UNIT - 1 : MATRICES

Cross-Multiplication Method Shortcut

$$x_1 + x_2 + x_3 = 0 \quad \text{--- (1)}$$
$$x_1 - x_2 + x_3 = 0 \quad \text{--- (2)}$$

Start with coeff of x_2

2 3 1 2 (these are coeff of x_2, x_3, x_1, x_2)

$$\begin{array}{c|cccc} \text{coeff of} & x_2 & x_3 & x_1 & x_2 \\ \hline eq^n \text{ (1)} & 1 & 1 & 1 & 1 \\ eq^n \text{ (2)} & -1 & 1 & 1 & -1 \end{array}$$

$x_1 \quad x_2 \quad x_3$

$$\frac{x_1}{1+1} = \frac{x_2}{1-1} = \frac{x_3}{-1-1}$$

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{x_2}{c_1a_2 - c_2a_1} = \frac{x_3}{a_1b_2 - a_2b_1}$$

$$\begin{array}{c|cccc} \text{coeff of} & x_2 & x_3 & x_1 & x_2 \\ \hline eq^n \text{ (1)} & b_1 & c_1 & a_1 & b_1 \\ eq^n \text{ (2)} & b_2 & c_2 & a_2 & b_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k \text{ (say)}$$

$$\Rightarrow x_1 = k; \; x_2 = 0; \; x_3 = -k$$

let $k = 1$

$\therefore x_1 = 1; \; x_2 = 0; \; x_3 = -1.$

Matrix : rectangular arrangement of entities in rows & columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{m \times n}$$

(m = no. of rows
n = no. of columns)

Various Types of Matrices

(1) **Row Matrix** : A matrix that has only one row and any no. of columns.

$[\quad]_{1 \times n}$ (n = no. of columns)

eg) $[1 \; 2 \; 3 \; 4 \; 5]_{1 \times 5}$

② **Column Matrix :** A matrix that has only one column and any no. of rows.

$$\begin{bmatrix} \\ \end{bmatrix}_{m \times 1} \quad (m = \text{no. of rows})$$

eg) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$.

③ **Null Matrix / Zero Matrix :** Any matrix in which all the elements are zero.

eg) $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

eg) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

eg) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

eg) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

④ **Square Matrix :** A matrix that has equal no. of rows and columns.
$(m = n)$

eg) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$

⑤ **Diagonal Matrix :** A square matrix is called a diagonal matrix if all it's non-diagonal elements are zero.

eg) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3 \times 3}$

⑥ **Scalar Matrix :** It is a diagonal matrix in which all the diagonal elements are equal to a scalar (say 'k')

eg) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3 \times 3}$.

⑦ Identity Matrix: A diagonal matrix in which all the diagonal elements are unity and non-diagonal elements are zero.

eg)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

⑧ Symmetric Matrix: A square matrix in which:

$$a_{ij} = a_{ji}$$
$$A = A^T$$

eg)
$$\begin{bmatrix} a_{11} & 1 & -2 \\ 1 & a_{22} & -1 \\ -2 & -1 & a_{33} \end{bmatrix}_{3\times3} \Rightarrow \begin{pmatrix} a_{12} = 1 = a_{21} \\ a_{13} = -2 = a_{31} \\ a_{23} = -1 = a_{32} \end{pmatrix}.$$

⑨ Skew-symmetric Matrix: A square matrix in which:

$$a_{ij} = -a_{ji}$$
$$A = -A^T$$
$$a_{11} = a_{22} = a_{33} = \cdots = 0$$

eg)
$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}_{3\times3}$$

⑩ Transpose of a Matrix: If in a given matrix A, we interchange the rows & the corresponding columns, the new matrix obtained is called the Transpose of matrix A and is denoted by A' or A^T.

eg)
$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix} = A$$

$$A^T = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

(11) Triangular Matrix

U.T.M. (Upper Triangular Matrix)

$$\begin{bmatrix} a_{11} & 1 & 2 \\ 0 & a_{22} & 3 \\ 0 & 0 & a_{33} \end{bmatrix}$$

(elements below leading diagonal are zero)

L.T.M. (Lower Triangular Matrix)

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 1 & a_{22} & 0 \\ 2 & 3 & a_{33} \end{bmatrix}$$

(elements above leading diagonal are zero).

(12) Orthogonal Matrix

$$\boxed{A \cdot A^T = I}$$

** # Eigen Values & Eigen Vectors

- Determinants have some value but matrix doesn't have any value.

