$z = f(x, y)$ 17/07/2023

The partial differnt'l cofficient are denoted as follows:

→ $\frac{\partial z}{\partial x} = p$ , $\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial z}{\partial y} = q$ , $\frac{\partial^2 z}{\partial x \partial y} = s$ , $\frac{\partial^2 z}{\partial y^2} = t$.

# Formation of PDE :-

By eliminatⁿ of arbitary const. | By eliminatⁿ of arbitary $f^n$.

- If the no. of arbit. const to be eliminated is less than or equal to the no. of independent variables, we get a differntⁿ eqⁿ of first order.
- If arbitary const. is greater than variable then we get higher order eqⁿ greater than 1.
- we may get different PDE.

Que :- form the PDF by eliminating the arbit. const from (i) $z = ax^3 + by^3$.

Soln :- $\frac{\partial z}{\partial x} = 3ax^2 = p$ , $\frac{\partial z}{\partial y} = 3by^2 = q$

$\Rightarrow a = \frac{p}{3x^2}$ $\Rightarrow b = \frac{q}{3y^2}$.

Hence,

$z = \frac{px^3}{3x^2} + \frac{qy^3}{3y^2} \Rightarrow \boxed{z = \frac{1}{3}(px + qy)}$ Ans

(ii) $z = ax + by + ab$.

$\frac{\partial z}{\partial x} = a = p$ , $\frac{\partial z}{\partial y} = q = b$ $\boxed{z = px + qy + pq}$

(iii) $z = (x^2+a)(y^2+b)$

$p = 2x(y^2+b)$ $q = (x^2+a)2y$

$p = 2xy^2 + 2xb$ $q = 2x^2y + 2ay$

$b = \frac{p - 2xy^2}{2x}$ $a = \frac{q - 2x^2y}{2y}$.

$z = \left(x^2 + \frac{q - 2x^2y}{2y}\right)\left(y^2 + \frac{p - 2xy^2}{2x}\right)$.

OR

$y^2 + b = \frac{p}{2x}$ , $\frac{q}{2y} = x^2 + a$

$z = \frac{p}{2x} \times \frac{q}{2y} \Rightarrow \boxed{z = \frac{pq}{4xy}}$ Answer.

Q: Obtain the PDE of all sphere whose center lies on $z=0$ and whose radius is constant and equal to $r$

Sol: eqn of sphere.

$$(x-a)^2+(y-b)^2+(z-0)^2=r^2$$

wrong method

$$\frac{\partial z}{\partial x}=\frac{\cancel{2}(x-a)}{\cancel{2}z}=\frac{x-a}{z}=p \Rightarrow a=x-pz$$

$$\Rightarrow -a=pz-x$$

$$\frac{\partial z}{\partial y}=\frac{\cancel{2}(y-b)}{\cancel{2}z}=\frac{y-b}{z}=q \Rightarrow b=y-qz$$

$$\Rightarrow (x-pz-x)^2+(y-qz-y)^2+z^2=r^2$$

$$pz^2+qz^2+z^2=r^2$$

$$\Rightarrow$$

$$=(x-a)^2+(y-b)^2+z^2=r^2 \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial}{\partial x}(x-a)^2+\frac{\partial}{\partial x}(y-b)^2+\frac{\partial}{\partial x}z^2=r^2$$

$$\Rightarrow 2(x-a)+0+2z\frac{\partial z}{\partial x}=0$$

$$\cancel{2}(x-a)=-\cancel{2}zp$$

$$(x-a)=-zp$$

Similly. w.r.t y.

$$0 + 2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$2(y-b) = -2zq$$

$$(y-b) = -zq$$

putting in eqn ①.

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$z^2p^2 + z^2q^2 + z^2 = r^2$$

$$\underline{z^2(p^2+q^2+1) = r^2.}$$

<u>Que</u> form PDE by elimination of arbitrary $f^n$ from

(i) $z = f(x^2+y^2)$

$$\frac{\partial z}{\partial x} = p = f'(x^2+y^2)2x \quad -①, \quad \frac{\partial z}{\partial y} = q = f'(x^2+y^2)2y \quad -②$$

$$\frac{p}{q} = \frac{x}{y} \Rightarrow py = xq \Rightarrow \boxed{xq - py = 0}$$

(ii) $z = f(x+ct) + \phi(x-ct)$

$$p = f'(x+ct) + \phi'(x-ct), \; q = f'(x+ct)c - \phi'(x-ct)c$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct), \; \frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct)$$

$$\boxed{\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}}$$

<u>Answer</u>

$z = f(2x+y) + g(3x-y)$

$$\frac{\partial z}{\partial x} = 2f'(2x+y) + 3g'(3x-y) = p$$

$$\frac{\partial z}{\partial y} = f'(2x+y) - g'(3x-y) = q$$

$$\frac{\partial^2 z}{\partial x^2} = 4f''(2x+y) + 9g''(3x-y) = r \quad -(1)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(2x+y) + g''(3x-y) = t \quad -(2)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2f''(2x+y) - 3g''(3x-y) = s \quad -(3)$$

