CHAPTER 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

- Method of Separation of Variables
- The Vibration of String (Wave equation)
- One Dimensional Heat Flow
- Two Dimensional Heat Flow
- Heat Flow in Infinite Plates

Chapter - 3

Applications of Partial Differential Equations (Boundary Value Problems)

In many physical and Engineering problems, we always seek a solution of the differential equations, whether it is ordinary or partial, which satisfies some specified conditions called the boundary conditions. Any differential equation together with these boundary conditions is called boundary value problem. In the case of ordinary differential equations, first we get the general solution which contains the arbitrary constants and then we determine these constants from the initial values. But in the case of partial differential equations we get solutions involving both arbitrary constants and arbitrary functions. Hence it is difficult for us to adjust these constants and functions so as to satisfy the given boundary conditions. So we adopt a method known as ***"method of separation of variables"*** for solving linear partial differential equations so as it satisfy all or some of the given boundary conditions. In this method, right from the beginning, we try to find out the particular solutions of the given partial differential equation which satisfy the boundary conditions and then adjust them till the remaining conditions are also satisfied. The method is explained in the forthcoming article.

3.1 METHOD OF SEPARATION OF VARIABLES

If z be the dependent variable and x, y independent variables in the differential equation, then we assume the solution to be the product of two functions, one of them a function of x alone and the other a function of y alone. In this way, the solution of the differential equation is converted into the solution of ordinary differential equations. The procedure will be clear from the following solved examples.

EXAMPLE 1

Using the method of separation of variables solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ ***where*** $u(x, 0) = 6e^{-3x}$.

Solution

Given $$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad \dots (1)$$

Here u is a function of x and t.

$$\therefore \text{ Let } u = X(x)\,T(t) \quad \dots(2)$$

be the solution of the given differential equation where X is a function of x only and T is a function of t only.

Differentiating (2) partially w.r.t. x and t we get

$$\frac{\partial u}{\partial x} = X'T \quad \dots(3)$$

$$\frac{\partial u}{\partial t} = XT' \quad \dots(4)$$

Substituting (2), (3) and (4) in (1) we get

$$X'T = 2\,XT' + XT$$

i.e., $$X'T = X\,(2T' + T)$$

Separating the variables, we get

$$\frac{X'}{X} = \frac{(2T' + T)}{T} = k \text{ (constant)}$$

i.e., $$\frac{X'}{X} = K \text{ and } \frac{(2T' + T)}{T} = k$$

i.e., $$X' - kX = 0 \text{ and } \frac{2T' + T}{T} = k$$

i.e., $$\frac{dX}{dx} = kX \text{ and } \frac{dT}{dt} = \frac{1}{2}(k-1)\,T$$

By using variable separable method, we get

$$\frac{dX}{X} = k\,dx \text{ and } \frac{dT}{T} = \frac{1}{2}(k-1)\,dt$$

Integrating we get $$\log X = kx + \log a$$

and $$\log T = \frac{1}{2}(k-1)\,t + \log b$$

i.e., $$\frac{X}{a} = e^{kx} \text{ and } \frac{T}{b} = e^{\frac{1}{2}(k-1)t}$$

i.e., $$X = ae^{kx} \text{ and } T = be^{\frac{1}{2}(k-1)t}$$

$$\therefore \quad u = ab\,e^{kx}\cdot e^{\frac{1}{2}(k-1)t}$$

$$\therefore \quad u(x,t) = ab\,e^{kx}\cdot e^{\frac{1}{2}(k-1)t} \quad \dots(5)$$

Putting $t = 0$ in (5) we get

$$u(x,0) = ab\,e^{kx} \quad \dots(6)$$

But $$u(x,0) = 6\,e^{-3x} \quad \dots(7)$$

$\therefore$ From (6) and (7), $ab = 6, k = -3$ ... (8)

Substituting (8) in (5) we get

$$u = 6e^{-(3x+2t)}$$

■ **EXAMPLE 2** ■

Solve by using method of separation of variables the equation *[Nov. '86 ECE]*

$$2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0.$$

● **Solution**

Given $$2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0 \quad \dots(1)$$

$$\text{Let } z = X(x)\cdot Y(y) \quad \dots(2)$$

be the solution of (1).

Differentiating (2) partially w.r.t. 'x' and 'y' we get

$$\frac{\partial z}{\partial x} = X'Y \quad \dots(3)$$

$$\frac{\partial z}{\partial y} = XY' \quad \dots(4)$$

Substituting (3) and (4) in (1) we get

$$2x\,X'Y - 3y\,XY' = 0$$

i.e., $$2x\,X'Y = 3y\,XY'$$

Separating the variables we get

$$\frac{2x\,X'}{X} = \frac{3y\,Y'}{Y} = K \text{ (constant)}$$

i.e., $$\frac{2x\,X'}{X} = K \text{ and } \frac{3y\,Y'}{Y} = K$$

i.e., $$2x\,X' = KX \text{ and } 3yY' = KY$$

i.e., $$2x\frac{dX}{dx} - KX = 0 \quad \dots(5)$$

and $$3y\frac{dY}{dy} - KY = 0 \quad \dots(6)$$

Now consider $$2x\frac{dX}{dx} - KX = 0 \quad \dots(5)$$

i.e., $$(2x\,D - K)\,X = 0$$

This is an ordinary differential equation with variable coefficients.

$$\text{put } x = e^z \text{ or } z = \log x$$

$$\therefore \quad xD = D' \text{ where } D' = \frac{d}{dz}$$

$\therefore$ Equation (5) becomes

$$(2D' - K)\,X = 0$$

i.e., $$2\frac{dX}{dz} = KX$$

i.e., $$2\frac{dX}{X} = k\,dz$$

Integrating we get, $\log X = \frac{k}{2}z + \log c_1$

i.e., $\log \frac{X}{c_1} = \frac{k}{2}z, \quad \frac{X}{c_1} = e^{\frac{k}{2}z}$ (or) $X = c_1 e^{\frac{k}{2}\log x}$

$$\therefore X = c_1 x^{\frac{k}{2}}$$

Now consider equation (6)

i.e., $$3y\frac{dY}{dy} - KY = 0$$

i.e., $$(3yD - k)Y = 0$$

This is a differential equation with variable coefficients

$$\therefore \text{ put } y = e^z, \log y = z$$

$$\therefore yD = D', \quad D' = \frac{d}{dz}$$

Equation (7) becomes

$$3D'Y - kY = 0$$

i.e., $$3\frac{dY}{dz} = kY$$

i.e., $$\frac{dY}{Y} = \frac{k}{3}\,dz$$

Integrating we get, $\log Y = \frac{k}{3}z + \log c_2$

$$\log\frac{Y}{c_2} = \frac{k}{3}z \text{ or } Y = c_2 e^{\frac{k}{3}z} = c_2 e^{\frac{k}{3}\log y}$$

$$\therefore Y = c_2 y^{\frac{k}{3}} \qquad \text{... (B)}$$

Substituting (A) and (B) in (2) we get

$$z = c_1 c_2 x^{\frac{k}{2}} y^{\frac{k}{3}}$$

i.e., $$z(x, y) = A x^{3b} y^{2b}$$

where $A = c_1 c_2$ and $k = 6b$.

EXERCISES

1. Solve the equation $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0, y) = 8e^{-3y}$ by the method of separation of variables.
[Ans. $u(x, y) = 8e^{-12x-3y}$]

2. Find a solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = X(x)\cdot Y(y)$. Hence find the value of $u(x, y)$ by using the conditions $u = 0$ and $\frac{\partial u}{\partial x} = 1 + e^{-3y}$, when $x = 0$
$\left[\text{Ans. } u = \frac{1}{\sqrt{2}}\sin h\sqrt{2}\,x + e^{-3y}\sin x\right]$

3. Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ by the method of separation of variables.
[Ans. $u(x, y) = (A e^{\alpha x} + B e^{\beta x})e^{-ky}$ where $\alpha = 1 + \sqrt{1+k}$, $b = 1 - \sqrt{1+k}$]

4. Solve by the method of separation of variables the equation
$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$
$\left[\text{Ans. } u(x, y) = 4e^{\frac{3}{2}y - x}\right]$

5. Solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0, y) = 8e^{-3y} + 4e^{-5y}$
[Ans. $u(x, y) = 8e^{-12x-3y} + 4e^{-20x-5y}$]

6. Solve $\frac{\partial u}{\partial x} + 4 = \frac{\partial u}{\partial t}$ with $u = 4e^{-3x}$ when $t = 0$ using method of separation of variables.
[Ans. $u = 4e^{-3x-2t}$]

7. Solve $2\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$ by the method of separation of variables.
$\left[\text{Ans. } u = (Ae^{\sqrt{k\cdot x}} + Be^{-\sqrt{k\cdot y}})e^{2ky}\right]$

8. Using the method of variables solve the equation
$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ and $u = e^{-5y}$ when $x = 0$
[Ans. $u = e^{2x-5y}$]

Classification of Partial Differential Equations of the second order

Let a second order partial differential equation in the function u of the two independent variables x, y be of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{... (1)}$$

This equation is linear in the second order terms but the term $f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ may be linear or nonlinear. In the former case, equation (1) is said to be linear, in the latter case to be quasi-linear.

Equation (1) is classified as elliptic, parabolic, or hyperbolic at the points of a given region R depending on whether

$B^2 - 4AC < 0$ [elliptic equation]

$B^2 - 4AC = 0$ [parabolic equation]

$B^2 - 4AC > 0$ [hyperbolic equation]

Examples : The well known examples are given below

Elliptic type	*Parabolic type*	*Hyperbolic type*
1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (*Laplace's equation in two dimension*)	1. $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$ (*one dimensional heat flow equation*)	1. $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2}$ (*one dimensional wave equation*)
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ (*Poisson's equation*)		

Classify the following equations:

EXAMPLE 1

$x^2 f_{xx} + (1 - y^2) f_{yy} = 0$ for $-1 < y < 1, -\infty < x < \infty$

$$\text{Here } A = x^2, \quad B = 0, \quad C = 1 - y^2$$

$$\therefore B^2 - 4AC = -4x^2(1 - y^2) = 4x^2(y^2 - 1)$$

x^2 is always +ve in $-\infty < x < \infty$.

In $-1 < y < 1$, $y^2 - 1$ is negative.

$\therefore B^2 - 4AC = -$ ve $(x \neq 0)$

$\therefore$ The equation is **elliptic.**

If $x = 0$, $B^2 - 4AC = 0$, the equation is parabolic. When $y > 1, y < -1$, then $B^2 - 4AC > 0$, the equation is **hyperbolic.**

EXAMPLE 2

$x f_{xx} + y f_{yy} = 0, \quad x > 0, \quad y > 0$

Here $A = x, C = y$

$\therefore B^2 - 4AC = -4xy = -$ ve when $x > 0, y > 0$.

$\therefore$ The equation is **elliptic.**

EXAMPLE 3

$f_{xx} - 2 f_{xy} = 0, \quad x > 0, \quad y > 0.$

Here $A = 1, \ B = -2, \ C = 0.$

$B^2 - 4AC = 4 - 0 = +$ve $\quad \therefore$ **Hyperbolic.**

EXAMPLE 4

$u_{xx} - 2u_{xy} + u_{yy} = 0$

Here $A = 1, B = -2, C = 1$

$B^2 - 4AC = 4 - 4 = 0 \quad \therefore$ **Parabolic.**

EXAMPLE 5

$f_{xx} + 2f_{xy} + 4f_{yy} = 0, \quad x > 0, \quad y > 0$

Hence $A = 1, \ B = 2, \ C = 4$

$B^2 - 4AC = 4 - 16 = -$ve

$\therefore$ **Elliptic.**

> *NOTE : To classify the differential equations the region is very important.*
>
> *For example the PDE*
>
> *$xf_{xx} + f_{yy} = 0$ is (i) elliptic if $x > 0$*
>
> *(ii) parabolic if $x = 0$*
>
> *(iii) Hyperbolic if $x < 0$*

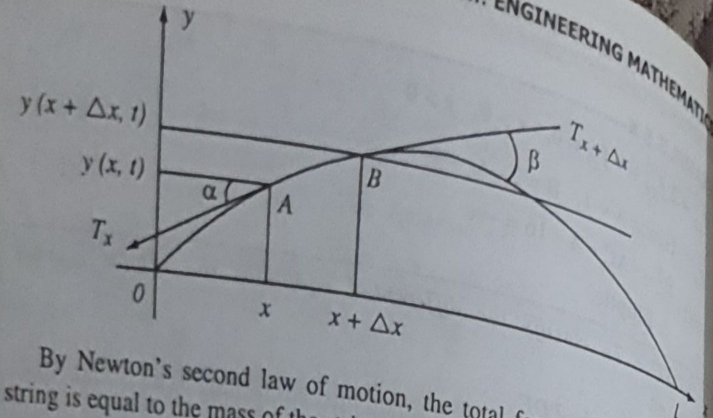
3.2 THE VIBRATING STRING

As our first physical application of partial differential equation, let us derive the equation governing small transverse vibrations of an elastic string. which is stretched to a length 'l' and then fixed at the end points 'o' and 'l' on the x - axis (see fig.). Suppose that the string is pulled back vertically a distance that is very small compared to the length 'l' and released at time $t = 0$, causing it to vibrate. Our problem is to determine the displacement $y(x, t)$ of the point on the string that is 'x' units away from the end 'O' at any time $t > 0$. When we derive a differential equation, which corresponds to a given physical problem, we usually make some simplifying assumptions in order that the resulting equation does not become too complicated. In our present case we make the following assumptions.

1. **The mass of the string per unit length is constant.**
2. **The string is perfectly elastic and does not offer any resistance to bending.**
3. **The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.**
4. **The string performs a small transverse motion in a vertical plane, that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string remain small in absolute value.**

From these assumptions we may expect that the solution $y(x, t)$ of the differential equation to be obtained will reasonably well describe small vibrations of the string.

To derive the differential equation we consider the forces acting on a small portion Δx of the string (see fig.)



By Newton's second law of motion, the total force acting on this piece of string is equal to the mass of the string multiplied by its acceleration.

i.e.,

$$\text{Force} = \text{mass} \times \text{acceleration}$$
$$= m \times a$$
$$= (m\,\Delta x)\,\frac{\partial^2 y}{\partial t^2} \quad \dots (1)$$

Here to find acceleration we take partial derivative of 'y', w.r.t. 't' because y is a function of two variables. We assume in this equation that the string is moving only in the xy-plane and that each particle in the string moves only vertically.

Let T_x and $T_{x+\Delta x}$ be the tension vectors at the end points of the given segment AB ($= \Delta x$). These forces are applied tangentially since the string offers no resistance to bending. Since there is no motion in the x - direction, the x - components of the tension vectors must coincide.

But the horizontal components of T_x and $T_{x+\Delta x}$ are $T_x \cos\alpha$ and $T_{x+\Delta x} \cos\beta$ respectively.

$$\therefore T_x \cos\alpha = T_{x+\Delta x} \cos\beta = T \quad \dots (2)$$

[∵ The horizontal components of the tension must be coincide]. Similarly, in vertical direction we have two forces, namely the vertical components $-T_x \sin\alpha$ and $T_{x+\Delta x} \sin\beta$ of T_x and $T_{x+\Delta x}$; here the minus sign appears because that component at A is directed downward. By Newton's Second law the resultant of these two forces is equal to the mass '$m\Delta x$' of the portion times the acceleration $\frac{\partial^2 y}{\partial x^2}$, evaluated at some point between x and $x + \Delta x$.

Hence $T_{x+\Delta x} \sin\beta - T_x \sin\alpha = m\,\Delta x\,\frac{\partial^2 y}{\partial t^2}$ ... (3)

Dividing each term in (3) by the corresponding term in (2) we get

$$\frac{T_{x+\Delta x} \sin\beta}{T_{x+\Delta x} \cos\beta} - \frac{T_x \sin\alpha}{T_x \cos\alpha} = \frac{m}{T}\,\Delta x\,\frac{\partial^2 y}{\partial t^2}$$

i.e.

$$\tan\beta - \tan\alpha = \frac{m}{T}\,\Delta x.\,\frac{\partial^2 y}{\partial t^2} \quad \dots (4)$$

Here

$$\tan\alpha = \text{slope at } x = \left(\frac{\partial y}{\partial x}\right)_x$$

and

$$\tan\beta = \text{slope at } x + \Delta x = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$$

∴ From (4), we get

$$\frac{1}{\Delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \frac{m}{T}\,\frac{\partial^2 y}{\partial t^2}$$

Let $\Delta x \to 0$, we obtain in the limit case the equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{m}{T}\,\frac{\partial^2 y}{\partial t^2}$$

(or)

$$\frac{\partial^2 y}{\partial t^2} = c^2\,\frac{\partial^2 y}{\partial x^2}$$

$$\text{where } c^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass}}$$

This equation is often called ***"One dimensional Wave equation"***

NOTE : As yet we have not made use of the fact that the string is fixed at its end points. We can write these boundary conditions as $y(0, t) = y(l, t) = 0, t > 0$. In addition to the above, we have not taken into account the initial distortion of the string and the fact that it was at rest when released. These initial conditions may be written as $y(x, 0) = f(x), 0 \le x \le l$

$$\left[\frac{\partial y(x, t)}{\partial t}\right]_{t=0} = 0$$

where $f(x)$ is the initial position of the string and $\left.\frac{\partial y(x, t)}{\partial t}\right|_{t=0}$ is the velocity of the point 'x' units away from the origin at time 't'.

The direct method we use to solve this type of problem is due to d'Alembert. Since it is only rarely possible to apply this method, we develop a method which is applicable to us.

The simplest method for solving an ordinary differential equation is variable separable method. Although we have now two independent variables, we can nevertheless adapt the technique to a partial differential equation of the above form. The method is best explained by means of the following.

3.3 SOLUTION OF THE WAVE EQUATION (METHOD OF SEPARATION OF VARIABLES)

[Nov. '91 C...]

We know that one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots (1)$$

Let $y(x, t) = X(x)\,T(t)$ ...(2)

be the solution of the given equation where X is a function of 'x' only and T is a function of 't' only.

Differentiating (2) partially with respect to 'x' and 't' we get,

$$\frac{\partial^2 y}{\partial x^2} = X''T \text{ and } \frac{\partial^2 y}{\partial t^2} = XT''$$

Substituting these values in (1) we get

$$XT'' = a^2 X''T$$

i.e.,
$$\frac{X''}{X} = \frac{T''}{a^2 T} = k \text{ (By separating the variables)}$$

$$\therefore \frac{X''}{X} = k \text{ and } \frac{T''}{a^2 T} = k$$

i.e.,
$$X'' - kX = 0 \quad \ldots (3)$$

and
$$T'' - ka^2 T = 0 \quad \ldots (4)$$

The equations (3) and (4) are ordinary differential equations the solution of which depends on the value of k. There are three cases arises.

Case (i) : Let k be positive *i.e.*, $k = p^2$

[Here p^2 is always positive whether p is +ve or −ve]

Now equations (3) and (4) become

$$X'' - p^2 X = 0 \text{ and } T'' - p^2 a^2 T = 0$$

i.e.,
$$\frac{d^2X}{dx^2} - p^2 X = 0 \text{ and } \frac{d^2T}{dt^2} - p^2 a^2 T = 0$$

The auxiliary equations are

$$m^2 - p^2 = 0 \text{ and } m^2 - a^2 p^2 = 0$$

i.e.,
$$m = \pm p \text{ and } m = \pm ap$$

$$\therefore X = c_1 e^{px} + c_2 e^{-px} \quad \ldots (5)$$

and
$$T = c_3 e^{pat} + c_4 e^{-pat} \quad \ldots (6)$$

Substituting (5) and (6) in (2) we get

$$y(x, t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$$

Case (ii) : When k is negative, *i.e.*, $k = -p^2$. [Here whether p is + ve or − ve p^2 is always + ve. $\therefore -p^2$ is always − ve].

Now equations (3) and (4) become

$$X'' + p^2 X = 0 \text{ and } T'' + a^2 p^2 T = 0$$

i.e.
$$\frac{d^2X}{dx^2} + p^2 X = 0 \text{ and } \frac{d^2T}{dt^2} + a^2 p^2 T = 0$$

The auxiliary equations are

$$m^2 + p^2 = 0 \text{ and } m^2 + a^2 p^2 = 0$$

i.e.
$$m = \pm ip \text{ and } m = \pm iap$$

$$\therefore X = (c_5 \cos px + c_6 \sin px) \quad \ldots (7)$$

and
$$T = (c_7 \cos pat + c_8 \sin pat) \quad \ldots (8)$$

Substituting (7) and (8) in (2) we get

$$y(x, t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$$

Case (iii) : When $k = 0$, the equations (3) and (4) become

$$X'' = 0 \text{ and } T'' = 0$$

i.e.,
$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2T}{dt^2} = 0$$

Solving these equations we get

$$X = c_9 x + c_{10} \quad \ldots (9)$$

$$T = c_{11} t + c_{12} \quad \ldots (10)$$

Substituting (9) and (10) in (2) we get,

$$y(x, t) = (c_9 x + c_{10})(c_{11} t + c_{12})$$

Thus depending upon the value of k, the various possible solutions of the wave equation are

$$y(x, t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat}) \quad \ldots (11)$$

$$y(x, t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat) \quad \ldots (12)$$

$$y(x, t) = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad \ldots (13)$$

Now let us choose the correct solution which satisfies the boundary conditions of the given problem. In general, in the problems of vibration of strings the two boundary or end conditions viz. $y(0, t) = 0$, $y(l, t) = 0$ are always fixed because the ends $x = 0$ and $x = l$ are fixed. Hence by applying these two conditions in the above solution we have to select the correct one which is suitable for our problem.

Now consider solution (11)

$$y(x, t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$$

Applying the condition $y(0, t) = 0$, we get $\quad [\because e^0 = 1]$

$$y(0, t) = (c_1 + c_2)(c_3 e^{pat} + c_4 e^{-pat}) = 0$$

Here $c_3 e^{pat} + c_4 e^{-pat} \neq 0$ because it is defined for all $t > 0$. ...(A)

$$\therefore c_1 + c_2 = 0$$

Applying the condition $y(l, t) = 0$, we get

$$y(l, t) = (c_1 e^{pl} + c_2 e^{-pl})(c_3 e^{pat} + c_4 e^{-pat}) = 0$$

Here also $c_3 e^{pat} + c_4 e^{-pat} \neq 0$ because it is defined for all $t > 0$

$$\therefore \; c_1 e^{pl} + c_2 e^{-pl} = 0$$

Solving (A) and (B) we get

$$c_1 = 0 \text{ and } c_2 = 0$$

Substituting (C) in (11) we get

$$y(x, t) = 0 \text{ which is a trivial solution.}$$

$\therefore$ Solution (11) is not the correct solution for our problem.

Now take the solution (13).

$$y(x, t) = (c_9 x + c_{10})(c_{11} t + c_{12})$$

Applying the condition $y(0, t) = 0$, we get

$$y(0, t) = c_{10}(c_{11} t + c_{12}) = 0$$

i.e., $$c_{10} = 0$$

$$\therefore y(x, t) = c_9 x (c_{11} t + c_{12}) \quad \ldots (D)$$

Applying the condition $y(l, t) = 0$ we get

$$y(l, t) = c_9 l (c_{11} t + c_{12}) = 0$$

Here $l \neq 0$ and $c_{11} t + c_{12} \neq 0$

$$\therefore c_9 = 0 \quad \ldots (E)$$

Substituting (D) and (E) in (13) we get

$$y(x, t) = 0 \text{ which is again a trivial solution,}$$

$\therefore$ Solution (13) is also not the correct solution.

$\therefore$ The correct solution is

$$y(x, t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$$

NOTE : In problems of vibration of strings we always take the following as the correct solution.

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$$

Here using of the constants c_1, c_2, c_3 and c_4 instead of c_5, c_6, c_7 and c_8 is only for convenience.

NOTE : We can choose the correct solution as follows : Out of the above three types of solutions we have to choose the correct one which is consistent with the physical nature of the problem. Since we are dealing with problem on vibrations, y must be a periodic function of x and t. Therefore, we choose the solution which contains the trigonometric terms since sine and cosine functions are periodic in nature. Hence the correct solution is

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$$

EXAMPLE 1

A string is stretched and fastened to two points $x = 0$ and $x = l$ apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t. [Nov. '87 civil, Nov. '90 civil]

Solution

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions

(i) $y(0, t) = 0$ for all $t > 0$

(ii) $y(l, t) = 0$ for all $t > 0$

(iii) $\frac{\partial y(x, 0)}{\partial t} = 0$ ($\because$ initial velocity is zero)

(iv) $y(x, 0) = k(lx - x^2)$

Now the correct solution which satisfies our boundary conditions is given by

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \ldots (1)$$

[Refer to the previous article]

Applying condition (i) in (1), we get

$$y(0, t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

i.e., $c_1 = 0$ and $c_3 \cos pat + c_4 \sin pat \neq 0$

Putting $c_1 = 0$ in (1) we get

$$y(x, t) = c_2 \sin px (c_3 \cos pat + c_4 \sin pat) \quad \ldots (2)$$

Applying condition (ii) in (2) we get

$$y(l, t) = c_2 \sin pl (c_3 \cos pat + c_4 \sin pat) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$ [$\because$ it is defined for all t]

Therefore either $c_2 = 0$ or $\sin pl = 0$.

Suppose if we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial solution.

$\therefore$ we take $\sin pl = 0$ [$\because \sin n\pi = 0$]

i.e., $pl = n\pi$

i.e., $\boxed{p = \frac{n\pi}{l}}$ [n being an integer]

Now substituting $p = \frac{n\pi}{l}$ in (2) we get

$$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi a t}{l} + c_4 \sin \frac{n\pi a t}{l} \right) \quad \ldots (3)$$

Before applying condition (iii), differentiating (3) partially w.r.t. 't'.

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-c_3 \frac{n\pi a}{l}\sin\frac{n\pi at}{l} + c_4 \frac{n\pi a}{l}\cos\frac{n\pi at}{l}\right)$$

Applying condition *(iii)* we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi a}{l}\right) = 0$$

Here $c_2 \neq 0$ (already explained)

$$\sin\frac{n\pi x}{l} \neq 0 \quad (\because \text{ it is defined for all } x)$$

and

$$\frac{n\pi a}{l} \neq 0 \quad (\because \text{ all are constants})$$

$$\therefore c_4 = 0$$

$\therefore$ Substituting $c_4 = 0$ in (3) we get

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$
$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

where $c_n = c_2 c_3$ ... (4)

Since the partial differential equation (wave equation) is linear any linear combination of solutions (or sum of the solutions) of the form (4) with $n = 1, 2, 3$ ... is also a solution of the equation *(super position principle).*

$\therefore$ The most general solution of (4) can be written as

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}} \quad \text{... (5)}$$

Applying the boundary condition *(iv)* in (5) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = k(lx - x^2) \quad \text{... (6)}$$

To find c_n expand $k(lx - x^2)$ in a half-range Fourier sine series in the interval $(0, l)$

i.e.,

$$f(x) = k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{... (7)}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From (6) and (7) we get $\boxed{b_n = c_n}$

$$\therefore c_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx = \frac{2k}{l}\left[(lx - x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-\frac{2\cos n\pi}{\frac{n^3\pi^3}{l^3}} + \frac{2}{\frac{n^3\pi^3}{l^3}}\right] \quad [\because \sin n\pi = 0]$$

$$c_n = \frac{4kl^2}{n^3\pi^3}[1 - (-1)^n]$$

$$\therefore c_n = 0 \text{ if } n \text{ is even} = \frac{8kl^2}{n^3\pi^3} \text{ if } n \text{ is odd.}$$

Substituting c_n in (5) we get

$$y(x,t) = \sum_{1,3,5}^{\infty} \frac{8kl^2}{\pi^3 n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

EXAMPLE 2

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_0 \sin^3\frac{\pi x}{l}$. If it is released from rest from this position find the displacement y at any distance x from one end at any time t.

