4 \ldots$$

#check the power on x, if it is 1 then open 2 steps of binomial series, if it is 2 then open 3 steps.

$$PI = [1-(2D+D^2)+(2D+D^2)^2 \ldots]x$$

$$= [1-(2D+D^2)]x$$

$$= x - 2D(x) - D^2(x)$$

$$= (x-2)$$

$$PI = \frac{1}{4}\left[1-\left(\frac{-D}{2}+\frac{D^2}{4}\right)+\left(\frac{-D}{2}+\frac{D^2}{4}\right)^2 \ldots\right]x^2$$

$$= \frac{1}{4}\left[1+\frac{D}{2}-\cancel{\frac{D^2}{4}}+\cancel{\frac{D^2}{4}}+\frac{D^4}{16}-\frac{D^3}{4}\right]x^2$$

$$= \frac{1}{4}\left[1+\frac{D}{2}+\frac{D^4}{16}-\frac{D^3}{4}\right]x^2$$

$$= \frac{1}{4}\left[1+\frac{D}{2}\right]x^2$$

$$= \frac{1}{4}[x^2+x]$$

Case IV $PI = \frac{1}{f(D)} \cdot e^{ax} \cdot \phi(x)$ → $\sin ax$ or $\cos ax$, x^n

e^{ax} ↓ BSF, $\phi(x)$ ↓ Army

$$= e^{ax}\left[\frac{1}{f(D+a)}\right] \cdot \phi(x)$$

Ques: $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x} \cdot \sin x$

$m = 2, 5, 0$

$$PI = \frac{1}{D^3 - 7D^2 + 10D} \cdot e^{2x} \cdot \sin x$$

$$= e^{2x}\left[\frac{1}{(D+2)^3 - 7(D+2)^2 + 10(D+2)}\right]\sin x$$

$$= e^{2x}\left[\frac{1}{D^3 + 8 + 6D^2 + 12D - 7(D^2 + 4 + 4D) + 10D + 20}\right]\sin x$$

$$= e^{2x}\left[\frac{1}{D^3 - D^2 - 6D}\right]\sin x$$

$$= e^{2x}\left[\frac{1}{D^3 - D^2 - 6D}\right]\sin x$$

$$= e^{2x}\left[\frac{1}{-D + 1 - 6D}\right]\sin x = e^{2x}\left[\frac{1}{1 - 7D}\right]\sin x$$

$$= e^{2x}\left[\frac{(1+7D)}{(1-7D)(1+7D)}\right]\sin x = e^{2x}\left[\frac{1+7D}{1-49D^2}\right]\sin x$$

$$= e^{2x}\left[\frac{1+7D}{1+49}\right]\sin x = \frac{e^{2x}}{50}\left[7D+1\right]\sin x$$

$$= \frac{e^{2x}}{50}\left[7D\sin x + \sin x\right] = \frac{e^{2x}}{50}\left[7\cos x + \sin x\right]$$

Case V $x^n \cdot \sin ax$ or $x^n \cdot \cos ax$

UNIT-2 (Differential calculus)

I.D.V → ~~Indefinite~~ Independent Variable
D.V → Dependent Variable

1 I.D.V → Ordinary derivative
2 & more I.D.V → Partial derivative

Radius of curvature :- (ρ)

Cartesian form (x,y)	Parametric (x,y)	Polar form (r,θ)
→ Only 1 eqn, two variables	Two eqn, two variable	$r = f(\theta)$
→ IDV and DV not defined.	Both are DV & have t, v, u → IDV common IDV.	$y = r\sin\theta$
→ $y = f(x)$ or $x = f(y)$	$x = f(t)$	$x = r\cos\theta$
	$y = f(t)$	

formula : Cartesian form

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

when, $x \to$ IDV
$y \to$ DV
$\boxed{y = f(x)}$

or

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

when $x \to$ DV
$y \to$ IDV
$\boxed{x = f(y)}$

Here $\frac{d^2y}{dx^2} \neq 0 \longrightarrow$ It can never be zero.

Formula : Parametric form –

$$\boxed{\rho = \frac{[x_1^2 + y_1^2]^{3/2}}{x_1 y_2 - y_1 x_2}}$$

↳ $x_1 = \frac{dx}{d\theta}$ or $\frac{dx}{dt}$ ↳ $x_2 = \frac{d^2x}{d\theta^2}$ or $\frac{d^2x}{dt^2}$

↳ $y_1 = \frac{dy}{d\theta}$ or $\frac{dy}{dt}$ ↳ $y_2 = \frac{d^2y}{d\theta^2}$ or $\frac{d^2y}{dt^2}$

Formula : Polar form –

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - r r_2 + 2r_1^2}}$$

- $r_1 = \frac{dr}{d\theta}$
- $r_2 = \frac{d^2r}{d\theta^2}$

r ← dependent (always) because radius will always be same.

