# Chapter – 2

# Fourier Series

The development of the techniques of Fourier Series has a long history involving a great many individuals and the investigation of many different physical phenomena. By the examination of the motion of a vibrating string in 1748, L. Euler started the modern history of the Fourier analysis. But Jean Papliste Joseph, a French mathematician made a substantial contribution to the development of Fourier Series. His work led to the development, to a big step forward.

Fourier was born on March 21, 1768, in Auxerre, France. His revolutionary discoveries, although not completely appreciated during his own life time, have had a major impact on the development of mathematics and still are great importance in an extremely wide range of scientific and engineering disciplines.

Besides a mathematician, he was also an active politician. After the French revolution he was very active and he became an associate of Napolean Bonoparte, accompanied him on his expeditions to Egypt and in 1802 was appointed by Bonoparte to the position of *perfect* of a region of France. While serving as *perfect* Fourier developed his ideas on trigonometric series.

The physical motivation for Fourier's work was the phenomenon of heat propagation and diffusion. By 1807, Fourier had completed a substantial portion of his work on heat diffusion and on December 21, 1807, he presented these results to the Institute de France. In his work, Fourier had found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body.

Four distinguished mathematicians and scientists were appointed to examine the 1807 paper of Fourier. Three of four, S.F. Laxrox, G. Monge and P.S. Laplace were in favour of the publication of the paper, but the Fourth, J.L. Lagrange remained adament in the rejection of trigonometric series that he had put forth 50 years earlier because of this Fourier's paper never appeared.

After several other attempts to have his work accepted and published by the Institute de France, Fourier undertook the writing of another version of his work. This book was published in 1822, 15 years after Fourier had first presented his results to the Institute de France.

Towards the end of his life Fourier received some of the recognitions he deserved but the most significant tribute to him has been the enormous impact of his work on so many disciplines within the field of Maths, Science and Engineering.

Before we attempt to study the Fourier Series the following basic definitions related to functions, limits and continuity become indispensable.

## ❑ PERIODIC FUNCTIONS

**A function $f(x)$ is said to have a period T if for all $x$, $f(x+T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the period of $f(x)$.**

### ■ EXAMPLE 1 ■

We know that $f(x) = \sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ Therefore the function has periods $2\pi$, $4\pi$, $6\pi$, etc. However, $2\pi$ is the least value and therefore is the period of $f(x)$.

Similarly $\cos x$ is a periodic function with the period $2\pi$ and $\tan x$ has period $\pi$.

### ■ EXAMPLE 2 ■

The period of $\sin nx$ and $\cos nx$ where $n$ is a positive integer is $\frac{2\pi}{n}$.

*i.e.*, the period of $\sin 2x$ is $\pi$, $\sin 3x$ is $\frac{2\pi}{3}$, etc.

**A constant** has any positive number as a period.

## ❑ LIMIT OF A FUNCTION

A function $f(x)$ is said to tend to a limit '$l$' as $x$ tends to '$a$' if to each given $\varepsilon > 0$, there exists a positive number $\delta$ such that $|f(x) - l| < \varepsilon$ when $0 < |x - a| < \delta$.

This is denoted by $\underset{x \to a}{\text{Lt}} f(x) = l$.

## ❑ LEFT-HAND AND RIGHT-HAND LIMITS

$f(x)$ is said to tend to $l$ as $x$ tends to '$a$' through values less than $a$, if to each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ when $a - \delta < x < a$ and is denoted by $f(a-0) = \underset{x \to a-0}{\text{Lt}} f(x)$ is called right-hand limit.

Similarly if $f(x)$ tends to $l$ as $x$ tends to '$a$' through values which are greater than $a$, if there exist $\delta > 0$ such that $|f(x) - l| < \varepsilon$ when $a < x < a + \delta$, then $f(x)$ is said to tend to $l$ from the right and is denoted by $f(a+0) = \underset{x \to a+0}{\text{Lt}} f(x)$ is called right-hand limit.

To find the left-hand limit *i.e.*, $f(a-0)$ we first put $x = a - h$ in $f(x)$ and then take the limit as $h \to 0$.

Thus $\quad f(a-0) = \underset{h \to 0}{\text{Lt}}\ f(a-h).$

To find $f(a+0)$ we first put $x = a + h$ in $f(x)$ and then take the limit as $h \to 0$..

Thus $\quad f(a+0) = \underset{h \to 0}{\text{Lt}}\ f(a+h)$

## EXAMPLE 1

*If $f(x) = x \sin \frac{1}{x}$ find $(0-0)$ and $f(0+0)$.*

**Solution**

Here $f(0-0)$ is left hand limit and $f(0+0)$ is right hand limit.

Now, $$f(0-0) = \underset{h \to 0}{\text{Lt}}\ (0-h) \sin\left(\frac{1}{0-h}\right)$$

$$= \underset{h \to 0}{\text{Lt}}\ h \sin \frac{1}{h} = 0$$

Now, $$f(0+0) = \underset{h \to 0}{\text{Lt}}\ (0+h) \sin\left(\frac{1}{0+h}\right)$$

$$= \underset{h \to 0}{\text{Lt}}\ h \sin \frac{1}{h} = 0.$$

## EXAMPLE 2

*If $f(x) = 2^{\frac{1}{x-1}}$ find $f(1-0)$ and $f(1+0)$.*

**Solution**

$$f(1-0) = \underset{h \to 0}{\text{Lt}}\ 2^{\frac{1}{1-h-1}} = \underset{h \to 0}{\text{Lt}}\ 2^{-\frac{1}{h}}$$

(Replace $x$ by $1-h$) $\quad = 2^{-\infty} = 0$

$$f(1+0) = \underset{h \to 0}{\text{Lt}}\ 2^{\frac{1}{1+h-1}} = \underset{h \to 0}{\text{Lt}}\ 2^{\frac{1}{h}}$$

$$= 2^{\infty} = \infty$$

## EXAMPLE 3

*Find $f(a-0)$ and $f(a+0)$ for the function*

$$f(x) = \begin{cases} \frac{x^2}{a} - a & \text{for } 0 < x < a \\ 0 & \text{for } x = a \\ a - \frac{a^3}{x^2} & \text{for } x > a \end{cases}$$

**Solution**

$$f(a-0) = \underset{h \to 0}{\text{Lt}}\left[\frac{(a-h)^2}{a} - a\right] = \frac{a^2}{a} - a = 0$$

[Replace $x$ by $a-h$]

$$f(a+0) = \underset{h\to 0}{\text{Lt}}\left[a - \frac{a^3}{(a+h)^2}\right]$$

$$= a - \frac{a^3}{a^2} = 0$$

**EXAMPLE 4**

*Find $f(a-0)$ and $f(a+0)$ for the function*

$$f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x \geq a \end{cases}$$

**Solution**

$$f(a-0) = \underset{h\to 0}{\text{Lt}} -1 = -1$$

$$f(a+0) = \underset{h\to 0}{\text{Lt}} 1 = 1$$

**EXAMPLE 5**

*Find $f(0-0)$ and $f(0+0)$ for the function*

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

**Solution**

$$f(0-0) = \underset{h\to 0}{\text{Lt}} -\pi = -\pi$$

$$f(0+0) = \underset{h\to 0}{\text{Lt}} (0+h) = 0$$

## CONTINUOUS FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if given $\varepsilon > 0$, however small, we can find a number $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ when $|x - a| < \delta$ and is denoted by $\underset{x\to a}{\text{Lt}} f(x) = f(a)$. *i.e.,* $\underset{x\to a}{\text{Lt}} f(x)$ exists if $\underset{x\to a-0}{\text{Lt}} f(x)$ and $\underset{x\to a+0}{\text{Lt}} f(x)$ exists and are equal.

$f(x)$ is said to be continuous in an interval $(a, b)$ if it is continuous at every point of the interval.

## DISCONTINUOUS FUNCTION

A function $f(x)$ is said to be discontinuous at a point if it is not continuous at that point.

## PIECEWISE CONTINUOUS FUNCTION

A function $f(x)$ is said to be piecewise continuous in an interval if (*i*) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and (*ii*) the limits of $f(x)$ as $x$ approaches the end points of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has atmost a finite number of finite discontinuities. An example of a piecewise continuous function is shown in fig.2. The values of the left hand limit $f(x-0)$ and right hand limit $f(x+0)$ at the point $x$ are as indicated in figure 2. In fig.1, the function has discontinuity at $x = a$. If $f(x)$ is expressed as a Fourier series, the value of the series at $x = a$ is $\frac{1}{2}$ [BE + CE].