[Ap '86 Civil, Nov '91 Civil, Ap '86 ECE]

Solution

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem we have the following boundary conditions

(i) $y(0, t) = 0$, for all $t > 0$

(ii) $y(l, t) = 0$, for all $t > 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0$, for all x. (Initial velocity is zero)

(iv) $y(x, 0) = y_0 \sin^3\frac{\pi x}{l}$

As in the previous example the solution for y satisfying the first three boundary conditions is

$$y(x,t) = c_n \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l}$$

The most general solution can be taken as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{... (1)}$$

Applying condition *(iv)* in (1) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l}$$

We know that $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$$\therefore \sin^3\frac{\pi x}{l} = \frac{1}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

From (2) and (3) we get

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

i.e., $c_1 \sin\frac{\pi x}{l} + c_2\sin\frac{2\pi x}{l} + c_3\sin\frac{3\pi x}{l} + \ldots\ldots = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$

By equating like coefficients on either side, we get

$c_1 = \frac{3y_0}{4}$, $c_3 = \frac{-y_0}{4}$, $c_2 = 0, c_3 = 0, c_4 = 0 \ldots\ldots$

Substituting these values of $c_1, c_2, c_3 \ldots\ldots$ in (1),

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

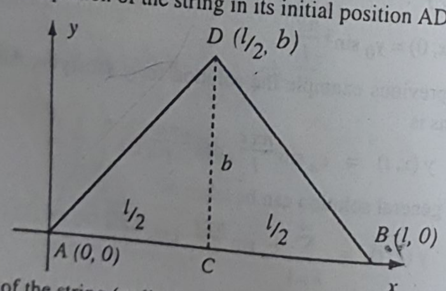
> ***NOTE : In the above problem for finding the value of c_n there is no need for half range sine series expansion because $\sin^3\frac{\pi x}{l}$ contains only two terms.***

EXAMPLE 3

A string is tightly stretched and its ends are fastened at two points $x = 0$ and $x = l$. The mid point of the string is displaced transversely through a small distance 'b' and the string is released from rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion. [Anna Univ. Apr. 2001]

Solution

First we find the equation of the string in its initial position ADB (see fig.)



The equation of the string (or line)

AD is

$$\frac{x-0}{0-\frac{l}{2}} = \frac{y-0}{0-b} \qquad \left[\because CD = b\right]$$

$$\left[\text{Using } \frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}\right]$$

i.e.

$$-bx = -\frac{l}{2}y$$

$$\therefore y = \frac{2bx}{l},\ 0 < x < \frac{l}{2}$$

$\therefore$ The equation of the string in the interval $\left(0, \frac{l}{2}\right)$ is $\frac{2bx}{l}$

The equation of the string (or line) DB is

$$\frac{x-\frac{l}{2}}{\frac{l}{2}-l} = \frac{y-b}{b-0}$$

i.e., $b\left(x-\frac{l}{2}\right) = -\frac{l}{2}(y-b)$

i.e., $y - b = \frac{bl-2bx}{l}$

i.e., $y = \frac{bl-2bx}{l} + b = \frac{2b}{l}(l-x)$

$\therefore$ The equation of the string in the interval $\left(\frac{l}{2}, l\right)$ is

$$y = \frac{2b}{l}(l-x)$$

Hence initially the displacement of the string is in the form

$$y(x,0) = \frac{2bx}{l},\ 0 < x < \frac{l}{2}$$
$$= \frac{2b}{l}(l-x),\ \frac{l}{2} < x < l$$

Now the wave equation is ... (1)

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0$ for every x in $(0, l)$

(iv) $y(x,0) = \frac{2bx}{l},\ 0 < x < \frac{l}{2}$

$\qquad = \frac{2b}{l}(l-x),\ \frac{l}{2} < x < l.$

The solution of the wave equation (1) satisfying the boundary conditions (ii) and (iii) is

$$y(x,t) = c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

[Refer to Ex

Applying condition (iv) in (2) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = f(x)\ \text{(say)}$$

where
$$f(x) = \frac{2bx}{l},\ 0 < x < \frac{l}{2}$$
$$= \frac{2b}{l}(l-x),\ \frac{l}{2} < x < l$$

Now to find 'c_n' expand the function $f(x)$ in a half-range Fourier sine series in the interval $0 < x < l$.

Then
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad \ldots(4)$$

From (3) and (4) we get $\boxed{c_n = b_n}$

Therefore
$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2bx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2b}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4b}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{4b}{l^2}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4b}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$c_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \ldots(5)$$

Substituting (5) in the most general solution (2), we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

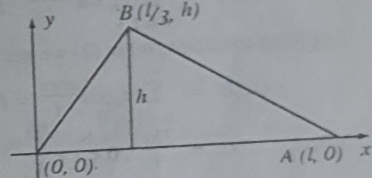
EXAMPLE 4

A taut string of length l has its ends $x = 0, x = l$ fixed. The point where $x = \frac{l}{3}$ is drawn aside a small distance h, the displacement $y(x,t)$ satisfies $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$. Determine $y(x,t)$ at any time t.

Solution

First we find the equation of the string in its initial position.

The equation of the line (or string) OB



$$\frac{x-0}{0-\frac{l}{3}} = \frac{y-0}{0-h}$$

i.e., $$\frac{-3x}{l} = -\frac{y}{h}$$

i.e., $$y = \frac{3xh}{l}$$

$\therefore$ The equation of the string in its initial position in $\left(0, \frac{l}{3}\right)$ is

$$y = \frac{3xh}{l}$$

The equation of the line (or string) BA is $\dfrac{x-\frac{l}{3}}{\frac{l}{3}-l} = \dfrac{y-h}{h-0}$

i.e., $$\frac{y-h}{h} = \frac{3x-l}{-2l}$$

i.e., $$y-h = \frac{h(3x-l)}{-2l}$$

$$\therefore y = \frac{h(l-3x)}{2l} + h = \frac{3h}{2l}(l-x)$$

$\therefore$ The equation of the string in its initial position in $\left(\frac{l}{3}, l\right)$ is

$$y = \frac{3h}{2l}(l-x)$$

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0, t) = 0$ for every $t > 0$

(ii) $y(l, t) = 0$ for every $t > 0$

(iii) $\frac{\partial y(x, 0)}{\partial t} = 0$ for every x in $(0, l)$

(iv) $y(x, 0) = \frac{3xh}{l}, 0 < x < \frac{l}{3}$

$= \frac{3h}{2l}(l-x), \frac{l}{3} < x < l$

Applying the first three boundary conditions in the correct solution of (1), we get

$$y(x, t) = c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

[Refer to ex. (1)]

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad \dots(2)$$

Applying condition (iv) in (2) we get

$$y(x, 0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = f(x) \text{ (say)} \quad \dots(3)$$

where $f(x) = \frac{3hx}{l}, 0 < x < \frac{l}{3}$

$= \frac{3h}{2l}(l-x), \frac{l}{3} < x < l$

Now to find c_n expand $f(x)$ in a half-range Fourier sine series.

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \dots(4)$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

From (3) and (4) we get $\boxed{c_n = b_n}$

$$\therefore c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{l} \frac{3h}{2l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6h}{l^2}\left[\left\{x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/3} + \frac{1}{2}\left\{(l-x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/3}^{l}\right]$$

$$= \frac{6h}{l^2}\left[-\frac{\frac{l}{3}\cos\frac{n\pi}{3}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi}{3}}{\frac{n^2\pi^2}{l^2}} + \frac{1}{2}\cdot\frac{2l}{3}\frac{\cos\frac{n\pi}{3}}{\frac{n\pi}{l}} + \frac{1}{2}\frac{\sin\frac{n\pi}{3}}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{6h}{l^2}\left[\frac{3}{2}\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right] = \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$\therefore c_n = \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3} \quad \dots(5)$$

Substituting (5) in (2) we get

$$y(x, t) = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

EXAMPLE 5

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially displaced in a sinusoidal arc of height y_0 and then released from rest. Find the displacement 'y' at any distance 'x' from one end at time t.

[MKU, Nov. 98, Apr. 97]

Solution

The wave equation is $\frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2}$.

From the given problem we have following boundary conditions.

(i) $y(0, t) = 0$

(ii) $y(l, t) = 0$

(iii) $\frac{\partial y(x,0)}{\partial t} = 0$

(iv) $y(x,0) = y_0 \sin\frac{\pi x}{l}$ [sinusoidal arc of height y_0]

The most general solution after applying conditions (a), (b), (c), we get,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{[Refer to Ex : 1]}$$

Applying condition (iv) in (1), we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin\frac{\pi x}{l}$$

i.e., $c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + \ldots\ldots = y_0 \sin\frac{\pi x}{l}$

Equating like coefficients, we get,

$c_1 = y_0$, $c_2 = c_3 = \ldots\ldots = 0$.

Substituting $c_1 = y_0$ and $c_2 = c_3 = \ldots\ldots 0$ in (1) we get,

$$y(x,t) = c_1 \sin\frac{\pi x}{l}$$

$$= y_0 \sin\frac{\pi x}{l}\cos\frac{\pi at}{l}$$

EXAMPLE 6

An elastic string is stretched between two points at a distance π apart. In its equilibrium position the string is in the shape of the curve $f(x) = k(\sin x - \sin^3 x)$. Obtain $y(x,t)$ the vertical displacement if y satisfies the equation $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$.

[Apr. 97, Nov. 99]

Solution

Given $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$ ... (1)

The boundary conditions are,

(a) $y(0,t) = 0$

(b) $y(\pi,t) = 0$

(c) $\frac{\partial y(x,0)}{\partial t} = 0$

(d) $y(x,0) = k(\sin x - \sin^3 x)$

The most general solution of (1) satisfying conditions (a), (b) and (c) is,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

Applying condition (d) in (2), we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin nx = k(\sin x - \sin^3 x)$$

i.e. $c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + c_4 \sin 4x + \ldots\ldots$

$$= k\left\{\sin x - \frac{1}{4}(3\sin x - \sin 3x)\right\} \quad \ldots (3)$$

$$= k\frac{1}{4}(\sin x + \sin 3x)$$

Equating like coefficients on both sides of (3), we get,

$c_1 = \frac{k}{4}$, $c_2 = 0$, $c_3 = \frac{k}{4}$, $c_4 = c_5 = 0 \ldots\ldots$

Substituting these values of c's in (2), we get,

$$y(x,t) = \frac{k}{4}\sin x\cos t + \frac{k}{4}\sin 3x\cos 3t$$

EXAMPLE 7

Find the displacement of any point of a string, if it is length $2l$ and vibrating between fixed end points with initial velocity zero and initial displacement given by

[BDN, Oct. 99, Apr. 98]

$$f(x) = \frac{kx}{l} \text{ in } 0 < x < l$$

$$= 2k - \frac{kx}{l} \text{ in } l < x < 2l$$

Solution

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ ... (1)

From the given problem we have the following boundary conditions.

(a) $y(0,t) = 0$; (b) $y(L,t) = 0$ where $L = 2l$;

(c) $\frac{\partial y(x,0)}{\partial t} = 0$

(d) $y(x,0) = f(x) = \begin{cases} \frac{2kx}{L}, & 0 < x < \frac{L}{2} \\ 2k - \frac{2kx}{L}, & \frac{L}{2} < x < L \end{cases}$

The most general solution of (1) after applying conditions (a), (b) and (c) is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L} \quad \ldots (2)$$

Applying condition (d) in (2) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x) \quad \ldots (3)$$

To find c_n expand $f(x)$ in a half-range Fourier sine series in $(0, L)$

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

From (3) and (4), we get,

$$c_n = b_n$$

Now $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\int_0^{L/2}\frac{2kx}{L}\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}\frac{2k}{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{4k}{L^2}\left[\left\{x\frac{\left(-\cos\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} - \frac{\left(-\sin\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right\}_0^{L/2} + \left\{(L-x)\frac{\left(-\cos\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + \frac{\left(-\sin\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right\}_{L/2}^{L}\right]$$

$$\therefore b_n = \frac{8K}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \therefore c_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \text{... (5)}$$

Substituting (5) in (2), we get,

$$y(x, t) = \sum_{n=1}^{\infty}\frac{8K}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \quad L = 2l$$

■ **EXAMPLE 8** ■

A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = kx^2(l - x)$, where 'k' is a constant, and then released from rest. Find the displacement of any point 'x' of the string at any time $t > 0$.

● **Solution**

The wave equation is $$\frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2} \qquad \text{... (1)}$$

From the given problem the boundary conditions are

(i) $y(0, t) = 0 \;\forall\; t > 0$

(ii) $y(l, t) = 0, \;\forall\; t > 0$

(iii) $\dfrac{\partial y(x, 0)}{\partial t} = 0$, x is in $(0, l)$

(iv) $y(x, 0) = kx^2(l - x)$, $0 \le x \le l$

The correct solution of (1) is

$$y(x, t) = (c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat) \qquad \text{... (2)}$$

After applying conditions (i), (ii) and (iii) in (2), we get the most general solution is

$$y(x, t) = \sum_{1}^{\infty} c_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad \text{... (A)}$$

At time $t = 0$, the string is in the shape of

$$f(x) = kx^2(l - x) \qquad \text{... (3)}$$

i.e., $$y(x, 0) = kx^2(l - x)$$

Applying (3) in (1), we get

$$y(x, 0) = \sum_{1}^{\infty} c_n\sin\frac{n\pi x}{l} = kx^2(l - x) = \sum_{1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

$$\boxed{c_n = b_n}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2k}{l}\int_0^l (lx^2 - x^3)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[(lx^2 - x^3)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2lx - 3x^2)\times\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (2l - 6x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right) + 6\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^4\pi^4}{l^4}}\right)\right]_0^l$$

$$= \frac{2k}{l}\times\frac{l^3}{n^3\pi^3}\left[-4l\cos n\pi - 2l\right] = \frac{-2kl^3}{n^3\pi^3}\left[1 + 2(-1)^n\right] \qquad \text{... (4)}$$

Substituting (4) in (A), we get

$$y(x, t) = \sum_{1}^{\infty}\frac{-2kl^3}{n^3\pi^3}\left[1 + 2(-1)^n\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

● **EXERCISES**

1. A uniform elastic string of length 60 cm. is subjected to a constant tension of 2 kg. If the ends are fixed and the initial displacement is $y(x, 0) = 60x - x^2$, $0 < x < 60$, while the initial velocity is zero, find the displacement function $y(x, t)$. *[Nov. '87 ECE]*

   $$\left[\textit{Ans. } y(x, t) = \frac{8l^2}{\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\right]$$

   where $l = 60$ and $a^2 = \dfrac{T}{m} = \dfrac{2000}{m}$, T is tension '$m$' is mass per unit length of the string.

2. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$ the string is given a shape defined by $f(x) = \mu x(l - x)$ where μ is a constant and then released. Find the displacement of the string at any time 't'. [Ap. '87 Mech. Ap. '86 M...]

$$\left[\textbf{Ans. } y(x, t) = \frac{8\mu l^2}{\pi^3} \sum_{n=1, 3}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}\right]$$

3. A string is stretched and fastened to two points 'l' apart. Motion is started by displacing the string in the form $y = k \sin\frac{\pi x}{l}$ from which it is released at time $t = 0$. Find the displacement of any point at a distance x from one end at time 't'. [Nov. '87 Mech]

$$\left[\textbf{Ans. } y(x, t) = k \cos\frac{\pi at}{l} \sin\frac{\pi x}{l}\right]$$

4. The points of trisection of a string are pulled aside through a distance b on opposite sides of the position of equilibrium and the string is released from rest. Find the displacement of the string at any subsequent time. [Ap. '90 Mech]

$$\left[\textbf{Ans. } y(x, t) = \frac{9b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{2n\pi}{3} \sin\frac{2n\pi x}{l} \cos\frac{2n\pi at}{l}\right]$$

5. A uniform string of line density ρ is stretched to tension ρc^2 and executes small transverse vibrations in a plane through the undisturbed line of the string. The ends $x = 0$ and $x = l$ of the string are fixed. The string is released from rest in the position $y = f(x)$ at any subsequent time 't', if $f(x) = \frac{4c}{l^2} x(l - x)$ find $y(x, t)$ [Nov. '88 ECE, Nov. '89 Civil]

Hint : Tension is given by ρc^2, so the wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

where $c^2 = \frac{T}{\rho}$

$$\left[\textbf{Ans. } y(x, t) = \frac{32c}{\pi^2} \sum_{n=1, 3, 5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}\right]$$

6. A uniform string of density ρ stretched to the tension ρa^2, executes small transverse vibrations in a plane through the undisturbed line of the string. The ends $x = 0$ and $x = l$ are fixed. The string is at rest with the point $x = b$ drawn aside through a small distance 'd' parallel to the y - axis and released at time $t = 0$. Find the displacement of the string at any time 't'. [Ap. '90 ECE, Ap. '86 Mech., Ap. '88 Civil]

$$\left[\textbf{Ans. } y(x, t) = \frac{2dl^2}{b(l - b)\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}\right]$$

7. A string is stretched between two fixed points $x = 0$ and $x = l$ and is released from rest from the initial position given by

$$f(x) = \frac{2kx}{l}, \quad 0 < x < \frac{l}{2}$$
$$= \frac{2k}{l}(l - x), \quad \frac{l}{2} < x < l.$$

Find the deflection of the string at any time 't'. [Nov. '89 ECE. Ap. '91 ECE, Nov. '91 ECE]

$$\left[\textbf{Ans. } y(x, t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \cos\frac{n\pi at}{l} \sin\frac{n\pi x}{l}\right]$$

8. A taught string of length $2l$ is fastened at both ends. The mid point of the string is taken to a height 'b' and then released from rest in that position. Find the displacement of any point x at any time 't'. [Ap. '91 Mech.]

$$\textbf{Ans. } y(x, t) = \frac{4b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{2l} \cos\frac{(2n-1)\pi at}{2l}$$

9. Solve the boundary value problem $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$

$y(0, t) = y(5, t) = 0,\ y(x, 0) = 0,\ \left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x)$

if (i) $f(x) = 5 \sin \pi x$

(ii) $f(x) = 3 \sin 2\pi x - 2 \sin 5\pi x$

$$\left[\textbf{Ans. } (i)\ y(x, t) = \frac{5}{2\pi} \sin \pi x \sin 2\pi t.\right.$$

$$\left.(ii)\ y(x, t) = \frac{3}{4\pi} \sin 2\pi x \sin 4\pi t - \frac{1}{5\pi} \sin 5\pi x \sin 10\pi t\right]$$

5.4 PROBLEMS ON VIBRATING STRING WITH NON-ZERO INITIAL VELOCITY

In the previous article we allow the string to vibrate by taking it to some position say $f(x)$ and then released from rest. Therefore in that case the initial velocity $\frac{\partial y(x, t)}{\partial t}$ at $t = 0$ is zero. We may also allow the string to vibrate by giving some velocity to the string in its equilibrium position. This initial velocity is given to each and every point in the string from 0 to l and hence it may be a function of x say $g(x)$. Because the string is in its equilibrium position and hence there is no displacement at time $t = 0$ therefore we have $y(x, t) = 0$ at $t = 0$ *i.e.*, $y(x, 0) = 0$ for every x. Now the boundary conditions are

(i) $y(0, t) = 0$ *for all* $t > 0$

(ii) $y(l, t) = 0$ *for all* $t > 0$

(iii) $y(x, 0) = 0$ *for all* x *in* $(0, l)$

(iv) $\frac{\partial y(x, 0)}{\partial t} = g(x)$ *for all* x *in* $(0, l)$

EXAMPLE 1

If a string of length 'l' is initially at rest in its equilibrium position and each of its points is given the velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}, \; 0 < x < l$$

Determine the displacement function y (x, t).

[Ap. '86 Civil, Ap. '90 Civil Ap. '90 ECE]

Solution

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ ... (1)

The boundary conditions are

(i) $y(0, t) = 0$ for every $t > 0$

(ii) $y(l, t) = 0$ for every $t > 0$

(iii) $y(x, 0) = 0$ for all x in $(0, l)$ since the string has no displacement in the equilibrium position.

(iv) $\frac{\partial y(x, 0)}{\partial t} = V_0 \sin^3 \frac{\pi x}{l}$ for all x in $(0, l)$

Solving equation (1) and applying the boundary conditions (i) and (ii) we get the correct solution

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \ldots (2)$$

Now applying condition (i) in (2) we get

$$y(0, t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

$$\therefore c_1 = 0 \quad \ldots (3)$$

Substituting (3) in (2) we get

$$y(x, t) = c_2 \sin px (c_3 \cos pat + c_4 \sin pat) \quad \ldots (4)$$

Applying condition (ii) in (4) we get

$$y(l, t) = c_2 \sin pl (c_3 \cos pat + c_4 \sin pat) = 0$$

Here $c_2 \neq 0$ because if $c_2 = 0$ we get only trivial solution.

Also $c_3 \cos pat + c_4 \sin pat \neq 0$ because it is true for all $t > 0$.

$\therefore \sin pl = 0$ i.e., $pl = n\pi$ [$\because \sin n\pi = 0$]

i.e., $\boxed{p = \frac{n\pi}{l}}$ ... (5)

Substituting (5) in (4) we get

$$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \quad \ldots (6)$$

Applying condition (iii) in (6) we get

$$y(x, 0) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 = 0$$

[$\because$ it is defined for all x]

Here as usual $c_2 \neq 0$, $\sin \frac{n\pi x}{l} \neq 0$

$$\therefore c_3 = 0 \quad \ldots (7)$$

Substituting (7) in (6) we get

$$y(x, t) = c_2 \sin \frac{n\pi x}{l} \cdot c_4 \sin \frac{n\pi at}{l}$$

$$y(x, t) = c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \text{ where } c_n = c_2 c_4.$$

By superposition principle, we can write the most general solution as

$$\boxed{y(x, t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}} \quad \ldots (8)$$

Before applying the condition (iv), find $\frac{\partial y(x, t)}{\partial t}$. Partially differentiating (8) w.r.t. 't' we get

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad \ldots (9)$$

Putting $t = 0$ in (9) we get

$$\frac{\partial y(x, 0)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l}$$

$$= v_0 \sin^3 \frac{\pi x}{l} \quad \text{[By condition (iv)]}$$

i.e., $$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = \frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) \quad \ldots (10)$$

$$\left[\sin^3 x = \frac{3 \sin x - \sin 3x}{4}\right]$$

Here to find c_n, we need not expand the R.H.S. of (10) in a half-range Fourier sine series because it contains only two terms.

From (10) we get

$$c_1 \frac{\pi a}{l} \sin \frac{\pi x}{l} + c_2 \frac{2\pi a}{l} \cdot \sin \frac{2\pi x}{l} + c_3 \frac{3\pi a}{l} \sin \frac{3\pi x}{l} + \ldots\ldots$$

$$= \frac{3v_0}{4} \sin \frac{\pi x}{l} - \frac{v_0}{4} \sin \frac{3\pi x}{l}$$

By equating like coefficients we get

$$c_1 \frac{\pi a}{l} = \frac{3v_0}{4}, \quad c_2 = 0, \quad c_3 \frac{3\pi a}{l} = -\frac{v_0}{4}, \quad c_4 = 0, \quad c_5 = 0, \ldots$$

$$\therefore c_1 = \frac{3v_0}{4}\frac{l}{\pi a} \text{ and } c_3 = -\frac{v_0 l}{12\pi a}$$, the remaining c_n's are zero.

Substituting these values of c's in (8) we get

$$y(x, t) = \frac{3v_0 l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi a t}{l}$$

EXAMPLE 2

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l - x)$, then show that

$$y(x, t) = \frac{8\lambda l^3}{\pi^4 a}\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n a \pi t}{l}$$

[Nov. '86 Civil, Ap. '89 ECE]

Solution

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ ... (1)

The boundary conditions are

(i) $y(0, t) = 0$ for all $t > 0$

(ii) $y(l, t) = 0$ for all $t > 0$

(iii) $y(x, 0) = 0$ for all x in $(0, l)$

[$\because$ The string at $t = 0$ is in its equilibrium position and hence there is no displacement]

(iv) $\frac{\partial y(x, 0)}{\partial t} = \lambda x(l - x)$ for every x in $(0, l)$

Applying the first three boundary conditions in the correct solution of (1) as in the previous example we get the most general solution as

$$y(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \quad \ldots (2)$$

Differentiating (2) partially w.r.t. 't' we get

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l} \quad \ldots (3)$$

Putting $t = 0$ in (3) we get

$$\frac{\partial y(x, 0)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l} = \lambda x(l - x) \quad \ldots (4)$$

[By condition (iv)]

Now to find c_n, expand $\lambda x(l - x)$ in a half-range Fourier sine series, we get

$$\lambda x(l - x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \ldots (5)$$

From (4) and (5) we get

$$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Equating like co-efficients, we get

$$c_n \frac{n\pi a}{l} = b_n$$

$$\therefore \boxed{c_n = b_n \frac{l}{n\pi a}} \quad \ldots (6)$$

But $b_n = \frac{2}{l}\int_0^l \lambda x(l - x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2\lambda}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{-2\cos n\pi}{\frac{n^3\pi^3}{l^3}} + \frac{2}{\frac{n^3\pi^3}{l^3}}\right]$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$\therefore b_n = 0$ when 'n' is even

$= \frac{8\lambda l^2}{n^3\pi^3}$ when 'n' is odd ... (7)

Substituting (7) in (6) we get

$$C_n = b_n\frac{l}{n\pi a} = \frac{8\lambda l^2}{n^3\pi^3}\frac{l}{n\pi a} = \frac{8\lambda l^3}{n^4\pi^4 a} \quad \ldots (8)$$

Substituting (8) in (2) we get

$$y(x, t) = \sum_{n=1,3,5}^{\infty}\frac{8\lambda l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

■ **EXAMPLE 3** ■

A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v where

$$v = \frac{cx}{l} \text{ in } 0 < x < l$$

$$= \frac{c}{l}(2l - x) \text{ in } l < x < 2l$$

x being the distance from an end point. Find the displacement of the string at any time.

[Ap. '87 ECE, Ap. '87 Civil, Nov. '88 Mech.]

● **Solution**

The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$ ... (1)

Let $$2l = \text{L for convenience}$$

Therefore the boundary conditions are

(i) $y(0, t) = 0$ for all $t > 0$

(ii) $y(\text{L}, t) = 0$ for all $t > 0$

(iii) $y(x, 0) = 0$ for all x in $(0, \text{L})$

(iv) $$\frac{\partial y(x, 0)}{\partial t} = \begin{cases} \dfrac{2cx}{\text{L}}, & 0 < x < \dfrac{\text{L}}{2} \\ \dfrac{2c}{\text{L}}(\text{L} - x), & \dfrac{\text{L}}{2} < x < \text{L} \end{cases}$$

After applying the first three boundary conditions in the correct solution of (1) we get the most general solution as

$$y(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\text{L}} \sin\frac{n\pi at}{\text{L}} \qquad \text{... (2)}$$

Partially differentiating (2) w.r.t. 't' we get

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{\text{L}} \sin\frac{n\pi x}{\text{L}} \cos\frac{n\pi at}{\text{L}} \qquad \text{... (3)}$$

Putting $t = 0$ in (3) we get

$$\frac{\partial y(x, 0)}{\partial t} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{\text{L}} \sin\frac{n\pi x}{\text{L}} = f(x) \text{ (say)} \qquad \text{... (4)}$$

where $$f(x) = \begin{cases} \dfrac{2cx}{\text{L}} \text{ in } 0 < x < \dfrac{\text{L}}{2} \\ \dfrac{2c}{\text{L}}(\text{L} - x) \text{ in } \dfrac{\text{L}}{2} < x < \text{L} \end{cases}$$

To find c_n expand $f(x)$ in a half-range Fourier sine series.