(*) Eigen Values (only for square matrix) [No. of eigen values = order of square matrix]

(1) For a given square matrix A, $(A-\lambda I)$ matrix is called the Characteristic matrix or Eigen matrix, where λ is scalar and I is the unit matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}_{3\times 3}$$

$A - \lambda I =$ (a form)

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$\Rightarrow [A-\lambda I] =$ (eigen matrix)

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

(2) The determinant $|A-\lambda I|$ when expanded will give a polynomial which we call as characteristic Poly. (or) Eigen Poly. of matrix A.

$$[A-\lambda I] = |A-\lambda I|$$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix}$$

(3) The eqⁿ $|A-\lambda I| = 0$ is called the ~~characteris~~ characteristic eqⁿ or Eigen eqⁿ of Matrix A.

$$|A-\lambda I| = 0$$

i.e. $$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

** Shortcut to solve Eigen eqⁿ of square matrix A

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \quad \text{— (A)}$$

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

- a_1 = sum of leading diagonal elements of A
  $= 2+3+2$
  $= 7$
- a_2 = sum of minors of leading diagonal elements of A
  $= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$
  $= (6-1) + (4-2) + (6-2)$
  $= 5 + 2 + 4$
  $= 11$

• $a_3 = |A|$

$$= \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= 2(6-1) - 2(2-2) + 1(1-6)$$
$$= 2(5) - 2(0) + 1(-5)$$
$$= 10 - 5$$
$$= 5$$

→ Put values of a_1, a_2 and a_3 in eqⁿ (A).

$$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0. \quad \text{— (B)}$$

Put $\lambda = 1$

$$(1)^3 - 7(1)^2 + 11(1) - 5 = 0$$
$$\Rightarrow 0 = 0$$

$\therefore \lambda = 1$ is a root of eqⁿ (B).

★ To solve cubic eqⁿ (B) →

M-I) Long-division Method.

$$\lambda - 1 \,\overline{\big)\, \lambda^3 - 7\lambda^2 + 11\lambda - 5}$$

The quotient that we get is the quadratic form.

M-II) Factorisation.

$$\lambda^2(\lambda-1) - 6\lambda(\lambda-1) + 5(\lambda-1)$$
$$\Rightarrow (\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$$

Solve to get 'λ'.

★★ M-III) Shortcut!

Always write 0 here.

1	1	−7	11	−5
	$\boxed{0}$	1	−6	5
	1	−6	5	0
coeff of	λ^2	λ	const.	

[×1 Multiplying with the no. outside].

$$\therefore \lambda^2 - 6\lambda + 5 = 0$$

Solve to get other 2 values of λ.

> For UTM, the diagonal elements are the Eigen values (No need to make Eigen eqⁿ)

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(2-\lambda)(3-\lambda)=0$

$\Rightarrow \lambda = 1, 2, 3.$

(A) Eigen Vectors

- can never be a NULL matrix.
- Always a column matrix.
- 2 Eigen Vectors can't be same.

① Non-Symmetric Matrices with Non-Repeated Eigen Values.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$[A - \lambda I] X = 0.$

Q $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$; Eigen values = 1, 2, 3.

$[A - \lambda I] X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (A)}$$

⊙ put $\lambda = 1$ in (A)

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_3 = 0 \quad —①$

same eqⁿ ⇓ "let" $\left[\begin{array}{l} x_1 + x_2 + x_3 = 0 \quad —② \\ x_1 + x_2 + x_3 = 0 \quad —③ \end{array}\right.$

} 3 simultaneous eqⁿ multiplication of matrix and then equal matrix से आएंगी।

Put ① in ②

$\therefore \quad x_1 + x_2 = 0$

$\Rightarrow \quad x_1 = -x_2$

let $x_1 = k$

$\therefore x_2 = -k$

(where $k = 1, 2, 3, \dots, n$)

let $k = 1$

$\therefore x_1 = 1 \,; \quad x_2 = -1$

when $\lambda = 1$, $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

⊙ Put $\lambda = 2$ in Ⓐ

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + x_3 = 0 \quad —①$

$x_1 + x_3 = 0 \quad —②$

$2x_1 + 2x_2 + x_3 = 0 \quad —③$

coeff of	x_2	x_3	x_1	x_2
eq^n 1	0	1	1	0
eq^n 3	2	1	2	2

$$\frac{x_1}{(0-2)} = \frac{x_2}{(2-1)} = \frac{x_3}{(2-0)}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-2} = k \text{ (say)}$$

$\therefore x_1 = 2k;\ x_2 = -k;\ x_3 = -2k$
where $(k = 1, 2, 3, \ldots, n)$

let $k = 1$
$\therefore x_1 = 2;\ x_2 = -1;\ x_3 = -2.$

When $\lambda = 2$, $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

- Put $\lambda = 3$ in eq^n (A)

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_3 = 0$ — (1)

$x_1 - x_2 + x_3 = 0$ — (2)

$x_1 + x_2 = 0$ — (3)

coeff of	x_2	x_3	x_1	x_2
eq^n ①	0	1	2	0
eq^n ③	1	0	1	1

$$\frac{x_1}{0-1} = \frac{x_2}{1-0} = \frac{x_3}{2-0}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-2} = k \text{ (say)}$$

where $(k = 1, 2, 3, \ldots, n)$

$\therefore x_1 = k ;\ x_2 = -k ;\ x_3 = -2k$

let $k = 1$

$\therefore x_1 = 1 ;\ x_2 = -1 ;\ x_3 = -2$

when $\lambda = 3$, $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

First value को -ve aur fraction आने से बचाइए। Try to make it +ve integer.

$$\therefore \boxed{X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \text{ are the Eigen Vectors of matrix A.}}$$

➤ Suppose 3 eq^n are like :→

$x_2 + 4x_3 = 0$ —①

$x_2 - 6x_3 = 0$ —②

$x_3 = 0$ —③

$\Rightarrow x_3 = x_2 = 0$

x_1 eq^n mein नहीं है तो 'let' krenge.

let $x_1 = k$

let $k = 1$ $\therefore \boxed{x_1 = 1}$

So, $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

② Non-Symmetric Matrix with repeated Eigen Values

Case-1) If 3 eqⁿs that are made after taking first Eigen value are same.