Add eq^n (1) and (3)

$$r + s = 6f''(2x+y) + 6g''(3x-y)$$

$$r + s = 6[f''(2x+y) + g''(3x-y)]$$

$$\boxed{r + s = 6t}$$

Que :- $z = yf(x) + xg(y)$

$$\frac{\partial z}{\partial x} = yf'(x) + g(y) = p, \quad \frac{\partial z}{\partial y} = f(x) + xg'(y) = q.$$

$$\frac{\partial^2 z}{\partial x^2} = yf''(x) = r, \quad \frac{\partial^2 z}{\partial y^2} = xg''(y) = t.$$

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) = s$$

$$s = \frac{p - g(y)}{y} + \frac{q - f(x)}{x}$$

$$sxy = (px + qy) - (xg(y) + yf(x))$$

$$\boxed{sxy = px + qy - z} \text{ Ans}$$

## Elimination of arbitary $f^n$ of type $\phi(u,v)=0$

$$\phi(u,v) = 0 \quad (1)$$

Now, $Pp + Qq = R \quad (2)$ { Lagrange's $eq^n$ }

$$P = \frac{\partial(u,v)}{\partial(y,z)}, \quad Q = \frac{\partial(u,v)}{\partial(z,x)}, \quad R = \frac{\partial(u,v)}{\partial(x,y)}$$

( p & q are known as $\frac{dz}{dx}$ and $\frac{dz}{dy}$ )

Method 2 – $u = f(v)$ or $v = f(u)$

Que:- $\phi(x^2+y^2+z^2, lx+my+nz)=0$

$$\phi(u,v)=0$$

where $u = x^2+y^2+z^2$, $v = lx+my+nz$

PDE: $Pp + Qq = R. \quad (1)$

$$P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} \Rightarrow \begin{vmatrix} 2y & 2z \\ m & n \end{vmatrix} \Rightarrow P = 2(ny - mz)$$

$$Q = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ n & l \end{vmatrix} \Rightarrow Q = 2(lz - nx)$$

$$R = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ l & m \end{vmatrix} \Rightarrow R = 2(m\cancel{x} - ly)$$

Put values in eqn ①

$$z[(ny-nz)p+(lz-nx)q]=z(mx-ly)$$

$$(ny-mz)p+(lz-nx)q=mx-ly.$$

<u>Method 2</u>:-

$$u=f(v)$$

$$x^2+y^2+z^2=f(lx+my+nz)$$

Partially differentiate wrt $x$

$$2x+2z\frac{\partial z}{\partial x}=f'(lx+my+nz)\left(l+n\frac{\partial z}{\partial x}\right)$$

$$2(x+zp)=f'(lx+my+nz)(l+np) \quad -②$$

diff. wrt $y$.

$$2(y+zq)=f'(lx+my+nz)(m+nq) \quad -③$$

$$② \div ③$$

$$\frac{x+zp}{y+zq}=\frac{l+np}{m+nq}$$

$$mx+nzqp+xnq+mzp=ly+npy+lqz+pqnz.$$

$$\Rightarrow \boxed{(ny-mz)p+(lz-nx)q=mx-ly} \text{ Answer}$$

# Non-linear PDE of 1st order.

## Types of Sol$^n$

Complete Sol$^n$ /Comp. Integral | Singular Sol$^n$ | General Sol$^n$/Integral. /Particular Intgl

**TYPE-1:** $\boxed{F(p,q)=0}$, f. ①

$z = ax + by + c$ ②

$\frac{\partial z}{\partial x} = \boxed{p=a}$, $\frac{\partial z}{\partial y} = \boxed{q=b}$

$F=(a,b)=0.$

$b = \phi(a)$

Singular Sol$^n$

$z = ax + \phi(a)y + c$

$\frac{\partial z}{\partial a} = 0 = x + \phi'(a)y.$

$\frac{\partial z}{\partial c} = 0 = 1$ — Absurd

not possible.

General Sol$^n$ :-

$z = ax + \phi(a)y + c.$ — ③

put $c = \psi(a)$

$z = ax + \phi(a)y + \psi(a)$ — ④

$0 = x + \phi'(a)y + \psi'(a)$ — ⑤

Let $\sqrt{p} + \sqrt{q} = 1$

$F(p, q) = 0$ — (1)

The complete solⁿ of eqⁿ is $z = ax + by + c$ — (2)

$p = \frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = q = b.$

$\Rightarrow \sqrt{a} + \sqrt{b} = 1$

$\{\therefore \sqrt{b} = 1 - \sqrt{a}\}$

$b = (1 - \sqrt{a})^2$

$z = ax + (1 - \sqrt{a})^2 y + c$ — (3) {Complete solution}

for singular solⁿ.

$\frac{\partial z}{\partial a} = 0, \quad \& \frac{\partial z}{\partial c} = 0 = 0 + 0 + 1$

$\Rightarrow 0 = 1$ — Absurd.

Hence no singular solⁿ.

For General solⁿ

$c = \varphi(a)$

$z = ax + (1 - \sqrt{a})^2 y + \varphi(a)$ — (4)

diff. eqⁿ (4) partially w.r.t. a,

Eliminating eqⁿ ④ and ⑤, we get general solⁿ.

Ques. $p+q=1$

$F(p,q)=0$

$z = ax+by+c$ — ①

$\frac{\partial z}{\partial x} = p = a$, $\frac{\partial z}{\partial y} = q = b$.