θ ← can change
↳ Independent

* curvature = $\frac{1}{\text{Radius of curvature } (\rho)}$

Derivate

(unit 2) O.D. $\left(\frac{d}{dx}\right)$ only one IDV involved

One/two variable in equ^n

Partial (unit 4) Two or more IDV involved $\left(\frac{\partial}{\partial x}\right)$

Three or more variables.

conversion $\longrightarrow$ cartesian to Parametric form

1. $y^2 = 4ax$ (Parabola) $x^2 = 4ay$

$x = at^2$ $\quad$ $x = 2at$

$y = 2at$ $\quad$ $y = at^2$

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\sin\theta$ or $a\cos\theta$

$y = b\cos\theta$ or $b\sin\theta$

3. $\sqrt{x} + \sqrt{y} = 1$

$x = \sin^4\theta$ or $\cos^4\theta$

$y = \cos^4\theta$ $\quad$ $\sin^4\theta$

4. $x^2 + y^2 = r^2$

$x = r\sin\theta$ $\quad$ $r\cos\theta$

$y = r\cos\theta$ or $r\sin\theta$

5. $x^{2/3} + y^{2/3} = a^{2/3}$

$x = a\sin^3\theta$ or $a\cos^3\theta$

$y = a\cos^3\theta$ $\quad$ $a\sin^3\theta$

centre of curvature

Represented by $\longrightarrow (\bar{x}, \bar{y})$

$$\bar{x} = x - \frac{[y_1(1+y_1^2)]}{y_2}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

where, $y_1 = \frac{dy}{dx}$ & $y_2 = \frac{d^2y}{dx^2}$

circle of curvature —

$$(x-\bar{x})^2 + (y-\bar{y})^2 = (\rho)^2$$

EVALUTES & INVALUTES

Let Q be the centre of curvature of a given curve "C" at the point P on it. when P moves on curve "C" and takes different positions, Q will also take diff. position and move on another curve C'. This curve C' is called the evalute of curve C.

Thus evalute can be defined as the locus of the centre of curvature when C' is the evaluate of curve C

C is called the invalute of the curve C'.

Ans Find the Evolute of parabola $x^2 = 4ay$.

Parametric form – $x = 2at$, $y = at^2$

$x = 2at$ $\qquad$ $y = at^2$

$\frac{dx}{dt} = 2a$ $\qquad$ $\frac{dy}{dt} = 2at$

$$y_1 = \frac{dy}{dx} = \frac{2at}{2a} = t$$

$\rightarrow y_1 = t$

$$\rightarrow y_2 = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]\frac{dt}{dx} = \frac{d(t)}{dt}\frac{1}{2a} = \frac{1}{2a}$$

Find centre of curvature

WKT, $\bar{x} = x - \frac{[y_1(1+y_1^2)]}{y_2}$

$$\cancel{\bar{x} = 2at - \frac{[t(1+t^2)]}{1} = 2at - t - t^3}$$

$$\bar{x} = 2at - \frac{[t(1+t^2)]2a}{1}$$

$$= 2at - [t+t^3]2a$$

$$= \cancel{2at} - \cancel{2at} - 2at^3$$

$$\bar{x} = -2at^3 \quad \text{— (A)}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= at^2 + \left(\frac{1+t^2}{1/2a}\right)$$

$$= at^2 + 2a + 2at^2$$

$$\bar{y} = 3at^2 + 2a \quad \text{— (B)}$$

Removing t & merging to make a single equ.

$$t^3 = \frac{-\bar{x}}{2a} \qquad t^2 = \frac{\bar{y} - 2a}{3a}$$

$$\cancel{\therefore \frac{-\bar{x}}{2at} = \frac{\bar{y}-2a}{3a}} \qquad t^6 = \frac{(\bar{x})^2}{4a^2}, \quad t^6 = \frac{(\bar{y}-2a)^3}{27a^3}$$

centre of curvature

$$\frac{(\bar{x})^2}{(2a)^2} = \frac{(\bar{y}-2a)^3}{27a^3}$$

$$27a^{\cancel{3}}(\bar{x})^2 = (\bar{y}-2a)^3 4\cancel{a^2}$$

Replace $\bar{x} = x$ & $\bar{y} = y$

where

$C' \rightarrow 27ax^2 = (y-2a)^3 4$ —— evalute //

#

Envelope :

Method of fin
curves. Let
of curves.

Method 1: Her

$f(x, y, \omega)$

$\frac{\partial f(x,y,\omega)}{\partial \omega}$

Method 2: G

x^2+4

y^2+4

Az^2+

(i) B^2-

(ii) B^2-

(iii) B^2

$A\omega^2+$

(x,y)

B^2-4

... of curvature at the point $(\frac{1}{4}, \frac{1}{4})$ on the

Envelope :

Method of finding the equ. of the envelope of a family of curves. Let $f(x, y, \omega) = 0$ be the equ. of given family of curves, where ω is parameter.