Q $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, (Eigen values $= -3, -3, 5$)

$[A - \lambda I]X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (A)}$$

⊙ Put $\lambda = 5$ in eqⁿ (A) (First take that Eigen value that is NOT repeated)

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 7x_1 - 2x_2 + 3x_3 = 0$ — ①

$x_1 - 2x_2 - 3x_3 = 0$ — ②

$x_1 + 2x_2 + 5x_3 = 0$ — ③

coeff of	x_2	x_3	x_1	x_2
eqⁿ ②	-2	-3	1	-2
eqⁿ ③	2	5	1	2

$$\frac{x_1}{-10+6} = \frac{x_2}{-3-5} = \frac{x_3}{2+2}$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-8} = \frac{x_3}{4}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k \text{ (say)}$$

$\Rightarrow x_1 = k;\ x_2 = 2k;\ x_3 = -k$

where $(k = 1, 2, 3, \ldots\ldots, n)$

let $k = 1$

$\therefore x_1 = 1;\ x_2 = 2;\ x_3 = -1$

when $\lambda = 5$, $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

• Put $\lambda = -3$ in eq^n (A) (first repeated Eigen Value)

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$ —— (1)

$x_1 + 2x_2 - 3x_3 = 0$ —— (2)

$x_1 + 2x_2 - 3x_3 = 0$ —— (3)

$\therefore x_1 + 2x_2 - 3x_3 = 0$

(1 eq^n aur 3 unknown variables, kisi ek ko zero let krlijiye)

Let $x_2 = 0$

$\therefore x_1 - 3x_3 = 0$

$\Rightarrow x_1 = 3x_3$

Let $x_3 = k$

$\therefore x_1 = 3k$ (where $k = 1, 2, 3, \ldots\ldots, n$)

let $k = 1$

$\therefore x_1 = 3$

$x_3 = 1$

When $\lambda = -3$, $X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

- put $\lambda = -3$ in eq^n (A) [2nd repeated Eigen Value]

⋮

$x_1 + 2x_2 - 3x_3 = 0$

(पिछली बार $x_2 = 0$ किया था तो इस बार x_1, x_3 mein se kisi ko 0 करेंगे)

let $x_3 = 0$

$\therefore x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

let $x_2 = k$; $x_1 = -2k$

(where $k = 1, 2, 3, \ldots n$)

★ Always let k = +ve value = +1 (preferably)

let $k = 1$

$\therefore x_2 = 1$; $x_1 = -2$

When $\lambda = -3$, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ $\boxed{X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ are the Eigen Vectors of matrix A.}}$ //

(P.T.O)

Case-2) If 3 eq^ns that are made after taking first Eigen Value are NOT same.

Q $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$; (Eigen values = 2, 2, 3)

$[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (A)}$$

- Put $\lambda = 3$ in eq^n (A) [Non-repeated Eigen Value]

$$\begin{bmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_2 + x_3 = 0 \quad \text{— (1)}$$

$$x_1 + 3x_2 + 2x_3 = 0 \quad \text{— (2)}$$

$$3x_1 + 5x_2 + 4x_3 = 0 \quad \text{— (3)}$$

coeff of	x_2	x_3	x_1	x_2
eq^n (1)	2	1	0	2
eq^n (2)	3	2	1	3

$$\frac{x_1}{(4-3)} = \frac{x_2}{(1-0)} = \frac{x_3}{(0-2)}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2} = k \text{ (say)}$$

$\Rightarrow x_1 = k;\ x_2 = k;\ x_3 = -2k$

where $(k = 1, 2, 3, \cdots\cdots, n)$.

let $k = 1$

$\therefore x_1 = 1;\ x_2 = 1;\ x_3 = -2$

when $\lambda = 3$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

⊙ Put $\lambda = 2$ in eqⁿ Ⓐ [First repeated Eigen Value]

$$\begin{bmatrix} 3-2 & 10 & 5 \\ -2 & -3-2 & -4 \\ 3 & 5 & 7-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 10x_2 + 5x_3 = 0 \quad \text{—} (1)$

$2x_1 + 5x_2 + 4x_3 = 0 \quad \text{—} (2)$

$3x_1 + 5x_2 + 5x_3 = 0 \quad \text{—} (3)$

coeff of	x_2	x_3	x_1	x_2
eqⁿ (2)	5	4	2	5
eqⁿ (3)	5	5	3	5

$$\frac{x_1}{25-20} = \frac{x_2}{12-10} = \frac{x_3}{10-15}$$

$$\Rightarrow \frac{x_1}{5} = \frac{x_2}{2} = \frac{x_3}{-5} = k \text{ (say)}$$

$\therefore x_1 = 5k;\ x_2 = 2k;\ x_3 = -5k$

(where $k = 1, 2, 3, \cdots\cdots, n$)

let $k = 1$

$\therefore x_1 = 5;\ x_2 = 2;\ x_3 = -5$

When $\lambda = 2$, $X_2 = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$

★★

When $\lambda = 2$, $X_3 = \begin{bmatrix} \text{Dependent} \\ \text{Eigen} \\ \text{Vector} \\ X_2 \end{bmatrix}$

③ Symmetric Matrices with Non-Repeated Eigen Values

↳ Always use cross-multiplication method.