***NOTE : The value of the Fourier series at $x = \lambda$ or $x = 2\pi + \lambda$ (both are points of discontinuity) is $\frac{1}{2}[f(\lambda+0)+f(2\pi+\lambda-0)]$.***

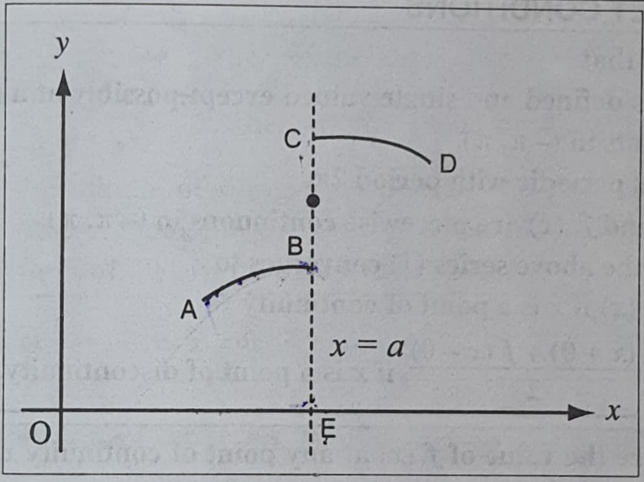



*Fig. 1*

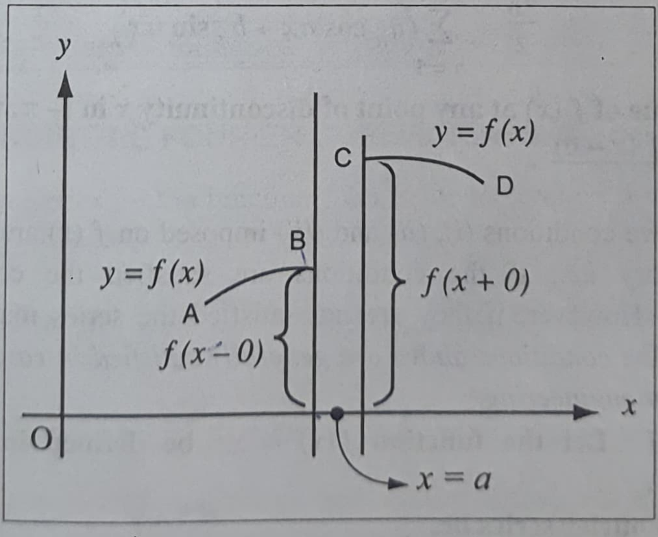



*Fig. 2*

## ❑ FOURIER SERIES

If $f(x)$ is a periodic function and satisfies Dirichlet conditions (to be described in subsequent article), then it can be represented by an infinite series called **Fourier Series** as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx))$$

**where $a_0$, $a_n$ and $b_n$ are called FOURIER COEFFICIENTS.**

Equation (1) is called *trigonometric form of Fourier Series* as we have another form called *complex or exponential form of Fourier Series* which will be studied at the end of this chapter.

## ❑ DIRICHLET CONDITIONS

**Suppose that**

**(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(-\pi, \pi)$.**

**(ii) $f(x)$ is periodic with period $2\pi$.**

**(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-\pi, \pi)$.**

**Then the above series (1) converges to**

**(a) $f(x)$ if $x$ is a point of continuity**

**(b) $\frac{f(x+0) + f(x-0)}{2}$ if $x$ is a point of discontinuity.**

**Therefore the value of $f(x)$ at any point of continuity $x$ in $(-\pi, \pi)$ is given by**

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

**The value of $f(x)$ at any point of discontinuity $x$ in $(-\pi, \pi)$ is given by $\frac{f(x+0) + f(x-0)}{2}$.**

The above conditions *(i)*, *(ii)* and *(iii)* imposed on $f(x)$ are sufficient but not necessary *i.e.,* if the conditions are satisfied the convergence is guaranteed. However, if they are not satisfied the series may or may not converge. *The conditions above are generally satisfied in cases which arise in science or engineering.*

*NOTE 1* : **Let the function $f(x) = x^2$ be defined in the interval $<x<\pi$.**

**Let its Fourier series be,**

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad \ldots (1)$$

**At $x = 0$ ('0' lies within $-\pi < x < \pi$ and hence it is a point of ontinuity) the sum of the Fourier series (1) equal to $f(0)$ (which is zero ere).**

$$i.e., \; f(0) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

**NOTE 2 : Suppose $f(x)$ is defined in the interval $-\pi < x < \pi$, then at the points of discontinuities (at $x = \pi$ or $x = -\pi$), the sum of the Fourier series is equal to the arithmetic mean of the value of $f(x)$ at $x = \pi$ and $x = -\pi$.**

*i.e.*, **Sum of the series at $x = \pi$ is equal to $\dfrac{f(\pi) + f(-\pi)}{2}$**

**NOTE 3 : Suppose the function $f(x)$ is defined by $f(x) = \begin{cases} -\pi & \text{in} -\pi < x < 0 \\ x & \text{in } 0 < x < \pi \end{cases}$, then at the point of discontinuity $x = 0$ (which is in the middle of the given interval) the sum of the Fourier series converges (or equals) to the average value of the right hand limit and left hand limit of the given function at $x = 0$.**

*i.e.*, **Sum of the series at $x = 0$ is equal to $\dfrac{f(0+0) + f(0-0)}{2}$**

**Here sum of the series $= \dfrac{0 - \pi}{2} = \dfrac{-\pi}{2}$**

**NOTE 4 : If $x$ is a point of continuity, then the sum of the Fourier series is equal to $f(x)$ *i.e.*,**

**Sum of the series $= f(x)$**

## ❑ DETERMINING THE FOURIER COEFFICIENTS $a_0$, $a_n$ AND $b_n$

The Fourier Series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx \\ b_n &= \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx \end{aligned}\right\} \quad \ldots (1)$$

The values of $a_0$, $a_n$, $b_n$ are known as **Euler's Formulae.** To establish these formulae, the following results will be required.

1. $\int_c^{c+2\pi} \cos nx\, dx = 0\ (n \neq 0)$

2. $\int_c^{c+2\pi} \sin nx\, dx = 0\ (n \neq 0)$

3. $\int_c^{c+2\pi} \cos mx \cos nx\, dx = 0\ (m \neq n)$

4. $\int_c^{c+2\pi} \cos^2 nx\, dx = \pi\ (n \neq 0)$

5. $\int_c^{c+2\pi} \sin mx \cos nx\, dx = 0\ (m \neq n)$

6. $\int_c^{c+2\pi} \sin nx \cos nx\, dx = 0$

7. $\int_c^{c+2\pi} \sin mx \sin nx\, dx = 0\ (m \neq n)$

8. $\int_c^{c+2\pi} \sin^2 nx = \pi\ (n \neq 0)$

## PROOF OF EULER FORMULAE

Let $f(x)$ be represented in the interval $(c, c + 2\pi)$ by the Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{... (1)}$$

To find the coefficients $a_0$, $a_n$, $b_n$, we assume that the series (1) can be integrated term by term from $x = c$ to $x = c + 2\pi$.

To find $a_0$, integrate both sides of (1) from $x = c$ to $x = c + 2\pi$. Then,

$$\int_c^{c+2\pi} f(x)\, dx = \frac{1}{2} a_0 \int_c^{c+2\pi} dx + \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{a_0}{2} [x]_c^{c+2\pi} + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nx\, dx$$

$$= \frac{1}{2} a_0 [(c + 2\pi - c) + 0 + 0] \quad \text{[by (1) and (2)]}$$

$$= a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$$

To find $a_n$, **multiply** each side of (1) by cos $nx$ and integrate from $x = c$ to $x = c + 2\pi$. Then

$$\int_c^{c+2\pi} f(x) \cos nx\, dx = \frac{1}{2} a_0 \int_c^{c+2\pi} \cos nx\, dx + \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx\, dx$$

$$+ \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx\, dx$$

$$= \frac{1}{2} a_0 (0) + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nx \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nx \cos nx\, dx$$

$$= 0 + \pi a_n + 0 \quad \text{[by integrals (1), (4) and (6)]}$$

$$\therefore a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx$$

To find $b_n$, multiply each side of (1) by sin $nx$ and integrate from $x = c$ to $x = c + 2\pi$. Then

$$\int_c^{c+2\pi} f(x) \sin nx\, dx = \frac{a_0}{2} \int_c^{c+2\pi} \sin nx\, dx +$$

$$+ \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) \sin nx\, dx + \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} b_n \sin nx \right) \sin nx\, dx$$

$$= 0 + 0 + \pi b_n \quad \text{[by integrals (2), (6) and (8)]}$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx$$

***COROLLARY 1 :*** **Putting $c = 0$, the interval becomes $0 < x < 2\pi$, and the formula (1) reduces to**

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

*COROLLARY 2 :* **Putting $c = -\pi$, the intervals becomes $-\pi < x < \pi$ and the formula (1) becomes**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

*NOTE :* **The following two results are very useful for this chapter and solving of boundary value problems.**

$$\cos n\pi = (-1)^n, \text{ when '}n\text{' is an integer.}$$
$$\sin n\pi = 0, \text{ when '}n\text{' is an integer.}$$

**EXAMPLE 1**

*Expand $f(x) = (\pi - x)^2$ in $(-\pi, \pi)$.*

**Solution**

We know that a Fourier series for the function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx + \sum_{1}^{\infty} b_n \sin x \quad \ldots (A)$$

Here, $f(x) = (\pi - x)^2$ in $(-\pi, \pi)$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2\,dx = \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{(2\pi)^3}{-3}\right] = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{8\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(0) - \left(\frac{-4\pi\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi\cos n\pi}{n^2}\right]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n} \qquad \dots (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi-x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0 - 0 + \frac{2\cos n\pi}{n^3}\right) - \left(\frac{-(2\pi)^2\cos n\pi}{n} - 0 + \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{2\cos n\pi}{n^3}\right) - \left(\frac{-4\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2\cos n\pi}{n}\right]$$

$$\boxed{b_n = \frac{4\pi}{n}(-1)^n} \qquad \dots (3)$$

From (1), (2) and (3), we get

$$a_0 = \frac{8\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = \frac{4\pi}{n}(-1)^n$$

Substituting these values in (A), we get

$$f(x) = \frac{4\pi^2}{3} + \sum_{1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{1}^{\infty}\frac{4\pi}{n}(-1)^n \sin nx$$

$$= \frac{4\pi^2}{3} + 4\sum_{1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 4\pi\sum_{1}^{\infty}\frac{(-1)^n \sin nx}{n}$$

## EXAMPLE 2

***Obtain the Fourier Series for***

$$f(x) = 1 + x + x^2 \text{ in } (-\pi, \pi).$$

*Deduce that* $\frac{1}{1^2} + \frac{1}{2^2} + \dots\dots = \frac{\pi^2}{6}$.