Eq. $p+q=1$

$\Rightarrow a+b=1$

$\Rightarrow b = 1-a$.

$z = ax + (1-a)y + c$ — ③ — Complete solⁿ

for singular solⁿ:

$\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial c} = 0 + 0 + \boxed{1 = 0}$ — Absurd.

Hence no singular solⁿ.

for General solⁿ.

$c = \varphi(a)$

$z = ax + (1-a)y + \varphi(a)$ — ④

diff wrt a.

$0 = x - y + \varphi'(a)$ — ⑤

eliminating eqⁿ ④ and eqⁿ ⑤ we get general solⁿ

Que:- $p^2+q^2=npq$ — (1)

$f(p,q)=0$

Complete soln eqn is $z=ax+by+c$ — (2)

$p=a \quad q=b$

$\Rightarrow a^2+b^2=nab$

$\Rightarrow b^2-nab+a^2=0$

$\Rightarrow b=\frac{na\pm\sqrt{n^2a^2-4a^2}}{2}=\frac{a}{2}[n\pm\sqrt{n^2-4}]$

$z= ax+\frac{a}{2}(n\pm\sqrt{n^2-4})y+c$ . — (3)

$\frac{\partial z}{\partial a}=0=x+\frac{1}{2}(n\pm\sqrt{n^2-4}]y.$

$\frac{\partial z}{\partial c}=0+0+1=0$ (Absurd). $\rightarrow$ No Singular Sol$^n$

General Sol$^n$:-

Put $c=\psi(a)$.

$z=ax+\frac{a}{2}(n\pm\sqrt{n^2-4})y+\psi(a)$. — (4)

$0=x+\frac{1}{2}(n\pm\sqrt{n^2-4})y+\psi'(a)$ — (5)

eliminating 'a' eqn (4) and (5), we get general sol$^n$.

Eq:- $p+q=pq$, $pq=1$.

$\Rightarrow$ Complete solⁿ :- $z = ax + by + c$

$a+b=ab \Rightarrow b-ab+a$

$\Rightarrow$ ~~b=ab-a~~ $b(1-a) = -a \Rightarrow b = \frac{-a}{1-a} = \frac{a}{a-1}$

$\Rightarrow z = ax + \frac{a}{(a-1)}y + c$

Singular $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial c} = 1 = 0$ Absurd

General solⁿ $c = \phi(a)$

$z = ax + \frac{a}{(a-1)}y + \phi(a)$

$0 = x + \frac{\cancel{a}(-1)-\cancel{a}}{(a-1)^2}y + \phi'(a)$

$0 = x - \frac{y}{(a+1)^2} + \phi(a)$.

$pq = 1 \Rightarrow z = ax + by + c$ complete solⁿ

$ab = 1 \Rightarrow b = \frac{1}{a} \Rightarrow z = ax + \frac{1}{a}y + c$

for singular, $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial c} = 1 = 0$ Absurd

for general put $c = \phi(a)$

$z = ax + \frac{y}{a} + \phi(a)$

w.r.t. a

$0 = x + y\log a + \phi'(a)$

By solving these eqⁿ

## Type II :- Clairaut's form

$$z = px + qy + f(p,q)$$

$$z = ax + by + f(a,b) \quad \text{—} ① \quad \{\text{complete sol}^n\}$$

$$\frac{\partial z}{\partial a} = 0, \quad \frac{\partial z}{\partial b} = 0 \quad \text{— Singular sol}^n$$

$$b = \phi(a).$$

differentiating eq$^n$ → general sol$^n$

**Que** :- $z = px + qy + p^2 - q^2$. —① which satisfies the Clairaut form

Hence complete ~~integral~~ sol$^n$ is :-

$$z = ax + by + a^2 - b^2 \quad \left\{ \begin{matrix} p = a \\ q = b \end{matrix} \right\} \quad \text{—} ②$$

For singular sol$^n$ :-

$$0 = \frac{\partial z}{\partial a} = x + 2a, \quad \frac{\partial z}{\partial b} = y - 2b = 0$$

$$\Rightarrow a = -x/2, \quad b = y/2$$

Putting values of a and b in eq$^n$ ②

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} \Rightarrow \boxed{4z = y^2 - x^2} \quad \text{—} ③$$

Put $b = \phi(a)$ into eq$^n$ ②. General sol$^n$

$$z = ax + \phi(a)y + a^2 - [\phi(a)]^2 \quad \text{—} ④$$

diff w.r.t a

$$0 = x + y\phi'(a) + 2a - 2\phi(a)\cdot\phi'(a) \quad \text{—} ⑤$$

Que:- $z = px + qy + \sqrt{1+p^2+q^2}$ — (1)

complete ~~integral~~ soln eqn is:- $z = ax + by + \sqrt{1+a^2+b^2}$ — (2)

for singular soln:-

$$\frac{\partial z}{\partial a} = 0 = x + \frac{a}{\sqrt{1+a^2+b^2}} \Rightarrow x = -\frac{a}{\sqrt{1+a^2+b^2}}.$$

$$\frac{\partial z}{\partial b} = 0 = y + \frac{b}{\sqrt{1+a^2+b^2}} \Rightarrow y = \frac{-b}{\sqrt{1+a^2+b^2}}.$$

$$\Rightarrow x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2} \Rightarrow 1-(x^2+y^2) = 1-\left(\frac{a^2+b^2}{1+a^2+b^2}\right)$$

$$\Rightarrow 1-x^2-y^2 = \frac{1}{1+a^2+b^2} \Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow x = -a\sqrt{1-x^2-y^2}, \quad y = -b\sqrt{1-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}, \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Putting in eqn (2).