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\text{L}} \qquad \text{... (5)}$$

From (4) and (5) we get,

$$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{\text{L}} \sin\frac{n\pi x}{\text{L}} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\text{L}}$$

Equating like co-efficients we get

$$c_n \frac{n\pi a}{\text{L}} = b_n$$

i.e., $$\boxed{c_n = \frac{b_n \text{L}}{n\pi a}} \qquad \text{... (6)}$$

Now $$b_n = \frac{2}{\text{L}} \int_0^{\text{L}} f(x) \sin\frac{n\pi x}{\text{L}}\, dx$$

$$= \frac{2}{\text{L}} \left[\int_0^{\text{L}/2} \frac{2cx}{\text{L}} \sin\frac{n\pi x}{\text{L}}\, dx + \int_{\text{L}/2}^{\text{L}} \frac{2c}{\text{L}}(\text{L} - x) \sin\frac{n\pi x}{\text{L}}\, dx \right]$$

$$= \frac{4c}{\text{L}^2} \left[\left\{ x \left(\frac{-\cos\frac{n\pi x}{\text{L}}}{\frac{n\pi}{\text{L}}} \right) - 1 \left(\frac{-\sin\frac{n\pi x}{\text{L}}}{\frac{n^2\pi^2}{\text{L}^2}} \right) \right\}_0^{\text{L}/2} + \left\{ (\text{L} - x) \left(\frac{-\cos\frac{n\pi x}{\text{L}}}{\frac{n\pi}{\text{L}}} \right) - (-1) \left(\frac{-\sin\frac{n\pi x}{\text{L}}}{\frac{n^2\pi^2}{\text{L}^2}} \right) \right\}_{\text{L}/2}^{\text{L}} \right]$$

$$= \frac{4c}{\text{L}^2} \left[-\frac{\frac{\text{L}}{2}\cos\frac{n\pi}{2}}{\frac{n\pi}{\text{L}}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{\text{L}^2}} + \frac{\frac{\text{L}}{2}\cos\frac{n\pi}{2}}{\frac{n\pi}{\text{L}}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{\text{L}^2}} \right]$$

$$= \frac{4c}{\text{L}^2} \cdot \frac{2\sin\frac{n\pi}{2}}{n^2\pi^2} \text{L}^2 = \frac{8c\sin\frac{n\pi}{2}}{n^2\pi^2}$$

$$\therefore b_n = \frac{8c\sin\frac{n\pi}{2}}{n^2\pi^2} \qquad \text{... (7)}$$

Substituting (7) in (6) we get

$$c_n = \frac{8c\sin\frac{n\pi}{2}}{n^2\pi^2} \frac{\text{L}}{n\pi a}$$

$$c_n = \frac{8c\,L \sin\frac{n\pi}{2}}{n^3\pi^3 a}$$

Replace L by $2l$ we get

$$c_n = \frac{16cl \sin\frac{n\pi}{2}}{n^3\pi^3 a}$$

Substituting (8) in (2), we get

$$y(x,t) = \sum_{n=1}^{\infty} \frac{16cl \sin\frac{n\pi}{2}}{n^3\pi^3 a} \cdot \sin\frac{n\pi x}{2l} \sin\frac{n\pi at}{2l}$$

EXAMPLE 4

A uniform string of length 'l' is struck in such a way that an initial velocity of V_0 is imparted to the portion of the string between $\frac{l}{4}$ and $\frac{3l}{4}$ while the string is in its equilibrium position. Find the displacement of the string at any time.

[MSU, Nov. 9?]

Solution

The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$ ... (1)

The boundary conditions are,

(i) $y(0,t) = 0,\ t>0$ (ii) $y(l,t) = 0,\ t>0$, (iii) $y(x,0) = 0$

$$\text{(iv)}\quad \frac{\partial y(x,0)}{\partial t} = \begin{cases} 0, & 0<x<\frac{l}{4} \\ v_0, & \frac{l}{4}\le x\le \frac{3l}{4} \\ 0, & \frac{3l}{4}<x\le l \end{cases}$$

The most general solution of (1) which satisfies conditions (i), (ii) and (iii) is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad \text{... (2)}$$

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \cdot \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad \text{... (3)}$$

Applying condition (iv) in (3), we get,

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \cdot \sin\frac{n\pi x}{l} = f(x) \quad \text{... (3a)}$$

To find C_n, expand $f(x)$ in a half range Fourier sine series.

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{... (3b)}$$

From (3a) and (3b), we get,

i.e., $C_n \dfrac{n\pi a}{l} = b_n$

$$\boxed{C_n = \frac{b_n l}{n\pi a}} \quad \text{... (4)}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[0 + \int_{l/4}^{3l/4} v_0 \sin\frac{n\pi x}{l}\,dx + 0\right]$$

$$= \frac{2v_0}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_{\frac{l}{4}}^{\frac{3l}{4}}$$

$$= \frac{-2v_0}{n\pi}\left[\cos\frac{3n\pi}{4} - \cos\frac{n\pi}{4}\right]$$

$$= \frac{2v_0}{n\pi}\left[2\sin\frac{n\pi}{2} \sin\frac{n\pi}{4}\right]$$

$$\therefore C_n = \frac{4v_0}{an^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi}{4} \cdot l$$

Substituting this value of C_n in (2), we get,

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4v_0 l}{an^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi}{4} \cdot \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

EXAMPLE 5

Find the displacement of a tightly stretched string of length 7 cms vibrating between fixed end points if initial displacement is 10 sin $\left(\frac{3\pi x}{7}\right)$ and initial velocity is 15 sin $\left(\frac{9\pi x}{7}\right)$.

Solution : The vibration of string equation is

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \quad \text{... (1)}$$

Let $l = 7$ cm

From the given problem we have the following boundary conditions.

(i) $y(0, t) = 0$

(ii) $y(l, t) = 0$

(iii) $y(x, 0) = 10 \sin\left(\frac{3\pi x}{l}\right)$

(iv) $\frac{\partial y(x, 0)}{\partial t} = 15 \sin\left(\frac{9\pi x}{l}\right)$

The correct solution of (1) which satisfies the above boundary conditions is

$$y(x, t) = (C_9 \cos px + C_{10} \sin px)(C_{11} \cos pat + C_{12} \sin pat) \quad \dots (2)$$

Applying condition (*i*) in (2), we get

$$y(0, t) = C_9 (C_{11} \cos pat + C_{12} \sin pat) = 0$$

$$C_9 = 0 \quad \dots (3)$$

Substituting (3) in (2), we get

$$y(x, t) = C_{10} \sin px (C_{11} \cos pat + C_{12} \sin pat) \quad \dots (4)$$

Applying condition (ii) in (4), we get

$$y(l, t) = C_{10} \sin pl (C_{11} \cos pat + C_{12} \sin pat) = 0$$

$$\sin pl = 0 \quad [\because C_{10} \neq 0]$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l} \quad \dots (5)$$

Substituting (5) in (4), we get

$$y(x, t) = C_{10} \sin \frac{n\pi x}{l}\left(C_{11} \cos \frac{n\pi at}{l} + C_{12} \sin \frac{n\pi at}{l}\right)$$

$$= \sin \frac{n\pi x}{l}\left(C_n \cos \frac{n\pi at}{l} + d_n \sin \frac{n\pi at}{l}\right)$$

$$C_n = C_{10} C_{11} \ ; \ d_n = C_{10} C_{12}$$

The most general solution can be written as

$$y(x, t) = \sum_{n=1}^{\infty}\left(C_n \cos \frac{n\pi at}{l} + d_n \sin \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \quad \dots (6)$$

Applying condition (iii) in (6), we get

$$y(x, 0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = 10 \sin \frac{3\pi x}{l}$$

i.e., $C_1 \sin \frac{\pi x}{l} + C_2 \sin \frac{2\pi x}{l} + C_3 \sin \frac{3\pi x}{l} + \dots\dots = 10 \sin \frac{3\pi x}{l}$

Equating like coefficients, we get

i.e., $\boxed{C_3 = 10} \quad \dots (7)$

From (6), we get

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty}\left(-C_n \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + d_n \cdot \frac{n\pi a}{l} \cos \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \quad \dots (8)$$

Applying condition (iv) in (8), we get

$$\frac{\partial y(x, 0)}{\partial t} = \sum_{n=1}^{\infty} d_n \frac{n\pi a}{l} \cdot \sin \frac{n\pi x}{l} = 15 \sin \frac{9\pi x}{l}$$

[Equating like coefficients]

i.e., $d_9 \frac{9\pi a}{l} = 15$

$$\boxed{d_9 = \frac{15l}{9\pi a}} \quad \dots (9)$$

The remaining d's viz., $d_1 = d_2 = \dots\dots\dots\dots = 0$

Substituting (7) and (9) in (6), we get

$$y(x, t) = 10 \cos \frac{3\pi at}{l} \sin \frac{3\pi x}{l} + \frac{15l}{9\pi a} \sin \frac{9\pi at}{l} \cdot \sin \frac{9\pi x}{l}, \quad \text{where } l = 7 \text{ cm.}$$

EXAMPLE 6

Solve the following boundary value problem of vibration of string :

(i) $y(0, t) = 0$

(ii) $y(l, t) = 0$

(iii) $\frac{\partial y(x, 0)}{\partial t} = x(x - l), \quad 0 < x < l$

(iv) $y(x, 0) = x \quad \text{in} \quad 0 < x < \frac{l}{2}$

$\qquad = l - x \quad \text{in} \quad \frac{l}{2} < x < l$

Solution

The vibration of string equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \dots (1)$$

The correct solution of (1) which satisfies the given differential equation is

$$y(x, t) = (C_9 \cos px + C_{10} \sin px)(C_{11} \cos pat + C_{12} \sin pat) \quad \dots (2)$$

Applying condition (*i*) in (2), we get

$$y(0, t) = C_9 (C_{11} \cos pat + C_{12} \sin pat) = 0$$

$\therefore C_9 = 0$

Substituting $C_9 = 0$ in (2), we get

$$y(x,t) = C_{10} \sin px\,(C_{11} \cos pat + C_{12} \sin pat)$$

Applying condition (ii) in (4), we get

$$y(l,t) = C_{10} \sin pl\,(C_{11} \cos pat + C_{12} \sin pat) = 0$$

$$C_{10} \neq 0$$

$$\therefore \sin pl = 0 \Rightarrow pl = n\pi$$

$$\text{i.e., } p = \frac{n\pi}{l}$$

Substituting (5) in (4), we get

$$y(x,t) = C_{10} \sin \frac{n\pi x}{l}\left(C_{11} \cos \frac{n\pi at}{l} + C_{12} \sin \frac{n\pi at}{l}\right)$$

$$= \sin \frac{n\pi x}{l}\left(C_n \cos \frac{n\pi at}{l} + d_n \sin \frac{n\pi at}{l}\right)$$

where $C_n = C_{10} C_{11}$; $d_n = C_{10} C_{12}$;

The most general solution can be written as

$$y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\left(C_n \cos \frac{n\pi at}{l} + d_n \sin \frac{n\pi at}{l}\right)$$

Applying condition (iv) in (6), we get

$$y(x,0) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\,(C_n)$$

$$= f(x) = \begin{cases} x, & \left(0, \frac{l}{2}\right) \\ l - x, & \left(\frac{l}{2}, l\right) \end{cases}$$

To find C_n, expand RHS of (7) in a half range sine series.

i.e., $$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

i.e., $$\boxed{C_n = b_n}$$

Where $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l}\,dx\right]$$

$$b_n = \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} \quad \text{when '}n\text{' is odd.}$$

$$\boxed{C_n = \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2}} \quad \text{... (8)}$$

Partially differentiating (6), w.r.t. 't', we get

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\left(-C_n \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + d_n \frac{n\pi a}{l} \cos \frac{n\pi at}{l}\right) \quad \text{... (9)}$$

Applying condition (iii) in (9), we get

$$\frac{\partial y(x,0)}{\partial t} = \sum_{1}^{\infty} \sin \frac{n\pi x}{l} \cdot d_n \frac{n\pi a}{l} = x(x-l)$$

$$= \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\therefore \boxed{d_n \frac{n\pi a}{l} = a_n}$$

$$a_n = \frac{2}{l}\int_0^l x(x-l) \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(x^2 - lx)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2x-l)\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{-8l^2}{n^3\pi^3} \text{ if } n \text{ is odd.}$$

$$\therefore d_n = \frac{-8l^3}{n^4\pi^4 a^2}, \; n \text{ is odd} \quad \text{... (10)}$$

Substituting (8) and (10) in (6), we get

$$y(x,t) = \sum_{1,3,5}^{\infty} \sin \frac{n\pi x}{l}\left\{\frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} \cos \frac{n\pi at}{l} - \frac{8l^3}{n^4\pi^4 a^2} \sin \frac{n\pi at}{l}\right\}$$

■ EXAMPLE 7 ■

Solve the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}, \quad y(0, t) = y(5, t) = 0, \qquad \text{[MSU, Apr}$$

$$y(x, 0) = 0, \quad \frac{\partial y(x, 0)}{\partial t} = 3\sin 2\pi x - 2\sin 5\pi x$$

● Solution

The heat flow equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

where $a = 2$.

The boundary conditions are,

(a) $y(0, t) = 0$

(b) $y(l, t) = 0$ where $l = 5$

(c) $y(x, 0) = 0$

(d) $\dfrac{\partial y(x, 0)}{\partial t} = 3\sin 2\pi x - 2\sin 5\pi x$

After applying conditions *(a)*, *(b)* and (c),

We get the most general solution of (1) as

$$y(x, t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \qquad \ldots(2)$$

Partially differentiating (2), w.r.t. 't', we get,

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\;\frac{n\pi a}{l}$$

Applying condition *(d)* in (3), we get

$$\frac{\partial y(x, 0)}{\partial t} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\,\frac{n\pi a}{l}$$

$$= 3\sin 2\pi x - 2\sin 5\pi x$$

Replacing l by 10, we get,

$$c_1\sin\frac{\pi x}{10}\cdot\frac{\pi a}{10} + \ldots + c_{20}\sin\frac{20\pi x}{10}\,\frac{20\pi a}{10} \ldots + c_{50}\sin\frac{50\pi x}{10}\cdot\frac{50\pi a}{10} + \ldots\ldots$$

$$= 3\sin 2\pi x - 2\sin 5\pi x$$

i.e., $c_1\sin\dfrac{\pi x}{10}\cdot\dfrac{\pi a}{10} + \ldots\ldots + c_{20}\sin 2\pi x\; 2\pi a + \ldots\ldots$

$$\ldots\ldots + c_{50}\sin 5\pi x\; 5\pi a + \ldots\ldots$$

$$= 3\sin 2\pi x - 2\sin 5\pi x$$

Equating like coefficients, we get,

$$c_{20}\,2\pi a = 3$$

$$c_{20} = \frac{3}{2\pi a} = \frac{3}{4\pi} \quad [\because a = 2] \qquad \ldots(4)$$

$$c_{50}\,5\pi a = -2$$

$$c_{50} = \frac{-2}{5\pi a} = -\frac{1}{5\pi} \quad [\because a = 2] \qquad \ldots(5)$$

The remaining values of c's are zero.

(i.e., $c_1 = c_2 = 0$ ...... except c_{20} and c_{50})

Substituting (4) and (5) in (2), we get,

$$y(x, t) = c_{20}\sin\frac{20\pi x}{10}\sin\frac{20\pi at}{10} + c_{50}\sin\frac{50\pi x}{10}\sin\frac{50\pi at}{10}$$

[Replacing l by 10]

$$y(x, t) = \frac{3}{4\pi}\sin 2\pi x\sin 4\pi t - \frac{1}{5\pi}\sin 5\pi x\sin 10\pi t$$

● EXERCISES

1. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l - x)$, find the transverse displacement at any point of the string at any time 't'. *[Nov. '86 ECE, Ap. '91 Civil]*

   [Ans. put $\lambda = 3$ in Ex (2)]

2. A string of length $2l$ is tightly stretched and fixed at its ends points $(0, 0)$ and $(2l, 0)$ of the xy plane. It is made to vibrate transversely in the xy plane by giving to each of its points a transverse velocity v in the xy plane where v is given by

$$v = kx \text{ for } 0 < x < l$$
$$= k(2l - x) \text{ for } l < x < 2l$$

   Find the expression for the transverse displacement of the string at any time 't'.

$$\left[\textit{Ans. } y(x, t) = \frac{16kl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}\right]$$

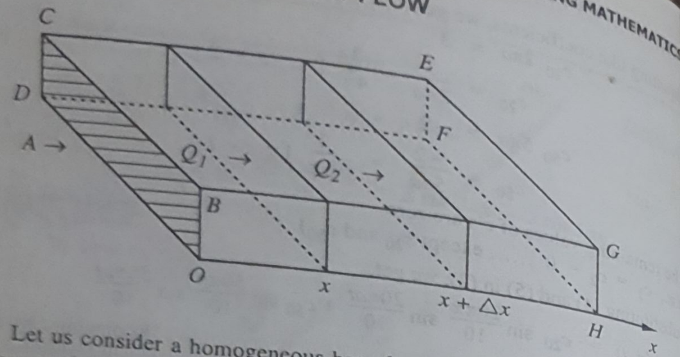
3. A taut string of length 20 cms fastened at both ends is displaced from its position of equilibrium, by imparting to each of its points an initial velocity given by

$$V = x \text{ in } 0 < x < 10$$
$$= 20 - x \text{ in } 10 < x < 20$$

   x being the distance from one end. Determine the displacement at any subsequent time.

$$\left[\textit{Ans. } y(x, t) = \frac{1600}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{20}\sin\frac{n\pi at}{20}\right]$$

3.5 ONE-DIMENSIONAL HEAT FLOW



Let us consider a homogeneous bar of uniform cross-section. Let the area of cross section be A cm^2. Let us assume that the sides of the bar are insulated so that the loss or gain of heat from the sides by conduction or radiation is negligible. Take one end of the bar as the origin and the direction of flow as the positive x - axis. Let us assume that, same amount of heat is applied at all points of the face OBCD. Since the remaining four faces OBGH, EFGH, CDFE, ODFH are insulated the heat flow is taken place in only one direction (x - direction) and perpendicular to the area A. Let ρ be the density (gm./cm^3), 's' the specific heat (cal/gr.des) and k thermal conductivity (cal. cm. des. cm.) of the material. Now the temperature u (x, t) at any point of the bar depends on the distance 'x' of the point from one end and at time 't'.

Now consider the cross section at x. The amount of heat flowing through this section of the bar per sec depends on the area A of the cross section, the conductivity 'k' and the temperature gradient $\dfrac{\partial u}{\partial x}$.

Therefore Q_1, the quantity of heat flowing into the section at a distance $x = -kA\left(\dfrac{\partial u}{\partial x}\right)_x$ per sec. (Here the negative sign is due to the fact that the temperature decreases with increase in distance from the hot end).

Similarly Q_2, the quantity of heat flowing out of the section at a distance

$$x + \Delta x = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} \text{ per sec.}$$

Therefore the amount of heat retained by the slab with thickness Δx is

$$Q_1 - Q_2 = -KA\left(\frac{\partial u}{\partial x}\right)_x + KA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$= KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per sec} \quad \dots (1)$$

The rate of increase of heat in the section x and $x + \Delta x$

$$= s\rho A\,\Delta x\,\frac{\partial u}{\partial t} \quad \dots (2)$$

where s is the specific heat and ρ be the density of the material.

Now from (1) and (2) we get

$$s\rho A\,\Delta x\,\frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

i.e., $$s\rho\frac{\partial u}{\partial t} = \frac{k\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]}{\Delta x}.$$

Taking limit as $\Delta x \to 0$, we get

$$s\rho\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

i.e., $$\frac{\partial u}{\partial t} = \frac{K}{s\rho}\cdot\frac{\partial^2 u}{\partial x^2}$$

or $$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$$, where $\alpha^2 = \dfrac{k}{s\rho}$ *is known as diffusivity of the material of the bar.*

Hence the ***"one-dimensional heat flow equation"*** is

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}}$$

Solution of the one-dimensional heat equation (Method of separation of variables). [Nov. '91 ECE]

We know that the one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \quad \dots (1)$$

Let $$u = X(x)\cdot T(t) \quad \dots (2)$$

be the solution of (1) where X is a function of x only and T is a function t only.

Differentiating (2) partially w.r.t. 't' we get

$$\frac{\partial u}{\partial t} = XT' \quad \dots (3)$$

Differentiating (2) partially w.r.t 'x' twice we get

$$\frac{\partial^2 u}{\partial x^2} = X''T \quad \dots (4)$$

Substituting (3) and (4) in (1) we get

$$XT' = \alpha^2 X''T$$

Separating the variables we get

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = K \text{ (say)}$$

i.e., $$\frac{T'}{\alpha^2 T} = K \text{ and } \frac{X''}{X} = K$$

i.e., $$T' - K\alpha^2 T = 0$$

and $$X'' - KX = 0$$

The equations (6) and (7) are ordinary differential equations the solution of which depend on the value of K.

Therefore consider the following cases.

Case (i) : Let $K = 0$

$\therefore$ Equations (6) and (7) becomes

$$T' = 0 \text{ and } X'' = 0$$

i.e., $$\frac{dT}{dt} = 0 \text{ and } \frac{d^2X}{dx^2} = 0$$

Integrating we get,

$$T = c_1 \text{ and } X = c_2 x + c_3$$

$\therefore$ The solution of (1) is $u(x, t) = c_1 (c_2 x + c_3)$ ... (8)

Case (ii) : Let K be positive

i.e., $K = p^2$ [Here k is always positive whether p is +ve or –ve]

$\therefore$ The differential equations (6) and (7) become

$$T' - p^2 \alpha^2 T = 0 \text{ and } X'' - p^2 X = 0$$

i.e., $$\frac{dT}{dt} - p^2 \alpha^2 T = 0 \text{ and } \frac{d^2X}{dx^2} - p^2 X = 0$$

i.e., $$\frac{dT}{dt} = p^2 \alpha^2 T \text{ and } m^2 - p^2 = 0 \quad \text{[Auxiliary equation]}$$

i.e., $$\frac{dT}{T} = p^2 \alpha^2 \, dt \text{ and } m = \pm p$$

i.e., $$\log T = \alpha^2 p^2 t + \log c_3 \text{ and } X = (c_4 e^{px} + c_5 e^{-px})$$

i.e., $$T = c_3 e^{\alpha^2 p^2 t} \text{ and } X = c_4 e^{px} + c_5 e^{-px}$$

Substituting the values of X and T in (1) we get

$\therefore$ $$u(x, t) = c_3 e^{\alpha^2 p^2 t} (c_4 e^{px} + c_5 e^{-px}) \quad \text{... (9)}$$

Case (iii) : Let K be negative

i.e., $K = -p^2$ [Here K is always –ve whether p is +ve or –ve]

$\therefore$ The differential equations (6) and (7) become

$$T' + p^2 \alpha^2 T = 0 \text{ and } X'' + p^2 X = 0$$

i.e., $$\frac{dT}{dt} = -p^2 \alpha^2 T \text{ and } \frac{d^2X}{dx^2} + p^2 X = 0$$

i.e., $$\frac{dT}{T} = -p^2 \alpha^2 \, dt \text{ and } m^2 + p^2 = 0 \quad \text{[Auxiliary equation]}$$

$$\log T = -p^2 \alpha^2 t + \log c_6 \text{ and } m = \pm ip$$

$$T = c_6 e^{-p^2 \alpha^2 t} \text{ and } X = c_7 \cos px + c_8 \sin px$$

$$u(x, t) = c_6 e^{-p^2 \alpha^2 t} (c_7 \cos px + c_8 \sin px) \quad \text{... (10)}$$

From the above three cases the different possible solutions of the heat-equation are

$$u(x, t) = c_1 (c_2 x + c_3) \quad \text{... (8)}$$

$$u(x, t) = c_3 e^{\alpha^2 p^2 t} (c_4 e^{px} + c_5 e^{-px}) \quad \text{... (9)}$$

$$u(x, t) = c_6 e^{-\alpha^2 p^2 t} (c_7 \cos px + c_8 \sin px) \quad \text{... (10)}$$

Out of these three solutions we have to choose the correct solution which satisfies the physical nature of the problem. In this chapter, we are dealing with the problems on heat conduction. According to the Law of Thermodynamics, when time 't' increases the temperature $u(x, t)$ will not increase. For example, if water is heated for many hours the maximum temperature will not increase from 100°C (The boiling point of water).

Now consider the solution

$$u(x, t) = c_3 e^{\alpha^2 p^2 t} (c_4 e^{px} + c_5 e^{-px})$$

Here if t increases then $u(x, t)$ is also increases.

i.e., if we allow $t \to \infty$ then $u(x, t) \to \infty$.

In other words, if we put $t = \infty$ in the above solution we get [$\because e^{\infty} = \infty$]

$$u(x, t) = \infty$$

This is in contradiction with the Law of Thermodynamics. Hence this solution is not suitable for our problems on heat conduction. Next consider the solution $u(x, t) = c_1 (c_2 x + c_3)$. In general, in problems on heat conduction we want to find the temperature $u(x, t)$ at any distance 'x' and at any time 't'. But in this solution, we have $u(x, t)$ as a function of x alone. Therefore it is not possible to measure the temperature at any time 't'. Hence this solution is also not suitable for our problems on heat conduction.

> *NOTE : At steady state conditions only we can use the solution*
>
> $$u(x, t) = c_1 (c_2 x + c_3)$$
> $$u(x) = ax + b$$
> $$c_1 c_2 = a, \quad c_1 c_3 = b$$

Therefore, the correct solution which is suitable for our problems on one dimensional heat flow is given by

$$u(x, t) = c_6 e^{-\alpha^2 p^2 t} (c_7 \cos px + c_8 \sin px)$$

(or) $$\boxed{u(x, t) = (A \cos px + B \sin px) \, e^{-\alpha^2 p^2 t}}$$ [$\therefore c_6 c_7 = A \; c_6 c_8 = B$]

NOTE : We can always take the correct solution for one dimensional heat flow equation as

$$\boxed{u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t}}$$

The exponential factors in the above solution causes $u(x, t)$ to approach zero as t tends to infinity.

■ **EXAMPLE 1** ■

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ *subject to the boundary conditions* $u(0, t) = 0,\ u(l, t) = 0\ u(x, 0) = x.$

Solution

When we solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ by the method of separation of variables we get three types of solutions [Refer to previous article]. Here the given equation is a one dimensional heat flow equation and therefore the correct solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \quad \dots (1)$$

The boundary conditions are

(i) $u(0, t) = 0$ for all $t > 0$

(ii) $u(l, t) = 0$ for all $t > 0$

(iii) $u(x, 0) = x$ for all x in $(0, l)$

Applying condition (i) in (1) we get

$$u(0,t) = A\, e^{-\alpha^2 p^2 t} = 0$$

$\therefore A = 0$ since $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ it is defined for all 't']

Substituting $A = 0$ in (1) we get

$$u(x,t) = B\sin px\, e^{-\alpha^2 p^2 t} \quad \dots (2)$$

Applying condition (ii) in (2) we get

$$u(l,t) = B\sin pl\, e^{-\alpha^2 p^2 t} = 0$$

Here $B \neq 0$ since if $B = 0$ we get trivial solution.