(Never use 'let' (99%))

Q Find the Eigen values & Eigen Vectors for the matrix :-

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow [A - \lambda I] = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$[A - \lambda I] = |A - \lambda I|$$

Now, $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

- Let $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$ — (A)

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

- $a_1 = 8 + 7 + 3$
  $= 18$

- $a_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$
  $= (21-16) + (24-4) + (56-36)$
  $= 5 + 20 + 20$
  $= 45$

- $a_3 = |A|$
  $= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$
  $= 8(21-16) + 6(-18+8) + 2(24-14)$
  $= 8(5) + 6(-10) + 2(10)$
  $= 40 - 60 + 20$
  $= 0$

→ Put values of a_1, a_2, a_3 in eqn (A).

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$

$\Rightarrow \lambda(\lambda - 15)(\lambda - 3) = 0$

$\Rightarrow$ $\boxed{\lambda = 0, 3, 15. \text{ are the Eigen values of matrix A}}$ //.

* **To find Eigen Vectors of matrix A :→**

$[A-\lambda I]X=0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (B)}$$

⊙ Put $\lambda = 0$ in eqⁿ (B)

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 - 3x_2 + x_3 = 0 \quad \text{— (1)}$$

$$6x_1 - 7x_2 + 4x_3 = 0 \quad \text{— (2)}$$

$$2x_1 - 4x_2 + 3x_3 = 0 \quad \text{— (3)}$$

coeff of	x_2	x_3	x_1	x_2
eqⁿ (1)	-3	1	4	-3
eqⁿ (2)	-7	4	6	-7

$$\frac{x_1}{-12+7} = \frac{x_2}{6-16} = \frac{x_3}{-28+18}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{-10} = \frac{x_3}{-10}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k \text{ (say)}$$

$\therefore x_1 = k;\ x_2 = 2k;\ x_3 = 2k$

where $(k = 1, 2, 3, \dots, n)$

let $k=1$

$\therefore x_1 = 1;\ x_2 = 2;\ x_3 = 2$

When $\lambda = 0$, $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0 \quad \text{--- (1)}$$
$$3x_1 - 2x_2 + 2x_3 = 0 \quad \text{--- (2)}$$
$$x_1 - 2x_2 = 0 \quad \text{--- (3)}$$

coeff of	x_2	x_3	x_1	x_2
eqⁿ (1)	-6	2	5	-6
eqⁿ (3)	-2	0	1	-2

$$\frac{x_1}{(0+4)} = \frac{x_2}{(2-0)} = \frac{x_3}{(-10+6)}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k \text{ (say)}$$

$\therefore x_1 = 2k;\ x_2 = k;\ x_3 = -2k$

where $(k = 1, 2, 3, \ldots\ldots, n)$

let $k = 1$

$x_1 = 2;\ x_2 = 1;\ x_3 = -2$

When $\lambda = 3$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

⊙ Put $\lambda = 15$ in eqⁿ (B).

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 7x_1 + 6x_2 - 2x_3 = 0 \quad -(1)$$

$$3x_1 + 4x_2 + 2x_3 = 0 \quad -(2)$$

$$x_1 - 2x_2 - 6x_3 = 0 \quad -(3)$$

coeff of	x_2	x_3	x_1	x_2
eqⁿ (2)	4	2	3	4
eqⁿ (3)	-2	-6	1	-2

$$\frac{x_1}{-24+4} = \frac{x_2}{2+18} = \frac{x_3}{-6-4}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{20} = \frac{x_3}{-10}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k \text{ (say)}$$

$\therefore x_1 = 2k$; $x_2 = -2k$; $x_3 = k$

(where $k = 1, 2, 3, \ldots\ldots n$)

let $k = 1$

$\therefore x_1 = 2$; $x_2 = -2$; $x_3 = 1$

when $\lambda = 15$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$$\boxed{\therefore X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \text{ are the Eigen Vectors of matrix } A.}$$

☆ There is NO Eigen vector of Identity or Scalar matrix since it becomes NULL matrix.

★★ **Q** $\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = A$

Find the Eigen Values and the Eigen Vectors of matrix A.

$$A-\lambda I = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow [A-\lambda I] = \begin{bmatrix}1-\lambda & 1 & 0\\ 0 & 1-\lambda & 0\\ 0 & 0 & 1-\lambda\end{bmatrix}$$

$$[A-\lambda I] = |A-\lambda I|$$

Now, $|A-\lambda I| = 0$

i.e.- $\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 0$

⊙ Let $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

$$A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

- $a_1 = 1+1+1$
  $= 3$
- $a_2 = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} + \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} + \begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}$

No need for all this

$\because$ it is an U.T.M, so, the diagonal elements of A are the eigen values.