**Solution**

The Fourier series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx + \sum_{1}^{\infty} b_n \sin nx \qquad \dots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x+1)\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3}+\frac{\pi^2}{2}+\pi+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\pi\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{2\pi^3}{3}+2\pi\right] \quad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x+1)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2+x+1)\left(\frac{\sin nx}{n}\right)-(2x+1)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi+1)\cos n\pi}{n^2}-\frac{(-2\pi+1)\cos n\pi}{n^2}\right]$$

$$= \frac{1}{n^2\pi}\left[4\pi\cos n\pi\right] = \frac{4}{n^2}(-1)^n \quad \ldots(3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2+x+1)\left(\frac{-\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(\pi^2+\pi+1)\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}+\frac{(\pi^2-\pi+1)\cos n\pi}{n}-\frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{\cos n\pi}{n\pi}\left[-\pi^2-\pi-1+\pi^2-\pi+1\right]$$

$$b_n = \frac{(-1)^n}{n\pi}\left[-2\pi\right] = \frac{2(-1)^{n+1}}{n} \quad \ldots(4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = x^2+x+1 = \frac{1}{2\pi}\left(\frac{2\pi^3}{3}+2\pi\right)+\sum_{1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \sum_{1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx \quad \ldots(5)$$

Here $x=\pi$ is the end point of the interval. Therefore, the sum of the Fourier series at $x=\pi$ is equal to the average of the values of $f(x)$ at $x=\pi$ and at $x=-\pi$.

$\therefore$ Putting $x = \pi$ in (5), we get the Fourier series,

$\frac{\pi^2}{3} + 1 + 4 \sum_{1}^{\infty} \frac{1}{n^2}$ converges to average of $f(\pi) + f(-\pi)$

$$i.e., \quad \frac{\pi^2}{3} + 1 + 4 \sum_{1}^{\infty} \frac{1}{n^2} = \frac{f(\pi) + f(-\pi)}{2}$$

$$= \frac{(\pi^2 + \pi + 1) + (\pi^2 - \pi + 1)}{2}$$

$$= \pi^2 + 1$$

$$i.e., \quad \frac{\pi^2}{3} + 1 + 4 \sum_{1}^{\infty} \frac{1}{n^2} = \pi^2 + 1$$

$$\Rightarrow 4 \sum_{1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$= \frac{2\pi^2}{3}$$

$$\therefore \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

**EXAMPLE 3**

*Express $f(x) = x \sin x$ as a Fourier Series in $0 \le x \le 2\pi$*

*[Apr. 91, 87, Nov. 91]*

**Solution**

We know that a Fourier Series for the function $f(x)$ in the interval $[0, 2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{...(A)}$$

Here $f(x) = x \sin x$

Now $a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x \sin x\, dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1\cdot(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi] = -2 \quad [\because \sin 2\pi = 0, \cos 2\pi = 1] \quad \ldots(1)$$

$$\therefore \quad \boxed{a_0 = -2} \quad \ldots(1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$\left[\because \cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]\right]$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

(Since $\sin 2(n+1)\pi = 0$, $\sin 2(n-1)\pi = 0$ and $\cos 2(n+1)\pi = 1$, $\cos 2(n-1)\pi = 1$, whether $n$ is odd or even.)

$$a_n = -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \text{ provided } n \neq 1 \quad \ldots(2)$$

when $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\, dx$$

$$[\because \sin 2x = 2\sin x\cos x]$$

$$= \frac{1}{2\pi}\left\{x\left(-\frac{\cos 2x}{2}\right) - 1\cdot\left(-\frac{\sin 2x}{4}\right)\right\}_0^{2\pi} = \frac{1}{2\pi}[-\pi] = -\frac{1}{2} \quad \ldots(3)$$

$$\therefore \quad \boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\cdot\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$\therefore \quad \boxed{b_n = 0 \text{ provided } n \neq 1} \qquad \ldots (4)$$

when $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1 \cdot \left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \pi \qquad \ldots (5)$$

From (A) we get

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

From (1), (2), (3), (4) and (5) we get

$$a_0 = -2, \qquad a_n = \frac{2}{n^2 - 1}, \ (n \neq 1) \quad a_1 = -\frac{1}{2}, \qquad b_n = 0, \ b_1 = \pi$$

Substituting these values in (A) we get,

$$f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \cos nx$$

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2 - 1}\cos 2x + \frac{2}{3^2 - 1}\cos 3x + \ldots$$

**EXAMPLE 4**

***Express $f(x) = (\pi - x)^2$ as a Fourier Series of period $2\pi$ in the interval $0 < x < 2\pi$ Hence deduce the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$***

*[Nov. 2000, Apr. 89, Mech]*

**Solution**

We know that a Fourier Series for the function $f(x)$ in the interval $[0, 2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (A)$$

Here $f(x) = (\pi - x)^2$ in $[0, 2\pi]$

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\, dx$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$\therefore \quad \boxed{a_0 = \frac{2\pi^2}{3}} \quad \ldots (1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi \cos 2n\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{4}{n^2} \qquad [\because \cos 2n\pi = 1]$$

$$\boxed{a_n = \frac{4}{n^2}} \quad \ldots (2)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2 \cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = 0 \qquad (\because \cos 2n\pi = 1)$$

$$\therefore \quad \boxed{b_n = 0} \quad \ldots (3)$$

From (1), (2) and (3) we get $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2}$, $b_n = 0$.

Substituting these values in (A) we get,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx$$

$$f(x) = (\pi - x)^2 = \frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right] \quad \ldots (4)$$

Here 0 is a point of discontinuity which is an end point of the interval $0 < x < 2\pi$.

$\therefore$ The value of Fourier series at $x = 0$ is equal to the average value of $f(x)$ at the end points.

*i.e.,* The series converges to $\frac{f(0) + f(2\pi)}{2}$.

Putting $x = 0$ in (4), we get the Fourier series,

$$\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right] = \frac{f(0) + f(2\pi)}{2}$$

$$= \frac{\pi^2 + \pi^2}{2} = \pi^2$$

*i.e.,* $$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{3\pi^2 - \pi^2}{12}$$

*i.e.,* $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

## EXAMPLE 5

***Find the Fourier Series for the function $f(x) = e^x$ defined in $(-\pi, \pi)$.***

*[Nov. 91, 86, Civil, Apr. 89, ECE]*

**Solution**

We know that a Fourier Series for the function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(A)$$

Here $f(x) = e^x$

Now $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$$

$$= \frac{1}{\pi}[e^x]_{-\pi}^{\pi} = \frac{1}{\pi}[e^{\pi} - e^{-\pi}]$$

$$\therefore \boxed{a_0 = \frac{2}{\pi}\sin h\pi} \qquad \ldots(1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2 + n^2}(1\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{(1^2 + n^2)\pi}[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi]$$

$$= \frac{\cos n\pi\,(e^{\pi} - e^{-\pi})}{\pi(1 + n^2)} = \frac{2(-1)^n \sin h\pi}{\pi(1 + n^2)} \qquad \ldots(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi(1+n^2)}[-ne^{\pi}\cos n\pi + ne^{-\pi}\cos n\pi]$$

$$= \frac{n(-1)^n[e^{-\pi}-e^{\pi}]}{\pi(1+n^2)} = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)} \quad \ldots(3)$$

Now we have

$$a_0 = \frac{2\sinh\pi}{\pi}, \quad a_n = \frac{2(-1)^n\sinh\pi}{\pi(1+n^2)}, \quad b_n = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)}$$

Substituting these values in (A) we get,

$$f(x) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos nx - \sum_{n=1}^{\infty} n\frac{(-1)^n}{1+n^2}\sin nx\right]$$

**■ EXAMPLE 6 ■**

***Determine the Fourier Series expansion of $x + x^2$ in the interval $(-\pi, \pi)$ and hence deduce the sum of series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$*** *[Anna Univ. Nov. 2001]*

**● Solution**

We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(A)$$

Here $f(x) = x + x^2$ in $[-\pi, \pi]$

Now $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2}{3}\pi^2$$

$$\therefore \boxed{a_0 = \frac{2\pi^2}{3}} \quad \ldots(1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1+2\pi)\cos n\pi}{n^2} - \frac{(1-2\pi)\cos n\pi}{n^2}\right]$$

$$= \frac{\cos n\pi}{n^2\,\pi}\,[1+2\pi-1+2\pi]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n} \qquad \ldots (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(\pi+\pi^2)(-\cos n\pi)}{n} + \frac{2\cos n\pi}{n^3} + \frac{(-\pi+\pi^2)\cos n\pi}{n} - \frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{\cos n\pi}{n\pi}\,[-\pi-\pi^2-\pi+\pi^2] = -\frac{2}{n}(-1)^n \qquad \ldots (3)$$

$$\therefore \quad \boxed{b_n = -\frac{2}{n}(-1)^n} \qquad [\because \cos n\pi = (-1)^n]$$

Hence we have $a_0 = \dfrac{2\pi^2}{3}$, $a_n = \dfrac{4}{n^2}(-1)^n$, $b_n = -\dfrac{2}{n}(-1)^n$.

Substituting these values in (A), we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$$

i.e., $$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right] \qquad \ldots (4)$$

Here $x = \pi$ is a point of discontinuity which is an end point of the given interval $-\pi < x < \pi$. Therefore, the value of Fourier series at $x = \pi$ is equal to the average value of $f(-\pi)$ and $f(\pi)$.

i.e., Putting $x = \pi$ in (4), we get,

$$\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{f(\pi)+f(-\pi)}{2} = \frac{\pi+\pi^2+\pi^2-\pi}{2}$$

$$= \pi^2$$

$$\therefore \sum_{n=1}^{\infty}\frac{4}{n^2} = \pi^2-\frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

i.e.,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

**EXAMPLE 7**

*Find the Fourier Series for f (x) if*

$$\left.\begin{aligned} f(x) &= -\pi \text{ in } -\pi<x<0 \\ &= x \text{ in } 0<x<\pi \end{aligned}\right\}$$

*Deduce that* $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$ *[Apr. 86, Civil, Apr. 86, Mech]*

**Solution**

The Fourier Series for the function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad \ldots(A)$$

Now
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\,dx+\int_{0}^{\pi}f(x)\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\,dx+\int_{0}^{\pi}x\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]$$

$$\therefore \boxed{a_0 = -\frac{\pi}{2}} \quad \ldots(1)$$

Now
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nx\,dx+\int_{0}^{\pi}f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\cos nx\,dx+\int_{0}^{\pi}x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi\sin nx}{n}\right)_{-\pi}^{0}+\left\{\frac{x\sin nx}{n}-\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos n\pi - 1)\right] = \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$\boxed{\begin{aligned} \therefore a_n &= 0 \quad \text{when } n \text{ is even} \\ &= \frac{-2}{n^2\pi} \quad \text{when } n \text{ is odd} \end{aligned}} \quad \ldots (2)$$

Now $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \sin nx \, dx + \int_{0}^{\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi \cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi}{n}(-1)^n - \frac{\pi(-1)^n}{n}\right]$$

$$\boxed{b_n = \frac{1}{n}[1 - 2(-1)^n]} \quad \ldots (3)$$

Substituting (1), (2) and (3) in (A), we get

$$f(x) = \frac{-\pi}{4} + \sum_{n=1,3,5}^{\infty} \frac{-2}{n^2\pi} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n}[1-2(-1)^n] \sin nx$$

i.e., $\quad f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

$$+ 3 \sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} \quad \ldots (3a)$$

Putting $x = 0$ in (3a), we get

$$f(0) = \frac{-\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \quad \ldots (4)$$

Here 0 is a point of discontinuity which is in the middle of the interval.