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}.$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} \Rightarrow \boxed{z = \sqrt{1-x^2-y^2}}$$

$$\Rightarrow z^2 = 1-x^2-y^2 \Rightarrow \boxed{x^2+y^2+z^2 = 1}$$

put $b = \phi(a)$ for General soln in eqn (2)

$$z = ax + \phi(a)y + \sqrt{1+a^2+(\phi(a))^2} \quad \text{— (3)}$$

diff eqn (3) wrt $a$.

$$0 = x + \phi'(a)y + \frac{2a + 2\phi(a)\phi'(a)}{2\sqrt{1+a^2+(\phi(a))^2}} \quad \text{— (4)}$$

eliminating $a$ from eqn (3) & (4) we get General soln

Ques $z = px + qy + pq^2$

Complete sol$^n$:- $z = ax + by + a^2b^2$ —①

$\frac{\partial z}{\partial a} = 0 = x + 2ab^2$ —②, $\frac{\partial z}{\partial b} = 0 = y + 2a^2b$ —③

$\Rightarrow x = -2ab^2$ $\Rightarrow y = -2a^2b$

$\Rightarrow \frac{x}{b} = -2ab.$ $\Rightarrow \frac{y}{a} = -2ab.$

$\Rightarrow \frac{x}{b} = \frac{y}{a} = \frac{1}{k}$ (say).

$a = ky$, $b = kx$

Putting in eq$^n$ ②

$\cancel{x} = -2ky k^2 x^2 = -2k^3 x^{\cancel{2}} y.$

$\Rightarrow k^3 = -\frac{1}{2xy}.$

Put value of a and b in eq$^n$ ①.

$z = kyx + kxy + k^2x^2k^2y^2$

$z = 2kxy + kx^2y^2k^3$

$z = 2kxy + kx^{\cancel{2}}y^{\cancel{2}}\left(\frac{-1}{2\cancel{x}\cancel{y}}\right)$

$\Rightarrow z = 2kxy - \frac{kxy}{2} \Rightarrow z = \frac{3}{2}kxy$

$z^3 = \frac{27}{8}k^3x^3y^3 \Rightarrow z^3 = \frac{27}{8}\left(\frac{-1}{2\cancel{x}\cancel{y}}\right)x^{\cancel{3}}y^{\cancel{3}}$

$z^3 = \frac{-27}{16}x^2y^2 \Rightarrow 16z^3 = -27x^2y^2.$

$\Rightarrow \boxed{16z^3 + 27x^2y^2 = 0}$ {Singular sol$^n$}.

**Type III :-** (a) $f(z,p,q)=0$.

Let $z=f(u)$ where $u=x+ay$.

$z=f(x+ay)$.

$$p=\frac{\partial z}{\partial x}=\frac{dz}{\partial u}\times\left(\frac{\partial u}{\partial x}\right)\Rightarrow\frac{\partial z}{\partial u}\times 1\Rightarrow\boxed{\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}=p}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right)\Rightarrow\frac{\partial z}{\partial u}\times a\Rightarrow\boxed{\frac{\partial z}{\partial y}=a\frac{\partial z}{\partial u}=q}$$

**Ques :-** solve $p(1+q)=qz$.

Let $z=f(u)=f(x+ay)$

$$p=\frac{\partial z}{\partial u},\quad q=a\frac{\partial z}{\partial u}$$

$$\Rightarrow\frac{\partial z}{\partial u}\left(1+a\frac{\partial z}{\partial u}\right)=az\left(\frac{\partial z}{\partial u}\right)$$

$$\Rightarrow 1+\frac{a\,\partial z}{\partial u}=az$$

$$\Rightarrow\frac{a\,\partial z}{\partial u}=az-1\Rightarrow\int\frac{a\,\partial z}{az-1}=\int\partial u$$

$$\Rightarrow\log(az-1)=u+c$$

$$\Rightarrow\boxed{\log(az-1)=x+ay+c}$$ - complete solⁿ. — (1)

Singular solⁿ, $\frac{\partial z}{\partial a},\ \frac{\partial z}{\partial c}=1=0$ — absurd.

General, $c=\phi(a)$. ~~and~~ diff. wrt $a$, → By eliminatg. eqⁿ (2) & (3)

$\log(az-1)=x+ay+\phi(a)\Rightarrow\frac{z}{\ldots}=y+\phi'(a)$ — (3) -

, $F(x, p, q) = 0$

Assume, $dz = pdx + qdy$.

Put $q = a$.

$$dz = pdx + ady$$

$$F(x, p, a) = 0 \Rightarrow \boxed{p = \phi(x, a)}$$

(c) $F(y, p, q) = 0$, $dz = pdx + qdy$.