$\therefore$ $\boxed{\text{Eigen values of matrix A} = 1, 1, 1}$ //.

* To find Eigen Vectors of matrix A :→

$[A-\lambda I]\,X=0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (A)}$$

- Put $\lambda=1$ in eq^n (A)

$$\begin{bmatrix} 1-1 & 1 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2=0$

let $x_1=k$ (where $k=1,2,3,\ldots,n$)

let $x_3=0$

let $k=1$

$\therefore x_1=1$

when $\lambda=1$, $X_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- Put $\lambda=1$ in eq^n (A)

⋮

$x_2=0$

let $x_1=0$

let $x_3=k$ (where $k=1,2,3,\ldots,n$)

let $k=1$

$\therefore x_3=1$

when $\lambda=1$, $X_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

** When $\lambda = 1$, $X_3 = \begin{bmatrix} \text{Dependent} \\ \text{Eigen} \\ \text{vector} \\ X_1 \text{ or } X_2 \end{bmatrix}$

$$\therefore \boxed{X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \text{Dependent} \\ \text{Eigen} \\ \text{Vector} \\ X_1 \text{ or } X_2 \end{bmatrix} \text{ are the Eigen vectors of matrix } A.}$$

//.

(4) Symmetric Matrices with Repeated Eigen Values

Q Find the Eigen Vectors of the following matrix:→

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad ;(\text{Eigen values} = 1, 3, 3)$$

$$[A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— (A)}$$

⊙ Put $\lambda = 1$ in eq^n (A) [Non- Repeated Eigen Value]

$$\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 3-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_3 = 0 \quad \text{— (1)}$$

$$x_2 = 0 \quad \text{— (2)}$$

$$x_1 + x_3 = 0 \quad \text{— (3)}$$

[$\because$ ① & ③ are same, so use 'let'
agr ① & ③ same नहीं होती, तो cross-multiplication method
shortcut use होगा]

$$x_1 + x_3 = 0$$

$$x_1 = -x_3$$

let $x_1 = k$,

$\therefore x_3 = -k$

where $(k = 1, 2, 3, \ldots, n)$

let $k = 1$

$\therefore x_1 = 1;\ x_3 = -1$

when $\lambda = 1$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

⊙ Put $\lambda = 3$ in eqⁿ ⓒ :→

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 3-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_3 = 0$ — ①

$x_1 - x_3 = 0$ — ②

Let $x_2 = 0$

$$x_1 = x_3$$

let $x_3 = k$ $\therefore x_1 = k$

(where $k = 1, 2, 3, \ldots, n$)

let $k = 1$

$\therefore x_1 = 1;\ x_3 = 1$

when $\lambda = 3$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

★ Symmetric Matrix with repeated Eigen values mein 3rd Eigen vector कभी dependent नहीं हो सकता (100%)

★★ Agr symmetric matrix with Repeated Eigen Values hai तो 3rd Eigen Vector following method से निकालो।

"ORTHOGONAL FORM"

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

Orthogonal Pair of Eigen Vectors

$X_1 \cdot X_2^T = 0$

$X_2 \cdot X_3^T = 0$

$X_3 \cdot X_1^T = 0$

★ Orthogonal Form

$X_1 \cdot X_3^T = 0$ — ①

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [l \; m \; n] = 0$

$l + 0 \cdot m - n = 0$ — Ⓐ

$X_2 \cdot X_3^T = 0$ — ②

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [l \; m \; n] = 0$

$l + 0 \cdot m + n = 0$ — Ⓑ

coeff of	m	n	l	m
eqn Ⓐ	0	-1	1	0
eqn Ⓑ	0	1	1	0

$\frac{l}{0-0} = \frac{m}{-1-1} = \frac{n}{0-0}$

$\Rightarrow \frac{l}{0} = \frac{m}{-2} = \frac{n}{0}$

$\Rightarrow \frac{l}{0} = \frac{m}{1} = \frac{n}{0} = k \text{ (say)}$

$\therefore l = 0$; $m = k$; $n = 0$

(where $k = 1, 2, 3, \ldots, n$)

let $k = 1$

$\therefore m = 1$

when $\lambda = 3$, $X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\therefore X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ are the Eigen vectors of matrix } A.}$

Diagonalisation by orthogonal Transformation / Reduction

** Agr orthogonal transformation solve kr rhe hai तो matrix symmetric होनी चाहिए ।

Q Diagonalise the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation.