Therefore, the sum of the Fourier series is equal to the average of right hand limit and left hand limit of the given function at $x = 0$.

i.e., $\quad f(0) = \frac{f(0-0) + f(0+0)}{2} = \frac{-\pi + 0 + 0 + 0}{2} = \frac{-\pi}{2} \quad \ldots (5)$

[Refer to Note 3]

From (4) and (5) we get,

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)$$

i.e., $$\frac{-\pi}{4} = -\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)$$

i.e., $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

## EXAMPLE 8

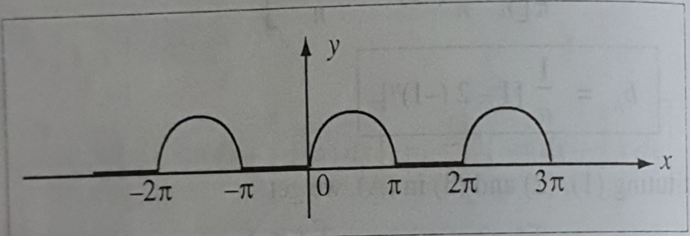
If $\left.\begin{array}{ll} f(x)=0 & -\pi \le x \le 0 \\ \quad\;\; = \sin x & 0 \le x \le \pi \end{array}\right\}$

Prove that $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$

Hence show that

(i) $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\ldots = \dfrac{1}{2}$ (ii) $\dfrac{1}{1\cdot 3}-\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\ldots = \dfrac{\pi-2}{4}$

[Anna Univ. Apr. 2001]




**Solution**

The graph of the given function is shown below:

The Fourier Series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(A)$$

Now $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin x\,dx\right] = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}$$

$$a_0 = \frac{2}{\pi} \quad \ldots(1)$$

Now $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} \sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{0}^{\pi} \{\sin(n+1)x + \sin(1-n)x\}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(-\frac{\cos(n+1)x}{n+1}\right)_0^{\pi} - \left(\frac{\cos(-n)x}{1-n}\right)_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{1}{n+1} + \frac{\cos(1-n)x\pi}{n-1} - \frac{1}{n-1}\right]$$

When $n$ is odd $\quad \cos(n+1)\pi = 1, \quad \cos(n-1)\pi = 1$

When $n$ is even $\quad \cos(n+1)\pi = -1, \quad \cos(n-1)\pi = -1.$

∴ When $n$ is odd,

$$a_n = \frac{1}{2\pi}\left[-\frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n-1} - \frac{1}{n-1}\right] = 0$$

$$\boxed{a_n = 0} \qquad \dots (2)$$

When $n$ is even

$$a_n = \frac{1}{2\pi}\left[\frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n-1} - \frac{1}{n-1}\right] = \frac{-2}{\pi(n^2-1)}$$

i.e., $$\boxed{a_n = \frac{-2}{\pi(n^2-1)}} \quad (n \neq 1) \qquad \dots (3)$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx = \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = 0$$

i.e., $$\boxed{a_1 = 0} \qquad \dots (4)$$

Now $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} \sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx \sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0$$

$\therefore \boxed{b_n = 0}$ provided $n \neq 1$ ...(5)

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

$\therefore \boxed{b_1 = \frac{1}{2}}$ ...(6)

Substituting (1), (2), (3), (4), (5) and (6) in (A), we get

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^2 - 1} \quad ...(7)$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots\right] + \frac{1}{2}\sin x$$

Putting $x = 0$ in (7) we get

*i.e.,* $$f(0) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{1}^{\infty} \frac{1}{4n^2-1}$$

[$x = 0$ is point of continuity. Hence the Fourier series converges to $f(0)$]

*i.e.,* $$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{1}^{\infty} \frac{1}{4n^2-1}$$

*i.e.,* $$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

*i.e.,* $$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

Putting $x = \frac{\pi}{2}$ in (7), we get,

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2-1}$$

i.e.,
$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2 - 1}$$

[Here $x = \frac{\pi}{2}$ is a point of continuity. Therefore, the Fourier series converges to $f\left(\frac{\pi}{2}\right)$.]

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \dots\right)$$

i.e.,
$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7}$$

## EXAMPLE 9

*If $-\pi < x < \pi$ and $\alpha$ is not an integer show that*

$$\cos \alpha x = \frac{\sin \pi\alpha}{\pi\alpha} + \sum_{n=1}^{\infty} (-1)^n \frac{2\pi \sin \pi\alpha}{\pi(\alpha^2 - n^2)} \cos nx. \qquad \text{[Apr. 92]}$$

### Solution

Here $f(x) = \cos \alpha x$.

We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \dots(A)$$

Now
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos \alpha x\,dx = \frac{1}{\pi}\left[\frac{\sin \alpha x}{\alpha}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi\alpha}[\sin \alpha\pi + \sin \alpha\pi]$$

$$\therefore \quad \boxed{a_0 = \frac{2 \sin \alpha\pi}{\alpha\pi}} \qquad \dots(1)$$

Here $\sin \alpha\pi \neq 0$, since $\alpha$ is not an integer.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos \alpha x \cdot \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\cos(\alpha + n)x + \cos(\alpha - n)x\}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(\alpha+n)x}{\alpha+n} + \frac{\sin(\alpha-n)x}{\alpha-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n} + \frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\sin(\alpha+n)\pi}{\alpha+n} + \frac{2\sin(\alpha-n)\pi}{\alpha-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(\alpha-n)\sin(\alpha+n)\pi + (\alpha+n)\sin(\alpha-n)\pi}{(\alpha+n)(\alpha-n)}\right]$$

$$= \frac{\alpha}{\pi}\left[\frac{\sin(\alpha+n)\pi + \sin(\alpha-n)\pi}{\alpha^2-n^2}\right]$$

$$= \frac{\alpha}{\pi}\left[\frac{\sin\alpha\pi\cos n\pi + \cos\alpha\pi\sin n\pi + \sin\alpha\pi\cos n\pi - \cos\alpha\pi\sin n\pi}{\alpha^2-n^2}\right]$$

$$= \frac{\alpha}{\pi}\left[\frac{2\sin\alpha\pi\cos n\pi}{\alpha^2-n^2}\right] \qquad [\because \sin n\pi = 0]$$

$$\therefore \quad \boxed{a_n = \frac{2\alpha\sin\alpha\pi(-1)^n}{\pi(\alpha^2-n^2)}} \qquad \ldots(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\alpha x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{\sin(\alpha+n)x - \sin(\alpha-n)x}{2}\right\}dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(\alpha+n)x}{\alpha+n} + \frac{\cos(\alpha-n)x}{\alpha-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\left\{\frac{-\cos(\alpha+n)\pi}{\alpha+n} + \frac{\cos(\alpha-n)\pi}{\alpha-n}\right\} - \left\{\frac{-\cos(\alpha+n)\pi}{\alpha+n} + \frac{\cos(\alpha-n)\pi}{\alpha-n}\right\}\right]$$

$$= \frac{1}{2\pi}\times 0 = 0$$

$$\boxed{\therefore b_n = 0}$$

Substituting (1), (2) and (3) in (A), we get

$$\cos\alpha x = \frac{\sin\alpha\pi}{\pi\alpha} + \sum_{n=1}^{\infty}\frac{2\alpha(-1)^n\sin\alpha\pi}{\pi(\alpha^2-n^2)}\cos nx$$

$$= \frac{\sin\alpha\pi}{\pi\alpha} + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\alpha\pi}{(\alpha^2-n^2)}\cos nx$$

**EXAMPLE 10**

*If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ by $f(x) = \frac{1}{12}(3x^2 - 6x\pi + 2\pi^2)$. Prove that $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ and hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$*

**Solution**

We know that the Fourier series for $f(x)$ in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \dots (1)$$

**To find $a_0$**

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{12\pi}\int_0^{2\pi} (3x^2 - 6x\pi + 2\pi^2)\, dx$$

$$= \frac{1}{12\pi}\left[x^3 - 3x^2\pi + 2\pi^2 x\right]_0^{2\pi} = \frac{1}{12\pi}\left[8\pi^3 - 12\pi^3 + 4\pi^3\right]$$

$$a_0 = 0 \qquad \dots (2)$$

**To find $a_n$**

$$a_n = \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{12}(3x^2 - 6x\pi + 2\pi^2) \cos nx\, dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x - 6\pi)\left(\frac{-\cos nx}{n^2}\right) + 6\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[\frac{(12\pi - 6\pi)}{n^2} + \frac{6\pi}{n^2}\right] = \frac{1}{12\pi}\left[\frac{12\pi}{n^2}\right]$$

$$\therefore a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{12}(3x^2 - 6x\pi + 2\pi^2) \cdot \sin nx\, dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(\frac{-\cos nx}{n}\right) - (6x - 6\pi)\left(\frac{-\sin nx}{n^2}\right) + 6\left(\frac{\cos nx}{n^3}\right)\right]$$

$$= \frac{1}{12\pi}\left[(12\pi^2 - 12\pi^2 + 2\pi^2)\left(\frac{-1}{n}\right) + \frac{6}{n^3} + \frac{2\pi^2}{n} - \frac{6}{n^3}\right]$$

$$b_n = 0 \quad \ldots($$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad \ldots($$

Here $x = 0$ is a point of discontinuity which is one end of the given interv Therefore, the sum of Fourier series (5) at $x = 0$ is the average value of $f(x)$ at end points of the given interval.