Method 1: Here, $x, y, \omega \rightarrow$ Independent
$\therefore$ Real

$$f(x, y, \omega) = 0$$

$$\frac{\partial f(x, y, \omega)}{\partial \omega} = 0$$

Method 2: Quadratic form

$$x^2 + 4x + 2 = 0$$
$$y^2 + 4y + 2 = 0$$
$$Az^2 + Bz + C = 0$$

(i) $B^2 - 4AC > 0$ ✗
(ii) $B^2 - 4AC < 0$ ✗
(iii) $B^2 - 4AC = 0$ ✓

$$A\omega^2 + B\omega + C = 0$$
$$\begin{array}{ccc} | & | & | \\ (x,y) & (x,y) & (x,y) \end{array}$$

$$B^2 - 4AC = 0$$

Find the envelope of family of lines $y = mx + \frac{a}{m}$ — ①

$f(x, y, m) = 0$

$y - mx - \frac{a}{m} = 0$ — ②

Method 1

$$\frac{\partial}{\partial m}\left[y - mx - \frac{a}{m}\right] = 0$$

$$0 - x + \frac{a}{m^2} = 0$$

$$x = \frac{a}{m^2}$$

$$m^2 = \frac{a}{x} \quad \text{— ③}$$

from equ. ①

$$y^2 = \left(mx + \frac{a}{m}\right)^2$$

$$y^2 = m^2x^2 + \frac{a^2}{m^2} + 2\cancel{m}x\frac{a}{\cancel{m}}$$

$$y^2 = \frac{a}{\cancel{x}}x^{\cancel{2}} + \frac{a^{\cancel{2}}}{\cancel{x}}\cancel{x} + 2xa$$

$$y^2 = ax + ax + 2ax$$

$$\boxed{y^2 = 4ax}$$

Method 2.

$$y = mx + \frac{a}{m} \quad \text{— ①}$$

$$my = m^2x + a$$

$$m^2x - my + a = 0$$

$$(x)m^2 + (-y)m + (a) = 0$$

$$B^2 - 4AC = 0$$

$$(-y)^2 - 4(x)(a) = 0$$

$$y^2 - 4ax = 0$$

$$\boxed{y^2 = 4ax}$$

Ques 1 find the radius of curvature at the point $(\frac{1}{4}, \frac{1}{4})$ on the curve $\sqrt{x} + \sqrt{y} = 1$

$\sqrt{x} + \sqrt{y} = 1$ —— (1)

Differentiating both sides wrt. x

$$\frac{1}{2}\frac{1}{\sqrt{x}} + \frac{1}{2}\frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \text{—— (2)}$$

$$\left(\frac{dy}{dx}\right)_{\frac{1}{4}, \frac{1}{4}} = \frac{-\sqrt{1/4}}{\sqrt{1/4}} \quad \text{—— (3)}$$

$$= -1$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$= \frac{-\left[\sqrt{x}\frac{d}{dx}\sqrt{y} - \sqrt{y}\frac{d}{dx}\sqrt{x}\right]}{(\sqrt{x})^2}$$

$$= \frac{-\left[\frac{1}{2}\frac{\sqrt{x}}{\sqrt{y}}\left(\frac{dy}{dx}\right) - \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}}\right]}{x}$$

$$= \frac{-\left[\frac{1}{2}\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{y}}}\left(+\frac{\cancel{\sqrt{y}}}{\cancel{\sqrt{x}}}\right) + \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}}\right]}{x}$$

$$= \frac{\left(1 + \frac{\sqrt{y}}{\sqrt{x}}\right)}{2x}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} \quad \text{—— (4)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\frac{1}{4}, \frac{1}{4}} = \frac{\sqrt{1/4} + \sqrt{1/4}}{2 \times \frac{1}{4}\sqrt{\frac{1}{4}}}$$

$$= \frac{2\cancel{\sqrt{1/4}}}{2 \cdot \frac{1}{4} \cdot \cancel{\sqrt{1/4}}} = 4 \quad \text{—— (5)}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \left[\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right]_{\frac{1}{4}, \frac{1}{4}}$$

$$= \frac{\left[1 + (-1)^2\right]^{3/2}}{4}$$

$$= \frac{2^{3/2}}{4} = \frac{8^{1/2}}{4} = \frac{\sqrt{8}}{4}$$

$$\rho = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\boxed{\rho = \frac{1}{\sqrt{2}}}$$

Ques 2 find the radius of curvature at the point "θ" on the curve $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$.

Meth MI Parametric form —

$$x_1 = \frac{d(x)}{d\theta} = \frac{d}{d\theta}[3a\cos\theta - a\cos 3\theta] = -3a\sin\theta + 3a\sin 3\theta$$

$$= 3a(\sin 3\theta - \sin\theta)$$

$$x_2 = \frac{d(x_1)}{d\theta} = 3a[3\cos 3\theta - \cos\theta]$$

$$y_1 = \frac{d(y)}{d\theta} = \frac{d}{d\theta}[3a\sin\theta - a\sin 3\theta] = 3a\cos\theta - 3a\cos 3\theta$$

$$= 3a(\cos\theta - \cos 3\theta)$$

$$y_2 = \frac{d(y_1)}{d\theta} = 3a[-\sin\theta - (-\sin 3\theta)\cdot 3]$$

$$= 3a[-\sin\theta + \sin 3\theta \cdot 3] = 3a[3\sin 3\theta - \sin\theta]$$

$$\rho = \frac{[x_1^2 + y_1^2]^{3/2}}{x_1 y_2 - y_1 x_2}$$

$$= \frac{\left\{[3a[\sin 3\theta - \sin\theta]]^2 + [3a[\cos\theta - \cos 3\theta]]^2\right\}^{3/2}}{[3a(\sin 3\theta - \sin\theta)][3a(3\sin 3\theta - \sin\theta] - [3a(\cos\theta - \cos 3\theta]\,[3a(3\cos 3\theta - \cos\theta)]}$$

$$= \frac{(3a)^3[(\sin 3\theta - \sin\theta)^2 + (\cos\theta - \cos 3\theta)^2]^{3/2}}{9a^2[(\sin 3\theta - \sin\theta)(3\sin 3\theta - \sin\theta) - (\cos\theta - \cos 3\theta)(3\cos 3\theta - \cos\theta)]}$$

UNIT-(2)

DIFFERENTIAL CALCULUS

1 I.D.V. → Ordinary Derivative
2 or more I.D.V → Partial Derivative

$\frac{N}{D}$ N → D.V.
D → I.D.V.