$e^{-\alpha^2 p^2 t} \neq 0$ since it is true for all 't'.

Hence $\sin pl = 0$ i.e., $pl = n\pi$

$$\therefore \quad \boxed{p = \frac{n\pi}{l}}$$

Substituting $p = \dfrac{n\pi}{l}$ in (2) we get

$$u(x,t) = B\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

The most general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \dots (3)$$

[By superposition principle]

Applying condition (iii) in (3) we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = x \quad \dots (4)$$

To find B_n expand x in $(0, l)$ in a half-range Fourier sine series.

i.e. $$x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \dots (5)$$

From (4) and (5) we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{i.e.,} \quad \boxed{B_n = b_n}$$

Now
$$b_n = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= -\frac{2l}{n\pi}\cos n\pi = \frac{2l}{n\pi}(-1)^{n+1} \quad [\because \cos n\pi = (-1)^n]$$

$$\therefore B_n = \frac{2l}{n\pi}(-1)^{n+1} \quad \dots\dots (6)$$

Substituting (6) in (3) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

■ **EXAMPLE 2** ■

Find the solution to the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ *that satisfies the conditions*

(i) $u(0, t) = 0$ (ii) $u(l, t) = 0$ *for* $t > 0$

(iii) $u(x, 0) = x$ *for* $0 < x < \dfrac{l}{2}$

$= l - x$ *for* $\dfrac{l}{2} < x < l$ [MU. Oct. 2000, Ap. '90 Mech.]

Solution

As in the previous example, the correct solution of the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ and satisfying the conditions (i) and (ii) is

$$u(x,t) = B \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

The most general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \qquad \ldots (1)$$

Applying condition (*iii*) in (1) we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x)$$

Where

$$f(x) = x \text{ for } 0 < x < \frac{l}{2} \qquad \ldots (2)$$

$$= l - x \text{ for } \frac{l}{2} < x < l$$

To find B_n expand $f(x)$ in $(0, l)$ in a half-range Fourier sine series we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (3)$$

From (2) and (3) we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e., $\boxed{B_n = b_n}$

Now $b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

$$= \frac{2}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l-x) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \ldots (4)$$

$$\therefore B_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Substituting (4) in (1) we get

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

EXAMPLE 3

A homogeneous rod of conducting material of length 'l' units has ends kept at zero temperature and the temperature at the centre is T and falls uniformly to zero at the two ends. Find u (x, t). [BDN, Nov. 97, Apr. 99]

Solution

The one dimensional heat flow equation is, ... (1)

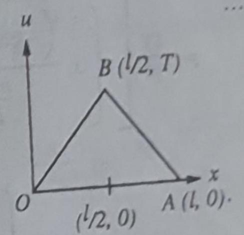
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are,

(*a*) $u(0,t) = 0$ for all t

(*b*) $u(l,t) = 0$ for all t

(*c*) $u(x,0) = \begin{cases} \dfrac{2Tx}{l}, & 0 < x < \dfrac{l}{2} \\ \dfrac{2T}{l}(l-x), & \dfrac{l}{2} < x < l \end{cases}$



[From graph,

The equation of OB is $u = \dfrac{2Tx}{l}$

The equation of AB is $\dfrac{x-l}{\frac{l}{2}-l} = \dfrac{u-0}{T-0}$; $u = \dfrac{2T}{l}(l-x)$]

The correct solution of (1) satisfying conditions (*a*) and (*b*) is,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \qquad \ldots (2)$$

Applying condition (*c*) in (2), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x) \qquad \ldots (3)$$

To find B_n, expand $f(x)$ in a half range Fourier sine series in $(0, l)$.

i.e.,
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

From (3) and (4), we get,

$$\therefore \boxed{B_n = b_n}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2Tx}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2T}{l}(l-x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4T}{l^2}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4T}{l^2}\left[\left\{x \frac{\left(-\cos \frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - \frac{\left(-\sin \frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right\}_0^{l/2} + \left\{(l-x) \frac{\left(-\cos \frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (-1) \frac{\left(-\sin \frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right\}_{l/2}^{l}\right]$$

$$= \left[\frac{4T}{l^2} \frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right]$$

$$\therefore B_n = \frac{8T}{n^2\pi^2} \sin \frac{n\pi}{2} \qquad \ldots (6)$$

Substituting (6) in (2), we get,

$$u(x, t) = \frac{8T}{\pi^2} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \sin \frac{n\pi}{2}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

EXERCISES

1. Solve $\frac{\partial \theta}{\partial t} = \alpha^2 \frac{\partial^2 \theta}{\partial t^2}$ given that

(i) θ is finite when $t \to \infty$

(ii) $\theta = 0$ when $x = 0$ and $x = \pi$ for all values of t

(iii) $\theta = x$ from $x = 0$ to $x = \pi$ when $t = 0$ [Nov. '87 Civil]

$$\left[\textit{Ans. } \theta(x, t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx\, e^{-\alpha^2 n^2 t}\right]$$

2. Solve the one dimensional heat flow equation $\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}$ by the variable separable method and using the suitable solution find U subject to the conditions $U(0, t) = 0$, $U(l, t) = 0$, $U(x, 0) = lx - x^2$ [Ap. '90 Civil]

$$\left[\textit{Ans. } U(x, t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8l^2}{n^3\pi^3} \sin \frac{n\pi x}{l}\, e^{\frac{-n^2\pi^3 c^2 t}{l^2}}\right]$$

3. A rod of length l is heated so that its ends A and B are at zero temperature. If initially its temperature is given by $u = \frac{cx(l-x)}{l^2}$ find the temperature at time t.

$$\left[\textit{Ans. } u(x, t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8c}{n^3\pi^3} \sin \frac{n\pi x}{l}\, e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}}\right]$$

4. Solve $\alpha^2 U_{xx} = u_t$, $0 \le x \le L$, $t > 0$

$u(0, t) = 0$, $u(L, t) = 0$, $0 \le t$,

$u(x, 0) = kx(L - x)$ $(k > 0)$, $0 \le x \le L$ [Nov. '89 Civil]

$$\left[\textit{Ans. } u(x, t) = \frac{8kL^2}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{L}\, e^{\frac{-n^2\pi^2 c^2 t}{L^2}}\right] \text{ where } c^2 = \frac{1}{\alpha^2}$$

5.6 STEADY STATE CONDITIONS AND ZERO BOUNDARY CONDITIONS

Suppose a rod is heated at both ends by a constant temperature $c_1°$ and $c_2°$. After some time the temperature in the rod remains constant. Hence there is no change in temperature in the rod if time t varies. Hence the temperature function $u(x, t)$ is independent of time t. $u(x, t)$ is a function of x alone. In other words $u(x, t)$ is independent of time t. ***This state in which the temperature does not vary with respect to time t is called steady state. Therefore when steady state condition exists, u (x, t) becomes u (x).***

■ **EXAMPLE 1** ■

A rod of length l has its ends A and B kept at 0°C and 100° C until steady state condition prevail. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained, find the temperature $u(x, t)$ at a distance x from A and at time t. [Anna Univ. 2001]

Solution

We know that the heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \ldots (1)$$

In the given problem the end A is at 0°C and the end B is at 100°C when steady state conditions prevails. Now we have to find the temperature distribution $u(x, t)$ when steady state conditions exists. In otherwords we have to find $u(x, t)$ before the end B is reduced to 0°C. We know that when steady state conditions prevail the temperature function $u(x, t)$ is a function of x alone. Hence it is free from time 't'. Therefore the heat flow equation (1) becomes.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only}\right]$$

since u is a function of x only, the above equation can be written as $\frac{d^2u}{dx^2} = 0$ $(\alpha \neq 0)$.

$$\left[\therefore \text{If } y = f(x) \text{ then there is no difference between } \frac{dy}{dx} \text{ and } \frac{\partial y}{\partial x}\right]$$

Hence when steady state conditions prevails the heat flow equation becomes

$$\frac{d^2u}{dx^2} = 0 \quad \ldots (2)$$

and the boundary conditions are

(i) $u(0) = 0$

(ii) $u(l) = 100$

The solution of (2) is $u(x) = ax + b$ ... (3)

Now applying condition (i) in (3) we get

$$u(0) = 0 + b \quad \text{i.e., } b = 0$$

Putting $b = 0$ in (3) we get $u(x) = ax$ ... (4)

Applying condition (ii) in (4) we get

$$u(l) = al = 100$$

$$\therefore a = \frac{100}{l} \quad (5)$$

Substituting (5) in (4) we get

$$\boxed{u(x) = \frac{100\,x}{l}}$$

Therefore in the steady state the temperature function is given by $u(x) = \frac{100\,x}{l}$. Now the end B is reduced to zero. At this stage the steady state is changed into unsteady state. For this unsteady state the initial temperature distribution is $u(x) = \frac{100\,x}{l}$ (i.e., the temperature distribution reached at the steady state becomes initial temperature distribution for the unsteady state)

Now the heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{n\pi}{l} - p$

The new boundary conditions are

(a) $u(0, t) = 0$ for all $t > 0$

(b) $u(l, t) = 0$ for all $t > 0$

(c) $u(x, 0) = \frac{100\,x}{l}$ for x in $(0, l)$

The general solution of $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ and satisfying condition (a) and (b) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \text{[As explained earlier]} \quad \ldots (6)$$

Applying condition (c) in (6) we get

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{100\,x}{l} \quad \ldots (7)$$

To find B_n expand $\frac{100\,x}{l}$ in $(0, l)$ in half-range Fourier sine series we get,

$$\frac{100\,x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \ldots (8)$$

From (7) and (8) we get

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\therefore \boxed{B_n = b_n}$$

Now $b_n = \frac{2}{l}\int_0^l \frac{100\,x}{l} \sin \frac{n\pi x}{l}\, dx$

$$= \frac{200}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l = \frac{200}{l^2}\left[\frac{-l^2 \cos n\pi}{n\pi}\right]$$

$$= \frac{200}{n\pi}(-1)^{n+1} \quad [\because \cos n\pi = (-1)^n]$$

Substituting (9) in (6) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

• EXERCISES

1. A rod of length l has its ends A and B kept at 0°C and 120°C respectively until steady state conditions prevail. If the temperature at B is reduced to 0°C and kept so while that of A is maintained find the temperature distribution in the rod. [Nov. '91 ECE]

$$\left[\text{Ans. } u(x,t) = \frac{240}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \right]$$

5.7 STEADY STATE CONDITIONS AND NON-ZERO BOUNDARY CONDITIONS

EXAMPLE 1

A bar, 10 cm. long, with insulated sides, has its ends A and B kept at 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the subsequent temperature at any point of the bar at any time.

[Anna Univ. Apr. 2001]

• Solution

The equation of heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \ldots (1)$$

When the steady state conditions prevail the temperature function $u(x, t)$ is a function of x alone. In other words $u(x, t)$ is independent of time 't'. Hence equation (1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \left[\because \frac{\partial u}{\partial t} = 0\right]$$

Because 'u' is a function of 'x' alone the above equation becomes $\frac{d^2u}{dx^2} = 0$ which is an ordinary differential equation. If we integrate this equation twice we get the solution

$$u(x) = ax + b \quad \ldots (2)$$

When the steady state condition exists the boundary conditions are

(a) $u(0) = 20$

(b) $u(l) = 40$ (Take length of the rod $l = 10$ for convenience)

Applying condition (a) in (2) we get

$$u(0) = b = 20 \quad \ldots (3)$$

Equation (2) becomes

$$u(x) = ax + 20$$

Applying condition (b) in (3) we get

$$u(l) = al + 20 = 40 \quad \ldots (4)$$

i.e., $$al = 20$$

i.e., $$a = \frac{20}{l} \quad \ldots (5)$$

Substituting (4) in (3) we get

$$u(x) = \frac{20x}{l} + 20$$

Hence in the steady state, the temperature function is given by

$$u(x) = \frac{20x}{l} + 20$$

Now the temperature at A is raised to 50°C and the temperature at B is lowered to 10°C. That is, the steady state is changed to unsteady state. For this unsteady state the initial temperature distribution is given by

$$u(x,0) = \frac{20x}{l} + 20$$

For unsteady state we have the following boundary conditions.

(i) $u(0, t) = 50$

(ii) $u(l, t) = 10$

(iii) $u(x, 0) = \frac{20x}{l} + 20$

where $l = 10$ cm. By using these conditions we can not determine $u(x, t)$. For, the correct solution of one dimensional heat flow equation is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \quad \ldots (5.a)$$

Substituting (i) and (ii) in (5.a) we get

$$u(0,t) = A e^{-\alpha^2 p^2 t} = 50 \quad \ldots (5.b)$$

$$u(l,t) = (A\cos pl + B\sin pl)\, e^{-\alpha^2 p^2 t} = 10 \quad \ldots (5.c)$$

From (5.b) and (5.c) it is not possible to find the constants A and B. Since we have infinite number of values for A and B. Therefore in this case we split the solution $u(x, t)$ into two parts.

i.e., $$u(x,t) = u_s(x) + u_t(x,t) \quad \ldots (6)$$

where $u_s(x)$ is a solution of the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ and is a function of x alone and satisfying the conditions $u_s(0) = 50$ and $u_s(l) = 10$ and $u_t(x, t)$ is a transient solution satisfying (6) which decreases as t increases.

To find $u_s(x)$

we have $u_s(x) = a_1 x + b_1$

Applying the condition $u_s(0) = 50$ in (7) we get

$$u_s(0) = b_1 = 50$$

$$\therefore \quad u_s(x) = a_1 x + 50$$

Applying the condition $u_s(l) = 10$ in (8) we get

$$u_s(l) = a_1 l + 50 = 10$$

i.e., $$a_1 = \frac{-40}{l}$$

Substituting (9) in (8) we get

$$u_s(x) = -\frac{40x}{l} + 50$$

To find $u_t(x, t)$

We assume that $u_t(x, t)$ is a transient solution of $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ and satisfy the equation

$$u(x, t) = u_s(x) + u_t(x, t)$$

$$\therefore u_t(x, t) = u(x, t) - u_s(x)$$

Now we have to find the boundary conditions for $u_t(x, t)$

Putting $x = 0$ in (11) we get

$$u_t(0, t) = u(0, t) - u_s(0)$$

$$= 50 - 50 \qquad [u(0, t) = 50,\ u_s(0) = 50]$$

$$\therefore u_t(0, t) = 0 \qquad \ldots (12)$$

Putting $x = l$ in (11) we get

$$u_t(l, t) = u(l, t) - u_s(l)$$

$$= 10 - 10 = 0$$

$$\therefore u_t(l, t) = 0 \qquad \ldots (13)$$

Putting $t = 0$ in (11) we get

$$u_t(x, 0) = u(x, 0) - u_s(x)$$

$$= \left(\frac{20x}{l} + 20\right) - \left(\frac{-40x}{l} + 50\right)$$

[From (10) and condition (iii)]

$$u_t(x, 0) = \frac{60x}{l} - 30 \qquad \ldots (14)$$

Now for the function $u_t(x, t)$ we have the following boundary conditions.

(a_1) $u_t(0, t) = 0$

(b_1) $u_t(l, t) = 0$

(c_1) $u_t(x, 0) = \frac{60x}{l} - 30$

Solving the equation (1) for $u_t(x, t)$ by the method of separation of variables we get the solution of the form

$$u_t(x, t) = (A \cos px + B \sin px)\, e^{-\alpha^2 p^2 t} \qquad \ldots (15)$$

Applying condition (a_1) and (b_1) in (15) we get the most general solution

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \qquad \ldots (16)$$

Applying condition (c_1) in (16) we get

$$u_t(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{60x}{l} - 30 \qquad \ldots (17)$$

To find B_n expand $\frac{60x}{l} - 30$ in a half range Fourier sine series

i.e. $$\frac{60x}{l} - 30 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \ldots (18)$$

From (17) and (18) we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{l} \int_0^l \left(\frac{60x}{l} - 30\right) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{60}{l^2} \int_0^l (2x - l) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{60}{l^2} \left[(2x - l)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) \right]_0^l$$

$$= \frac{60}{l^2}\left[-\frac{l^2}{n\pi} \cos n\pi - \frac{l^2}{n\pi}\right] = -\frac{60}{n\pi}\,[1 + (-1)^n]$$

$$\therefore B_n = 0 \text{ if } n \text{ is odd}$$

$$= -\frac{120}{n\pi} \text{ if } n \text{ is even}$$

$$\therefore u_t(x, t) = \sum_{n = 2, 4, \ldots}^{\infty} -\frac{120}{n\pi} \sin \frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

Now replace l by 10, we get

$$u_t(x, t) = \sum_{n=2,4,\ldots}^{\infty} -\frac{120}{n\pi} \sin \frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

But $u(x, t) = u_s(x) + u_t(x, t)$

$$= -4x + 50 + \sum_{n=2,4,\ldots}^{\infty} -\frac{120}{n\pi} \sin \frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

• EXERCISES

1. The ends A and B of a rod of 30 cms. long have their temperatures kept 20°C and 80°C respectively until steady state conditions prevail. The temperature at the end B is then suddenly reduced to 60°C and of the end A is raised to 40°C and maintained so. Find $u(x, t)$. [Nov. '91 Mech.]

$$\left[Ans.\ u(x, t) = \frac{2x}{3} + 40 - \frac{40}{\pi} \sum_{1}^{\infty} \sin \frac{n\pi x}{15}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{225}} \right]$$

2. The ends A and B of a rod, 30 cms. long, have their temperatures kept at 10°C and 100°C respectively, until steady state conditions prevail. The temperature at A is raised to 20°C and that at B is lowered to 80°C suddenly and these temperatures are maintained thereafter. Find the subsequent temperatures distribution in the rod. [Nov. '90 Mech.]

3. Two ends A and B of a rod of length 20 cms. have the temperature at 30°C and 80°C respectively until the steady-state conditions prevail. Then the temperatures at the ends A and B are changed to 40°C and 60°C respectively. Find $u(x, t)$. [Anna Univ. Nov. '01]

$$\left[Ans.\ u(x, t) = x + 40 - \frac{20}{\pi} \sum_{1}^{\infty} (2 \cos n\pi + 1) \sin \frac{n\pi x}{20}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{400}} \right]$$

4. The ends A and B of a rod 50 cm. long and insulated sides are maintained at 0°C and 100°C until steady-state is reached. The temperature at A is suddenly raised to 30°C and that at B reduced to 75°C and then maintained. Find $u(x, t)$. [Ap. '91 Civil]

5. A rod, 30 cm. long, has its ends A and B kept at 20°C and 80°C respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find $u(x, t)$. [Nov. '86 ECE]

6. A bar of length 10 cm. has its ends A and B kept at 50°C and 100°C until steady state conditions prevail. The temperature at A is then suddenly raised to 90°C and at the same instant that at B is lowered to 60°C and the end temperature are thereafter maintained. Find the temperature at a distance x from one end at time t. [Nov. '88 Mech.]

$$\left[Ans.\ u(x, t) = -3x + 90 - \frac{80}{\pi} \sum_{1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{5}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{25}} \right]$$

DEFINITIONS :

Temperature gradient : Consider a bar of uniform cross section of length 'x'. Let the two ends of the rod are maintained at temperatures u_1 and u_2 where $u_1 > u_2$. The quantity $\frac{u_1 - u_2}{x}$ represents the rate of fall of temperature with respect to distance. This rate of change of temperature with respect to distance is called the ***temperature gradient and is denoted by*** $\frac{\partial u}{\partial x}$.

Fourier law of heat conduction : The rate at which heat flows across an area A at a distance x from one end of a bar is given by

$$Q = -KA \left(\frac{\partial u}{\partial x} \right)_x$$

K is thermal conductivity and $\left(\frac{\partial u}{\partial x} \right)_x$ means the temperature gradient at x.

Thermally insulated ends : If there will be no heat flow passes through the ends of the bar then that two ends are said to be thermally insulated.

By Fourier law we have $Q = 0$ at both ends

i.e., $-KA \left(\frac{\partial u}{\partial x} \right)_x = 0$ at both ends.

i.e., $\left(\frac{\partial u}{\partial x} \right)_x = 0$ at both ends.

i.e., $\left(\frac{\partial u}{\partial x} \right)_{at\ x=0} = 0$ and $\left(\frac{\partial u}{\partial x} \right)_{at\ x=l} = 0$

In other words if the end say $x = 0$ thermally insulated then we have $\left(\frac{\partial u}{\partial x} \right)_{at\ x=0} = 0$ and if the end $x = l$ is thermally insulated we have $\left(\frac{\partial u}{\partial x} \right)_{at\ x=l} = 0$.

❑ THERMALLY INSULATED ENDS

(One end is thermally insulated)

■ **EXAMPLE 2** ■

Solve the problem of heat conduction in a rod given that

(i) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (ii) u *is finite as* $t \to \infty$

(iii) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$, *and* $x = l, t > 0$ (iv) $u = lx - x^2$, *for* $t = 0, 0 \le x \le l$

[Nov. '88 ECE, Ap. '89 Mech., Ap. '86 Civil]

● **Solution**

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \text{ (Given)} \quad \ldots (1)$$

On solving this equation by the method of separation of variables applying condition (*ii*) we get the correct solution of the form (as explained earlier)

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t}$$

Now condition (*iii*) can be written as

$$\frac{\partial u(0,t)}{\partial x} = 0$$

and $$\frac{\partial u(l,t)}{\partial x} = 0 \qquad \ldots (2)$$

It is convenient for us to apply condition (*iii*) if we find partial derivative of (2) w.r.t. 'x'. Differentiating (2) partially w.r.t. 'x' we get

$$\frac{\partial u(x,t)}{\partial x} = (-Ap\sin px + Bp\cos px)\, e^{-\alpha^2 p^2 t} \qquad \ldots (3)$$

Now applying condition (2*a*) in (3) we get

$$\frac{\partial u(0,t)}{\partial x} = Bp \cdot e^{-\alpha^2 p^2 t} = 0$$

Here $p \neq 0$ and $e^{-\alpha^2 p^2 t} \neq 0$ since it is valid for all $t > 0$.

Hence $B = 0 \qquad \ldots (4)$

Substituting (4) in (3) we get

$$\frac{\partial u(x,t)}{\partial x} = -Ap\sin px \cdot e^{-\alpha^2 p^2 t} \qquad \ldots (5)$$

Applying condition (2*b*) $\left(i.e., \frac{\partial u(l,t)}{\partial x} = 0\right)$ in (5) we get

$$\frac{\partial u(l,t)}{\partial x} = -Ap\sin pl\, e^{-\alpha^2 p^2 t} = 0$$

Here $A \neq 0$. If $A = 0$, already we have $B = 0$, then we get a trivial solution.

$$\therefore \sin pl = 0 \;\; i.e., pl = n\pi \text{ or } \boxed{p = \frac{n\pi}{l}} \qquad \ldots (6)$$

Substituting $B = 0$ and $p = \frac{n\pi}{l}$ in (2) we get

$$u(x,t) = A\cos\frac{n\pi x}{l} \cdot e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \qquad \ldots (7)$$

By superposition principle we get the most general solution as

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \qquad \ldots (8)$$

Applying condition (*iv*) in (8) we get

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l} = lx - x^2 \qquad [\because e^0 = 1]$$

i.e., $$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = lx - x^2 \qquad \ldots (9)$$

To find A_0 and A_n expand $lx - x^2$ in a half-range cosine series because in L.H.S. of (9) we have cosine series.

For $$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \ldots (10)$$

From (9) and (10) we get

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Equating like co-efficients we get

$$\boxed{A_0 = \frac{a_0}{2} \text{ and } A_n = a_n} \qquad \ldots (10.a)$$

Now $$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\, dx = \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right] = \frac{l^2}{3}$$

i.e., $$a_0 = \frac{l^2}{3}$$

Hence $$\boxed{A_0 = \frac{a_0}{2} = \frac{l^2}{6}} \qquad \ldots (11)$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l\cos n\pi}{\frac{n^2\pi^2}{l^2}} - \frac{l}{\frac{n^2\pi^2}{l^2}}\right]$$

So $$a_n = \frac{-2l^2}{n^2\pi^2}[1 + (-1)^n]$$

$$\therefore a_n = 0 \text{ when '}n\text{' is odd}$$

$$= \frac{-4l^2}{n^2\pi^2} \text{ when } n \text{ is even}$$

$$\therefore A_n = \frac{-4l^2}{n^2\pi^2} \quad \text{(by (10.}a\text{) when } n \text{ is even}$$

Substituting (11) and (12) in (8) we get

$$u(x, t) = \frac{l^2}{6} + \sum_{n=2,4}^{\infty} \frac{-4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l} \cdot e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \text{... (13)}$$

EXAMPLE 3

Solve the following boundary value problem

(i) $\frac{\partial u}{\partial t} = \frac{1}{a^2}\frac{\partial^2 u}{\partial x^2}$, $0 < x < 5$ (ii) $\frac{\partial u(0, t)}{\partial x} = 0$

(iii) $\frac{\partial u(5, t)}{\partial x} = 0$ (iv) $u(x, 0) = x$

Solution

One dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2} \text{ is given.}$$

Let $\frac{1}{\alpha^2} = c^2$ for convenience.