Putting $x = 0$ in (5), we get,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{f(0) + f(2\pi)}{2} = \frac{\frac{\pi^2}{6} + \frac{\pi^2}{6}}{2} = \frac{\pi^2}{6}$$

i.e.,

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \ldots$$

**EXAMPLE 11**

*Prove that for $-\pi < x < \pi$,*

$$\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} \ldots$$

[Nov. '9

**Solution**

$$\text{Let } f(x) = x(\pi^2 - x^2)$$

$$\text{Now } f(-x) = -x(\pi^2 - x^2) = -[x(\pi^2 - x^2)] = -f(x)$$

Hence the given function is an odd function. Therefore its Fourier series given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots($$

**To find $b_n$**

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x(\pi^2 - x^2)}{12} \sin nx\, dx$$

$$= \frac{2}{\pi} \cdot \frac{1}{12} \int_0^{\pi} x(\pi^2 - x^2) \sin nx \, dx$$

[odd function × odd function = even function]

$$= \frac{1}{6\pi}\left[(\pi^2 x - x^3)\left(\frac{-\cos nx}{n}\right) - (\pi^2 - 3x^2)\left(\frac{-\sin nx}{n^2}\right) + (-6x)\left(\frac{\cos nx}{n^3}\right) - (-6)\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{1}{6\pi}\left[\frac{-6\pi \cos n\pi}{n^3}\right] = \frac{-\cos n\pi}{n^3} = \frac{-(-1)^n}{n^3}$$

$$= \frac{(-1)^{n+1}}{n^3} \qquad \ldots (2)$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx$$

$$\therefore \frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3}$$

## EVEN AND ODD FUNCTIONS

*A function $f(x)$ is said to be even if $f(-x) = f(x)$.*

*Example :* $x^2$, $\cos x$, $\sin^2 x$, $|x|$, $x \sin x$ are even functions.

*A function $f(x)$ is said to be odd if $f(-x) = -f(x)$.*

*Example :* $x^3$, $\sin x$, $\tan^3 x$ are odd functions.

Knowing that a function $f(x)$ is even or odd, can help us avoid unnecessary work in computing the Fourier coefficients of $f(x)$ which is based on the following facts.

**NOTE 1 : The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.**

**NOTE 2 :** $\int_{-\pi}^{\pi} f(x)\, dx = 0$ **if $f(x)$ is an odd function.**

**(or)**

$\int_{-\pi}^{\pi} f(x)\, dx = 2\int_0^{\pi} f(x)\, dx$, **if $f(x)$ is an even function.**

Similarly $\int_{-l}^{l} f(x)\,dx = 0$, if $f(x)$ is an odd function.

(or)

$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$ if $f(x)$ is an even function.

NOTE 3 : When $f(x)$ is an even function, the Euler's coefficients becomes

[Anna Univ. Nov. 2001]

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

[$\because$ Both $f(x)$ and $\cos nx$ are even, the product $f(x)\cos nx$ is also even]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

[$\because$ $f(x)$ is even, $\sin nx$ is odd, the product $f(x)\sin nx$ is odd function.]

*Therefore, if a function $f(x)$ is even, its Fourier expansion contains only cosine terms.*

i.e., $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \\ a_n &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \end{aligned}}$$

NOTE 4 : If $f(x)$ is an odd function, then its Fourier expansion contains only sine terms

i.e., $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$,

where $b_n = \dfrac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$





Since $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$ [$\because$ $f(x)$ is odd]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

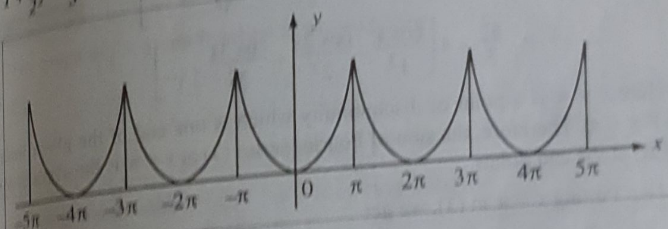
NOTE 5 : Even and odd functions cases can be used only when the function $f(x)$ is defined in $(-\pi, \pi)$ or $(-l, l)$.

Suppose $f(x) = x^2$ or $f(x) = x$ are defined in $(0, 2\pi)$ or $(0, 2l)$, we cannot use the above results.

**EXAMPLE 12**

*Obtain the Fourier series of period $2\pi$ for the function $f(x) = x^2$ in $(-\pi, \pi)$. Deduce the sum of the series*

*$1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ and $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$* [Nov. 89, Apr. 89, Civil]




**Solution**

Given $f(x) = x^2$. Here $f(x) = f(-x) = x^2$

Hence $f(x)$ is an even function.

Therefore the Fourier coefficient $b_n = 0$. Now the Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad [\because b_n = 0] \qquad \ldots (A)$$

Now we have to find $a_0$ and $a_n$.

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2 \qquad \ldots (1)$$

$$\boxed{a_0 = \frac{2}{3}\pi^2} \qquad \ldots (1)$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi \cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^n$$

i.e. $\boxed{a_n = \frac{4}{n^2}(-1)^n}$

Substituting (1) and (2) in (A), we get

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

i.e. $$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right]$$

$$\therefore x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right] \quad \dots(3)$$

Here $x = \pi$ is a point of discontinuity which is one end of the given interval $-\pi < x < \pi$. Therefore, the sum of Fourier series (3) at $x = \pi$ is the average value of $f(x)$ at the end points.

i.e., Putting $x = \pi$ in (3), we get

$$\frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2}\right] = \frac{f(-\pi)+f(\pi)}{2}$$

i.e. $$\frac{\pi^2}{3} + 4\left[\sum_1^{\infty}\frac{1}{n^2}\right] = \frac{\pi^2+\pi^2}{2} = \pi^2$$

$$4\sum_1^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\sum_1^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e. $$\frac{1}{1^2} + \frac{1}{2^2} + \dots\dots = \frac{\pi^2}{6}$$

Since $x = 0$ is a point of continuity, the Fourier series (3) converges to $f(0)$.
Putting $x = 0$ in (3), we get,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots = \frac{\pi^2}{12}$$



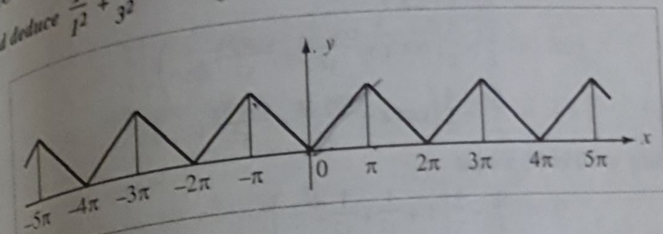


Since $x = 0$ is a point of continuity, the Fourier series (3) converges to $f(0)$.
Putting $x = 0$ in (3), we get,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots = \frac{\pi^2}{12}$$

**EXAMPLE 13**

***Obtain the Fourier Series to represent the function $f(x) = |x|, -\pi < x < \pi$ and deduce $\frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$.*** [Nov. 90, Civil, Apr. 87, Mech.]




**Solution**

The graph of the given function is shown below :

Given $$f(x) = |x|$$

$$\therefore f(-x) = |-x| = |x|$$

Hence $$f(x) = f(-x) = |x|$$

$\therefore$ The given function $f(x) = |x|$ is an even function.

$\therefore$ The Fourier coefficient $b_n = 0$.

Hence the Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad [\because b_n = 0] \quad \dots(1)$$

Now $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx \quad [\because \text{In } (0,\pi)\ |x| = x]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi \quad \dots(2)$$

i.e. $\boxed{a_0 = \pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \quad [\text{In } (0,\pi),\ |x| = x]$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

i.e.
$$\boxed{\begin{aligned} a_n &= 0, \text{ if } n \text{ is even} \\ &= \frac{-4}{n^2\pi}, \text{ if } n \text{ is odd} \end{aligned}}$$

Substituting (2) and (3) in (1) we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$

i.e.
$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) \quad \dots (4)$$

Putting $x = 0$ in (4) we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

i.e.
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

*NOTE : $x = 0$ is a point of continuity in $-\pi < x < \pi$*

**■ EXAMPLE 14 ■**

*Obtain the Fourier Series expansion of f (x) given by*

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$ *and hence deduce that*

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

*[Apr. 91 Civil, Nov. 87, Mech.]*

**● Solution**

In $-\pi \le x \le 0$, i.e., $0 \le -x \le \pi$.

$$\therefore f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

$\therefore f(x)$ is an even function.

Hence the Fourier coefficient $b_n = 0$. Now the Fourier Series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots (A)$$





Now
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0 \quad \dots (1)$$

i.e. $\boxed{a_0 = 0}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

i.e.,
$$\begin{aligned} a_n &= 0, \text{ when } n \text{ is even} \\ &= \frac{8}{\pi^2 n^2}, \text{ when } n \text{ is odd} \end{aligned} \quad \dots (2)$$

Substituting (1) and (2) in (A), we get,

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} \frac{8}{n^2\pi^2}\cos nx$$

i.e.,
$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] \quad \dots (3)$$

Putting $x = 0$ in (3) we get,

$$f(0) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots\right]$$

i.e.,
$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

i.e.,
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

*NOTE : Here $x = 0$ is a point of continuity. Therefore, the Fourier series converges to $f(0)$.*

**■ EXAMPLE 15 ■**

*If 'a' is neither zero nor an integer, find the Fourier series expansion of period $2\pi$ for the function $f(x) = \sin ax$, in $-\pi \le x \le \pi$* *[Nov. 90]*

**● Solution**

Here $f(x) = \sin ax$ is an odd function.