Let $p = a$. $dz = adx + qdy$.

$$F = (y, a, q) \Rightarrow \boxed{q = \phi(y, a)}$$

---

Que :- Solve, $p = 2qx$.

$dz = pdx + qdy$.

put $q = a$, $\Rightarrow p = 2ax$.

$\int dz = \int 2axdx + ady$.

$\boxed{z = ax^2 + ay + c} \rightarrow$ Complete solⁿ — ①

for Singular $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial c} = 0 = 1 \Rightarrow$ Absurd $\rightarrow$ No solⁿ.

for General solⁿ :- Put $c = \phi(a)$.

$$z = ax^2 + ay + \phi(a) \quad \text{— ②}$$

diff wrt $a$.

$$\frac{\partial z}{\partial a} = 0 = x^2 + y + \phi'(a) \quad \text{— ③}$$

eliminating eqⁿ ② & ③ we got General solⁿ.

Que:- $pq = y$.

Assume $p = a$.

$aq = y$.

$$\boxed{\frac{y}{a} = q}$$

Put $dz = p\,dx + q\,dy$.

$$\int dz = \int a\,dx + \int \frac{y}{a}\,dy$$

$$\boxed{z = ax + \frac{y^2}{2a} + c}$$ → complete solⁿ.

for singular solⁿ - $\frac{\partial z}{\partial c} = \boxed{0 = 1}$ Absurd - No solⁿ

for General solⁿ :- put $c = \phi(a)$

$$z = ax + \frac{y^2}{2a} + \phi(a) \quad (2)$$

diff w.r.t $a$.

$$0 = x - \frac{y^2}{2a^2} + \phi'(a) \quad (3)$$

By eliminating $a$ we get our General solⁿ

Que:- $q = px + p^2$.

Que:- $q(p^2 z + q^2) = 4$

Que:- $z = p^2 + q^2$

## Type IV :- Separable Eqn

$f(x,p) = \phi(y,q) = a$

$f(x,p) = a \quad , \quad \phi(y,q) = a$

$p = f(x,a), \quad q = \phi(y,a) \cdot \; dz = pdx + qdy$

---

**Que** :- $p^2 y(1+x^2) = qx^2$.

$\Rightarrow \dfrac{p^2(1+x^2)}{x^2} = \dfrac{q}{y} = a$ — (1)

$f(x,p) = \phi(y,q) = a$

$\Rightarrow p^2 = \dfrac{ax^2}{1+x^2} \Rightarrow \boxed{p = \dfrac{x\sqrt{a}}{\sqrt{1+x^2}}, \; q = ay}$

$\int dz = \int \dfrac{x\sqrt{a}}{\sqrt{1+x^2}} dx + \int ay\,dy$.

$\boxed{z = \sqrt{a(1+x^2)} + \dfrac{ay^2}{2} + c}$ - complete soln, (1)

$\Rightarrow$ No singular soln, $\dfrac{dz}{dc} = 0 = 1$ - Absurd.

for General soln, put $c = \phi(a)$

$z = \sqrt{a(1+x^2)} + \dfrac{ay^2}{2} + \phi(a)$. — (2)

diff wrt a.

$0 = \dfrac{\sqrt{1+x^2}}{2\sqrt{a}} + \dfrac{y^2}{2} + \phi'(a)$ — (3)

By eliminating 'a' eqn (2) and (3) we get general soln.

# Lagrange's Method, $Pp+Qq=R$

# Working Rule :-

* Auxiliary eq$^n$: $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

→ solve the above A·E, $u=c_1$, $v=c_2$.
→ $f(u,v)=0 \Rightarrow f(c_1,c_2)=0$ } I$^{st}$ Method

→ Method of multipliers ← II$^{nd}$ Method

Let $l,m,n$ may be constants of f$^n$ of $x,y,z$,
then we have, $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+Rn}$

→ $l,m,n$ are chosen in such a way that,
$lP+mQ+nR=0$ {den $=0$}.

Thus, $l\,dx+m\,dy+n\,dz=0 \Rightarrow u=c_1$
Smlly, $l_1dx+m_1dy+n_1dz=0 \Rightarrow v=c_2$

---

Que. find the general sol$^n$ of $px+qy=z$

$Pp+Qq=R \Rightarrow P=x,\ Q=y,\ R=z$.

A·E :- $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$ $\left\{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\right\}$

$\Rightarrow$ let $\frac{dx}{x}=\frac{dy}{y} \Rightarrow \log x=\log y+\log c_1$

$\Rightarrow \log c_1=\log x-\log y \Rightarrow \boxed{c_1=\frac{x}{y}}$

let $\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$

$\Rightarrow \log c_2 = \log y - \log z \Rightarrow \boxed{c_2 = y/z}$

$\Rightarrow$ Hence the general soln is :-

$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$

---

Ques $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$

$\Rightarrow (mz - ny)p + (nx - lz)q = ly - mx. \rightarrow$ is in lagrange's form.

$P = mz - ny,\ Q = nx - lz,\ R = ly - mx$

$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} =$

$\frac{x\,dx + y\,dy + z\,dz}{x(mz-ny) + y(nx-lz) + z(ly-mx)}$ {set of multipliers $x, y, z$}

$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0} \Rightarrow x\,dx + y\,dy + z\,dz = 0.$

Integrating.