$\therefore$ The above heat equation becomes

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{... (1)}$$

The correct solution of (1) is

$$u(x, t) = (A \cos px + B \sin px)\, e^{-c^2 p^2 t} \quad \text{... (2)}$$

Take $l = 5$ for convenience

$\therefore$ condition (iii) becomes $\frac{\partial u(l, t)}{\partial x} = 0$ ... (3)

Partially differentiating (2) w.r.t. 'x' we get

$$\frac{\partial u(x, t)}{\partial x} = (-Ap \sin px + Bp \cos px)\, e^{-c^2 p^2 t} \quad \text{... (4)}$$

Applying condition (*ii*) in (4) we get

$$\frac{\partial u(0, t)}{\partial x} = B\, p \cdot e^{-c^2 p^2 t} = 0$$

Hence $B = 0$ (For reasons refer to previous example) ... (5)

Substituting (5) in (4), We get

$$\frac{\partial u(x, t)}{\partial x} = -A\, p \sin px\, e^{-c^2 p^2 t} \quad \text{... (6)}$$

Applying condition (3) in (6) we get

$$\frac{\partial u(l, t)}{\partial x} = -A\, p \sin pl\, e^{-c^2 p^2 t} = 0$$

i.e. $\sin pl = 0$ since $A \neq 0$ and $p \neq 0$

$$\therefore pl = n\pi \text{ (or) } \boxed{p = \frac{n\pi}{l}} \quad \text{... (7)}$$

Substituting $B = 0$ and $p = \frac{n\pi}{l}$ in (2) we get

$$u(x, t) = A \cos\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2}{l^2}}$$

The most general solution can be written as

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \quad \text{... (8)}$$

Applying condition (*iv*) in (8) we get

$$u(x, 0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l} = x$$

$$\text{i.e., } A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = x \quad \text{... (9)}$$

To find A_0 and A_n expand the function 'x' in half-range Fourier cosine series in $(0, l)$

$$\text{i.e., } x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{... (10)}$$

From (9) and (10) we get

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

By equating like coefficients we get

$$\boxed{A_0 = \frac{a_0}{2} \text{ and } A_n = a_n} \quad \text{... (11)}$$

Now $a_0 = \frac{2}{l}\int_0^l x\, dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{2}{l}\left(\frac{l^2}{2}\right) = l$

$$\text{i.e., } a_0 = l \quad \therefore A_0 = \frac{l}{2} \quad \text{... (12)}$$

$$a_n = \frac{2}{l}\int_0^l x \cos\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}\cos n\pi - \frac{l^2}{n^2\pi^2}\right] = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore a_n = 0 \text{ when } n \text{ is even}$$

$$= -\frac{4l}{n^2\pi^2} \text{ when '}n\text{' is odd.}$$

$$\therefore A_n = 0 \text{ when } n \text{ is even}$$

$$= \frac{-4l}{n^2\pi^2} \text{ when '}n\text{' is odd.} \quad \ldots (13)$$

Substituting (12) and (13) in (8) we get

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4l}{n^2\pi^2} \cos\frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

Replacing $l = 5$ and $c^2 = \frac{1}{\alpha^2}$ we get

$$u(x, t) = \frac{5}{2} - \frac{20}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{5} \cdot e^{-\frac{n^2\pi^2 t}{\alpha^2 \cdot 25}}$$

EXERCISES

1. Find the solution of the equation
$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$, $0 \le x \le a, t \ge 0$ given that
(i) θ is bounded as $t \to \infty$
(ii) $\frac{\partial \theta}{\partial x} = 0$ for all values of t when $x = 0$ and $x = a$
(iii) $\theta = x(a - x)$ when $t = 0$ and $0 < x < a$. [Ap. '87 Civil]
[Ans. In worked example (2), page 3.59 replace u by θ and l by a]

2. The temperature at one end of a bar, 50cm., long with insulated sides, is kept at 0°C and that at the other end is kept at 100°C until steady state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution. [Ap. '90 ECE, Nov. '91]
$$\left[\textit{Ans. } u(x, t) = l - \frac{8l}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}\ \exp\left(-\frac{\alpha^2 n^2 \pi^2 t}{l^2}\right)\right]$$

3. Solve $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the boundary conditions
$\frac{\partial u(0, t)}{\partial x} = \frac{\partial u(l, t)}{\partial x} = 0$ and $u(x, 0) = kx$ [Ap. '90 Mech.]
$$\left[\textit{Ans. } u(x, t) = \frac{kl}{2} - \frac{4kl}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}\ \exp\left(-\frac{n^2\alpha^2\pi^2 t}{l^2}\right)\right]$$

TWO ENDS ARE THERMALLY INSULATED

When the two ends $x = 0$ and $x = l$ of a rod of length 'l' is thermally insulated then we have the following boundary conditions.

(i) $\left(\frac{\partial u}{\partial x}\right)_{\text{at } x=0} = 0$; (ii) $\left(\frac{\partial u}{\partial x}\right)_{\text{at } x=l} = 0$.

EXAMPLE 1

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ *subject to the following conditions*

(i) *u is finite when $t \to \infty$*
(ii) $\frac{\partial u}{\partial x} = 0$ *when $x = 0$ for all $t > 0$*
(iii) *$u = 0$ when $x = l$ for all $t > 0$*
(iv) *$u = u_0$ when $t = 0$ for all values of x between 0 and l.*
[Ap '88 Civil, Nov. '89 Mech.]

Solution

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Solving this equation by the method of separation of variables and applying condition (i) we get the solution of the form

$$u(x, t) = (A \cos px + B \sin px) \cdot e^{-\alpha^2 p^2 t} \quad \ldots (2)$$

Before applying condition (ii) we find the partial derivative of (2) w.r.t. x.

$$\frac{\partial u(x, t)}{\partial x} = (-Ap \sin px + Bp \cos px)\, e^{-\alpha^2 p^2 t} \quad \ldots (3)$$

Applying condition (ii) in (3) we get

$$\frac{\partial u(0, t)}{\partial x} = Bp \cdot e^{-\alpha^2 p^2 t} = 0$$

Here $p \ne 0$ and $e^{-\alpha^2 p^2 t} \ne 0$ (Refer to previous problem) ... (4)

$$\therefore B = 0$$

Substituting (4) in (2) we get

$$u(x, t) = A \cos px \cdot e^{-\alpha^2 p^2 t} \quad \ldots (5)$$

Applying condition (iii) viz $u(l, t) = 0$ in (5) we get

$$u(l, t) = A \cos pl \cdot e^{-\alpha^2 p^2 t} = 0$$

Here $A \ne 0$ and $e^{-\alpha^2 p^2 t} \ne 0$ for obvious reasons

Hence $\cos pl = 0$

$$pl = \text{an odd multiple of } \frac{\pi}{2} = (2n-1)\frac{\pi}{2}$$

i.e., $\boxed{p = \dfrac{(2n-1)\pi}{2l}}$

Substituting (6) in (5) we get

$$u(x,t) = A\cos\frac{(2n-1)\pi x}{2l}\cdot e^{-\frac{\alpha^2(2n-1)^2\pi^2 t}{4l^2}}$$

∴ The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n\cos\frac{(2n-1)\pi x}{2l}\,\exp\left[-\frac{\alpha^2(2n-1)^2\pi^2 t}{4l^2}\right]$$

Applying condition (*iv*) viz $u(x,0) = u_0$ in (7), we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n\cos\frac{(2n-1)\pi x}{2l} = u_0 \quad [\because e^0 = 1]$$

In the L.H.S. of (8) there is no sine or cosine series as we studied upto previous article. Hence to find the constants we have to adopt some fundam... principle.

$$u_0 = A_1\cos\frac{\pi x}{2l} + A_2\cos\frac{3\pi x}{2l} + \ldots\ldots + A_n\cos\frac{(2n-1)\pi x}{2l} + \ldots\ldots \quad \ldots(9)$$

Multiplying both sides of (9) by $\cos\dfrac{(2n-1)\pi x}{2l}$ and then integrating from 0 to l we get

$$\int_0^l u_0\cos\frac{(2n-1)\pi x}{2l}\,dx = \int_0^l A_1\cos\frac{\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}\,dx$$

$$+\int_0^l A_2\cos\frac{3\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}\,dx + \ldots + \int_0^l A_n\cos^2\frac{(2n-1)\pi x}{2l}\,dx + \ldots \quad \ldots(10)$$

We know that $\int_0^l \cos mx\cos nx\,dx = 0$ if $m \neq n$.

Applying this in equation (10) we get all terms in R.H.S. of (10) become zero except the term

$$\int_0^l A_n\cos^2\frac{(2n-1)\pi x}{2l}\,dx$$

Hence equation (10) reduces to

$$u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}\,dx = A_n\int_0^l \cos^2\frac{(2n-1)\pi x}{2l}\,dx \quad \ldots(11)$$

$$\text{L.H.S.} = u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}\,dx = u_0\left[\frac{\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right]_0^l$$

$$= \frac{2l u_0}{(2n-1)\pi}\cdot\sin(2n-1)\frac{\pi}{2} = \frac{2lu_0}{(2n-1)\pi}\sin\left(n\pi - \frac{\pi}{2}\right)$$

$$= \frac{2lu_0}{(2n-1)\pi}\left[\sin n\pi\cos\frac{\pi}{2} - \cos n\pi\sin\frac{\pi}{2}\right]$$

$$= \frac{2lu_0}{(2n-1)\pi}(-1)^{n+1} \quad \ldots(12)$$

$$\text{R.H.S.} = A_n\int_0^l \cos^2\frac{(2n-1)\pi x}{2l}\,dx$$

$$= A_n\int_0^l \frac{1}{2}\left[1 + \cos\frac{(2n-1)\pi x}{l}\right]dx \qquad \left[\because \cos^2 x = \frac{1+\cos 2x}{2}\right]$$

$$= \frac{A_n}{2}\left[x + \frac{\frac{\sin(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}}\right]_0^l = \frac{A_n}{2}[l] \quad \ldots(13)$$

From (12) and (13) we get

$$\frac{2lu_0(-1)^{n+1}}{(2n-1)\pi} = \frac{A_n l}{2} \quad \text{i.e.,}\quad A_n = \frac{4u_0(-1)^{n+1}}{(2n-1)\pi} \quad \ldots(14)$$

Substituting (14) in (7) we get

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{(2n-1)\pi x}{2l}\,\exp\left[-\frac{\alpha^2(2n-1)^2\pi^2 t}{4l^2}\right]$$

EXAMPLE 2

A uniform rod of length 'l' whose surface is thermally insulated is initially at a constant temperature k. At time t = 0 the left end is suddenly cooled to temperature zero while the right end is thermally insulated and these conditions are maintained at the ends. Find the temperature distribution in the rod at any subsequent time.

Solution

When constant temperature k exist the heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \text{ becomes } \frac{d^2u}{dx^2} = 0 \quad \ldots(1)$$

[Since temperature varies only with distance 'x' and not with time 't'].

The solution of (1) is ...(2)

$$u(x) = c_1 x + c_2$$

When constant temperature exists both ends $x = 0$ and $x = l$ is maintained temperature k.

(a) $u(0) = k$, (b) $u(l) = k$

Applying condition (a) in (2), we get,

$$u(0) = 0 + c_2 = k$$

Substituting $c_2 = k$ in (2), we get,

$$u(x) = c_1 x + k \qquad \dots (3)$$

Applying (b) in (3), we get

$$u(l) = c_1 l + k = k$$

i.e., $\boxed{c_1 = 0}$

$$u(x) = k \qquad \dots (4)$$

Now left end is suddenly cooled to 0°C and right end is thermally insulated.

∴ We have the following boundary conditions.

(i) $u(0, t) = 0$

(ii) $\dfrac{\partial u(l, t)}{\partial x} = 0$

(iii) $u(x, 0) = k$

Now the heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \dots (5)$$

The correct solution of (5) is,

$$u(x, t) = (A \cos px + B \sin px)\, e^{-\alpha^2 p^2 t} \qquad \dots (6)$$

Applying condition (1) in (6), we get,

$$u(0, t) = A \cdot e^{-\alpha^2 p^2 t} = 0$$

$$A = 0$$

Putting $A = 0$ in (6), we get,

$$u(x, t) = B \sin px\; e^{-\alpha^2 p^2 t} \qquad \dots (6a)$$

$$\frac{\partial u(x, t)}{\partial x} = p B \cos px\, e^{-\alpha^2 p^2 t} \qquad \dots (7)$$

Applying condition (ii) in (7), we get,

$$\frac{\partial u(l, t)}{\partial x} = p B \cos pl\; e^{-\alpha^2 \pi^2 t} = 0$$

$$\cos pl = 0 \;;\; pl = (2n-1)\frac{\pi}{2}$$

i.e., $$p = \frac{(2n-1)\pi}{2l} \qquad \dots (8)$$

...ing (8) in (6a), we get,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi x}{2l}\, e^{\frac{-\alpha^2 (2n-1)^2 \pi^2 t}{4l^2}} \qquad \dots (9)$$

...ing condition (iii) in (9), we get,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi x}{2l} = k$$

Now $$k = B_1 \sin \frac{\pi x}{2l} + B_2 \sin \frac{3\pi x}{2l} + \dots\dots + B_n \sin \frac{(2n-1)\pi x}{2l} + \dots \qquad \dots (10)$$

...ltiplying on both sides by $\sin \dfrac{(2n-1)\pi x}{2l}$ and then integrating from 0 to l, we get

$$\int_0^l k \sin \frac{(2n-1)\pi x}{2l}\, dx = \int_0^l B_n \sin^2 \frac{(2n-1)\pi x}{2l}\, dx$$

[The remaining integrals are zero.]

$$k\left[\frac{-\cos \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right]_0^l = B_n \int_0^l \frac{1}{2}\left[1 - \cos \frac{(2n-1)\pi x}{l}\right] dx$$

$$\left[\because \sin^2 x = \frac{1 - \cos 2x}{2}\right]$$

$$\frac{2lk}{(2n-1)\pi}[0 + 1] = \frac{B_n}{2}\left[x - \left(\frac{\sin \frac{(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}}\right)\right]_0^l$$

$$\frac{2lk}{(2n-1)\pi} = \frac{B_n l}{2} \quad \text{i.e., } B_n = \frac{4k}{(2n-1)\pi} \qquad \dots (11)$$

Substituting (11) in (9), we get,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2l}\, e^{\frac{-\alpha^2 (2n-1)^2 \pi^2 t}{4l^2}}$$

• EXERCISES

1. A bar with insulated sides is initially at temperature 0°C throughout. The end $x = l$ so that end $x = 0$ is kept at 0°C and heat is suddenly applied at the end $x = l$ so that $\dfrac{\partial u}{\partial x} = A$ for $x = l$ where A is a constant. Find the temperature function $u(x, t)$. [Ap. '86 Mech., Nov. '87 Mech.]

$$\left[\textit{Ans. } u(x, t) = Ax + \frac{8Al}{\pi^2} \sum_{n=1,3.}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi x}{2l}\, exp\left(\frac{n^2\pi^2\alpha^2 t}{4l^2}\right)\right]$$

2. Find the solution of the equation
$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \text{ such that}$$
   (i) v is finite when $t \to \infty$
   (ii) $\left.\begin{aligned}\frac{\partial v}{\partial x} &= 0 \text{ when } x = 0\\ v &= 0 \text{ when } x = l\end{aligned}\right\}$ for all t
   (iii) $v = v_0$ when $t = 0$ for all values of x between 0 and l.

   Note : Similar to the worked example (1). [Nov. '86 Civil]

3. Solve the one dimensional heat flow equation
$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \text{ subject to the following conditions}$$
   (a) $u(0, t) = 0$, for $t > 0$
   (b) $\frac{\partial u(l, t)}{\partial x} = 0$, for $t > 0$
   (c) $u(x, 0) = 50$, for $0 < x < l$. [Apr. '91 Common]

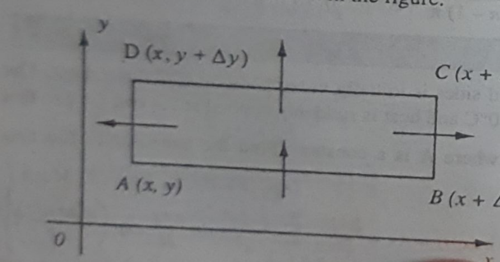
   Note : In worked example 1 replace u_0 by 50 we get answer.

❑ TWO DIMENSIONAL HEAT FLOW EQUATION

If the temperature distribution at any point is independent of the z-coordinate then the heat flow is called Two Dimensional heat flow. In otherwords in two dimensional heat flow, the heat flow is along curves instead of straight lines as mentioned in one dimensional heat flow.

Let us consider a rectangular sheet of heat conducting material of thickness λ, specific heat C, density ρ and thermal conductivity k. Let us assume that the faces of the sheet are insulated and that heat can enter and leave only through the edges of the sheet. Let $u(x, y)$ be any point on the face of the sheet, where $0 \le x \le l, 0 \le y \le m$.

Consider a rectangular element of the sheet with sides Δx and Δy. The co-ordinates of its corners are shown in the figure.



Now the quantity of heat that enters the plate per second from side AB is given by

$$Q_1 = -K\lambda\,\Delta x\left(\frac{\partial u}{\partial y}\right)_y \quad \ldots (1)$$

(Here the negative sign is due to the fact that the temperature decreases with increase in distance from the hot end)

The quantity of heat that enters the plate per second from side AD is given by

$$Q_2 = -K\lambda\,\Delta y\left(\frac{\partial u}{\partial x}\right)_x \quad \ldots (2)$$

The heat which flows out through the sides CD and BC per second is given by

$$-K\lambda\,\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} \quad \ldots (3)$$

and

$$-K\lambda\,\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} \quad \ldots (4)$$

respectively. Therefore the total gain of heat by the rectangular sheet ABCD per second

= Total heat flows in the sheet – Total heat flows out the sheet.

$$= \left\{-K\lambda\,\Delta x\left(\frac{\partial u}{\partial y}\right)_y - K\lambda\,\Delta y\left(\frac{\partial u}{\partial x}\right)_x\right\} - \left\{-K\lambda\,\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - K\lambda\,\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}\right\}$$

$$= K\lambda\left\{\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} + \Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \Delta x\left(\frac{\partial u}{\partial y}\right)_y - \Delta y\left(\frac{\partial u}{\partial x}\right)_x\right\}$$

$$= K\lambda\,\Delta x\,\Delta y\left\{\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y} + \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right\} \quad \ldots (5)$$

The rate of gain of heat energy in the sheet ABCD at any time t is approximately given by

$$C\rho\lambda\,\Delta x\,\Delta y\,\frac{\partial u}{\partial t} \quad \ldots (6)$$

From (5) and (6) we get,

$$K\lambda\,\Delta x\,\Delta y\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}\right\} = C\rho\lambda\,\Delta x\,\Delta y\,\frac{\partial u}{\partial t}$$

Dividing both sides by $\lambda\,\Delta x\,\Delta y$, we get

$$K\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}\right\} = C\rho\,\frac{\partial u}{\partial t}$$

Now taking limit as $\Delta x \to 0, \Delta y \to 0$, we get

$$K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = C\rho\,\frac{\partial u}{\partial t}$$

i.e.,

$$\frac{\partial u}{\partial t} = \frac{K}{\rho C}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad \ldots (7)$$

where $\alpha^2 = \frac{K}{\rho C}$ is the diffusivity

Equation (7) is called the *two-dimensional heat equation*. This equation gives the temperature distribution of the plane in the transient state.

COR : 1. When steady state exists, the temperature function u (x, t) is independent of t. Hence $\frac{\partial u}{\partial t} = 0$.

Hence the above equation reduces to $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

This equation is called Laplace's equation in two dimensions.

This equation is very important as we are restricted to study the two dimensional heat flow equation in steady state conditions.

COR : 2 When the heat flow is three dimensional i.e., when the stream lines are curves in space, we shall similarly derive the equation.

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

In the steady state conditions, the above equation reduces to

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0\right]$$

This equation is called Laplace's equation in three dimensions.

SOLUTION OF TWO DIMENSIONAL HEAT FLOW EQUATION

The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$$

Here 'u' is a function of x and y.

Let us solve (1) by using method of separation of variables.

Therefore let us assume that the solution of (1) is of the form

$$u = X(x) \cdot Y(y) \quad \ldots (2)$$

Differentiating equation (2) twice partially w.r.t. x and y we get,

$$\frac{\partial u}{\partial x} = X'Y \; ; \quad \frac{\partial^2 u}{\partial x^2} = X''Y \quad \ldots (3)$$

and $$\frac{\partial u}{\partial y} = XY' \; ; \quad \frac{\partial^2 u}{\partial y^2} = XY'' \quad \ldots (4)$$

Substituting (3) and (4) in (1) we get,

$$X''Y + XY'' = 0$$

$$X''Y = -XY''$$

i.e., $$\frac{X''}{X} = \frac{-Y''}{Y} = k \text{ (say)} \qquad \text{[By separating the variables]}$$

i.e., $$\frac{X''}{X} = k, \; \frac{-Y''}{Y} = k \quad \ldots (5)$$

i.e., $$X'' - kX = 0 \quad \ldots (6)$$

and $$Y'' + kY = 0$$

Equation (5) and (6) are ordinary differential equations the solution of which depends only on the value of k.

Case (i) : Let k be positive.

i.e., $k = p^2$ (Here k is always +ve whether p is +ve or –ve).

$\therefore$ Equation (5) and (6) becomes

$$X'' - p^2X = 0 \quad \text{and} \quad Y'' + p^2Y = 0$$

i.e., $$\frac{d^2X}{dx^2} - p^2X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + p^2Y = 0$$

$m^2 - p^2 = 0$ and $m^2 + p^2 = 0$ (Auxiliary equations)

$\therefore m = \pm p$ and $m = \pm ip$

$\therefore X = (c_1 e^{px} + c_2 e^{-px})$

and $Y = (c_3 \cos py + c_4 \sin py)$

$\therefore u(x, y) = (c_1 e^{px} + c_2 e^{-px}) \cdot (c_3 \cos py + c_4 \sin py)$ [Using (2)]

Case (ii) : Let k be negative.

i.e., $k = -p^2$ (k is always –ve whether p is +ve or –ve).

$\therefore$ Equations (5) and (6) becomes

$$X'' + p^2X = 0 \quad \text{and} \quad Y'' - p^2Y = 0$$

i.e., $$\frac{d^2X}{dx^2} + p^2X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} - p^2Y = 0$$

$$m^2 + p^2 = 0 \quad \text{and} \quad m^2 - p^2 = 0$$

(Auxiliary equations)

i.e., $m = \pm ip$ and $m = \pm p$

$\therefore X = (c_5 \cos px + c_6 \sin px)$

and $Y = (c_7 e^{py} + c_8 e^{-py})$

$\therefore u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$ [Using (2)]

Case (iii) : Let k be zero.

i.e., $k = 0$.

$\therefore$ Equations (5) and (6) becomes

$$X'' = 0 \quad \text{and} \quad Y'' = 0$$

i.e., $$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} = 0$$

Integrating we get,

$$X = c_9 x + c_{10} \quad \text{and} \quad Y = c_{11} y + c_{12}$$

$$\therefore u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$$

[Using (2)]

Hence the different solutions for equation (1) are

(i) $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

(ii) $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

(iii) $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Out of these three solutions we have to choose the correct solution which satisfies the given boundary conditions. Choosing of correct solution for a given problem is explained in the following examples.

■ EXAMPLE 1 ■

A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while the other three edges are kept at $0°$ C. Find the steady state temperature in the plate.

[Nov. 88, Mech., Nov. '91 ECE]

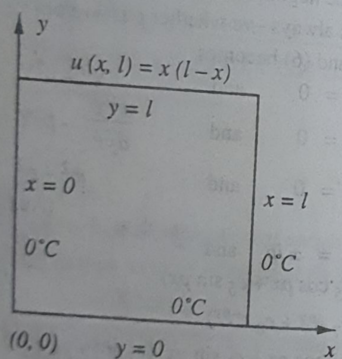
● Solution

Let us take the sides of the plate be $l = 20$. (For convenience)

Let $u(x, y)$ be the temperature at any point (x, y).

Then $u(x, y)$ satisfies the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$$



From the given problem we have the following boundary conditions.

(i) $u(0, y) = 0$ for $0 < x < l$

(ii) $u(l, y) = 0$ for $0 < y < l$

(iii) $u(x, 0) = 0$ for $0 \le x < l$

(iv) $u(x, l) = x(l - x)$ for $0 < x < l$

If we solve equation (1) by the method of separation of variables (as explained in previous article) we get the following solutions.

(a) $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

(b) $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

(c) $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Out of these three solutions we have to choose the correct solution which satisfies the above boundary conditions. Consider the solution (a),

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \ldots (2)$$

Applying condition (i) in (2), we get,

$$u(0, y) = (c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0 \quad \ldots (3)$$

i.e., $c_1 + c_2 = 0$

Applying condition (ii) in (2), we get,

$$u(l, y) = (c_1 e^{pl} + c_2 e^{-pl})(c_3 \cos py + c_4 \sin py) = 0 \quad \ldots (4)$$

i.e., $c_1 e^{pl} + c_2 e^{-pl} = 0$

Solving (3) and (4) we get $c_1 = 0$ and $c_2 = 0$.

Substituting these values of c_1 and c_2 in (2), we get,

$u(x, y) = 0$ which is a trivial solution.

$\therefore$ Solution (a) is not satisfying our boundary conditions.

Hence this solution (a) is not the correct solution.

Now consider the solution (c).

$$u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \ldots (5)$$

Applying condition (i) in (5) we get,

$$u(0, y) = c_{10}(c_{11} y + c_{12}) = 0$$

Here $c_{11} y + c_{12} \neq 0$ since it is true for all y.

$$\therefore c_{10} = 0 \quad \ldots (6)$$

Substituting (6) in (5), we get,

$$u(x, y) = c_9 x (c_{11} y + c_{12}) \quad \ldots (7)$$

Applying condition (ii) in (7) we get,

$$u(l, y) = c_9 l (c_{11} y + c_{12}) = 0$$

Here $l \neq 0$ $(l = 20)$ and $c_{11} y + c_{12} \neq 0$ since it is true for all y.

$$\text{Hence} \quad c_9 = 0 \quad \ldots (8)$$

Substituting (8) in (7) we get,

$u(x, y) = 0$ which is again a trivial solution.

Hence this solution is also not the correct solution.

Therefore the correct solution should be

$$u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

Applying condition (*i*) in (9) we get,

$$u(0, y) = c_5 (c_7 e^{py} + c_8 e^{-py}) = 0$$

i.e., $$c_5 = 0 \quad \ldots (10)$$

Substituting (10) in (9) we get,

$$u(x, y) = c_6 \sin px\,(c_7 e^{py} + c_8 e^{-py}) \quad \ldots (11)$$

Applying condition (*ii*) in (11) we get,

$$u(l, y) = c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py}) = 0$$

Here $c_7 e^{py} + c_8 e^{-py} \neq 0$ since it is true for all *y*.

Hence $c_6 = 0$ or $\sin pl = 0$.

If we take $c_6 = 0$ we get again a trivial solution.

$\therefore$ We take $\sin pl = 0$.

i.e., $$pl = n\pi \text{ (or) } p = \frac{n\pi}{l}$$

Substituting $p = \frac{n\pi}{l}$ in (11) we get, ... (12)

$$u(x, y) = c_6 \sin \frac{n\pi x}{l}\left(c_7 e^{\frac{n\pi y}{l}} + c_8 e^{\frac{-n\pi y}{l}}\right) \quad \ldots (13)$$

Applying condition (*iii*) in (13) we get,

$$u(x, 0) = c_6 \sin \frac{n\pi x}{l}(c_7 + c_8) = 0$$

i.e., $$c_7 + c_8 = 0 \text{ (or) } c_8 = -c_7 \quad \ldots (14)$$

Substituting (14) in (13), we get,

$$u(x, y) = c_7 c_6 \sin \frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}\right)$$

$$= c_6 c_7 \sin \frac{n\pi x}{l} \cdot 2 \sin h \frac{n\pi y}{l} \qquad [\because e^x - e^{-x} = 2 \sin hx]$$

$$\therefore u(x, y) = c_n \sin \frac{n\pi x}{l} \sin h \frac{n\pi y}{l} \qquad [2c_6 c_7 = c_n]$$

The most general solution can be taken as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin h \frac{n\pi y}{l} \quad \ldots (15)$$

Applying condition (*iv*) in (15) we get

$$u(x, l) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin h\, n\pi = x(l - x) \quad \ldots (16)$$

To find c_n expand $x(l-x)$ in a half-range Fourier sine series in the interval

$$x(l - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \ldots (17)$$

From (16) and (17) we get,

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin h\, n\pi = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \ldots (18)$$

Equating like coefficients of (18) we get,

$$c_n \sin h\, n\pi = b_n \text{ (or) } c_n = \frac{b_n}{\sin h\, n\pi} \quad \ldots (19)$$

Now,

$$b_n = \frac{2}{l}\int_0^l x(l - x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l}\left[\int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3} \cos n\pi + \frac{2l^3}{n^3\pi^3}\right] = \frac{4l^2}{n^3\pi^3}[1 - (-1)^n]$$

$$\therefore b_n = 0 \text{ when '}n\text{' is even}$$

$$= \frac{8l^2}{n^3\pi^3} \text{ when '}n\text{' is odd}$$

$\therefore$ From (19) we get, $c_n = \dfrac{8l^2}{n^3\pi^3 \sin h\, n\pi}$ when *n* is odd.