Hence the Fourier coefficients $a_0 = 0$, $a_n = 0$. Therefore the Fourier Series for the function. $f(x)$ is given by

2.35

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} [\cos (n-a)x - \cos (n+a)x] \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin (n-a)x}{n-a} - \frac{\sin (n+a)x}{n+a}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin (n-a)\pi}{n-a} - \frac{\sin (n+a)\pi}{n+a}\right] \qquad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi \cos a\pi - \cos n\pi \sin a\pi}{n-a} - \frac{\sin n\pi \cos a\pi}{n+a} - \frac{\cos n\pi \sin a\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} + \frac{(-1)^{n+1}\sin a\pi}{n+a}\right] \qquad [\because \sin n\pi = 0]$$

(Note that $\sin a\pi \neq 0$, since $a$ is not an integer.)

$$= \frac{(-1)^{n+1}\sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right] = (-1)^{n+1}\frac{2n \sin a\pi}{\pi(n^2-a^2)}$$

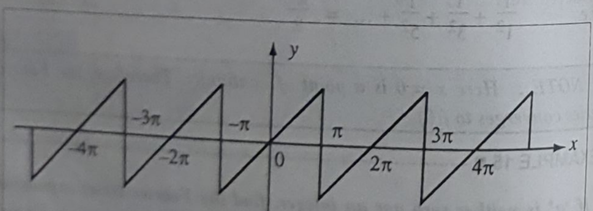
i.e., $$b_n = (-1)^{n+1}\frac{2n \sin a\pi}{\pi(n^2-a^2)} \qquad \ldots (B)$$

Substituting (B) in (A) we get

$$\therefore f(x) = \sin ax = \frac{2 \sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2-a^2}\sin nx$$

**EXAMPLE 16**

*Show that the Fourier Series for $f(x) = x$, $-\pi < x < \pi$ is given by $f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$* [Apr. 87, ECE]




**Solution**

The graph of the given function is shown as below:

Given $f(x) = x$

$\therefore f(-x) = -x$





$$f(x) = f(-x) = -f(x)$$

i.e., $\therefore f(x) = x$ is an odd function.

Hence $a_0 = 0, a_n = 0$.

Therefore the Fourier Series for the function. $f(x)$ is given by ... (1)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} - (-\pi)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{n}[-2\cos n\pi] = \frac{2}{n}(-1)^{n+1} \qquad \ldots (2)$$

i.e., $$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

Substituting (2) in (1), we get,

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$$

**EXAMPLE 17**

*Find the Fourier seires to represent the function $f(x) = |\sin x|$, $-\pi < x < \pi$* [Apr. '89, Apr. '91]

**Solution**

Here $\sin x$ is an odd function. But $|\sin x|$ is an even function. $\therefore$ The Fourier coefficient $b_n = 0$. $\therefore$ The Fourier series for

$$f(x) = |\sin x| \text{ becomes}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots (1)$$

To find $a_0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\,dx \qquad [\because \text{in } (0, \pi), |\sin x| = \sin x]$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}[1+1] = \frac{4}{\pi} \qquad \ldots (2)$$

**To find $a_n$**

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \qquad [\because f(x) \text{ is even}]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\sin x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx \qquad [\text{In } (0, \pi), |\sin x| = \sin x]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}\,[\sin (n+1)x - \sin (n-1)x]\,dx$$

[Using $2\cos A \sin B = \sin(A+B) - \sin(A-B)$]

$$= \frac{1}{\pi}\left[\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos (n+1)\pi}{n+1} + \frac{\cos (n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos n\pi \cos \pi}{n+1} + \frac{\cos n\pi \cos \pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$[\because \sin n\pi = 0]$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad [\because \cos \pi = -1]$$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi - (n+1)\cos n\pi + n - 1 - n - 1}{n^2-1}\right]$$

$$= \frac{1}{(n^2-1)\pi}\,[-2\cos n\pi - 2]$$

$$= \frac{-2}{(n^2-1)\pi}\,[1 + (-1)^n]$$

$$\therefore a_n = \begin{cases} 0, & \text{when '}n\text{' is odd} \\ \dfrac{-4}{\pi(n^2-1)}, & \text{when '}n\text{' is even provided } n \neq 1 \end{cases} \quad \ldots (3)$$

When $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} \sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{1}{2\pi}\,[-1+1] = 0 \quad \ldots (4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$





$$= \frac{2}{\pi} + \sum_{n=2,4}^{\infty} \frac{-4}{\pi(n^2-1)}\cdot \cos nx$$

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\sum_{n=2,4}^{\infty} \frac{\cos nx}{n^2-1}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right]$$

**EXAMPLE 18**

*Find the Fourier series for $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$.* *[Nov. '88, Apr. '92]*

**Solution**

Given $f(x) = |\cos x|$

$f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore$ The given function $f(x) = |\cos x|$ is an even function. Hence the Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots (1)$$

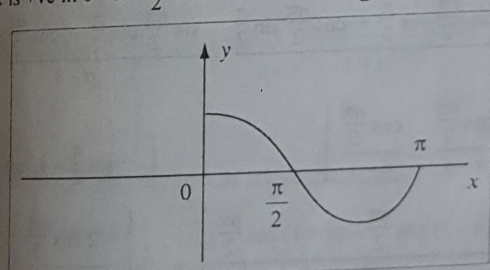
**To find $a_0$**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx \qquad [\because |\cos x| \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

[$\because \cos x$ is +ve in $0 < x < \frac{\pi}{2}$ and, $\cos x$ is –ve in $\frac{\pi}{2} < x < \pi$. See graph ]




$$= \frac{2}{\pi}\left\{[\sin x]_0^{\pi/2} + [-\sin x]_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}\,[1+1] \quad [\because \sin \pi = 0 \text{ and } \sin 0 = 0]$$

$$a_0 = \frac{4}{\pi} \quad \ldots (2)$$

**To find $a_n$**

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\cos nx\,dx \quad [\because |\cos x|\cos nx \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx\,dx + \int_{\pi/2}^{\pi} -\cos x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{1}{2}\{\cos(n+1)x+\cos(n-1)x\}\,dx - \int_{\pi/2}^{\pi}\frac{1}{2}\{\cos(n+1)x+\cos(n-1)x\,dx\}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}+\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{2\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2}+\cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n+1}+\frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2}-\cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}\right] \qquad \left[\because \sin\frac{\pi}{2}=1,\ \cos\frac{\pi}{2}=0\right]$$

$$= \frac{2}{\pi}\left[\frac{(n-1)\cos\frac{n\pi}{2}-(n+1)\cos\frac{n\pi}{2}}{n^2-1}\right] = \frac{2}{\pi}\left[\frac{-2\cos\frac{n\pi}{2}}{n^2-1}\right]$$

$$= \frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)} \quad [\text{provided } n\neq 1] \qquad \ldots(3)$$





When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\cos x|\cos x\,dx = \frac{2}{\pi}\int_0^{\pi}|\cos x|\cos x\,dx$$

$[\because |\cos x|\cos x$ is an even function $]$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2 x\,dx - \int_{\pi/2}^{\pi}\cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)_0^{\pi/2} - \left(\frac{x}{2}+\frac{\sin 2x}{4}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}-\frac{\pi}{2}+\frac{\pi}{4}\right] \qquad \ldots(4)$$

$$a_1 = 0$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cdot\cos nx$$

$$\therefore |\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\cos\frac{n\pi}{2}\cos nx}{(n^2-1)}$$

$a_0 = \frac{4}{\pi}$ 4/15

$a_n = \frac{-4\cos\left(\frac{n\pi}{2}\right)}{\pi(n^2-1)}$

$a_1 = 0$

**EXAMPLE 19**

*Prove that* $\sinh ax = \frac{2}{\pi}\left[\sinh a\pi \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot n\sin nx}{n^2+a^2}\right]$ *in* $(-\pi, \pi)$.

*[Nov. '87, Apr. '89]*

**Solution**

Let $f(x) = \sinh ax$. Clearly $\sinh ax$ is an odd function.

For $f(x) = \sinh ax = \frac{e^{ax}-e^{-ax}}{2}$.

$$f(-x) = \frac{e^{-ax}-e^{ax}}{2} = -\left[\frac{e^{ax}-e^{-ax}}{2}\right] = -f(x)$$

$\therefore$ Fourier series for $f(x)$ in $(-\pi, \pi)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ... (1)

$[\because f(x)$ is an odd function. Its Fourier coefficients $a_0$ and $a_n$ are zero$]$

**To find $b_n$**

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin h\,ax \sin nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\int_{0}^{\pi} \sin h\,ax\sin nx\,dx$$

[$\because$ odd function $\times$ odd function = even function]

$$= \frac{2}{\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{2}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} e^{ax}\sin nx\,dx - \int_0^{\pi} e^{-ax}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right\}_0^{\pi} - \left\{\frac{e^{-ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-n(-1)^n}{a^2+n^2}e^{a\pi} + \frac{n}{a^2+n^2} + \frac{n(-1)^n}{a^2+n^2}\cdot e^{-a\pi} - \frac{n}{a^2+n^2}\right]$$

$$= \frac{(-1)^n\cdot n}{\pi(n^2+a^2)}[-e^{a\pi}+e^{-a\pi}] = \frac{n(-1)^{n+1}}{\pi(n^2+a^2)}\cdot 2\sin h\,a\,\pi \qquad \ldots (2)$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{\pi(n^2+a^2)}\cdot \sin h\,a\pi\cdot\sin nx$$

*i.e.,* $\sin h\,ax = \frac{2}{\pi}\sin h\,a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n\sin nx}{n^2+a^2}$

**EXAMPLE 20**

***Prove that in the range $-\pi \le x \le \pi$,***

$$\cos h\,ax = \frac{2a}{\pi}\sin h\,a\pi\left[\frac{1}{2a^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2+a^2}\right] \qquad \textit{[Nov. '88]}$$