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a \Rightarrow \boxed{x^2 + y^2 + z^2 = c_1}$ {2a = const}

let take another set of multipliers $(l, m, n)$

$= \frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{l\,dx + m\,dy + n\,dz}{l(mz-ny) + m(nx-lz) + n(ly-mx)}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0 \Rightarrow \boxed{lx + my + nz = c_2}.$

Hence General integral is :-

$\phi(x^2 + y^2 + z^2,\ lx + my + nz) = 0.$

Que $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$

$Pp+Qq=R$

$P=x(z^2-y^2),\ Q=y(x^2-z^2),\ R=z(y^2-x^2)$

$A.E = \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)}=\frac{xdx+ydy+zdz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-x^2)}$$

$\Rightarrow xdx+ydy+zdz=0$. {x, y, z - 1st set of multi}

Integrating,

$\boxed{x^2+y^2+z^2=C_1}$

2nd set of multipliers $\{1/x, 1/y, 1/z\}$.

$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

$\Rightarrow \boxed{\log xyz=\log c_2} \Rightarrow \boxed{xyz=C_2}$

Hence General sol^n is $\phi(x^2+y^2+z^2, \log xyz)=0$

Q. $p\tan x+q\tan y=\tan z$

$\Rightarrow \frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z} \Rightarrow$ Hence General sol^n is $\int \frac{1}{\tan x}dx$

$\log(\sin x)=\log(\sin y)+\log c_1 \Rightarrow \phi(c_1,c_2)=0$

$c_1=\frac{\sin x}{\sin y},\ c_2=\frac{\sin y}{\sin z}$ $\quad \phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right)$ $\log(\sin x)$

Qu: $(y-xz)p+(yz-x)q=(x+y)(x-y)$

$$\frac{dx}{y-xz}=\frac{dy}{yz-x}=\frac{dz}{(x+y)(x-y)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\Rightarrow x\,dx+y\,dy+z\,dz=0 \Rightarrow x^2+y^2+z^2=c_1$

$\Rightarrow y\,dx+x\,dy+1\cdot dz=0$. {∴ $(x,y,1)$ - 2nd multiplier}

$\Rightarrow$ ~~$xy+yx+z=c_2=2xy+z$~~ wrong method: can't partially differentiate

$d(xy+z)=0 \Rightarrow \underline{xy+z=c_2}$

$xy$ → product rule

$$\underline{\phi(x^2+y^2+z^2,\ xy+z)=0}$$

Qu: $\frac{y^2z}{x}\cdot p+xzq=y^2$

$\Rightarrow y^2zp+x^2zq=xy^2$

$$\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{xy^2}$$

$x^2dx=y^2dy$

$\frac{x^3}{3}-\frac{y^3}{3}=c_1$

$\boxed{x^3-y^3=c_1}$

$\frac{dx}{y^2z}=\frac{dz}{xy^2}$

$x\,dx=z\,dz$

$\boxed{x^2-z^2=c_2}$

Hence general eqn is

$$\underline{\phi(x^3-y^3,\ x^2-z^2)=0}$$

## Homogeneous linear PDE of $n^{th}$ order with const. coefficient.

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial y\,\partial x^{n-1}}+\cdots a_n\frac{\partial^n z}{\partial y^n}=F(x,y).$$

$\Rightarrow \frac{\partial}{\partial x}=D,\ \frac{\partial}{\partial y}=D' \Rightarrow (a_0D^n+a_1D^{n-1}D'\cdots+D'^n)z=f(x,y).$

$f(D,D')z=f(x,y) \rightarrow$ then $\underline{A\cdot E}$ and,

find roots ... put $D=m,\ D'=1$

## ...les to find complementary $f^n$:-

1). when $m_1 \neq m_2$, then

$$C.F = f_1(y+m_1x) + f_2(y+m_2x) + \dots$$

2). when $m_1 = m_2$, then $C.F = f_1(y+mx) + x f_2(y+mx)$

if $m_1 = m_2 = m_3$, then $CF = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx)$

## Rules to find Particular Integral:-

I. Type I:- $\frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$, if $f(a,b) \neq 0$.

if den $= 0$, then diff. den. and multiply $x$ with num.

Type II:- $\frac{1}{f(D,D')} \sin(ax+by)$ / $\cos$ $= \frac{1}{f(D^2, DD', D'^2)} \sin/\cos(ax+by)$ $\left\{ \begin{array}{l} D^2 = -a^2 \\ D'^2 = -b^2 \\ D'D = -ab \end{array} \right.$

Type III:- $\frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$

Type IV:- $\frac{1}{f(D,D')} e^{ax+by} \cdot \phi(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} \cdot \phi(x,y)$

Type V $\frac{\sin ax \sin by}{f(D^2, D'^2)} = \frac{\sin ax \sin by}{f(-a^2, -b^2)}$ $\Big\}$ (same for cos)