D.V. → Dependent Variable
I.D.V. → Independent variable

RADIUS OF CURVATURE (ρ)

Cartesian form (x, y)

- 1 eq^n, 2 variables
- when it is/is NOT defined which is D.V. and which is I.D.V.
- $y = f(x)$ (or) $x = f(y)$

Parametric Form (x, y)

- 2 eq^n, 2 variables
- Both the variables are dependent but having common I.D.V.
- $x = f(t)$ or $f(\theta)$ or $f(u)$
  $y = f(t)$ or $f(\theta)$ or $f(u)$

Polar Form (r, θ)

- $r = f(\theta)$
- r → Dependent
  θ → Independent [Always]

★ Cartesian $\xrightarrow{to}$ Polar

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

■ Formula : Cartesian Form

$$\boxed{\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

$\frac{d^2y}{dx^2} \neq 0$

when, x → I.D.V
y → D.V
$\boxed{y = f(x)}$

$$\boxed{\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{d^2y}}}$$

$\frac{d^2x}{d^2y} \neq 0$

when, x → D.V.
y → I.D.V.
$\boxed{x = f(y)}$

"Think, what you say, and what you do are in harmony." —Mahatma Gandhi

Formula: Parametric Form

$$\boxed{\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}}$$

$x_1 = \frac{dx}{d\theta}$ or $\frac{dx}{dt}$; $x_2 = \frac{d^2x}{d\theta^2}$ or $\frac{d^2x}{dt^2}$

$y_1 = \frac{dy}{d\theta}$ or $\frac{dy}{dt}$; $y_2 = \frac{d^2y}{d\theta^2}$ or $\frac{d^2y}{dt^2}$

Formula: Polar Form

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - r r_2 + 2r_1^2}}$$

$r_1 = \frac{dr}{d\theta}$; $r_2 = \frac{d^2r}{d\theta^2}$

Curvature $= \frac{1}{\text{Radius of curvature}(\rho)}$

Q Find the radius of curvature at the pt. $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

Ans $x^3 + y^3 = 3axy$ — (1)

$\Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$

$\Rightarrow \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$ — (2)

$$\left(\frac{dy}{dx}\right)_{\frac{3a}{2}, \frac{3a}{2}} = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}$$

$$= \frac{\left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}$$

$$= \frac{\left(\frac{-3a^2}{4}\right)}{\left(\frac{3a^2}{4}\right)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\frac{3a}{2}, \frac{3a}{2}} = -1 \quad \text{——} \; (3)$$

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left[a\frac{dy}{dx} - 2x\right] - (ay - x^2)\left[2y\frac{dy}{dx} - a\right]}{(y^2 - ax)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\frac{3a}{2}, \frac{3a}{2}} = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left[-a - 3a\right] - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left[-3a - a\right.}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{(-4a - 4a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}$$

$$= \frac{-8a}{\left(\frac{3a^2}{4}\right)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\frac{3a}{2}, \frac{3a}{2}} = -\frac{32}{3a} \quad \text{——} \; (4)$$

Now, $$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$$

... when what you think, what you say, and what you do are in harmony." —Mahatma Gandhi

$$\rho = \left[\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}\right]_{\frac{3a}{2},\frac{3a}{2}}$$

$$\Rightarrow \rho = \frac{\left[1+(-1)^2\right]^{3/2}}{\left(\frac{-32}{3a}\right)}$$

$$\Rightarrow \rho = \frac{\cancel{2}\sqrt{2} \times 3a}{-\cancel{32}\,\cancel{16}\,8\sqrt{2}}$$

$$\Rightarrow \rho = \frac{-3a}{8\sqrt{2}}$$

$\because$ ρ can't be $-$ve

$$\therefore \boxed{|\rho| = \frac{3a}{8\sqrt{2}}}$$ //.