- Find Eigen Values : $\lambda = 1, -1, 4$
- Find Eigen Vectors: $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

- Modal matrix (M)

$M = [X_1 \ X_2 \ X_3]$

$$M = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Normalised Modal Matrix (N)

$$N = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

$$N^T = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

- Diagonal matrix (D)

$$D = \underbrace{N^T \cdot \underbrace{A \cdot N}_{①}}_{②}$$

Rough Work

- 1st column of M

1) $2 \quad -1 \quad 1$
2) $(2)^2 \quad (-1)^2 \quad (1)^2$
$\Rightarrow 4 \quad 1 \quad 1$
3) $4 + 1 + 1 = 6$
4) $\sqrt{6}$ (Divide by all the elements of X_1 in N)

- 2nd column of M

1) $0 \quad 1 \quad 1$
2) $(0)^2 \quad (1)^2 \quad (1)^2$
$\Rightarrow 0 \quad 1 \quad 1$
3) $0 + 1 + 1 = 2$
4) $\sqrt{2}$ (Divide by all the elements of N)

- 3rd column of M

1) $1 \quad 1 \quad -1$
2) $(1)^2 \quad (1)^2 \quad (-1)^2$
$\Rightarrow 1 \quad 1 \quad 1$
3) $1 + 1 + 1 = 3$
4) $\sqrt{3}$ (Divide by all the elements of N)

P.T.O.

$$D=\begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}}_{①}$$

②

$$D=\begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Write anything over here BUT atleast write something, It will not be checked what we have written is correct or not!

$$D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Trick

$$D=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$ ← It will be the ans

a = 1st eigen vector (for which 1st Eigen vector is found out)

b = 2nd eigen vector (for which 2nd Eigen vector is found out)

c = 3rd eigen vector (for which 3rd Eigen vector is found out)

Quadratic forms

* When quadratic form $\xrightarrow{\text{to}}$ matrix (ONLY square matrix will be made, $m \times n$ नहीं बनेगी)
* No. of unknown variables in quad. form = order of square matrix.

- $\lambda^2+\lambda_1+2=0$ (Not a quad. eqn, क्यूंकि λ_1 par power '2' nhi hai)
- $x_1^2+x_2^2+x_3^2+2x_1x_2+2x_2x_3+2x_3x_1=0$ (Yes, it's a quad. eqn)
- $x_1^2+x_2^2+2x_1x_2$ (Yes, it's a quad. form)

> कितने भी unknown variables ho par agr sab par power '2' hogi तो wo quadratic form/eqn hogi.

If → $\boxed{\text{2 unknown variables}}$ → max ___ steps [Symmetric Matrix]
→ max ___ steps [Non-Symmetric Matrix]

If → $\boxed{\text{3 unknown variables}}$ → max 6 steps [Symmetric Matrix] (98%)
→ max 9 steps [Non-Symmetric Matrix] (2%)

- Agr (min 3), 4, 5 steps hai तो 6 steps की form बनाओ
- Agr 7, 8 steps hai तो 9 steps की form बनाओ

(max 9 steps तक ही बनेगी form क्योंकि 3×3 matrix mein 9 elements honge)

$$Q = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$$
$$= x_1x_1 + x_2x_2 + x_3x_3 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$$
$$= x_{11} + x_{22} + x_{33} + 2x_{12} + 2x_{23} + 2x_{31}$$

(Put $x = a$)

$$Q = a_{11} + a_{22} + a_{33} + 2a_{12} + 2a_{23} + 2a_{31}$$

$\begin{bmatrix} a_{12} = a_{21} \\ a_{23} = a_{32} \\ a_{31} = a_{13} \end{bmatrix}$ kyuki symmetric matrix बनेगी condition-1 से.

$\boxed{Q = X^T A \cdot X}$ → where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ → column matrix of unknown variables.

1

2

$$\Rightarrow Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

[$a_{12} \Rightarrow$ coeff of $x_1x_2 = 2$ → divide by $2 = \frac{2}{2} = 1$

$\therefore \boxed{a_{12} = 1}$]

→ Jo variable अकेला है पर उसपर power '2' hai, uska coeff. diagonal elements min jaaeyga.

→ $x_1x_3 \Rightarrow$ iska coeff a_{13} and a_{31} par jaaeyga (if symmetric) divided by 2.

★ To form a quad. form from the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

$$Q = 2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_3x_1$$

(after multiplying coeff. of a_{12}, a_{23}, a_{31} by '2')

★★ Q Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the cannonical form by an orthogonal transformation. Also find the rank, index, signature and nature of the quadratic form. [15 marks]

Ans $Q = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

$$\boxed{Q = X^T A \cdot X}$$

$$Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- Find Eigen values of A : $\lambda = 1, 3, 0$
- Find Eigen vectors of A : $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
- Modal Matrix (M) : $M = [X_1 \ X_2 \ X_3]$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

- Normalised Modal Matrix (N) : $N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$

• **Diagonal Matrix (D)**: $D = N^T A \cdot N$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\because Q = X^T A \cdot X$

[Replace $X \rightarrow y$, $A \rightarrow D$] where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore \boxed{Q = y^T D \cdot y}$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\boxed{Q = y_1^2 + 3y_2^2 + 0 \cdot y_3^2}$ ← CANNONICAL FORM.

★★ $Q = ay_1^2 + by_2^2 + cy_3^2$

$a, b, c \rightarrow$ Eigen values taken in order

Rank (r) ⇒ • No. of non-zero rows
i.e. • No. of non-zero Eigen values

Index (p) ⇒ Avoid zero and −ve Eigen values
uske baad jitni bhi +ve Eigen values = p.