Que :- $(D^2-4DD'+4D'^2)z=0$

Sol :- $A.E$ :- $m^2-4m+4 \quad \{D=m, D'=1\}$

$\Rightarrow m^2-2m-2m+4$

$\Rightarrow (m-2)(m-2) \Rightarrow m=2,2.$

$CF = f_1(y+2x)+xf_2(y+2x)$

$P.I = 0.$

$Z = CF+PI \Rightarrow Z = \underline{\quad}$ Answer

Synthetic Division

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$m^2-4m+4=0.$

$m=2,2.$

Que :- $\frac{\partial^3 z}{\partial x^3} - \frac{3\partial^3 z}{\partial x^2 \partial y} + \frac{4\partial^3 z}{\partial y^3} = e^{x+2y}$

$D^3-3D^2D'+4D' = e^{x+2y}$

$A.E = m^3-3m^2+4=0 \Rightarrow \underline{m=-1,2,2}.$

$CF = f_1(y-x)+f_2(y+2x)+xf_3(y+2x).$

$PI = \frac{e^{x+2y}}{D^3-3D^2D'+4D'^3} \quad \left\{ \begin{array}{l} D=a=1 \\ D'=b=2 \end{array} \right\}$

$Z = CF+PI$

$PI = \frac{e^{x+2y}}{1-3(2)+4(8)} = \frac{e^{x+2y}}{1-6+32} \Rightarrow \frac{e^{x+2y}}{27}$

Que :- $(D^3-7DD'^2-6D'^3)z = e^{2x+3y}+x^2y+\sin(x+2y).$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$AE = m^3-7m-6=0$

$m^2-m+6=0.$

$m=-1,3,-2.$

$CF = f_1(y-x)+f_2(y-2x)$
$+f_3(y+3x).$

$$PI_1 = \frac{e^{x+y}}{D^3-7DD'^2-6D'^3} = \frac{e^{x+y}}{1-7-6} = \frac{e^{x+y}}{-12}.$$

$$PI_2 = \frac{x^2y}{D^3-7DD'^2-6D'^3} = [D^3-7DD'^2-6D'^3]^{-1}x^2y.$$

$$\Rightarrow \frac{1}{D^3\left[1-\left(\frac{7D'^2}{D^2}+\frac{6D'^3}{D^3}\right)\right]} x^2y$$

$$\Rightarrow \frac{1}{D^3}\left[1-\left(\frac{7D'^2}{D^2}+\frac{6D'^3}{D^3}\right)\right]^{-1} x^2y.$$

$$\Rightarrow \frac{1}{D^3}\left(1+\frac{7D'^2}{D^2}+\frac{6D'^3}{D^3}\right)x^2y \Rightarrow \frac{1}{D^3}(x^2y+0+0)$$

$$\Rightarrow \frac{x^5y}{3\times4\times5} \Rightarrow \underline{\frac{x^5y}{60}} \left\{\frac{1}{D} = \text{integration}\right\}.$$

$$\underline{PI_3} - \frac{\sin(x+2y)}{D^3-7DD'^2-6D'^3} = \frac{\sin(x+2y)}{D^2D-7DD'^2-6D'^2D'}$$

$$\{a^2=1,\ b^2=4\}\ (D^2=-a^2,\ D'^2=-b^2,\ DD'=-ab)$$

$$\Rightarrow \frac{\sin(x+2y)}{-1(D)+14D+24D'} = \frac{\sin(x+2y)}{-D+38D'} = \frac{D\sin(x+2y)}{-D^2+38DD'}$$

$$\Rightarrow \frac{\cos(x+2y)}{1+38(-2)} \Rightarrow \frac{\cos(x+2y)}{-75}\ \text{Ans}$$

Q. $(D^2-4D'^2)z = \cos 2x \cos 3y$ PI $= \dfrac{\cos 2x \cos 3y}{D^2-4D'^2}$

$AE = m^2-4=0$

$m = \mp 2.$

$\Rightarrow \dfrac{\cos 2x\cos 3y}{-4-4(-9)} = \dfrac{\cos 2x \cos 3y}{32}.$

$CF = f_1(y-2x) + f_2(y+2x)$

$z = CF + PI$ Ans

Ques. solve $(D^3+D^2D'-DD'^2-D'^3)z = e^x \cos 2y$

$AE = m^3-m^2-m-1=0$, $m = 1,-1,-1.$

$CF = f_1(y+x) + f_2(y-x) + x f_3(y-x)$

$\left\{ \begin{array}{l} e^x \\ \Rightarrow a=1 \\ b=0 \end{array} \right\}$

$PI = \dfrac{e^x \cos 2y}{D^3+D^2D'-DD'^2-D'^3} \Rightarrow e^x \dfrac{1}{(D+1)^3+(D+1)^2(D')-(D+1)D'^2-D'^3} \cdot \cos 2y$

$\Rightarrow e^x \left( \dfrac{\text{Real part of } e^{i2y}}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3} \right)$ $\left( \begin{array}{l} D = D+a \\ D' = D'+b \end{array} \right)$

$\Rightarrow e^x \left( \dfrac{\text{Real part of } e^{i2y}}{1+2i+4+8i} \right)$ $\left\{ \begin{array}{l} D=a=0 \\ D=b=2i \end{array}, e^{i2y} \right\}$

$\Rightarrow \dfrac{e^x}{5} \cdot \left( \dfrac{\text{Real part of } e^{i2y}}{1+2i} \right) \times \dfrac{(1-2i)}{(1-2i)}$