Q Find the radius of curvature at the pt 'θ' on the curve;

$$\left[\begin{array}{l} x = 3a\cos\theta - a\cos 3\theta \quad \text{--- (1)} \\ y = 3a\sin\theta - a\sin 3\theta \quad \text{--- (2)} \end{array}\right]$$

Ans (M-I)

$$x_1 = \frac{dx}{d\theta} = -3a\sin\theta + 3a\sin 3\theta$$

$$= 3a(\sin 3\theta - \sin\theta) \quad \text{--- (3)}$$

$$x_2 = \frac{d^2x}{d\theta^2} = 3a(3\cos 3\theta - \cos\theta) \quad \text{--- (4)}$$

$$y_1 = \frac{dy}{d\theta} = 3a\cos\theta - 3a\cos 3\theta$$

$$= 3a(\cos\theta - \cos 3\theta) \quad \text{--- (5)}$$

$$y_2 = \frac{d^2y}{d\theta^2} = 3a(-\sin\theta + 3\sin 3\theta)$$

$$= 3a(3\sin 3\theta - \sin\theta) \quad \text{--- (6)}$$

Now, $\rho = \dfrac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$

$$\Rightarrow \rho = \frac{\left[\{3a(\sin 3\theta - \sin\theta)\}^2 + \{3a(\cos\theta - \cos 3\theta)\}^2\right]^{3/2}}{\{3a(\sin 3\theta - \sin\theta)\}\{3a(3\sin 3\theta - \sin\theta)\} - \{3a(\cos\theta - \cos 3\theta)\}\{3a(3\cos 3\theta - \cos\theta)\}}$$

$$\Rightarrow \rho = \frac{((3a)^2)^{3/2}\left[\sin^2 3\theta + \sin^2\theta - 2\sin 3\theta \sin\theta + \cos^2\theta + \cos^2 3\theta - 2\cos 3\theta \cos\theta\right]^{3/2}}{9a^2\left[3\sin^2 3\theta - \sin\theta \sin 3\theta - 3\sin 3\theta \sin\theta + \sin^2\theta - 3\cos 3\theta \cos\theta + \cos^2\theta + 3\cos^2 3\theta - \cos 3\theta \cos\theta\right]}$$

$$\Rightarrow \rho = \frac{\cancel{27}^{3}\cancel{a^3}\left[1 + 1 - 2(\cos 3\theta \cos\theta + \sin 3\theta \sin\theta)\right]^{3/2}}{\cancel{9a^2}\left[3 + 1 - 4(\cos 3\theta \cos\theta + \sin 3\theta \sin\theta)\right]}$$

$$\Rightarrow \rho = \frac{3a\left[2(1 - \cos 2\theta)\right]^{3/2}}{4(1 - \cos 2\theta)}$$

$$\Rightarrow \rho = \frac{3a}{\cancel{4}2}(\cancel{2}\sqrt{2})(1 - \cos 2\theta)^{1/2}$$

$$\Rightarrow \rho = \frac{3a}{\sqrt{2}}(2\sin^2\theta)^{1/2}$$

$$\Rightarrow \rho = \frac{3a}{\cancel{\sqrt{2}}}(\cancel{\sqrt{2}})\sin\theta$$

$$\Rightarrow \boxed{\rho = 3a\sin\theta} \quad //.$$

(M-II) [Parametric form $\longrightarrow$ cartesian form]

$$x = 3a\cos\theta - a\cos 3\theta \quad \text{—(1)}$$

$$\frac{dx}{d\theta} = -3a\sin\theta + 3a\sin 3\theta$$

$$= 3a(\sin 3\theta - \sin\theta) \quad \text{—(2)}$$

... what you say, and what you do are in harmony." —Mahatma Gandhi

$2a\sin\theta)$

$$y = 3a\sin\theta - a\sin 3\theta \quad \text{—} (3)$$

$$\frac{dy}{d\theta} = 3a\cos\theta - 3a\cos 3\theta$$

$$= 3a(\cos\theta - \cos 3\theta) \quad \text{—} (4)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{3a}(\cos\theta - \cos 3\theta)}{\cancel{3a}(\sin 3\theta - \sin\theta)} \quad (\text{from } (2) \,\&\, (4))$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{2}\sin 2\theta \, \cancel{\sin\theta}}{\cancel{2}\cos 2\theta \, \cancel{\sin\theta}}$$

$$\Rightarrow \frac{dy}{dx} = \tan 2\theta \quad \text{—} (5)$$

Now, $$\boxed{\frac{d^2y}{dx^2} = \left[\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\right]\frac{d\theta}{dx}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left[\frac{d}{d\theta}(\tan 2\theta)\right]\frac{1}{3a(\sin 3\theta - \sin\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cancel{2}\sec^2 2\theta \, \frac{1}{3a(\cancel{2}\cos 2\theta \sin\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3 2\theta}{3a\sin\theta} \quad \text{—} (6)$$

Now, $$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$$

$$\boxed{\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$$

$$\boxed{\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$\Rightarrow \rho = \frac{\left[1+(\tan 2\theta)^2\right]^{3/2}}{\left(\frac{\sec^3 2\theta}{3a\sin\theta}\right)}$$ (from ⑤ & ⑥)

$$\Rightarrow \rho = \frac{(1+\tan^2 2\theta)^{3/2}}{\left(\frac{\sec^3 2\theta}{3a\sin\theta}\right)}$$

$$\Rightarrow \rho = \frac{(\sec^2 2\theta)^{3/2}}{\sec^3 2\theta}(3a\sin\theta)$$