**Solution**

Let $f(x) = \cos h\,ax$. Then its Fourier series is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad \ldots (1)$$





**To find $a_0$**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos h\,ax\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin h\,ax}{a}\right]_{-\pi}^{\pi} = \frac{1}{\pi a}[\sin h a\pi - \sin h(-a\pi)]$$

$$= \frac{1}{a\pi}[\sin h a\pi + \sin h a\pi] = \frac{2\sin h\,a\pi}{a\pi} \qquad \ldots (2)$$

**To find $a_n$**

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos h\,ax\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{2}\int_{-\pi}^{\pi}(e^{ax}+e^{-ax})\cos nx\,dx$$

$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx + \int_{-\pi}^{\pi}e^{-ax}\cos nx\,dx\right\}$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx + \int_{-\pi}^{\pi}e^{-ax}\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right\}_{-\pi}^{\pi} + \left\{\frac{e^{-ax}}{a^2+n^2}(-a\cos nx+n\sin nx)\right\}_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{ae^{a\pi}}{n^2+a^2}(-1)^n - \frac{e^{-a\pi}}{n^2+a^2}a(-1)^n - \frac{e^{-a\pi}}{n^2+a^2}\cdot a(-1)^n + \frac{e^{a\pi}}{n^2+a^2}a(-1)^n\right]$$

$$= \frac{(-1)^n\cdot a}{2\pi(n^2+a^2)}[2e^{a\pi}-2e^{-a\pi}] = \frac{a(-1)^n}{\pi(n^2+a^2)}\cdot 2\sin h\,a\pi$$

$$= \frac{2a(-1)^n\sin h\,a\pi}{\pi(n^2+a^2)} \qquad \ldots (3)$$

Since the given function $f(x) = \cos h\,ax$ is an even function, the Fourier coefficient

$$b_n = 0 \qquad \ldots (4)$$

$\left[\because f(x) = \cos h\, ax = \dfrac{e^{ax}+e^{-ax}}{2}\;;\; f(-x) = \dfrac{e^{-ax}+e^{ax}}{2} = f(x).\right.$ Hence $f(x)$ = $\cos h\, ax$ is an even function.]

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{\sin h a\pi}{\pi a} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sin h\, a\pi}{\pi(n^2+a^2)} \cdot \cos nx$$

$$= \frac{2a}{\pi} \sin h a\pi \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2+a^2}\right]$$

■ EXAMPLE 21 ■

***Obtain a Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$***

[Apr. '86, Nov. '88]

**● Solution**

Let $f(x) = \sqrt{1-\cos x}$. It is an even function.

For $f(x) = \sqrt{1-\cos x}$

$f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$

∴ Its Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots (1)$$

To find $a_0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\, dx \qquad [\because f(x) \text{ is an function}]$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2} \cdot \sin\left(\frac{x}{2}\right) dx \qquad \left[\because \frac{1-\cos x}{2} = \sin^2\frac{x}{2}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{4\sqrt{2}}{\pi}\left[-\cos\frac{\pi}{2} + 1\right] \qquad \left[\because \cos\frac{\pi}{2} = 0\right]$$

$$a_0 = \frac{4\sqrt{2}}{\pi} \quad \ldots (2)$$





To find $a_n$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{1-\cos x} \cdot \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2} \cdot \sin\frac{x}{2} \cos nx\, dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \frac{1}{2}\left\{\sin\left(\frac{1}{2}+n\right)x + \sin\left(\frac{1}{2}-n\right)x\right\} dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{1}{2}+n\right)x}{\frac{1}{2}+n} - \frac{\cos\left(\frac{1}{2}-n\right)x}{\frac{1}{2}-n}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{1}{2}+n\right)\pi}{\frac{1}{2}+n} - \frac{\cos\left(\frac{1}{2}-n\right)\pi}{\frac{1}{2}-n} + \frac{1}{\frac{1}{2}+n} + \frac{1}{\frac{1}{2}-n}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{\frac{1}{2}+n} + \frac{1}{\frac{1}{2}-n}\right]$$

$$\because \cos\left(\frac{1}{2}+n\right)\pi = \cos\frac{\pi}{2}\cos n\pi - \sin\frac{1}{2}\pi \sin n\pi$$

$$= 0 - 0 \left(\cos\frac{\pi}{2} = 0,\ \sin n\pi = 0\right) = 0$$

$$\cos\left(\frac{1}{2}-n\right)\pi = \cos\frac{\pi}{2}\cos n\pi + \sin\frac{1}{2}\pi\sin n\pi$$

$$= 0 + 0 = 0 = \frac{\sqrt{2}}{\pi}\left[\frac{\frac{1}{2}-n+\frac{1}{2}+n}{\frac{1}{4}-n^2}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{4}{1-4n^2}\right] \quad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\pi} \cdot \frac{4}{1-4n^2} \cos nx$$

$$\therefore \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{\cos nx}{4n^2-1}$$

**EXAMPLE 22**

*Expand $x(2\pi - x)$ as a Fourier Series in $(0, 2\pi)$. Deduce the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$*

[MU, Apr. '9...]

**Solution**

Let $f(x) = x(2\pi - x)$

The Fourier Series of $f(x)$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(1)$$

where
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\,dx$$
$$= \frac{1}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right]$$
$$= \frac{1}{\pi}\left[\frac{12\pi^3 - 8\pi^3}{3}\right] = \frac{1}{\pi}\cdot\frac{4\pi^3}{3}$$
$$a_0 = \frac{4\pi^2}{3} \quad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\cos nx\,dx$$
$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{-2\pi}{n^2} - \frac{2\pi}{n^2}\right] \quad [\because \sin 0 = 0, \sin n\pi = 0, \cos 2n\pi = 1]$$
$$= \frac{1}{\pi}\left(\frac{-4\pi}{n^2}\right)$$
$$a_n = \frac{-4}{n^2} \quad \ldots(3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\sin nx\,dx$$
$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{-2}{\pi}\left[\frac{1}{n^3} - \frac{1}{n^3}\right]$$
$$b_n = 0 \quad \ldots(4)$$





Substituting (2), (3) and (4) in (1), we get,

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}\cos nx$$
$$f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \quad \ldots(5)$$

Here '0' is a point of discontinuity which is an end point of the given interval $0 < x < 2\pi$. Then the value of Fourier series at $x = 0$ is the averge value fo $f(x)$ at the end points.

i.e., Putting $x = 0$ in (5), we get,

$$\frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{f(0)+f(2\pi)}{2} = 0$$

i.e.,
$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad i.e., \quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots = \frac{\pi^2}{6}$$

**EXAMPLE 23**

*Obtain the Fourier Series for*

$$f(x) = \begin{cases} \pi + 2x, & -\pi < x \le 0 \\ \pi - 2x, & 0 < x < \pi \end{cases}$$

*Deduct that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\infty = \frac{\pi^2}{8}$.* [MU, Oct. '97]

**Solution**

We know that the Fourier Series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(1)$$

where
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi + 2x)\,dx + \int_0^{\pi}(\pi - 2x)\,dx\right]$$
$$= \frac{1}{\pi}\left[\left\{\frac{(2x+\pi)^2}{4}\right\}_{-\pi}^{0} + \left\{\frac{(\pi - 2x)^2}{-4}\right\}_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\pi^2}{4} - \frac{\pi^2}{4} - \frac{\pi^2}{4} + \frac{\pi^2}{4}\right] \quad \ldots(2)$$
$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(\pi+2x)\cos nx\,dx + \int_{0}^{\pi}(\pi-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[\left\{(\pi+2x)\left(\frac{\sin nx}{n}\right) - (2)\left(\frac{-\cos nx}{n^2}\right)\right\}_{-\pi}^{0} + \left\{(\pi-2x)\left(\frac{\sin nx}{n}\right) - (-2)\left(\frac{-\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^2} - \frac{2\cos n\pi}{n^2} - \frac{2\cos n\pi}{n^2} + \frac{2}{n^2}\right] = \frac{4}{n^2\pi}\left[1-(-1)^n\right]$$

$$\therefore a_n = \left.\begin{cases} 0 \text{ when '}n\text{' is even} \\ \dfrac{8}{n^2\pi} \text{ when '}n\text{' is odd}\end{cases}\right\} \quad \ldots(3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+2x)\sin nx\,dx + \int_{0}^{\pi}(\pi-2x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{(\pi+2x)\left(\frac{-\cos nx}{n}\right) - (2)\left(\frac{-\sin nx}{n^2}\right)\right\}_{-\pi}^{0} + \left\{(\pi-2x)\left(\frac{-\cos nx}{n}\right) - (-2)\left(\frac{-\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n} - \frac{\pi\cos n\pi}{n} + \frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\right]$$

$$b_n = 0 \quad \ldots(4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{8}{n^2\pi}\cos nx \quad \ldots(5)$$

Put $x = 0$ in (5), we get

$$f(0) = \frac{8}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

$$\pi = \frac{8}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

*i.e.,* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

> ***NOTE :** Here '0' is a point of continuity. Therefore, the Fourier series converges to $f(0)$.*





**EXAMPLE 24** *Find the Fourier Series for $f(x) = e^{ax}$ in a Fourier Series in $(0, 2\pi)$.* **[MU. Apr. '96]**

**Solution**

The Fourier series of $f(x)$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad \ldots(1)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$$

$$= \frac{1}{a\pi}\left[e^{2a\pi}-1\right] = \frac{1}{a\pi}\left[\frac{e^{a\pi}-e^{-a\pi}}{e^{-a\pi}}\right]$$

$$= \frac{e^{a\pi}}{a\pi}\left[2\sin h\,a\pi\right] \quad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{(a^2+n^2)\pi}\left[ae^{2a\pi}-a\right] = \frac{a}{(a^2+n^2)\pi}\left[\frac{e^{a\pi}-e^{-a\pi}}{e^{-a\pi}}\right]$$

$$a_n = \frac{2ae^{a\pi}}{\pi(a^2+n^2)}\cdot\sin h\,a\pi \quad \ldots(3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{(a^2+n^2)\pi}\left[-ne^{2a\pi}+n\right] = \frac{n}{\pi(a^2+n^2)}\left[1-e^{2a\pi}\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left[\frac{e^{-a\pi}-e^{a\pi}}{e^{-a\pi}}\right]$$

$$b_n = \frac{-ne^{a\pi}}{\pi(n^2+a^2)}\,2\sin h\,a\pi \quad \ldots(4)$$

Substituting (2), (3) and (4) in (1), we get,

$$f(x) = \frac{e^{a\pi}\cdot\sin h\,a\pi}{a\pi} + \sum_{n=1}^{\infty}\frac{2ae^{a\pi}\sin h\,a\pi}{\pi(a^2+n^2)}\cos nx - \sum_{n=1}^{\infty}\frac{2n\,e^{a\pi}\sin h a\pi}{\pi(n^2+a^2)}\sin nx$$

**EXAMPLE 25**

*Expand in Fourier series of periodicity $2\pi$ of $f(x) = x^2$ for $0 < x < 2\pi$* [Mano, Nov.]