$\Rightarrow \dfrac{e^x \quad (1-2i) \cdot \text{Real part of } e^{i2y}}{5 \times (1+4)}$

$\Rightarrow \dfrac{e^x}{25} (1-2i) (\cos 2y + i \sin 2y)$

$PI \rightarrow \dfrac{e^x}{25} \cos 2y + 2 \sin 2y$ Ans $\boxed{z = CF + PI}$ Answer

Q:- $(D^2+DD'-6D'^2)z = \cos(2x+y)$

A: $PI = \dfrac{\cos(2x+y)}{D^2+DD'-6D'^2}$ $\left\{ \begin{array}{l} D=2 \\ D'=1 \\ D^2=-a^2=-4 \\ D'^2=-a^2=-1 \\ DD'=-ab=-2 \end{array} \right\}$

$PI = \dfrac{\cos(2x+y)}{-4-2+6} = 0$ ? then differentiate

$PI = x\left[\dfrac{\cos(2x+y)}{2D+D'}\right] \Rightarrow PI = x\left[\dfrac{D\cos(2x+y)}{2D^2+DD'}\right]$

$PI = x\left(\dfrac{-2\sin(2x+y)}{2(-4)-2}\right) \Rightarrow \dfrac{x}{+\cancel{10}\,5}\cancel{+2}\sin(2x+y)$

$PI = +\dfrac{x}{5}\sin(2x+y)$ Answer.

General rule:-

If RHS of any ~~fnd~~ fn$^g$ $f(x,y)$

$PI = \dfrac{F(x,y)}{f(D,D')}$

Resolve $f(D,D')$ into Partial fraction considering $f(D,D')$ as a $f^n$ of D alone,

$PI = \dfrac{1}{D-mD'}f(x,y) = \int F(x, \underbrace{c-mx}_{y \text{ replace}}).dx$

where ~~of~~ c is replaced by $y+mx$ after integration

Que:- $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

$AE = m^2 - m - 2 = 0 \Rightarrow m = -1, 2.$

$CF = f_1(y-x) + f_2(y+2x)$.

$PI = \dfrac{(y-1)e^x}{D^2 - DD' - 2D'^2} \Rightarrow \dfrac{(y-1)e^x}{(D-2D')(D+D')}$

$PI = \left(\dfrac{1}{D+D'}\right) \int e^x (c-2x-1)\,dx$ $\left\{ \begin{array}{l} y = c - 2x \\ m = 2 \\ \therefore \underline{D - 2D'}. \end{array} \right.$

$PI = \dfrac{1}{D+D'}\left[(c-2x-1)e^x + 2e^x\right] = \dfrac{1}{D+D'}(ce^x - 2e^x \cdot x - e^x + 2e^x)$

$PI = \dfrac{1}{D+D'}\left[ce^x - 2xe^x + e^x\right]$

$PI = \dfrac{1}{D+D'}\left((y+2x)e^x - 2xe^x + e^x\right)$

$PI = \dfrac{1}{D+D'}\left[ye^x + e^x\right] = \dfrac{1}{D-(-D)'}(y+1)e^x.$

~~$PI = \int e^x(c+x+1)dx$~~ $PI = \int e^x(c+x+1)\,dx$

$PI = ((c+x+1)e^x - e^x)$

$PI = ce^x + xe^x + e^x - e^x$

$PI = (c+x)e^x$ $\left\{ \begin{array}{l} m = -1 \\ c = y + mx \\ c = y - x) \end{array} \right.$

$PI = (y - \cancel{x} + \cancel{x})e^x$

$\boxed{PI = ye^x}$.

Que $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{6\partial^2 z}{\partial y^2} = y\cos x$

$\Rightarrow (D^2 + DD' - 6D'^2)z = y\cos x$

AE = $m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0 \Rightarrow m = -3, 2.$

CF = $f_1(y-3x) + f_2(y+2x).$

PI = $\frac{y\cos x}{D^2 + DD' - 6D'^2} \Rightarrow \frac{y\cos x}{(D+3D')(D-2D')}$

PI = $\frac{1}{(D+3D')}\int \frac{(c-2x)\cos x\, dx}{Dx}$ $\left\{ \frac{1}{D-2D'}, \; y = \begin{matrix} c - mx \\ c - 2x \end{matrix}, \; m = 2 \right.$

PI = $\frac{1}{(D+3D')}\left[(c-2x)\sin x - 2\cos x\right].$

$= \frac{1}{D+3D'}\left[y\sin x - 2\cos x\right]$

$\Rightarrow \frac{1}{D-(-3D')}\left[y\sin x - 2\cos x\right]$ $\left\{ \begin{matrix} m = -3 \\ y = c - mx \\ y = c + 3x \end{matrix} \right\}$

$\Rightarrow \int\left[(c+3x)\sin x - 2\cos x\right] dx$

$\Rightarrow -(c+3x)\cos x + 3\sin x - 2\sin x$

$\Rightarrow -(y - \cancel{3x} + \cancel{3x})\cos x + 3\sin x - 2\sin x$

$\Rightarrow \underline{-y\cos x + \sin x}$

Hence, $z = CF + PI$

$$\underline{z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x}$$

Answer !!