$$\Rightarrow \boxed{\rho = 3a\sin\theta}$$

Cartesian to Parametric Form Imp. Conversions

Cartesian Form	Parametric Form	
① Parabola: (a) $y^2 = 4ax$	(a) $x = at^2$, $y = 2at$	
(b) $x^2 = 4ay$	(b) $x = 2at$, $y = at^2$	
② Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\sin\theta$, $y = b\cos\theta$ (OR)	$x = a\cos\theta$, $y = b\sin\theta$
③ Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta$, $y = b\tan\theta$ (OR)	$x = a\operatorname{cosec}\theta$, $y = b\cot\theta$
④ $\sqrt{x} + \sqrt{y} = 1$	$x = \sin^4\theta$, $y = \cos^4\theta$	
⑤ $x^2 + y^2 = r^2$	$x = r\cos\theta$, $y = r\sin\theta$ (OR)	$x = r\sin\theta$, $y = r\cos\theta$
⑥ $x^2 - y^2 = r^2$	$x = r\sec\theta$, $y = r\tan\theta$ (OR)	$x = r\operatorname{cosec}\theta$, $y = r\cot\theta$
⑦ $x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\sin^3\theta$, $y = a\cos^3\theta$ (OR)	$x = a\cos^3\theta$, $y = a\sin^3\theta$

Q Show that the radius of curvature of the curve $r^n = a^n \sin(n\theta)$ at the pole (point) is $\rho = \frac{a^n}{(n+1)r^{n-1}}$

Ans

$$r^n = a^n \sin(n\theta) \quad \text{—— (1)}$$

$$\cancel{n r^{n-1} \frac{dr}{d\theta} = a^n (n) \cos(n\theta)}$$

$$\cancel{\Rightarrow r_1 = \frac{a^n \cos(n\theta)}{r^{n-1}}}$$

$$\cancel{\Rightarrow r_2 = \frac{a^n \cos(n\theta)}{r_1^{n-1}}}$$

$$\cancel{\Rightarrow r_2 = \frac{\cancel{a^n \cos(n\theta)}\ a^n (-n) \sin(n\theta)}{\left(\frac{a^n \cos(n\theta)}{r^{n-1}}\right)^{n-1}}}$$

$$\cancel{\Rightarrow r_2 = \frac{-n a^n \sin(n\theta)}{a^{n^2-n} \cos^{n-1}(n\theta)}\ r^{(n-1)^2}}$$

Not a good approach!

Better approach

$$\boxed{r^n = a^n \sin(n\theta)} \quad \text{—— (1)}$$

Take 'log' on both sides of eqn (1)

$$\log(r^n) = \log(a^n \sin(n\theta))$$

$$\Rightarrow n \log r = n \log a + \log(\sin(n\theta))$$

$$\Rightarrow \frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin(n\theta)}(n)\cos(n\theta)$$

$$\Rightarrow \frac{\cancel{n}}{r} r_1 = \frac{\cancel{n}}{\sin(n\theta)} \cos(n\theta) \qquad \left[r_1 = \frac{dr}{d\theta}\right]$$

$$\Rightarrow \boxed{r_1 = r\cot(n\theta)} \quad \text{—— (2)}$$

(these are 2 fn so applying $(uv)' = uv' + vu'$)

$$r_2 = r_1 \cot(n\theta) + r(-n)\operatorname{cosec}^2(n\theta) \qquad \left[r_2 = \frac{d^2r}{d\theta^2}\right]$$

$$\Rightarrow \boxed{r_2 = r\cot^2(n\theta) - nr\operatorname{cosec}^2(n\theta)} \quad \text{—— (3)}$$

$$\because \boxed{\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2}}$$

$$\therefore \rho = \frac{[r^2 + r^2\cot^2(n\theta)]^{3/2}}{r^2 - r[r\cot^2(n\theta) - nr\,\mathrm{cosec}^2(n\theta)] + 2r^2\cot^2(n\theta)}$$

$$\Rightarrow \rho = \frac{r^3\,\mathrm{cosec}^3(n\theta)}{r^2[1 - \cot^2(n\theta) + n\,\mathrm{cosec}^2(n\theta) + 2\cot^2(n\theta)]}$$

$$\Rightarrow \rho = \frac{r\,\mathrm{cosec}^3(n\theta)}{[1 + \cot^2(n\theta) + n\,\mathrm{cosec}^2(n\theta)]}$$

$$\Rightarrow \rho = \frac{r\,\mathrm{cosec}^3(n\theta)}{[\mathrm{cosec}^2(n\theta) + n\,\mathrm{cosec}^2(n\theta)]}$$

$$\Rightarrow \rho = \frac{r\,\mathrm{cosec}^{\cancel{3}}(n\theta)}{\cancel{\mathrm{cosec}^2(n\theta)}(1+n)}$$

$$\Rightarrow \boxed{\rho = \frac{r}{(n+1)}\,\mathrm{cosec}(n\theta)} \quad \text{— (A)}$$

from eq^n (1)

$$r^n = a^n \sin(n\theta)$$

$$\mathrm{cosec}(n\theta) = \frac{a^n}{r^n} \quad \text{— (4)}$$

put (4) in (A)

$$\therefore \rho = \frac{r}{(n+1)}\frac{a^n}{r^n}$$

$$\Rightarrow \boxed{\rho = \frac{a^n}{(n+1)\,r^{n-1}}} \longrightarrow \text{Hence, proved.}$$

Q Find the radius of curvature of the curve $r = a(1+\cos\theta)$. at $\theta = \frac{\pi}{2}$.