Signature (s)

$\boxed{s = 2p - r}$

★ for this ques:→
$r = 2$
$p = 2$
$s = 2(2) - 2 = 2$
Nature = Positive semi-definite.

Nature

★ Types of Nature :→

① Positive Definite: all eigen values are +ve (zero excluded)

② Negative Definite: all eigen values are −ve (zero excluded).

③ Positive semi-definite: first check '0'
agr '0' ke baad saari eigen values +ve hai.

④ Negative semi-definite: first check '0'
agr '0' ke baad saari eigen values −ve hai

⑤ Indefinite : If above 4 conditions are **NOT** satisfied, then indefinite.

NOTE: For Identity and Scalar Matrix
- only eigen values
- No eigen vectors (CAN'T find)
- No quadratic form (coz Eigen vectors CAN'T be find).

Cayley-Hamilton Theorem (C-H Theorem)

- Checks whether all the elements of matrix are correct or not.

*Statement: Every square matrix satisfies it's own characteristic or Eigen eq^n.

Q. Find the characteristic or Eigen eq^n of the matrix $A=\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$. Find A^{-1} and A^4. Verify C-H theorem.

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$$

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

$$a_1 = 1;\ a_2 = -4;\ a_3 = -4$$

$$\therefore \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

★ Verification of C-H Theorem

Replace $\lambda \to A$ and constant $\to$ (constant) $\times$ (I)

$$\therefore A^3 - A^2 - 4A + 4I = 0 \quad \text{— (A)}$$

Now, $A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$ — ①

$A^3 = A \cdot A^2$ or $A^2 \cdot A$

(right way to represent)

$$\therefore A^3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} \quad \text{— ②}$$

• Put ① & ② in eqⁿ Ⓐ

$$A^3 - A^2 - 4A + 4I = 0$$

$$\therefore \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 2 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LHS = RHS

Hence, verified C-H theorem//.

$$\boxed{A^{-1} = \frac{adj(A)}{|A|}}$$

★ For finding A^{-1}

Multiply & divide eqⁿ Ⓐ by A^{-1}

$$\frac{A^{-1}}{A^{-1}}(A^3 - A^2 - 4A + 4I) = 0$$

$$\Rightarrow A^{-1}A^3 - A^{-1}A^2 - 4A^{-1}A + 4A^{-1}I = 0$$

$$\Rightarrow A^2 - A - 4I + 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = -A^2 + A + 4I$$

$$\Rightarrow A^{-1} = \frac{1}{4}(-A^2 + A + 4I)$$

solve it!

★ For finding A^4

Multiply & divide eqⁿ Ⓐ by A.

$$\frac{A}{A}(A^3 - A^2 - 4A + 4I) = 0$$

$$\Rightarrow A^4 - A^3 - 4A^2 + 4A = 0$$

$$\Rightarrow A^4 = A^3 + 4A^2 - 4A$$

solve it!

P.T.O.

Q Find the characteristic eqⁿ of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, verify C–H Theorem.

And find A^{-1} and $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Ans $\boxed{\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0}$ ← Eigen eqⁿ of matrix A.

Replace $\lambda \to A$ and constant → (constant) × (I)

$\therefore A^3 - 5A^2 + 7A - 3I = 0$ — (A)

show LHS = RHS to verify C-H Theorem.

★ To find A^{-1}

Multiply & divide eqⁿ (A) by A^{-1}

$\therefore \frac{A^{-1}}{A^{-1}}(A^3 - 5A^2 + 7A - 3) = 0$

$\Rightarrow A^{-1}A^3 - 5A^{-1}A^2 + 7A^{-1}A - 3A^{-1} = 0$

$\Rightarrow A^2 - 5A + 7I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 5A + 7I$

$\Rightarrow A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$

solve to get A^{-1}.

⊙ Now divide $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ by $\underline{A^3 - 5A^2 + 7A - 3I}$ (Eigen eqⁿ).

$$
\begin{array}{r|l}
 & A^5 + A \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \cancel{A^8} - \cancel{5A^7} + \cancel{7A^6} - \cancel{3A^5} \\
 & (-)\quad (+)\quad (-)\quad (+) \\
\hline
 & \cancel{A^4} - \cancel{5A^3} + 8A^2 - 2A + I \\
 & \cancel{A^4} - \cancel{5A^3} + 7A^2 - 3A \\
 & (-)\quad (+)\quad (-)\quad (+) \\
\hline
 & A^2 + A + I \\
\hline
\end{array}
$$

P.T.O.

$\because$ Dividend = (Divisor × Quotient) + Remainder.

$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^5(0) + A(0) + A^2 + A + I$ (from eqn Ⓐ)

$= A^2 + A + I$

(solve to get the ans !).

Properties of Eigen Values

P-① Every square matrix and it's transpose have the same Eigen values.

eg) if $A \Rightarrow \lambda = 1, 2, 3$

$A^T \Rightarrow \lambda = 1, 2, 3$.