Substituting this value of c_n in (15), we get,

$$u(x, y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8l^2}{n^3\pi^3 \sin h n\pi} \sin \frac{n\pi x}{l} \sin h \frac{n\pi y}{l}$$

Replacing *l* by 20, we get,

$$u(x, y) = \frac{3200}{\pi^3} \sum_{n=1,3\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{20} \sin h \frac{n\pi y}{20} \cdot \frac{1}{\sin h\, n\pi}$$

■ **EXAMPLE 2** ■

Find the steady state temperature at any point of a square plate whose two adjacent edges are kept at 0° C and the other two edges are kept at the constant temperature 100° C. [Nov ...]

● Solution

We know that the temperature $u(x, y)$ at any point satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let the side of the square plate be l.

From the given problem we have the following boundary conditions.

(i) $u(x, 0) = 0$ for $0 < x < l$

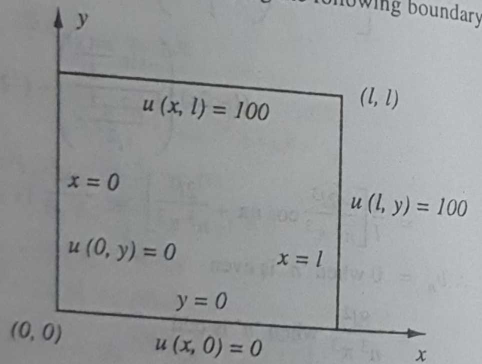
(ii) $u(l, y) = 100$ for $0 < y < l$

(iii) $u(x, l) = 100$ for $0 < x < l$

(iv) $u(0, y) = 0$ for $0 < y < l$

Let $u(x, y) = u_1(x, y) + u_2(x, y)$ ...(2)

where $u_1(x, y)$ and $u_2(x, y)$ are satisfying the following boundary conditions.



(a_1)	$u_1(0, y) = 0$	(a_2)	$u_2(x, 0) = 0$
(b_1)	$u_1(l, y) = 0$	(b_2)	$u_2(x, l) = 0$
(c_1)	$u_1(x, 0) = 0$	(c_2)	$u_2(0, y) = 0$
(d_1)	$u_1(x, l) = 100$	(d_2)	$u_2(l, y) = 100$

To find $u_1(x, y)$

When we solve the equation (1) and applying first three conditions via (a_1), (b_1) and (c_1) we get the most general solution as

$$u_1(x, y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin h\frac{n\pi y}{l} \quad \text{...(3)}$$

[Refer to example (1)]

Applying boundary condition (d_1) in (3), we get,

$$u_1(x, l) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin h\, n\pi = 100 \quad \text{...(4)}$$

To find c_n expand 100 in a half range Fourier sine series

$$\text{i.e.,}\quad 100 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{...(5)}$$

From (4) and (5) we get,

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin h\, n\pi = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Equating like coefficients we get,

$$c_n \sin h\, n\pi = b_n \quad \text{...(6)}$$

$$\therefore c_n = \frac{b_n}{\sin h\, n\pi}$$

Now,

$$b_n = \frac{2}{l}\int_0^l 100 \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l = \frac{200}{n\pi}[1-(-1)^n]$$

$$\therefore b_n = 0 \text{ when } n \text{ is even}$$

$$= \frac{400}{n\pi} \text{ when } n \text{ is odd} \quad \text{...(7)}$$

$$\therefore c_n = \frac{400}{n\pi \sin h\, n\pi} \text{ when 'n' is odd} \quad \text{...(8)}$$

Substituting (8) in (3), we get,

$$u_1(x, y) = \sum_{n=1,3...}^{\infty} \frac{400}{n\pi \sin h\, n\pi} \cdot \sin\frac{n\pi x}{l} \sin h\frac{n\pi y}{l}$$

To find $u_2(x, y)$

When we solve the equation (1) for u_2 we get the following solutions.

(a_3) $u_2(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

(b_3) $u_2(x, y) = (c_5 x + c_6)(c_7 y + c_8)$

(c_3) $u_2(x, y) = (c_9 \cos py + c_{10} \sin py)(c_{11} e^{px} + c_{12} e^{-px})$

Now applying condition (a_2) in (a_3) we get,

$$u_2(x, 0) = (c_1 \cos px + c_2 \sin px)(c_3 + c_4) = 0 \quad \text{...(9)}$$

$$\text{i.e.,}\quad c_3 + c_4 = 0$$

Applying condition (b_2) in (a_3) we get,

$$u_2(x, l) = (c_1 \cos px + c_2 \sin px)(c_3 e^{pl} + c_4 e^{-pl}) = 0 \quad \text{...(10)}$$

$$\text{i.e.,}\quad c_3 e^{pl} + c_4 e^{-pl} = 0$$

From (9) and (10) we get,

$$c_3 = c_4 = 0$$

Substituting (11) in (a_3) we get a trivial solution.

$\therefore$ Solution (a_3) is not the correct solution.

Similarly we can easily prove that the solution (b_3) is also not the co solution for this problem.

$\therefore$ The correct solution which satisfies our boundary conditions should be

$$u_2(x, y) = (c_9 \cos py + c_{10} \sin py)(c_{11} e^{px} + c_{12} e^{-px}) \quad \ldots (12)$$

Applying condition (a_2) in (12) we get,

$$u_2(x, 0) = c_9 (c_{11} e^{px} + c_{12} e^{-px}) = 0$$

i.e.,

$$c_9 = 0 \quad \ldots (13)$$

Substituting (13) in (12), we get,

$$u_2(x, y) = c_{10} \sin py \, (c_{11} e^{px} + c_{12} e^{-px}) \quad \ldots (14)$$

Applying condition (b_2) in (14), we get,

$$u_2(x, l) = c_{10} \sin pl \, (c_{11} e^{px} + c_{12} e^{-px}) = 0$$

i.e., $\sin pl = 0$ since $c_{10} \neq 0$ because if we take $c_{10} = 0$ again we get a trivi solution.

$$\therefore pl = n\pi \text{ (or) } p = \frac{n\pi}{l} \quad \ldots (15)$$

Substituting (15) in (14), we get,

$$u_2(x, y) = c_{10} \sin \frac{n\pi y}{l} \left(c_{11} e^{\frac{n\pi x}{l}} + c_{12} e^{\frac{-n\pi x}{l}} \right) \quad \ldots (16)$$

Applying condition (c_2) in (16), we get,

$$u_2(0, y) = c_{10} \sin \frac{n\pi y}{l} (c_{11} + c_{12}) = 0$$

i.e.,

$$c_{12} = -c_{11} \quad \ldots (17)$$

Substituting (17) in (16), we get,

$$u_2(x, y) = c_{10} \sin \frac{n\pi y}{l} \left(c_{11} e^{\frac{n\pi x}{l}} - c_{11} e^{\frac{-n\pi x}{l}} \right)$$

$$= c_{10} c_{11} \sin \frac{n\pi y}{l} \left(2 \sin h \frac{n\pi x}{l} \right)$$

$$= 2 c_{10} c_{11} \sin \frac{n\pi y}{l} \sin h \frac{n\pi x}{l} = c_n \sin \frac{n\pi y}{l} \sin h \frac{n\pi x}{l}$$

The most general solution can be written as

$$u_2(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi y}{l} \sin h \frac{n\pi x}{l} \quad \ldots (18)$$

Applying condition (d_2) in (18), we get,

$$u_2(l, y) = \sum_{1}^{\infty} c_n \sin \frac{n\pi y}{l} \sin h \, n\pi = 100 \quad \ldots (19)$$

To find c_n expand 100 in a half range sine series in the interval $(0, l)$.

i.e.,

$$100 = \sum_{1}^{\infty} b_n \sin \frac{n\pi y}{l} \quad \ldots (20)$$

From (19) and (20), we get,

$$c_n \sin h \, n\pi = b_n$$

$$\boxed{\therefore c_n = \frac{b_n}{\sin h n\pi}} \quad \ldots (21)$$

$$\text{But } b_n = \begin{cases} 0 & \text{when 'n' is even} \\ \dfrac{400}{n\pi} & \text{when 'n' is odd} \end{cases} \quad \ldots (22)$$

Substituting (22) in (21), we get,

$$c_n = \frac{400}{n\pi \sin h \, n\pi} \text{ when 'n' is odd} \quad \ldots (23)$$

Substituting (23) in (18), we get,

$$u_2(x, y) = \sum_{n=1,3,5}^{\infty} \frac{400}{n\pi \sin h n\pi} \sin \frac{n\pi y}{l} \sin h \frac{n\pi x}{l} \quad \ldots (24)$$

$$\therefore u(x, y) = u_1(x, y) + u_2(x, y)$$

$$= \frac{400}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n \sin h \, n\pi} \left\{ \sin \frac{n\pi x}{l} \sin h \frac{n\pi y}{l} + \sin \frac{n\pi y}{l} \sin h \frac{n\pi x}{l} \right\}$$

EXAMPLE 3

Solve the interior Dirichlet problem for a rectangle specified below :

$$u_{xx} + u_{yy} = 0$$

$$u(0, y) = 0, \; u(a, y) = 0, \; 0 < y < b$$

$$u(x, 0) = f(x), \; u(x, b) = 0, \; 0 < x < a$$

where $f(x) = \sin^3 \frac{\pi x}{a}$.

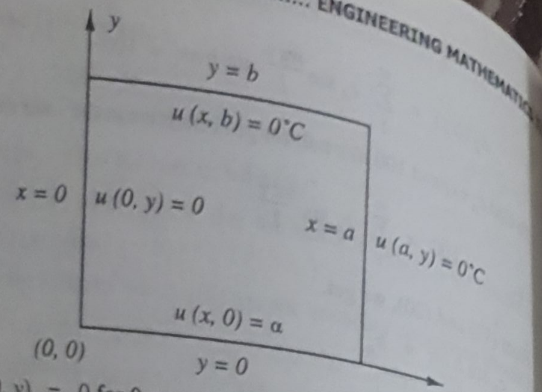
Solution

The given two dimensional heat flow equation is

$$u_{xx} + u_{yy} = 0 \quad \ldots (1)$$

i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is given

The boundary conditions are



(i) $u(0, y) = 0$ for $0 < y < b$

(ii) $u(a, y) = 0$ for $0 < y < b$

(iii) $u(x, b) = 0$ for $0 < x < a$

(iv) $u(x, 0) = \sin^3 \frac{\pi x}{a}, 0 < x < a$

Solving equation (1) we get three types of solutions as given below :

(a) $u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

(b) $u(x, y) = (c_5 \cos py + c_6 \sin py)(c_7 e^{px} + c_8 e^{-px})$

(c) $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Out of these three solutions the correct solution which satisfies our boundary conditions (as explained in Ex. [1]) is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \ldots (2)$$

Applying condition (i) in (2), we get,

$$u(0, y) = c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$$

i.e., $c_1 = 0$ ... (3)

Substituting (3) in (2), we get,

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \ldots (4)$$

Applying condition (ii) in (4), we get,

$$u(a, y) = c_2 \sin pa (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_2 \neq 0$ since if $c_2 = 0$ we get trivial solution.

Hence $\sin pa = 0$, i.e., $pa = n\pi$

(or) $p = \frac{n\pi}{a}$ ... (5)

Substituting (5) in (4), we get

$$u(x, y) = c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{\frac{-n\pi y}{a}} \right) \quad \ldots (6)$$

Applying condition (iii) in (6) we get,

$$u(x, b) = c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{\frac{n\pi b}{a}} + c_4 e^{\frac{-n\pi b}{a}} \right) = 0$$

Here $c_2 \neq 0$ and $\sin \frac{n\pi x}{a} \neq 0$.

Hence $c_3 e^{\frac{n\pi b}{a}} + c_4 e^{\frac{-n\pi b}{a}} = 0$

i.e. $c_4 e^{\frac{-n\pi b}{a}} = -c_3 e^{\frac{n\pi b}{a}}$

i.e. $c_4 = -c_3 e^{\frac{2n\pi b}{a}}$ ... (7)

Substituting (7) in (6) we get

$$u(x, y) = c_3 c_2 \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} \cdot e^{\frac{-n\pi y}{a}} \right) \quad \ldots (8)$$

$$= c_n \sin \frac{n\pi x}{a} \left(e^{\frac{2n\pi y}{a}} - e^{\frac{2n\pi b}{a}} \right) \cdot e^{\frac{-n\pi y}{a}}$$

The most general solution can be written as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \cdot e^{\frac{-n\pi y}{a}} \left(e^{\frac{2n\pi y}{a}} - e^{\frac{2n\pi b}{a}} \right) \quad \ldots (9)$$

Applying condition (iv) in (9), we get,

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \left(1 - e^{\frac{2n\pi b}{a}} \right) = \sin^3 \frac{\pi x}{a} \quad \ldots (10)$$

i.e., $\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \left(1 - e^{\frac{2n\pi b}{a}} \right) = \frac{1}{4} \left(3 \sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a} \right)$

i.e., $c_1 \sin \frac{\pi x}{a} \left(1 - e^{\frac{2\pi b}{a}} \right) + c_2 \sin \frac{2\pi x}{a} \left(1 - e^{\frac{4\pi b}{a}} \right) + c_3 \sin \frac{3\pi x}{a} \left(1 - e^{\frac{6\pi b}{a}} \right) + \ldots$

$$= \frac{1}{4} \left(3 \sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a} \right)$$

Equating like coefficients we get,

$$c_1 \left(1 - e^{\frac{2\pi b}{a}} \right) = \frac{3}{4}$$

i.e., $c_1 = \frac{3}{4\left(1 - e^{\frac{2\pi b}{a}} \right)}$

$$c_2 = 0$$

$$c_3\left(1-e^{\frac{6\pi b}{a}}\right) = -\frac{1}{4}$$

i.e.,

$$c_3 = -\frac{1}{4\left(1-e^{\frac{6\pi b}{a}}\right)}$$

$$c_4 = 0,\ c_5 = 0, \dots$$

Substituting these values of c's in (9), we get

$$u(x, y) = \frac{3}{4\left(1-e^{\frac{2b\pi}{a}}\right)} \sin\frac{\pi x}{a}\, e^{\frac{-\pi y}{a}}\left(e^{\frac{2\pi y}{a}} - e^{\frac{2\pi b}{a}}\right)$$

$$- \frac{1}{4\left(1-e^{\frac{6\pi b}{a}}\right)} \sin\frac{3\pi x}{a}\, e^{\frac{-3\pi y}{a}}\left(e^{\frac{6\pi y}{a}} - e^{\frac{6\pi b}{a}}\right)$$

■ EXAMPLE 4 ■

Find the steady temperature distribution at points in a rectangular plate with insulated faces the edges of the plate being the lines $x = 0, x = a, y = 0$ and $y = b$. When three of the edges are kept at temperature zero and the fourth at a fixed temperature α° C. [Apr. '89]

● Solution

Let $u(x, y)$ be the temperature distribution at any point in the plate. Then $u(x, y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (1)$$

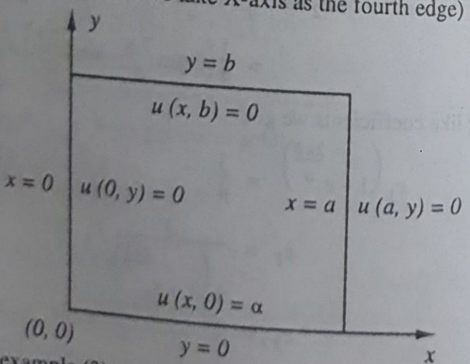
The boundary conditions are

(i) $u(0, y) = 0$

(ii) $u(a, y) = 0$

(iii) $u(x, b) = 0$

(iv) $u(x, 0) = \alpha$. [Here we take X-axis as the fourth edge)



As in worked example (3) we can write the most general solution as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\, e^{\frac{-n\pi y}{a}}\left(e^{\frac{2n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\right) \quad \dots (2)$$

Applying condition (iv) in (2) we get,

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\left(1-e^{\frac{2n\pi b}{a}}\right) = \alpha \quad \dots (3)$$

To find c_n expand α in a half-range Fourier sine series in the interval $(0, a)$.

i.e.,

$$\alpha = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \quad \dots (4)$$

From (3) and (4) we get

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\left(1-e^{\frac{2n\pi b}{a}}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$$

Equating like coefficients we get

$$c_n\left(1-e^{\frac{2n\pi b}{a}}\right) = b_n \quad \dots (5)$$

Now $b_n = \dfrac{2}{a}\displaystyle\int_0^a \alpha \sin\frac{n\pi x}{a}\, dx = \frac{2\alpha}{a}\left[\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right]_0^a = \frac{2\alpha}{n\pi}\left[1-(-1)^n\right]$

$\therefore b_n = 0$ when n is even

$= \dfrac{4\alpha}{n\pi}$ when n is odd.

Hence from (5) we get,

$c_n = 0$, when n is even

$$= \frac{4\alpha}{n\pi\left(1-e^{\frac{2n\pi b}{a}}\right)} \text{ when '}n\text{' is add} \quad \dots (6)$$

Substituting (6) in (2) we get

$$u(x, y) = \frac{4\alpha}{\pi}\sum_{n=1,3,5}^{\infty} \frac{1}{n\left(1-e^{\frac{2n\pi b}{a}}\right)} \cdot \sin\frac{n\pi x}{a} \cdot e^{\frac{-n\pi y}{a}}\left(e^{\frac{2n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\right)$$

EXAMPLE 5

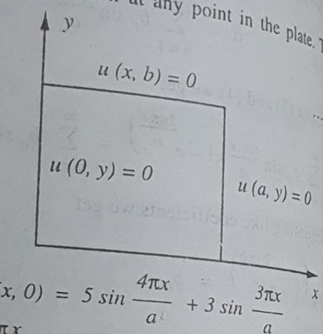
A rectangular plate is bounded by the lines $x = 0, x = a, y = 0, y = b$ and its edge temperatures are $u(0, y) = 0$, $u(a, y) = 0$, $u(x, b) = 0$, $u(x, 0) = 5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$. Find the temperature distribution.

[Nov. '94, Madras]

Solution

Let $u(x, y)$ be the temperature distribution at any point in the plate. Then $u(x, y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$$



From the given problem we have the following boundary conditions.

(a) $u(0, y) = 0$

(b) $u(a, y) = 0$

(c) $u(x, b) = 0$

(d) $u(x, 0) = 5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$

As in worked Ex (1) we can write the most general solution as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a}\, e^{-\frac{n\pi y}{a}} \left(e^{\frac{2n\pi y}{a}} - e^{\frac{2n\pi b}{a}} \right) \quad \ldots (2)$$

Applying condition (*d*) in (2) we get

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \left(1 - e^{\frac{2n\pi b}{a}} \right) = 5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$$

i.e., $c_1 \sin \frac{\pi x}{a} \left(1 - e^{\frac{2\pi b}{a}} \right) + c_2 \sin \frac{2\pi x}{a} \left(1 - e^{\frac{4\pi b}{a}} \right)$

$$+ c_3 \sin \frac{3\pi x}{a} \left(1 - e^{\frac{6\pi b}{a}} \right) + c_4 \sin \frac{4\pi x}{a} \left(1 - e^{\frac{8\pi b}{a}} \right) + \ldots\ldots$$

$$= 5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$$

Equating like coefficients on both sides, we get

$$c_3 \left(1 - e^{\frac{6\pi b}{a}} \right) = 3 \quad \text{i.e.,} \quad c_3 = \frac{3}{1 - e^{\frac{6\pi b}{a}}} \quad \ldots (3)$$

$$c_4 \left(1 - e^{\frac{8\pi b}{a}} \right) = 5 \quad \text{i.e.,} \quad c_4 = \frac{5}{1 - e^{\frac{8\pi b}{a}}} \quad \ldots (4)$$

$$\text{and } c_1 = c_2 = c_5 = c_6 = c_7 \ldots\ldots = 0 \quad \ldots (5)$$

Substituting (3), (4) and (5) in (2), we get

$$u(x, y) = \frac{3}{1 - e^{\frac{6\pi b}{a}}} \sin \frac{3\pi x}{a}\, e^{\frac{-3\pi y}{a}} \left(e^{\frac{6\pi y}{a}} - e^{\frac{6\pi b}{a}} \right) + \frac{5 \sin \frac{4\pi x}{a}}{1 - e^{\frac{8\pi b}{a}}} \cdot e^{\frac{-4\pi y}{a}} \left(e^{\frac{8\pi y}{a}} - e^{\frac{8\pi b}{a}} \right)$$

$$= \frac{3 \sin \frac{3\pi x}{a}}{e^{\frac{-3\pi b}{a}} - e^{\frac{3\pi b}{a}}}\, e^{\frac{-3\pi y}{a}}\, e^{\frac{-3\pi b}{a}} \left(e^{\frac{6\pi y}{a}} - e^{\frac{6\pi b}{a}} \right) + \frac{5 \sin \frac{4\pi x}{a}}{e^{-\frac{4\pi b}{a}} - e^{\frac{4\pi b}{a}}}\, e^{\frac{-4\pi y}{a}}\, e^{\frac{-4\pi b}{a}} \left(e^{\frac{8\pi y}{a}} - e^{\frac{8\pi b}{a}} \right)$$

$$= \frac{3 \sin \frac{3\pi x}{a}}{-2 \sin h \frac{3\pi b}{a}} \left[-2 \sin h \frac{3\pi}{a} (b - y) \right] + \frac{5 \sin \frac{4\pi x}{a}}{-2 \sin h \frac{4\pi b}{a}} \left[-2 \sin h \frac{4\pi}{a} (b - y) \right]$$

$$= 3 \sin \frac{3\pi x}{a} \sin h \frac{3\pi}{a} (b - y) \operatorname{cosec} h \frac{3\pi b}{a} + 5 \sin \frac{4\pi x}{a} \sin h \frac{4\pi}{a} (b - y) \operatorname{cosec} h \frac{4\pi b}{a}$$

EXAMPLE 6

Solve the BVP $u_{xx} + u_{yy} = 0$, $0 < x, y < \pi$ with $u(0, y) = u(\pi, y) = u(x, \pi) = 0$ and $u(x, 0) = \sin^3 x$. [Apr. 97]

Solution

The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$

The correct solution of (1) is,

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \ldots (2)$$

Applying first condition, we get,

$$u(0, y) = c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$$

i.e., $$c_1 = 0$$

Substituting (3) in (2), we get,

$$u(x, y) = c_2 \sin px \,(c_3 e^{py} + c_4 e^{-py}) \quad \dots(3)$$

Applying second condition in (4), we get,

$$u(\pi, y) = c_2 \sin p\pi \,(c_3 e^{py} + c_4 e^{-py}) = 0 \quad \dots(4)$$

i.e., $$\sin p\pi = 0$$

i.e., $$p\pi = n\pi$$

i.e., $$p = n$$

Substituting (5) in (4), we get,

$$u(x, y) = c_2 \sin nx \,(c_3 e^{ny} + c_4 e^{-ny}) \quad \dots(5)$$

Applying third condition in (6), we get,

$$u(x, \pi) = c_2 \sin nx \,(c_3 e^{n\pi} + c_4 e^{-n\pi}) = 0 \quad \dots(6)$$

i.e., $$c_3 e^{n\pi} + c_4 e^{-n\pi} = 0$$

i.e., $$c_4 e^{-n\pi} = -c_3 e^{n\pi}$$

$$c_4 = -c_3 e^{2n\pi} \quad \dots(7)$$

Substituting (7) in (6), we get,

$$u(x, y) = c_2 \sin nx \,[c_3 e^{ny} - c_3 e^{2n\pi} \cdot e^{-ny}]$$

$$= c_n \sin nx \,[e^{ny} - e^{2n\pi} e^{-ny}], \quad c_2 c_3 = c_n$$

The most general solution is,

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin nx \,[e^{ny} - e^{2n\pi} \cdot e^{-ny}] \quad \dots(8)$$

Applying last condition in (8), we get,

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin nx \,[1 - e^{2n\pi}] = \sin^3 x$$

i.e., $$c_1 \sin x\,(1 - e^{2\pi}) + c_2 \sin 2x\,(1 - e^{4\pi}) + c_3 \sin 3x\,(1 - e^{6\pi}) + \dots$$

$$= \frac{1}{4}\,[\sin 3x - 3 \sin x]$$

Equating like coefficients on both sides, we get,

$$c_1 (1 - e^{2\pi}) = \frac{-3}{4} \Rightarrow c_1 = -\frac{3}{4(1 - e^{2\pi})}$$

$$c_3 (1 - e^{6\pi}) = \frac{1}{4} \Rightarrow c_3 = \frac{1}{4(1 - e^{6\pi})}$$

$$c_2 = c_4 = c_5 = c_6 = \dots\dots = 0$$

Substituting these values of c's in (8), we get

$$u(x, y) = -\frac{3}{4(1 - e^{2\pi})}(e^{y} - e^{2\pi} \cdot e^{-y}) + \frac{1}{4(1 - e^{6\pi})} \sin 3x \,(e^{3y} - e^{6\pi} \cdot e^{-3y})$$

■ **EXAMPLE 7** ■

A rectangular plate is bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. It's surfaces are insulated. The temperature along $x = 0$ and $y = 0$ are kept at 0°C and others at 100°C. Find the steady state temperature at any point of the plate. [Nov. '87, ECE, Apr. '86]

● **Solution**

Let $u(x, y)$ be the temperature at any point in the plate. Then $u(x, y)$ satisfies the equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

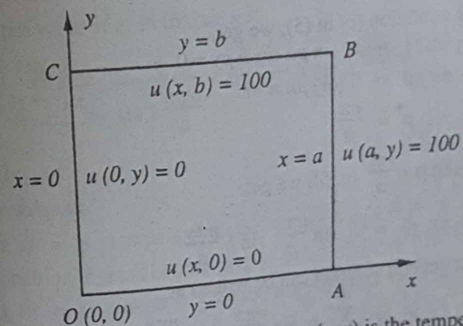
From the given problem we get the following boundary conditions.