Ans $r = a(1+\cos\theta)$

$$\Rightarrow r = a + a\cos\theta \quad \text{— (1)}$$

$$r_1 = \frac{dr}{d\theta} = 0 - a\sin\theta$$

$$\Rightarrow r_1 = -a\sin\theta \quad \text{— (2)}$$

Also, $\rho = (1 + u_1^2)^{3/2}$

$r_2 = -a\cos\theta$ — (2)

Now, $\boxed{\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2}}$

$$\therefore \rho = \frac{\left[(a + a\cos\theta)^2 + (-a\sin\theta)^2\right]^{3/2}}{(a + a\cos\theta)^2 - (a + a\cos\theta)(-a\cos\theta) + 2(-a\sin\theta)^2}$$

$$\Rightarrow \rho = \frac{(a^2)^{3/2}\left[1 + \cos^2\theta + 2\cos\theta + \sin^2\theta\right]^{3/2}}{a^2\left[1 + \cos^2\theta + 2\cos\theta + \cos^2\theta + \cos\theta + 2\sin^2\theta\right]}$$

$$\Rightarrow \rho = \frac{a\left[2(1 + \cos\theta)\right]^{3/2}}{\left[3(1 + \cos\theta)\right]}$$

$$\Rightarrow \rho = \frac{a\,(2)^{3/2}(1 + \cos\theta)^{1/2}}{3}$$

$$\Rightarrow \rho = \frac{a(2\sqrt{2})}{3}\left[2\cos^2(\theta/2)\right]^{1/2}$$

$$\Rightarrow \rho = \frac{2\sqrt{2}}{3}a(\sqrt{2})\cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \boxed{\rho = \frac{4a}{3}\cos\left(\frac{\theta}{2}\right)}$$

at $\theta = \pi/2$

$$\rho = \frac{4a}{3}\cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \rho = \frac{4a}{3}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \boxed{\rho = \frac{2\sqrt{2}a}{3}} \;//$$

P.T.O. →

Centre of curvature

Represented by $(\bar{x}, \bar{y})$

$(\bar{x}, \bar{y}) \rightarrow$ it is a pt. through which circle passes

$$\boxed{\bar{x} = x - \left[\frac{y_1(1+y_1^2)}{y_2}\right]}$$

$$\boxed{\bar{y} = y + \left[\frac{(1+y_1^2)}{y_2}\right]}$$

where, $\left(y_1 = \frac{dy}{dx} \;\&\; y_2 = \frac{d^2y}{dx^2}\right)$

Circle of curvature

$$\boxed{(x-\bar{x})^2 + (y-\bar{y})^2 = (\rho)^2}$$

★ Never put value of x and y in $(x-\bar{x})^2 + (y-\bar{y})^2 = (\rho)^2$

Q Find the eqn of the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Ans $\sqrt{x} + \sqrt{y} = \sqrt{a}$ —①

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}} \quad —②$$

$$\left(\frac{dy}{dx}\right)_{\frac{a}{4}, \frac{a}{4}} = -\sqrt{\frac{a/4}{a/4}} = -1$$

$$\Rightarrow (y_1)_{\frac{a}{4}, \frac{a}{4}} = -1 \quad —③$$

[where $y_1 = \frac{dy}{dx}$]

$$\cancel{y_2 = \frac{d^2y}{dx^2} = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right] \times \frac{dt}{dx}}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Also, $\rho = (1+y_1^2)^{3/2}$

$$\Rightarrow y_2 = -\frac{\left[\sqrt{x}\,\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\,\frac{1}{2\sqrt{x}}\right]}{x}$$

$$\Rightarrow y_2 = -\frac{\left[\frac{\sqrt{x}}{\sqrt{y}}\left(\frac{-\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{\sqrt{x}}\right]}{2x} \quad \text{(from ②)}$$

$$\Rightarrow y_2 = \frac{\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}{2x}$$

$$\Rightarrow (y_2)_{\frac{a}{4},\frac{a}{4}} = \frac{1+\frac{\sqrt{a/4}}{\sqrt{a/4}}}{2\left(\frac{a}{4}\right)} = \frac{2\times 2}{a} = \frac{4}{a}$$

$$(y_2)_{\frac{a}{4},\frac{a}{4}} = \frac{4}{a} \quad \text{—— ④}$$

Now, $\bar{x} = x - \left[\frac{y_1(1+y_1^2)}{y_2}\right]$

$$\Rightarrow \bar{x} = \frac{a}{4} - \left[\frac{(-1)(1+(-1)^2)}{4/a}\right] \quad \text{(from ③ \& ④)}$$

$$\Rightarrow \bar{x} = \frac{a}{4} + \frac{(\cancel{2})(a)}{\cancel{4}\,2}$$

$$\Rightarrow \boxed{\bar{x} = \frac{3a}{4}} \quad \text{—— ⑤}$$

$$\bar{y} = y + \left[\frac{1+y_1^2}{y_2}\right]$$

$$\Rightarrow \bar{y} = \frac{a}{4} + \left[\frac{1+(-1)^2}{4/a}\right] \quad \text{(from ③ \& ④)}$$

$$\Rightarrow \bar{y} = \frac{a}{4} + \frac{2\times a}{4}$$

$$\Rightarrow \boxed{\bar{y} = \frac{3a}{4}} \quad \text{—— ⑥}$$

Also, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$\Rightarrow \rho = \dfrac{[1+(-1)^2]^{3/2}}{4/a}$