P-② If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are the Eigen values of matrix A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots\ldots, \frac{1}{\lambda_n}$ are the Eigen values of A^{-1}.

eg) if $A \Rightarrow \lambda = 1, 2, 3$

$A^{-1} \Rightarrow \lambda = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

★★ [If $\lambda = 0$, A^{-1} ki eigen value solve नहीं होगी (then λ of A^{-1} will be NOT defined)]

P-③ If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are the Eigen Values of matrix A, then $\lambda_1^n, \lambda_2^n, \lambda_3^n, \ldots\ldots, \lambda_n^n$ are the Eigen values of A^n.

eg) if $A \Rightarrow \lambda = 1, -1, 2$

$A^2 \Rightarrow \lambda = (1)^2, (-1)^2, (2)^2$

$= 1, 1, 4$

$A^3 \Rightarrow \lambda = (1)^3, (-1)^3, (2)^3$

$= 1, -1, 8$

$A^n \Rightarrow \lambda = (1)^n, (-1)^n, (2)^n$

P-(4) If $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots \lambda_n$ are the Eigen values of matrix A, then $k\lambda_1, k\lambda_2, k\lambda_3, \ldots \ldots, k\lambda_n$ are the Eigen values of matrix kA.

eg) if $A \Rightarrow \lambda = 1, -1, 2$

$2A \Rightarrow \lambda = 2(1), 2(-1), 2(2)$
$= 2, -2, 4$

$3A \Rightarrow \lambda = 3(1), 3(-1), 3(2)$
$= 3, -3, 6$

$4A \Rightarrow \lambda = 4(1), 4(-1), 4(2)$
$= 4, -4, 8$; etc.

NOTE:
First solve Eigen values by $|A - \lambda I|$ and then find Eigen values of A^T, A^{-1}, A^n, kA using the properties of Eigen values.

P-(5) Eigen values of a $\boxed{\text{Triangular Matrix}}$ are just the diagonal elements of the matrix

L.T.M.
[Lower Triangular Matrix]

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

diagonal के नीचे भरा हुआ और upar zero

U.T.M.
[Upper Triangular Matrix]

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

diagonal के ऊपर भरा हुआ aur neeche zero.

☆ For a 2×2 matrix

$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix} \rightarrow$ L.T.M.

$\begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix} \rightarrow$ U.T.M.

P-(6)

- $\boxed{\text{Sum of the Eigen values of matrix A} = \text{Sum of the leading diagonal elements of the matrix A.}}$
- $\boxed{\text{Product of Eigen values of matrix } A = |A|}$

Q If 2 is an Eigen value of $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, find the other 2 Eigen values of matrix A.

Also find A^T, A^{-1}, A^2, $3A$.

Ans Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of matrix A.

$\lambda_1 + \lambda_2 + \lambda_3 =$ sum of leading diagonal elements of A.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 2 + 1 - 1$

$\Rightarrow \cancel{2} + \lambda_2 + \lambda_3 = \cancel{2}$

$\Rightarrow \lambda_2 = -\lambda_3$ — (1)

Also, $\lambda_1 \lambda_2 \lambda_3 = |A|$

$= -8$ — (2)

Put (1) in (2)

$\therefore (2)(\lambda_2)(-\lambda_2) = -8$

$\Rightarrow \cancel{-}\lambda_2^2 = \cancel{-}4$

$\Rightarrow \lambda_2 = \pm 2$

*(Either take $\lambda_2 = 2$ or $\lambda_2 = -2$)

Take $\lambda_2 = 2$

$\therefore \lambda_3 = -2$ (from (1))

So, $\boxed{\text{Eigen values of } A = 2, 2, -2}$

- Eigen values of $A^T = 2, 2, -2$
- Eigen values of $A^{-1} = \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$
- Eigen values of $A^2 = (2)^2, (2)^2, (-2)^2$
  $= 4, 4, 4$
- Eigen values of $3A = 3(2), 3(2), 3(-2)$
  $= 6, 6, -6$.

Ques) Determine the nature of the following quadratic form without reducing them to cannonical form.

$$Q = x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_1x_3$$

Ans $Q = X^T \cdot A \cdot X$

$$\Rightarrow Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$3 \times 3 \xrightarrow[\text{changed to}]{\text{can be}} 2 \times 2,\ 1 \times 1$

$\xrightarrow[\text{changed to}]{\text{can't be}} 4 \times 4,\ 5 \times 5, \ldots$

(क्यूकि elements नहीं है)

$1 \times 1 \Rightarrow [a_{11}]$

$2 \times 2 \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$3 \times 3 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$A \Rightarrow [a_{11}] = [1] = |1| = 1 = D_1 \xrightarrow{\text{sochiye}}$ Eigen value hai

$\Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (3-1) = 2 = D_2 \xrightarrow{\text{sochiye}}$ Eigen value hai.

$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 1(18-1) - 1(6-2) + 2(1-6)$

$= 17 - 4 - 10$

$= 3 = D_3 \xrightarrow{\text{sochiye}}$ Eigen value hai.

$\boxed{\text{Nature of } Q = \text{Positive Definite}}$ → (After judging D_1, D_2, D_3)

[Hum nature quadratic form का निकाल रहे हैं, matrix का नहीं]