(i) $u(x, 0) = 0$ for $0 < x < a$

(ii) $u(a, y) = 100$ for $0 < y < b$

(iii) $u(x, b) = 100$ for $0 < x < a$

(iv) $u(0, y) = 0$ for $0 < y < b$



Let $u(x, y) = u_1(x, y) + u_2(x, y)$, where (A) $u_1(x, y)$ is the temperature at any point when the edge BC is maintained at 100° C and the other three edges are maintained at 0° C and hence the boundary conditions for $u_1(x, y)$ are as follows :

(a) $u_1(x, 0) = 0$

(b) $u_1(0, y) = 0$

(c) $u_1(a, y) = 0$

(d) $u_1(x, b) = 100$

(B) $u_2(x, y)$ is the temperature distribution at any point when the edge AB is maintained at 100° C and the remaining edges are maintained at 0° C and hence the boundary conditions for $u_2(x, y)$ are as follows :

(a_1) $u_2(0, y) = 0$

(b_1) $u_2(x, 0) = 0$

(c_1) $u_2(x, b) = 0$

(d_1) $u_2(a, y) = 100$

To find $u_1(x, y)$:

The correct solution for $u_1(x, y)$ is

$$u_1(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

Applying condition (a) in (2), we get,

$$u_1(x, 0) = (c_1 \cos px + c_2 \sin px)(c_3 + c_4) = 0$$

i.e., $c_3 + c_4 = 0$

i.e., $c_4 = -c_3$ ... (3)

Substituting (3) in (2), we get,

$$u_1(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} - c_3 e^{-py})$$

$$u_1(x, y) = (c_1 \cos px + c_2 \sin px)\, 2c_3 \sin h\, py \qquad \text{... (4)}$$

Applying condition (b) in (4), we get,

$$u_1(0, y) = c_1 2c_3 \sin h\, py = 0$$

i.e., $c_1 = 0$. [$\because$ if $c_3 = 0$ we get trivial solution]

Substituting $c_1 = 0$ in (4) we get,

$$u_1(x, y) = 2c_2 c_3 \sin px \sin h\, py$$

$$u_1(x, y) = c_n \sin px \sin h\, py \qquad \text{... (5)}$$

where $2c_2 c_3 = c_n$

Applying condition (c) in (5), we get,

$$u_1(a, y) = c_n \sin pa \sin h\, py = 0$$

i.e., $\sin pa = 0$ (or) $pa = n\pi$

i.e., $p^* = \dfrac{n\pi}{a}$

Substituting $p = \dfrac{n\pi}{a}$ in (5), we get,

$$u_1(x, y) = c_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi y}{a}$$

The most general solution can be written as

$$u_1(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi y}{a} \qquad \text{... (6)}$$

Applying condition (d) in (6), we get,

$$u_1(x, b) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi b}{a} = 100 \qquad \text{... (7)}$$

To find c_n, expand 100 in a half-range Fourier sine series in $(0, a)$.

i.e., $$100 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \qquad \text{... (8)}$$

From (7) and (8) we get,

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \cdot \sin h \frac{n\pi b}{a} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$$

Equating like co-efficients we get

$$c_n \sin h \frac{n\pi b}{a} = b_n \qquad \text{... (9)}$$

(or) $$c_n = \frac{b_n}{\sin h \dfrac{n\pi b}{a}}$$

Now $$b_n = \frac{2}{a}\int_0^a 100 \sin \frac{n\pi x}{a}\, dx$$

$$= \frac{200}{a}\left[\frac{-\cos \dfrac{n\pi x}{a}}{\dfrac{n\pi}{a}}\right] = \frac{200}{n\pi}[(-1)^{n+1} + 1]$$

$\therefore b_n = 0$ when n is even ... (9a)

$b_n = \dfrac{400}{n\pi}$ when n is odd

Hence $$c_n = \frac{400}{n\pi \cdot \sin h \dfrac{n\pi b}{a}} \qquad [\because \text{ from (9)}]$$

Substituting this value of c_n in (6), we get,

$$u_1(x, y) = \sum_{n=1,3...}^{\infty} \frac{400}{n\pi} \cdot \frac{1}{\sin h \dfrac{n\pi b}{a}} \cdot \sin \frac{n\pi x}{a} \cdot \sin h \frac{n\pi y}{b} \qquad \text{... (10)}$$

To find $u_2(x, y)$:

Consider the solution ... (11)

$$u_2(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

Applying condition (b_1) in (11), we get,

$$u_2(x, 0) = (c_1 \cos px + c_2 \sin px)(c_3 + c_4) = 0 \qquad \text{... (12)}$$

i.e., $c_3 + c_4 = 0$

Applying condition (c_1) in (11), we get,

$$u_2(x, b) = (c_1 \cos px + c_2 \sin px)(c_3 e^{pb} + c_4 e^{-pb}) = 0 \qquad \text{... (13)}$$

i.e., $c_3 e^{pb} + c_4 e^{-pb} = 0$ ... (13a)

From (12) and (13), we get $c_3 = c_4 = 0$.

Substituting (13a) in (11), we get $u_2(x, y) = 0$ which is trivial.

$\therefore$ (11) is not the correct solution.

NOTE : The correct solution of $u_1(x, y)$ and $u_2(x, y)$ are not the same but they are different.

$\therefore$ We can take the correct solution as

$$u_2(x, y) = (c_1 \cos py + c_2 \sin py)(c_3 e^{px} + c_4 e^{-px})$$

Applying condition (a_1) in (14), we get,

$$u_2(0, y) = (c_1 \cos py + c_2 \sin py)(c_3 + c_4) = 0 \quad \dots (14)$$

i.e., $\quad c_4 = -c_3$

Substituting (15) in (14), we get,

$$u_2(x, y) = (c_1 \cos py + c_2 \sin py)\, c_3 (e^{px} - e^{-px}) \quad \dots (15)$$

i.e., $\quad u_2(x, y) = 2c_3 (c_1 \cos py + c_2 \sin py) \sin h\, px$

Applying condition (b_1) in (16), we get,

$$u_2(x, 0) = 2\, c_3\, c_1 \sin h\, px = 0 \quad \dots (16)$$

i.e., $\quad c_1 = 0 \quad$ [if $c_3 = 0$ we get trivial solution]

Substituting $c_1 = 0$ in (16), we get,

$$u_2(x, y) = 2\, c_3\, c_2 \sin py \sin h\, px \quad \dots (17)$$

Applying condition (c_1) in (17), we get,

$$u_2(x, b) = 2\, c_3\, c_2 \sin pb \cdot \sin h\, px = 0$$

Here $c_2 = c_3 \neq 0$.

$$\therefore \sin pb = 0 \text{ (or) } pb = n\pi$$

(or) $\quad p = \dfrac{n\pi}{b}$

Substituting $p = \dfrac{n\pi}{b}$ in (17), we get,

$$u_2(x, y) = 2\, c_2\, c_3 \sin \frac{n\pi y}{b} \sin h \frac{n\pi x}{b} = c_n \frac{\sin n\pi y}{b} \frac{\sin h\, n\pi x}{b}$$

The most general solution can be written as

$$u_2(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi y}{b} \sin h \frac{n\pi x}{b} \quad \dots (18)$$

Applying condition (d_1) in (18) we get,

$$u_2(a, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi y}{b} \sin h \frac{n\pi a}{b} = 100 \quad \dots (19)$$

To find c_n expand 100 in a half-range Fourier sine series in $(0, b)$.

i.e., $$100 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \quad \dots (20)$$

From (19) and (20), we get,

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi y}{b} \cdot \sin h \frac{n\pi a}{b} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b}$$

Equating like co-efficients, we get,

$$c_n \sin h \frac{n\pi a}{b} = b_n \quad \dots (21)$$

i.e., $$c_n = \frac{b_n}{\sin h \dfrac{n\pi a}{b}}$$

But $b_n = \dfrac{400}{n\pi}$ when n is odd (already proved)

$$\therefore c_n = \frac{400}{n\pi \sin h \dfrac{n\pi a}{b}}$$

Substituting this value of c_n in (18), we get,

$$u_2(x, y) = \sum_{n=1,3\dots}^{\infty} \frac{400}{n\pi} \cdot \frac{1}{\sin h \dfrac{n\pi a}{b}} \cdot \sin \frac{n\pi y}{b} \cdot \sin h \frac{n\pi x}{b} \quad \dots (22)$$

$$\therefore u(x, y) = u_1(x, y) + u_2(x, y)$$

$$= \frac{400}{\pi} \sum_{n=1,3\dots}^{\infty} \frac{1}{n} \left[\frac{1}{\sin h \dfrac{n\pi b}{a}} \cdot \sin \frac{n\pi x}{a} \cdot \sin h \frac{n\pi y}{b} + \frac{1}{\sin h \dfrac{n\pi a}{b}} \cdot \sin \frac{n\pi y}{b} \cdot \sin h \frac{n\pi x}{b} \right]$$

EXAMPLE 8

A rectangular plate is bounded by the lines $x = 0, x = a, y = 0$ and $y = b$ and the temperatures at the edges are given by

$$u(0, y) = y \text{ in } 0 < y < \frac{b}{2}$$

$$= b - y \text{ in } \frac{b}{2} < y < b$$

$$u(a, y) = 0 = u(x, b)$$

$$u(x, 0) = 5 \sin\left(\frac{4\pi x}{a}\right) + 3 \sin \frac{3\pi x}{a}$$

[Apr. '91 Madras]

Find the steady state temperature distribution inside the plate.

Solution

Let $u(x, y)$ be the temperature at any point in the plate. Then $u(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (1)$$

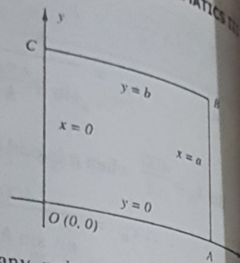
From the given problem we have the following boundary conditions

(i) $u(0, y) = \begin{cases} y, & 0 < y < \frac{b}{2} \\ b - y, & \frac{b}{2} < y < b \end{cases}$

(ii) $u(a, y) = 0,\ 0 < y < b$

(iii) $u(x, b) = 0,\ 0 < x < a$

(iv) $u(x, 0) = 5 \sin\left(\frac{4\pi x}{a}\right) + 3 \sin \frac{3\pi x}{a}$



Let $u(x, y) = u_1(x, y) + u_2(x, y)$

where (1) $u_1(x, y)$ is the temperature at any point when the edge OA is maintained at the temperature given by $5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$ and the other three edges are maintained at 0°C.

Hence the boundary conditions for $u_1(x, y)$ are as follows.

(a) $u_1(0, y) = 0$

(b) $u_1(a, y) = 0$

(c) $u_1(x, b) = 0$

(d) $u_1(x, 0) = 5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$

The solution for $u_1(x, y)$ is given by

$$u_1(x, y) = 3 \sin \frac{3\pi x}{a} \sin h \frac{3\pi (b - y)}{a} \operatorname{cosec} h \frac{3\pi b}{a} + 5 \sin \frac{4\pi x}{a} \sin h \frac{4\pi (b - y)}{a} \operatorname{cosec} h \frac{4\pi b}{a}$$

[Refer to Ex : 5 Page 3.86]

To find $u_2(x, y)$:

$u_2(x, y)$ is the temperature at any point when the edge OC is maintained at the temperature given by $u_2(0, y)$

$$f(y) = \begin{cases} y, & 0 < y < \frac{b}{2} \\ b - y, & \frac{b}{2} < y < b \end{cases}$$

and the other three edges are maintained at 0°C. Hence the boundary conditions for $u_2(x, y)$ are as follows:

(a_1) $u_2(x, 0) = 0,\ 0 < x < a$

(b_1) $u_2(x, b) = 0,\ 0 < x < a$

(c_1) $u_2(a, y) = 0,\ 0 < y < b$

(d_1) $u_2(0, y) = f(y),\ 0 < y < b$

When we solve equation (1) we have the following three solutions.

(a_2) $u_2(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

(b_2) $u_2(x, y) = (c_5 x + c_6)(c_7 y + c_8)$

(c_2) $u_2(x, y) = (c_9 \cos py + c_{10} \sin py)(c_{11} e^{px} + c_{12} e^{-px})$

Now we have to choose the correct solution which satisties the above boundary conditions.

Applying condition (a_1) in (a_2) we get

$$u_2(x, 0) = (c_1 \cos px + c_2 \sin px)(c_3 + c_4) = 0 \quad \ldots (2)$$

i.e., $c_3 + c_4 = 0 \quad \ldots (3)$

Applying condition (b_1) in (a_2) we get

$$u_2(x, b) = (c_1 \cos px + c_2 \sin px)(c_3 e^{-pb} + c_4 e^{pb}) = 0$$

i.e., $c_3 e^{-pb} + c_4 e^{pb} = 0 \quad \ldots (4)$

From (3) and (4) we get $c_3 = c_4 = 0$

Substituting $c_3 = c_4 = 0$ in (a_2) we get

$u_2(x, y) = 0$ which is trivial

Hence (a_2) is not the correct solution.

Similarly we can easily show that the solution (b_2) is also not the correct solution.

Hence the correct solution is

$$u_2(x, y) = (c_9 \cos py + c_{10} \sin py)(c_{11} e^{px} + c_{12} e^{-px}) \quad \ldots (5)$$

[Note that the correct solution of $u_1(x, y)$ and $u_2(x, y)$ are different]

Applying condition (a_1) in (5), we get

$$u_2(x, 0) = c_9 (c_{11} e^{px} + c_{12} e^{-px}) = 0 \quad \ldots (6)$$

i.e., $c_9 = 0$

Substituting (6) in (5) we get

$$u_2(x, y) = c_{10} \sin py (c_{11} e^{px} + c_{12} e^{-px}) \quad \ldots (7)$$

Applying condition (b_1) in (7), we get

$$u_2(x, b) = c_{10} \sin pb (c_{11} e^{px} + c_{12} e^{-px}) = 0$$

$c_{10} \neq 0$ [$\because$ If $c_{10} = 0$ we get trivial solution]

$\therefore \sin pb = 0$

i.e., $pb = n\pi$

$$p = \frac{n\pi}{b} \quad \ldots (8)$$

Substituting (8) in (7) we get

$$u_2(x, y) = c_{10} \sin \frac{n\pi y}{b} \left[c_{11} e^{\frac{n\pi x}{b}} + c_{12} e^{-\frac{n\pi x}{b}} \right] \quad \ldots (9)$$

Applying condition (c_1) in (9) we get

$$u_2(a, y) = c_{10}\sin\frac{n\pi y}{b}\left[c_{11}e^{\frac{n\pi a}{b}} + c_{12}e^{-\frac{n\pi a}{b}}\right] = 0$$

Here $c_{10} \neq 0$ (explained earlier) and

$\sin\frac{n\pi y}{b} \neq 0$ since y varies from 0 to b

$$\therefore c_{11}e^{\frac{n\pi a}{b}} + c_{12}e^{-\frac{n\pi a}{b}} = 0$$

$$c_{12}e^{-\frac{n\pi a}{b}} = -c_{11}e^{\frac{n\pi a}{b}} \quad i.e., \; c_{12} = -c_{11}e^{\frac{2n\pi a}{b}} \qquad \ldots(10)$$

Substituting (10) in (9) we get

$$u_2(x, y) = c_{10}\sin\frac{n\pi y}{b}\left[c_{11}e^{\frac{n\pi x}{b}} - c_{11}e^{\frac{2n\pi a}{b}}\cdot e^{-\frac{n\pi x}{b}}\right]$$

$$= c_{10}c_{11}\sin\frac{n\pi y}{b}\left[e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}}\cdot e^{-\frac{n\pi x}{b}}\right]$$

$$= c_n\sin\frac{n\pi y}{b}\left[e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}}\cdot e^{-\frac{n\pi x}{b}}\right]$$

The most general solution can be written as

$$u_2(x, y) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi y}{b}\left[e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}}\cdot e^{-\frac{n\pi x}{b}}\right] \qquad \ldots(11)$$

Applying condition (d_1) in (11) we get

$$u_2(0, y) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi y}{b}\left[1 - e^{\frac{2n\pi a}{b}}\right] = f(y) \qquad \ldots(12)$$

$$\text{where } f(y) = \begin{cases} y, & 0 < y < \frac{b}{2} \\ b - y, & \frac{b}{2} < y < b \end{cases}$$

To find c_n expand $f(y)$ in a half range Fourier Sine Series in the interval $(0, b)$

$$i.e., \; f(y) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi y}{b} \qquad \ldots(13)$$

$$\text{where } b_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

From (12) and (13) we get

$$\sum_{n=1}^{\infty} c_n\sin\frac{n\pi y}{b}\left(1 - e^{\frac{2n\pi a}{b}}\right) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi y}{b}$$

$$c_n\left(1 - e^{\frac{2n\pi a}{b}}\right) = b_n$$

$$c_n = \frac{b_n}{1 - e^{\frac{2n\pi a}{b}}} \qquad \ldots(14)$$

$$\text{Now } b_n = \frac{2}{b}\int_0^b f(y)\cdot\sin\frac{n\pi y}{b}\,dy$$

$$= \frac{2}{b}\left[\int_0^{b/2} y\sin\frac{n\pi y}{b}\,dy + \int_{b/2}^{b}(b - y)\sin\frac{n\pi y}{b}\,dy\right]$$

$$= \frac{2}{b}\left[\left\{y\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right) - 1\left(\frac{-\sin\frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right)\right\}_0^{b/2} + \left\{(b - y)\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right)\right\}_{b/2}^{b}\right]$$

$$= \frac{2}{b}\left[\frac{-\frac{b}{2}\cos\frac{n\pi}{2}}{\frac{n\pi}{b}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{b^2}} + \frac{\frac{b}{2}\cos\frac{n\pi}{2}}{\frac{n\pi}{b}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{b^2}}\right]$$

$$= \frac{2}{b}\times\frac{b^2}{n^2\pi^2}\cdot 2\sin\frac{n\pi}{2}$$

$$b_n = \frac{4b}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \ldots(15)$$

Substituting (15) in (14) we get

$$c_n = \frac{\frac{4b}{n^2\pi^2}\sin\frac{n\pi}{2}}{1 - e^{\frac{2n\pi a}{b}}} \qquad \ldots(16)$$

Substituting (16) in (11), we get

$$u_2(x, y) = \sum_{n=1}^{\infty}\frac{4b\sin\frac{n\pi}{2}}{n^2\pi^2\left(1 - e^{\frac{2n\pi a}{b}}\right)}\cdot\sin\frac{n\pi y}{b}\left(e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}}\cdot e^{-\frac{n\pi x}{b}}\right)$$

$$= \frac{4b}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2\left(-2\sinh\frac{n\pi a}{b}\right)}\cdot\sin\frac{n\pi y}{b}\left(e^{-\frac{n\pi a}{b}}\cdot e^{\frac{n\pi x}{b}}-e^{\frac{n\pi a}{b}-\frac{n\pi x}{b}}\right)$$

$$= \frac{4b}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2\left(-2\sinh\frac{n\pi a}{b}\right)}\cdot\sin\frac{n\pi y}{b}\left\{e^{-\frac{n\pi}{b}(a-x)}-e^{\frac{n\pi}{b}(a-x)}\right\}$$

$$= \frac{4b}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2\sinh\frac{n\pi a}{b}}\cdot\sin\frac{n\pi y}{b}\cdot\sinh\frac{n\pi}{b}(a-x)$$

$$= \frac{4b}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi y}{b}\cdot\sinh\frac{n\pi(a-x)}{b}\cdot\operatorname{cosech}\frac{n\pi a}{b}$$

$$\therefore u(x,y) = u_1(x,y)+u_2(x,y)$$

$$= 3\sin\frac{3\pi x}{a}\cdot\sinh\frac{3\pi(b-y)}{a}\cdot\operatorname{cosech}\frac{3\pi b}{a}$$

$$+5\sin\frac{4\pi x}{a}\cdot\sinh\frac{4\pi(b-y)}{a}\cdot\operatorname{cosech}\frac{4\pi b}{a}$$

$$+\frac{4b}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\cdot\sin\frac{n\pi y}{b}\cdot\sinh\frac{n\pi(a-x)}{b}\cdot\operatorname{cosech}\frac{n\pi a}{b}$$

EXERCISES

1. The three sides $x = 0, x = a, y = 0$ of a square plate bounded by the lines $x = 0, x = a, y = 0$ and $y = a$ are kept temperature $0°$ C. The side $y = a$ is kept at steady temperature given by $u(x, a) = bx(x - a), 0 \le x \le a$ where b is a constant. Find the steady-state temperature $u(x, y)$ in the plate. [Nov. '89 Mech.]

$$\left[\textit{Ans. } u(x,y) = \frac{-8ba^2}{\pi^3}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n^3\sinh n\pi}\cdot\sin\frac{n\pi x}{a}\cdot\sinh\frac{n\pi y}{a}\right]$$

2. A square plate is bounded by the lines $x = 0, y = 0, x = a, y = a$. Its surfaces are insulated and their temperatures along the edges $x = a$ and $y = a$ are each $100°$ C, while the other two edges are kept at $0°$ C. Find the steady state temperature distribution at any point on the plate. [Apr. '87 Mech.]

$$\left[\textit{Ans. } u(x,y) = \frac{400}{\pi}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n\sinh n\pi}\left\{\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}+\sin\frac{n\pi y}{a}\sinh\frac{n\pi x}{a}\right\}\right]$$

3. Find the solution of the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi, 0 < y < \pi$ given $u(0, y) = u(\pi, y) = u(x, \pi) = 0, u(x, 0) = \sin^2 x$. [Nov. '86 Civil, Nov. '88 Mech]

$$\left[\textit{Ans. } u(x,y) = \frac{-8}{\pi}\sum_{n=1,3\ldots}^{\infty}\frac{\sin nx\cdot\sinh n(\pi-y)}{n(n^2-4)\sinh n\pi}\right]$$

4. A function $u(x, y)$ satisfies the Laplace's equation in rectangular co-ordinates (x, y) and for the points within the rectangle $x = 0, x = a, y = 0, y = b$, it satisfies the conditions $u(0, y) = u(a, y) = u(x, b) = 0$ and $u(x, 0) = x(a - x); 0 < x < a$. Find $u(x, y)$.

$$\left[\textit{Ans. } u(x,y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{a}\cdot\frac{\sin\frac{(2n+1)\pi}{a}(b-y)}{\sinh\frac{(2n+1)\pi b}{a}}\right]$$

5. The temperature u is maintained at $0°$ C along three edges of a square plate of length 100 cm. and the fourth edge is maintained at a constant temperature $100°$ C until steady-state conditions prevail. Find an expression for the temperature u at any point (x,y).

$$\left[\textit{Ans. } u(x,y) = \frac{400}{\pi}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n\sinh n\pi}\cdot\sin\frac{n\pi x}{100}\cdot\sinh\frac{n\pi y}{100}\right]$$

INFINITE PLATES

EXAMPLE 1

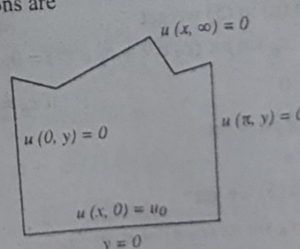
A long rectangular plate has its surfaces insulated and the two long sides as well as one of the short sides are maintained at 0°C. Find an expression for the steady state temperature $u(x, y)$ if the short side $y = 0$ is π cm long and is kept at u_0 °C. [B.E. Civil Apr. '86]

Solution

When steady state conditions exists, the two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots (1)$$

The boundary conditions are



(i) $u(0, y) = 0$ for all 'y'

(ii) $u(\pi, y) = 0$ for all 'y'

(iii) $u(\pi, \infty) = 0$ i.e., when $u \to 0$ as $y \to \infty$

(iv) $u(x, 0) = u_0$ for x in $(0, \pi)$

When we solve the equation (1) we get the following solutions

(a) $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

(b) $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

(c) $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Consider the first solution

Applying condition (i) in (a) we get

$$u(0, y) = (c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0$$

i.e., $c_1 + c_2 = 0$ ... (2)

Applying condition (ii) in (a) we get

$$u(\pi, y) = (c_1 e^{p\pi} + c_2 e^{-p\pi})(c_3 \cos py + c_4 \sin py) = 0$$

i.e., $c_1 e^{p\pi} + c_2 e^{-p\pi} = 0$ ... (3)

Solving (2) and (3) we get $c_1 = 0$ and $c_2 = 0$

Substituting $c_1 = 0$ and $c_2 = 0$ in (a), we get

$u(x, y) = 0$ which is a trivial solution

Hence solution (a) is not satisfying our conditions. Similarly we can easily prove that solution (c) is also not the correct solution.

∴ The correct solution is

$$u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \dots (4)$$

Applying condition (i) in (4) we get

$$u(0, y) = c_5 (c_7 e^{py} + c_8 e^{-py}) = 0$$

i.e., $c_5 = 0$

Substituting $c_5 = 0$ in (4), we get

$$u(x, y) = c_6 \sin px\, (c_7 e^{py} + c_8 e^{-py}) \quad \dots (5)$$

Applying condition (ii) in (5) we get

$$u(\pi, y) = c_6 \sin p\pi\, (c_7 e^{py} + c_8 e^{-py}) = 0.$$

Here $c_6 \neq 0$ [∵ if $c_6 = 0$ we get trivial solution]

∴ $\sin p\pi = 0$ or $p\pi = n\pi$

i.e., $p = n$

Substituting $p = n$ in (5) we get

$$u(x,y) = c_6 \sin nx\, (c_7 e^{ny} + c_8 e^{-ny}) \quad \dots (6)$$

Applying condition (iii) in (6), we get

$$u(x, \infty) = c_6 \sin nx\, (c_7 e^{\infty} + c_8 e^{-\infty}) = 0.$$

i.e., as $y \to \infty$, $u \to 0$ (condition (iii))

This is possible only when $c_7 = 0$ [∵ if $c_8 = 0$ we get $u \to \infty$]

Substituting $c_7 = 0$ in (6) we get

$$u(x, y) = c_6 \sin nx.\, c_8 e^{-ny}$$

The most general solution can be written as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin nx \cdot e^{-ny} \quad \dots (7)$$

Applying condition (iv) in (7) we get

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin nx = u_0 \quad \dots (8)$$

To find c_n, expand u_0 in a half-range Fourier sine series in $(0, \pi)$

i.e., $$u_0 = \sum_{n=1}^{\infty} b_n \sin nx \quad \dots (9)$$

From (8) and (9) we get

$$\sum_{n=1}^{\infty} c_n \sin nx = \sum_{n=1}^{\infty} b_n \sin nx$$

∴ $\boxed{c_n = b_n}$

$$\text{Now } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2u_0}{n\pi}[-\cos n\pi + 1] = \frac{2u_0}{n\pi}[((-1)^{n+1} + 1)]$$

∴ $b_n = 0$ when 'n' is even.

$b_n = \dfrac{4u_0}{n\pi}$ when 'n' is odd.

∴ $c_n = \dfrac{4u_0}{n\pi}$ when 'n' is odd.

Substituting this value of c_n in (7) we get

$$u(x,y) = \sum_{n=1,3,5\dots}^{\infty} \frac{4u_0}{n\pi} \cdot \sin nx \cdot e^{-ny}$$

EXAMPLE 2

An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge x = 0 is kept at temperature given by

$$u = \begin{cases} 20y, & \text{for } 0 \le y \le 5 \\ 20(10-y), & \text{for } 5 \le y \le 10 \end{cases}$$

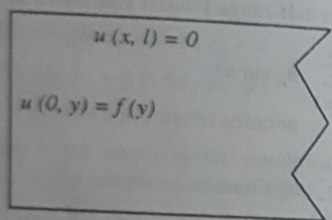
Find the steady state temperature distribution in the plate [Nov., Mech.]

Solution

The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$$

Let $l = 10$ cm for convenience. From the given problem we have the following boundary conditions.



(i) $u(x, 0) = 0$ for all x

(ii) $u(x, l) = 0$ for all x

(iii) $u(\infty, y) = 0$, i.e., when $x \to \infty$, $u \to 0$

(iv) $u(0, y) = \begin{cases} 2ly \text{ in } 0 \le y \le \frac{l}{2} \\ 2l(l-y) \text{ in } \frac{l}{2} \le y \le l \end{cases}$

When we solve equation (1) we get the following solutions.

(a) $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

(b) $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

(c) $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Consider the solution (b).

Applying condition (i) in (b) we get

$$u(x, 0) = (c_5 \cos px + c_6 \sin px)(c_7 + c_8) = 0$$

$$c_7 + c_8 = 0 \quad \ldots (2)$$

Applying condition (ii) in (b) we get

$$u(x, l) = (c_5 \cos px + c_6 \sin px)(c_7 e^{pl} + c_8 e^{-pl}) = 0$$

i.e., $c_7 e^{pl} + c_8 e^{-pl} = 0 \quad \ldots (3)$

Solving (2) and (3) we get $c_7 = 0 = c_8$.

Substituting $c_7 = 0$ and $c_8 = 0$ in (b) we get $u(x, y) = 0$ which is trivial solution.

Hence this solution is not the correct solution.

Similarly we can easily show that solution (c) is also not the correct solution for our problem.