$\Rightarrow \rho = \dfrac{(2)^{3/2}(a)}{4}$

$\Rightarrow \rho = \dfrac{2\sqrt{2}\,a}{4}$

$\Rightarrow \boxed{\rho = \dfrac{a}{\sqrt{2}}}$ —— (7)

eqⁿ of circle of curvature →

$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

('x' and 'y' की value put मत करिए kyuki eqⁿ ~~निकालनी~~ है)

$\Rightarrow \left(x - \dfrac{3a}{4}\right)^2 + \left(y - \dfrac{3a}{4}\right)^2 = \left(\dfrac{a}{\sqrt{2}}\right)^2$

$\Rightarrow x^2 + \dfrac{9a^2}{16} - 2(x)\left(\dfrac{3a}{4}\right) + y^2 + \dfrac{9a^2}{16} - 2(x)\left(\dfrac{3a}{4}\right) = \dfrac{a^2}{2}$

$\Rightarrow x^2 + y^2 - \dfrac{3a}{2}x - \dfrac{3a}{2}y + \dfrac{9a^2}{8} - \dfrac{a^2}{2} = 0$

$\Rightarrow \boxed{8x^2 + 8y^2 - 12ax - 12ay + 5a^2 = 0}$ ← eqⁿ of circle of curvature

EVALUTES AND INVALUTES

- Let Q be the centre of curvature of a given curve "C" at the point P on it. When P moves on the curve "C" and takes different positions, Q will also take different position and move on another curve C'. This curve C' is called the <u>evalute</u> of the curve C.
- Thus, evalute can be defined as the locus of the centre of curvature when C' is the evalute of the curve C.
- C is called the <u>invalute</u> of the curve C'.

Q Find the evalute of the parabola $x^2 = 4ay$.

Ans Curve C (Invalute) $\Rightarrow x^2 = 4ay$ — ①

Parametric Form:

$x = 2at$ — ②

$y = at^2$ — ③

$\boxed{P = (2at, at^2)}$

$x = 2at$; $y = at^2$

$\frac{dx}{dt} = 2a$ — ④ ; $\frac{dy}{dt} = 2at$ — ⑤

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2at}{2a} \quad (\text{from ④ \& ⑤})$$

$$\Rightarrow \frac{dy}{dx} = t = y_1 \quad \text{— ⑥}$$

$$y_2 = \frac{d^2y}{dx^2} = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]\left(\frac{dt}{dx}\right)$$

$$\Rightarrow y_2 = \left[\frac{d}{dt}(t)\right] \times \frac{1}{2a}$$

$$\Rightarrow y_2 = \frac{1}{2a} \quad \text{— ⑦}$$

Now, $\bar{x} = x - \left[\frac{y_1(1+y_1^2)}{y_2}\right]$

$$\Rightarrow \bar{x} = 2at - \left[\frac{t(1+t^2)}{1/2a}\right] \quad (\text{from ②, ⑥ \& ⑦})$$

$$\Rightarrow \bar{x} = 2at - 2at - 2at^3$$

$$\Rightarrow \boxed{\bar{x} = -2at^3} \quad \text{— ⑧}$$

$$\bar{y} = y + \left[\frac{1+y_1^2}{y_2}\right]$$

$$\Rightarrow \bar{y} = at^2 + \left[\frac{1+t^2}{1/2a}\right] \quad \text{(from ③, ⑥ \& ⑦)}$$

$$\Rightarrow \bar{y} = at^2 + 2a + 2at^2$$

$$\Rightarrow \boxed{\bar{y} = 2a + 3at^2} \quad \text{— ⑨}$$

[Our target is to remove the parametric variable 't' from eqⁿ ⑧ & ⑨]

$$\bar{x} = -2at^2 \quad \text{— Ⓐ}$$

$$\Rightarrow t^3 = \frac{\bar{x}}{-2a} \quad \text{— Ⓐ'}$$

$$\Rightarrow t^6 = \left(\frac{\bar{x}}{-2a}\right)^2 \quad \text{— Ⓐ''}$$

$$\bar{y} = 2a + 3at^2 \quad \text{— Ⓑ}$$

$$\Rightarrow t^2 = \frac{\bar{y} - 2a}{3a} \quad \text{— Ⓑ'}$$

$$\Rightarrow t^6 = \left(\frac{\bar{y} - 2a}{3a}\right)^3 \quad \text{— Ⓑ''}$$

From Ⓐ'' and Ⓑ''

$$\left(\frac{\bar{x}}{-2a}\right)^2 = \left(\frac{\bar{y} - 2a}{3a}\right)^3$$

$$\Rightarrow \frac{(\bar{x})^2}{4a^2} = \frac{(\bar{y} - 2a)^3}{27a^3}$$

$$\Rightarrow 27a(\bar{x})^2 = 4(\bar{y} - 2a)^3$$

(Replace $\bar{x} \rightarrow x$ and $\bar{y} \rightarrow y$.)

$$\therefore \boxed{27ax^2 = 4(y - 2a)^3} \leftarrow \text{EVALUTE!}$$