∴ The correct solution is

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \ldots (4)$$

Applying condition (i) in (4) we get

$$u(x, 0) = c_3 (c_1 e^{px} + c_2 e^{-px}) = 0$$

i.e., $c_3 = 0$

Substituting $c_3 = 0$ in (4) we get

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin py \quad \ldots (5)$$

Applying condition (ii) in (4) we get

$$u(x, l) = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin pl = 0.$$

Here $c_4 \ne 0$. ∴ $\sin pl = 0$ i.e., $pl = n\pi$

$$\therefore p = \frac{n\pi}{l} \quad \ldots (6)$$

Substituting (6) in (5) we get

$$u(x, y) = \left(c_1 e^{\frac{n\pi x}{l}} + c_2 e^{\frac{-n\pi x}{l}}\right) c_4 \sin \frac{n\pi y}{l} \quad \ldots (7)$$

Applying condition (iii) in (7) we get

$$u(\infty, y) = (c_1 e^{\infty} + c_2 e^{-\infty}) c_4 \sin \frac{n\pi y}{l} = 0$$

i.e., as $x \to \infty$, $u \to 0$ (by condition (iii))

This is possible only when $c_1 = 0$

(∵ if $c_2 = 0$, we get $u = \infty$ since $e^{\infty} = \infty$)

Substituting $c_1 = 0$ in (7) we get

$$u(x, y) = c_2 c_4 \sin \frac{n\pi y}{l} \cdot \frac{-n\pi x}{l}$$

$$= c_n \cdot \sin \frac{n\pi y}{l} \cdot e^{\frac{-n\pi x}{l}} \cdot (c_2 c_4 = c_n)$$

The most general solution can be written as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi y}{l} \cdot e^{\frac{-n\pi x}{l}}$$

Applying condition (*iv*) in (8), we get

$$u(0, y) = \sum_{n=1}^{\infty} c_n \cdot \sin \frac{n\pi y}{l} = f(y) \text{ (say)}$$

Where
$$f(y) = \begin{cases} 2ly, & 0 \le y \le \frac{l}{2} \\ 2l(l-y), & \frac{l}{2} \le y \le l \end{cases}$$

To find c_n expand $f(y)$ in a half-range Fourier sine series

i.e.,
$$f(y) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi y}{l} \qquad \dots(10)$$

From (9) and (10) we get $b_n = c_n$

Now
$$b_n = \frac{2}{l} \int_0^l f(y) \cdot \sin \frac{n\pi y}{l}\, dy$$

$$= \frac{2}{l}\left[\int_0^{l/2} 2ly \sin \frac{n\pi y}{l}\, dy + \int_{l/2}^{l} 2l(l-y) \cdot \sin \frac{n\pi y}{l}\, dy\right]$$

$$= 4\left[\left\{y\left(\frac{-\cos \frac{n\pi y}{l}}{\frac{n\pi}{l}}\right) - 1 \cdot \left(\frac{-\sin \frac{n\pi y}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{(l-y)\left(\frac{-\cos \frac{n\pi y}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin \frac{n\pi y}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= 4\left[\frac{-l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right]$$

$$= 4\left[\frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right]$$

$$\therefore c_n = \frac{8l^2}{n^2\pi^2} \cdot \sin \frac{n\pi}{2} \qquad \dots(11)$$

Substituting (11) in (8), we get

$$u(x,y) = \sum_{n=1}^{\infty} \frac{8l^2}{n^2\pi^2} \cdot \sin \frac{n\pi}{2} \cdot \sin \frac{n\pi y}{l} \cdot e^{\frac{-n\pi x}{l}}$$

Replacing l by 10 we get,

$$u(x,y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \sin \frac{n\pi}{2} \cdot \sin \frac{n\pi y}{10} \cdot e^{\frac{-n\pi x}{10}}$$

EXAMPLE 3

An infinitely long-plane uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth of this edge $x = 0$ is π, this end is maintained at temperature as $u = k(\pi y - y^2)$ at all points while the other edges are at zero temperature. Determine the temperature $u(x, y)$ at any point of the plate in the steady state if u satisfies Laplace equation. **[Oct. 99, Apr. 2000]**

Solution

The heat flow equation is
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \dots(1)$$

The boundary conditions are

(*i*) $u(x, 0) = 0$ for all x

(*ii*) $u(x, \pi) = 0$ for all x

(*iii*) $u(\infty, y) = 0$, *i.e.*, when $x \to \infty$, $u \to 0$

(*iv*) $u(0, y) = k(\pi y - y^2)$

After applying conditions (*a*), (*b*) and (*c*), we get the most general of (1),

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin ny\, e^{-nx} \qquad \dots(2)$$

Applying condition (iv) in (2), we get,

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin ny = k(\pi y - y^2) \qquad \dots(2a)$$

To find c_n, expand $k(\pi y - y^2)$ in a half range sine series in $(0, \pi)$.

i.e.,
$$k(\pi y - y^2) = \sum_{n=1}^{\infty} b_n \sin ny \qquad \dots(3)$$

From (2*a*) and (3), we get,

$$\boxed{c_n = b_n}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} k(\pi y - y^2) \sin ny\, dy$$

$$= \frac{2k}{\pi}\left[(\pi y - y^2)\left(\frac{-\cos ny}{n}\right) - (\pi - 2y)\left(\frac{-\sin ny}{n}\right) + (-2)\left(\frac{\cos ny}{n^3}\right)\right]_0^\pi$$

$$= \frac{2k}{\pi}\left[\frac{-2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4k}{n^3\pi}\left[1 - (-1)^n\right]$$

i.e., $b_n = 0$, when 'n' is even

$= \frac{8k}{n^3\pi}$, when 'n' is odd

$\therefore c_n = b_n$ [From (3)]

Substituting this value of c_n in (2), we get

$$u(x, y) = \sum_{n=1,3}^{\infty} \frac{8k}{n^3\pi} \cdot \sin n y\, e^{-nx}$$

■ EXAMPLE 4 ■

A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}$ in $0 < x < 8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edges are kept at 0°C, find the steady state temperature function $u(x, y)$.

● Solution

The heat flow equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

The boundary conditions are,

let $l = 8$

(i) $u(0, y) = 0$ for all y; (ii) $u(l, y) = 0$ for all y

(iii) $u(x, \infty) = 0$; (iv) $u(x, 0) = 100 \sin \frac{\pi x}{l}$ for x in $(0, l)$

The correct solution of (1) is,

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \ldots (2)$$

After applying condition (i), (ii) and (iii), we get the most general solution,

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}\, e^{\frac{-n\pi y}{l}} \quad \ldots (3)$$

Applying condition (iv) in (3), we get,

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = 100 \sin \frac{\pi x}{l}$$

i.e., $c_1 \sin \frac{\pi x}{l} + c_2 \sin \frac{2\pi x}{l} + \ldots\ldots = 100 \sin \frac{\pi x}{l}$

Equating like terms on both sides,

i.e., $c_1 = 100, \; c_2 = 0, \; c_3 = c_4 \ldots = 0$

Substituting $c_1 = 100$ and $c_2 = c_3 = \ldots\ldots = c_n = 0$ in (3), we get,

$$u(x, y) = 100 \sin \frac{\pi x}{l}\, e^{\frac{-\pi y}{l}}$$

● EXERCISES

1. Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with the boundary conditions $u(0,y) = u(\pi, y) = 0$ for all y, $u(x, \infty) = 0$ in $0<x<\pi$ and $u(x, 0) = u_0$ in $0<x<\pi$.

   [Ans. $u(x, y) = \frac{4u_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \cdot \sin nx \cdot e^{-ny}$]

2. A rectangular plate with insulated surfaces is 8 cm. wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}$ in $0 < x < 8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edges are kept at 0°C, find the steady state temperature function $u(x, y)$. *[Nov.'89]*

   [Ans. $u(x, y) = 100 \cdot \sin \frac{\pi x}{8} \cdot e^{\frac{-\pi y}{8}}$]

3. A long rectangular plate of width a cm. with insulated surface has its temperature is equal to zero on both the long sides and one of the shorter sides so that $u(0,y) = 0$, $u(a, y) = 0$, $u(x, \infty) = 0$ and $u(x, 0) = \lambda x$. Show that the steady state temperature within the plate is given by $u(x, y) = \frac{2a\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{\frac{-n\pi y}{a}} \cdot \sin \frac{n\pi x}{a}$. *[Apr.'86]*

4. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$

is given by $u = 20x$ for $0 \le x \le 5$ and $u = 20\,(10 - x)$ for $5 \le x \le 10$ and two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature at any point (x,y) is given by

$$u\,(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cdot\sin\frac{(2n-1)\pi x}{10}\cdot e^{\frac{-(2n-1)\pi y}{10}}$$

5. A long rectangular plate has its surface insulated and the two long sides as well as one of the short sides are kept at 0°C, while the other sides $u\,(x, 0) = 3x$ and the length being 5 cm. Find $u\,(x, y)$. [Nov. '87 M...]

[Nov. ...]

[*Ans.* $u\,(x, y) = \frac{30}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot e^{\frac{-n\pi y}{5}}\cdot\sin\frac{n\pi x}{5}$

6. A long rectangular plate of width l cm with insulated surfaces has its temperature u equal to zero on both the long sides and one of the short sides so that $u\,(0, y) = u\,(l, y) = u\,(x, \infty) = 0$ and $u\,(x, 0) = kx$. Find $u\,(x, y)$.

[*Ans.* $u\,(x, y) = \frac{2kl}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot e^{\frac{-n\pi y}{l}}\cdot\sin\frac{n\pi x}{l}$

Two Marks Q & A

1. ***Explain the initial and boundary value problems.*** *[Apr 95]*

Ans : In ordinary differential equations, first we get the general solution which contains the arbitrary constants and then we determine these constants from the given initial values. This type of problems is called *intial value problems*.

In many physical problems, we always seek a solution of the differential equations, whether it is ordinary or partial, which satisfies some specified conditions called boundary conditions. Any differential equations together with these boundary conditions is called boundary value problems.

2. ***Explain the method of separation of variables.*** *[Apr 95]*

Ans : In given a partial differential equation if z is the dependent variable and x,y independent variables then we assume the solution to be the product of two functions, one of them a function of x alone and the other a function of y alone. In this way, the solution of the partial differential equation is converted in to the solution of ordinary differential equations. This method is known as ***method of separation of variables.***

3. ***By the method of separation of variables solve $x^2 q + y^3 p = 0$*** *[Apr 95]*

Ans : Given $x^2\frac{\partial z}{\partial y} + y^3\frac{\partial z}{\partial x} = 0$ ...(1)

Let $z = X(x)\,Y(y)$ be the solution of (1)

Then $\frac{\partial z}{\partial x} = X'Y$; $\frac{\partial z}{\partial y} = XY'$ ... (3)

Substituting (3) in (1) we get

$$x^2 XY' + y^3 X'Y = 0$$

$$ie.,\quad x^2 XY' = -y^3 X'Y$$

$$\frac{Y'}{-y^3 Y} = \frac{X'}{x^2 X} = k\ (\text{say})$$

$$\text{i.e., } Y' = -ky^3 Y \quad ; \quad X' = kx^2 X$$

$$\frac{dY}{dy} = -ky^3 Y \quad ; \quad \frac{dX}{dx} = kx^2 X$$

$$\text{i.e., } \frac{dY}{Y} = -ky^3 dy \quad ; \quad \frac{dX}{X} = -kx^2\,dx$$

Integrating we get

$$\log Y = -\frac{ky^4}{4} \quad ; \quad \log X = k\frac{x^3}{3}$$

$$Y = e^{\frac{-ky^4}{4}} \quad ; \quad X = e^{k\frac{x^3}{3}} \quad \dots (4)$$

Substituting (4) in (2) we get

$$z = e^{k\frac{x^3}{3}} \cdot e^{\frac{-ky^4}{4}} = e^{\frac{k}{12}(4x^3 - 3y^4)}$$

$$= e^{c(4x^3 - 3y^4)}$$

4. *State the assumptions made in the derivation of one dimensional wave equation* [Nov 95, Apr 95]

Ans :

(i) The mass of the string per unit length is constant

(ii) The string is perfectly elastic and does not offer any resistance to bending.

(iii) The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected

iv) The string performs a small transverse motion in a vertical plane, that is every particle of the string moves strictly vertically so that the deflection and the slope at every point of the string remain small in absolute value.

5. *The one dimensional wave equation is ....* [Apr 95]

Ans : The one dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

6. *The one dimensional wave equation is* [Nov.95]

a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ b) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

c) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ d) $\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial y^2} = c\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}$

Ans : (c)

7. *The three possible solutions of $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ are ...*

Ans :

i) $u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

ii) $u(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat})$

iii) $u(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$

8. *Obtain the solution of one dimensional wave equation by the method of separation of variables (any one case)* [Apr 95]

Ans : The one dimensional wave equation is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \dots (1)$$

$$\text{Let } u = X(x) \cdot T(t) \quad \dots (2)$$

be the solution of (1).

Then $\frac{\partial^2 u}{\partial x^2} = X'' \cdot T, \quad \frac{\partial^2 u}{\partial t^2} = XT''$ ... (3)

Substituting (3) in (1) we get

$$a^2 X''T = XT''$$

$$\text{i.e.,} \quad a^2 \frac{X''}{X} = \frac{T''}{a^2 T} = k$$

$$\frac{X''}{X} = k, \quad \frac{T''}{a^2 T} = k$$

i.e., Let k be negative

$$\text{i.e., } k = -p^2$$

$$X'' - p^2 X = 0, \quad T'' - a^2 p^2 T = 0$$

Auxiliary equations are $m^2 - p^2 = 0$; $m^2 - a^2 p^2 = 0$

$$m = \pm p$$

$$X = (c_1 e^{px} + c_2 e^{-px}),$$

$$T = (c_3 \cos pat + c_4 \sin pat)$$

$$\therefore u(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos pat + c_4 \sin pat)$$

9. *The PDE of a vibrating string is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ what is a^2 ?* [Apr 95]

Ans : $a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass}}$

10. *Explain the various variables involved in one dimensional wave equation* [Apr, Nov 95]

Ans : One dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ Here x and t are the two variables. where x denotes length and t denotes time.

11. Write down the boundary conditions for the following boundary value problem "If a string of length 'l' initially at rest in its equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$ $0 < x < l$, determine the displacement function $y(x,t)$? [Nov 95]

Ans : The boundary conditions are

i) $y(0,t) = 0, t > 0$ ii) $y(l,t) = 0, t > 0$

iii) $y(x,0) = 0, 0 < x < l$ iv) $\frac{\partial y(x,0)}{\partial t} = v_0 \sin^3 \frac{\pi x}{l}, 0 < x < l.$

12. *Define temperature gradient*

Ans : Consider a bar of uniform cross section of length 'x' cm. Let the two ends of the rod are maintained at temperature u_1 and u_2 where $u_1 > u_2$. The quantity $\frac{u_1 - u_2}{x}$ represents the rate of fall of temperature w.r.t distance. This rate of change of temperature w.r.t distance is called the ***temperature gradient*** and is denoted by $\frac{\partial u}{\partial x}$.

13. *Define steady state temperature distribution.*

Ans : If the temperature will not change when time varies is called steady state temperature distribution.

14. *How many boundary conditions are required to solve completely* $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ [Nov 95]

Ans : Three conditions.

15. *State the laws assumed to derive the one dimensional heat equation* [Apr 95]

Ans :

1. Heat flows from a higher temperature to lower temperature.
2. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.
3. The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area.

16. *Say True or False : One dimensional heat conduction equation is* $\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$ [Apr.95]

Ans : False

17. *The steady state temperature of a rod of length 'l' whose ends are kept at 30° and 40° is …………* [Apr.95]

Ans : The heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ when steady state condition exist the heat flow equation becomes $\frac{d^2 u}{dx^2}$

$$u(x) = c_1 x + c_2 \quad \text{...(1)}$$

The end conditions are

$$u(0) = 30 \text{ ...(2)} \quad u(l) = 40 \quad \text{...(3)}$$

Substituting (2) in (1) we get $u(0) = c_2 = 30$

$$\text{ie., } \boxed{c_2 = 30}$$

$$\therefore u(x) = c_1 x + 30 \quad \text{...(4)}$$

Substituting (3) in (4) we get $u(l) = c_1 l + 30 = 40$

$$c_1 l = 40$$

$$c_1 = \frac{40}{l} \quad \text{...(5)}$$

Substituting (5) in (4) we get $u(x) = \frac{40x}{l} + 30$

18. *State one dimensional heat equation with the initial and boundary conditions* [Apr.95]

Ans : One dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ The boundary conditions are

a) $u(0, t) = k_1{}^\circ c$ for all $t > 0$ b) $u(l, t) = k_2{}^\circ c$ for all $t > 0$

The initial condition is $u(x, 0) = f(x), 0 < x < l.$

19. *What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation.* [Apr.95]

Ans : The correct solution of one dimensional wave equation is of periodic in nature. But the solution of heat flow equation is not in periodic in nature.

20. Give three possible solutions of the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

Ans : i) $u(x,t) = (c_1 x + c_2)$

ii) $u(x,t) = e^{\alpha^2 p^2 t}(c_3 e^{px} + c_4 e^{-px})$

iii) $u(x,t) = e^{-\alpha^2 p^2 t}(c_5 \cos px + c_6 \sin px)$

21. *Obtain the one dimensional heat flow equation from two dimensional heat flow equation for unsteady case.* [Nov 95]

Ans : The two dimensional heat flow equation for unsteady case is $\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ ... (1) When the stream of lines are parallel to the x–axis, the rate of change of the temperature $\frac{\partial u}{\partial y}$ in the direction of the y–axis is zero.

$\therefore$ (1) becomes $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ which is one dimensional heat flow equation.

22. *Express the boundary conditions in respect of insulated ends of a bar of length 'a' and also the initial temperature distribution.* [Apr 95]

Ans : i) $\frac{\partial u(0,t)}{\partial x} = 0$ ii) $\frac{\partial u(a,t)}{\partial x} = 0$ iii) $u(0,t) = f(x)$

23. *State Fourier law of heat conduction* [Apr 95]

Solution: The rate at which heat flows across an area A at a distance x from one end of a bar is given by $Q = -KA\left(\frac{\partial u}{\partial x}\right)_x$, k is thermal conductivity and $\left(\frac{\partial u}{\partial x}\right)_x$ means the temperature gradient at x.

24. *Write the solution of one dimensional heat flow equation, when the time derivative is absent.* [Apr. 97]

Ans : When time derivative is absent the heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} = 0$.

25. *When the ends of a rod length 20 cm are maintained at the temperature 10℃ and 20℃ respectively until steady state is prevailed. Determine the steady state temperature of the rod.* [BDN, Nov. 97]

Ans : When steady state condition exists the heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} = 0$$

i.e.,

$$u(x) = c_1 x + c_2 \qquad \dots (1)$$

The boundary conditions are

$$(a)\ u(0) = 10$$
$$(b)\ u(20) = 20$$

Applying *(a)* in (1), we get

$$u(0) = c_2 = 10$$

Substituting $c_2 = 10$ in (1), we get

$$u(x) = c_1 x + 10 \qquad \dots (2)$$

Applying *(b)* in (2), we get

$$u(20) = c_1(20) + 10 = 20$$
$$c_1 = \frac{10}{20} = \frac{1}{2}$$

Substituting $c_1 = \dfrac{1}{2}$ in (2), we get

$$u(x) = \frac{1}{2}x + 10$$

26. *State how many initial and boundary conditions are required to solve the PDE which represents the wave equation.* [Nov. 97]

Ans : Four

27. *If $y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$ is the solution of wave motion satisfying certain conditions, then what will be the solution satisfying $y(x, 0) = A \sin\left(\frac{\pi x}{l}\right), 0 \le x \le l$.* [Nov. 97]

Ans :

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

i.e., $b_1 \sin\dfrac{\pi x}{l} + b_2 \sin\dfrac{2\pi x}{l} + \dots\dots\dots = A \sin\left(\dfrac{\pi x}{l}\right)$

i.e., $b_1 = A,\ b_2 = b_3 = \dots\dots\dots = 0$

$$y(x,t) = A \sin\frac{\pi x}{l} \cos\frac{\pi at}{l}$$

28. *One dimensional heat flow equation is $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$. Say true or false.* [Nov. 97]

Ans : True

The given equation can be written as $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, which is of standard type. Here $\alpha = 1$.

29. *In steady state, two dimensional heat-flow equation in cartesian coordinate is ....* [Nov. 97]

Ans :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

30. *The PDE of one dimensional heat flow in steady state is given by*

(a) $\dfrac{\partial u}{\partial t} = 0$ (b) $\dfrac{\partial^2 u}{\partial y^2} = 0$ (c) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} = 0$

(d) $\dfrac{\partial^2 u}{\partial x^2} = 0$ (e) $\dfrac{\partial u}{\partial t} = 0$ [Nov. 97]

Ans : (d) $\dfrac{\partial^2 u}{\partial x^2} = 0$

31. *Explain the term steady state.* [Apr. 98]

Ans : When the heat flow is independent of time '*t*', it is called steady state. In steady state the heat flow is only w.r.t. the distance '*x*'.

32. *Obtain one dimensional heat flow equation from two dimensional heat flow for unsteady case.* [Apr. 98]

Ans : When unsteady state condition exists the two dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In one dimensional heat flow there will be no heat flow in *y*-direction and hence $\dfrac{\partial^2 u}{\partial y^2} = 0$.

∴ The heat flow equation becomes

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

33. *State the reason for choosing $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$ as a suitable solution for one dimensional wave flow.* [Nov. 97]

Ans : Since it is periodic.

35. *Write the boundary conditions for solving the string equation, if the string is subjected to initial displacement $f(x)$ and initial velocity $g(x)$.* [Apr. 98]

Ans : (i) $y(0, t) = 0$, (ii) $y(l, t) = 0$, (iii) $\dfrac{\partial y(x,0)}{\partial t} = g(x)$

(iv) $y(x, 0) = f(x)$

36. ***If the solution of one dimensional heat flow equation depends neither on Fourier cosine series nor on Fourier sine series, what would have been the nature of the end conditions.*** [Apr. 97]

Ans : One end should be thermally insulated and the other end is at zero temperature.

37. ***Given that the general solution of the one dimensional heat flow equation is $u(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin nx\, e^{-n^2 \alpha^2 t}$ and the initial temperature in the rod is $4 \sin^3 x$, find λ_n.***

Ans : $u(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin nx\, e^{-n^2 \alpha^2 t}$ ... (1)

Initial temperature is $4 \sin^3 x$

i.e., $$u(x, 0) = 4 \sin^3 x = 4 \frac{1}{4} [3 \sin x - \sin 3x]$$

Put $t = 0$ in (1), we get

$$u(x, 0) = \sum_{n=1}^{\infty} \lambda_n \sin nx = 3 \sin x - \sin 3x \text{ [From (2)]} \quad \dots (2)$$

i.e., $\lambda_1 \sin x + \lambda_2 \sin 2x + \lambda_3 \sin 3x + \dots\dots = 3 \sin x - \sin 3x$

i.e., $\lambda_1 = 3,\ \lambda_3 = -1,\ \lambda_2 = \lambda_4 = \lambda_5 = \dots\dots = 0$

38. ***If the end $x = 0$ is insulated in one dimensional heat flow problem will you get half range sine series or half range cosine series in the solution ?*** [Oct. 97]

Ans : No

39. ***If the ends of a string of length 'l' are fixed and the mid point of the string is drawn aside through a height 'h' and the string is released from rest, write the initial conditions.*** [Oct. 97]

Ans : (i) $y(0, t) = 0$, (ii) $y(l, t) = 0$, (iii) $\frac{\partial y(x, 0)}{\partial t} = 0$

(iv) $$y(x, 0) = \begin{cases} \frac{2hx}{l}, & 0 < x < \frac{l}{2} \\ \frac{2h}{l}(l - x), & \frac{l}{2} < x < l \end{cases}$$

40. ***In steady state conditions derive the solution of one dimensional heat flow equation.*** [Apr. 97]

Ans : When steady state conditions exist the heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \dots (1)$$

[∵ u will not change w.r.t. time 't'. Hence $\frac{\partial u}{\partial t} = 0$]

Solving (1), we get, $u(x) = c_1 x + c_2$

...at is meant by two dimensional flow ?

Ans : The heat flows in xy-direction.

...plain the term thermally insulated ends ?

Ans : If there will be no heat flow passes through the end of the bar then ...at two ends are said to be thermally insulated.

In one dimensional heat flow equation if the temperature function is independent of time, then the solution is $u = ax + b$. Say true or false. [Apr. 97]

Ans : True

In two dimensional heat flow, the temperature at any point is independent of .... coordinate. [Apr. 97]

Ans : Z-coordinate

When the ends of a rod is non-zero for one dimensional heat flow equation the temperature function $u(x, t)$ is modified as the sum of steady state and transient state temperatures. The transient part of the solution which

(a) increases with increase of time

(b) decreases with increase of time

(c) decreases with decrease of time

(d) increases with decrease of time

Ans : (b)

46. ***The rate at which heat flows across any areas is jointly proportional to the area and to the ....*** [Apr. 97]

Ans : Temperature gradient normal to the area.

47. A separable solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ satisfying the end conditions $y(0, t) = y(l, t) = 0$ is

(a) $\sin\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{2}\right)$ (b) $e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$

(c) $x(x - l) \sin \frac{n\pi xt}{l}$ (d) $\sin\left(\frac{n\pi ct}{l}\right) \cos\left(\frac{n\pi x}{l}\right)$

(e) None of these

Ans : (e) The correct solution is

$$y(x, t) = c_1 \sin \frac{n\pi x}{l} \left(c_2 \cos \frac{n\pi ct}{l} + c_3 \sin \frac{n\pi ct}{l}\right)$$

48. **Classify the following PDE.**

(a) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ (b) $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + xy$

Solution :

(a) Here A = 1, B = 0, C = − 1

$$B^2 - 4AC = 4 > 0 \quad \therefore \text{Hyperbolic equation}$$

(b) Here $B = 1$, $A = 0$, $C = 0$

$$B^2 - 4AC = 1 > 0$$

$\therefore$ Hyperbolic equation.

49. Classify the following second order PDE

(a) $4\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} - 6\dfrac{\partial u}{\partial x} - 8\dfrac{\partial u}{\partial y} - 16u = 0$

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$

Solution :

(a) $A = 4$, $B = 4$, $C = 1$

$$B^2 - 4AC = 4 - 16 < 0$$

$\therefore$ Elliptic equation.

(b) Here $A = 1$, $B = 0$, $C = 1$

$$B^2 - 4AC = 0 - 4 < 0 \qquad \text{[Elliptic]}$$

50 Classify the following partial differential equations.

(*a*) $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0.$

(*b*) $y^2 u_{xx} - u_{yy} + u_x^2 + u_y^2 + 7 = 0.$

Solution :

(a) Here $A = y^2$, $B = -2xy$, $C = x^2$

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0.$

The equation is parabolic equation.

(b) $A = y^2$, $B = 0$, $C = 1$

Now $B^2 - 4AC = y^2 - 1$

Here y^2 is always +ve.

When $-1 < y < 1$, $y^2 - 1$ is –ve.

$\therefore$ The equation is elliptic.

When $y > 1$, $y^2 - 1 > 0$

$\therefore$ The equation is Hyperbolic.

51. Classify the partial differential equation $u_{xx} + x\,u_{yy} = 0.$

Solution : Here $A = 1$, $B = 0$, $C = x$

$$B^2 - 4AC = -4x < 0. \quad \text{Elliptic equation.}$$