**Solution**

The Fourier series of $f(x)$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(1)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[8\pi^3]$

$$a_0 = \frac{8\pi^2}{3} \quad \dots(2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] \qquad [\because \cos 2n\pi = 1,\ \sin n\pi = 0]$$

$$a_n = \frac{4}{n^2} \quad \dots(3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$b_n = \frac{-4\pi}{n} \quad \dots(4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin nx.$$

**EXAMPLE 26**

*Determine the Fourier series expansion of $f(x) = \pi^2 - x^2$ in $(-\pi, \pi)$.*

**Solution**

Here $f(x) = \pi^2 - x^2$ is an even function.

$$\therefore b_n = 0$$

The Fourier series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots(1)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx \quad [\because \pi^2 - x^2 \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^3 - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\cdot\frac{2\pi^3}{3}$$

$$a_0 = \frac{4\pi^2}{3} \quad \dots(2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx \qquad \left[\because \begin{array}{r}\text{even function} \times \text{even function}\\ = \text{even function}\end{array}\right]$$

$$= \frac{2}{\pi}\left[(\pi^2 - x^2)\left(\frac{\sin nx}{n}\right) - (-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\pi\cos n\pi}{n^2}\right] = \frac{4(-1)^{n+1}}{n^2} \quad \dots(3)$$

Substituting (2) and (3) in (1) we get

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2}\cos nx$$

**EXAMPLE 27**

*Find a Fourier series to represent $x - x^2$ in the interval $(0, 2\pi)$.*

**Solution**

The Fourier series of $f(x) = x - x^2$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(1)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(x - x^2)\,dx$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{4\pi^2}{2} - \frac{8\pi^3}{3}\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{6\pi^2 - 8\pi^3}{3}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1-4\pi)}{n^2} - \frac{1}{n^2}\right] = \frac{-4\pi}{n^2\pi}$$

$$a_n = \frac{-4}{n^2} \quad \ldots (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{-\cos nx}{n} - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(2\pi - 4\pi^2)}{n} - \frac{2}{n^3} + \frac{2}{n^3}\right] \qquad [\because \cos 2n\pi = 1]$$

$$b_n = \frac{(4\pi - 2)}{n} \quad \ldots (4)$$

Substituting (2), (3) and (4) in (1) we get

$$f(x) = \frac{1}{2\pi}\left(\frac{6\pi^2 - 8\pi^3}{3}\right) + \sum_{n=1}^{\infty}\frac{-4}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{(4\pi-2)}{n}\sin nx$$

**■ EXAMPLE 28 ■**

*Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that $\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$.* [BDN, Apr. '95]

**● Solution**

The Fourier series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = \frac{-2\pi^3}{3\pi}$$





$$a_0 = \frac{-2\pi^2}{3} \quad \ldots (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1-2\pi)\cos n\pi}{n^2} - \frac{(1+2\pi)\cos n\pi}{n^2}\right] = \frac{1}{n^2\pi}[-4\pi\cos n\pi]$$

$$\text{i.e., } a_n = \frac{-4(-1)^n}{n^2} = \frac{4}{n^2}(-1)^{n+1} \quad \ldots (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(\pi-\pi^2)\cos n\pi}{n} - \frac{2\cos n\pi}{n^3} + \frac{(-\pi-\pi^2)\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{1}{n\pi}[-\pi\cos n\pi + \pi^2\cos n\pi - \pi\cos n\pi - \pi^2\cos n\pi]$$

$$= \frac{1}{n\pi}[-2\pi\cos n\pi]$$

$$b_n = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \quad \ldots (4)$$

Substituting (2), (3) and (4) in (1); we get

$$f(x) = \frac{-\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \quad \ldots (5)$$

Put $x = 0$ in (5) we get

$$f(0) = \frac{-\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$\text{But } f(0) = 0 \qquad [\because \text{'0' is a point of continuity}]$$

$$\therefore 0 = \frac{-\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$\text{i.e., } 4\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\right] = \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\text{i.e., } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \ldots = \frac{\pi^2}{12}$$

**EXAMPLE 29**

*Obtain the Fourier series of $f(x) = x \sin x$ in $(-\pi, \pi)$.*

*[Madras Nov. 2000, MKU, Apr. ...*

**Solution**

Here $f(x) = x \sin x$ is even function in $(-\pi, \pi)$.

$\therefore b_n = 0$

$\therefore$ The Fourier series of $f(x) = x \sin x$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots (1)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x\,dx = \frac{1}{\pi}\left[x(-\cos x) - (-\sin x)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi \cos\pi - \pi \cos\pi\right]$$

$$a_0 = 2 \quad [\because \cos\pi = -1] \qquad \ldots (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\left\{\sin(1+n)x + \sin(1-n)x\right\}dx$$

$$= \frac{1}{2\pi}\left[x\left\{\left(\frac{-\cos(1+n)x}{1+n}\right) - \left(\frac{-\sin(1+n)x}{1+n}\right)\right\}\right]_{-\pi}^{\pi}$$

$$+ \left\{x\left(\frac{-\cos(1-n)x}{1-n}\right) - \left(\frac{-\sin(1-n)x}{1-n}\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n} - \frac{\pi\cos(1+n)\pi}{1+n} - \frac{\pi\cos(1-n)\pi}{1-n} - \frac{\pi\cos(1-n)\pi}{1-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi\cos(1+n)\pi}{1+n} - \frac{2\pi\cos(1-n)\pi}{1-n}\right]$$

$$= \frac{-\cos\pi\cos n\pi}{1+n} - \frac{\cos\pi\cos n\pi}{1-n} \qquad [\because \sin n\pi = 0]$$

$$= (-1)^n\left[\frac{1}{1+n} + \frac{1}{1-n}\right] = (-1)^n\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$a_n = \frac{2(-1)^n}{1-n^2} \text{ provided } n \neq 1 \qquad \ldots (3)$$

when $n = 1$,

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \sin 2x\,dx$$





$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{2} - \frac{\pi}{2}\right] = -\frac{1}{2} \qquad \ldots (4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2}\cos nx$$

**EXAMPLE 30**

*Obtain the Fourier Series for $f(x) = e^{\mu x}$ in $(-\pi, \pi)$ where $\mu$ is not an integer. Using the Fourier Series for $e^{\mu x}$, develop the Fourier Series for $\cos \mu x$ in $(-\pi, \pi)$.* *[BDN, Nov. '97]*

**Solution**

The Fourier series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\mu x}\,dx = \frac{1}{\pi}\left[\frac{e^{\mu x}}{\mu}\right]_{-\pi}^{\pi} = \frac{1}{\mu\pi}\left[e^{\mu\pi} - e^{-\mu\pi}\right]$$

$$a_0 = \frac{2\sin h\,\mu\pi}{\mu\pi} \qquad \ldots (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\mu x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{\mu x}}{\mu^2+n^2}(\mu\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{\mu}{\pi(n^2+\mu^2)}\left[e^{\mu\pi}\cos n\pi - e^{-\mu\pi}\cos n\pi\right]$$

$$a_n = \frac{\mu(-1)^n}{\pi(n^2+\mu^2)}\cdot 2\sin h\,\mu\pi \qquad \ldots (3)$$

$$\left[\because \sin hx = \frac{e^x - e^{-x}}{2}\right]$$

$$b = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\mu x}\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{\mu x}}{\mu^2+n^2}(\mu\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(\mu^2+n^2)}\left[-e^{\mu\pi}\cdot n\cos n\pi + e^{-\mu\pi}\cdot n\cos n\pi\right]$$

$$b_n = \frac{n(-1)^n}{\pi(\mu^2+n^2)}(-2\sin h\mu\pi)$$

Substituting (2), (3) and (4) in (1) we get

$$f(x) = \frac{\sin h\mu\pi}{\mu\pi} + \sum_{n=1}^{\infty}\frac{\mu(-1)^n\cdot 2\sin h\mu\pi}{\pi(n^2+\mu^2)}\cdot\cos nx - \sum_{n=1}^{\infty}\frac{n(-1)^n\cdot 2\sin h\mu\pi}{\pi(n^2+\mu^2)}\cdot\sin nx$$

$$e^{\mu x} = \frac{\sin h\mu\pi}{\pi}\left[\frac{1}{\mu} + \sum_{n=1}^{\infty}\frac{2\mu(-1)^n\cdot\cos nx}{n^2+\mu^2} - \sum_{n=1}^{\infty}\frac{2n(-1)^n}{\pi(\mu^2+n^2)}\cdot\sin nx\right] \quad \ldots (5)$$

Replace $\mu$ by $-\mu$ in (5), we get

$$e^{-\mu x} = \frac{-\sin h\mu\pi}{\pi}\left[-\frac{1}{\mu} - \sum_{n=1}^{\infty}\frac{2\mu(-1)^n\cos nx}{n^2+\mu^2} - \sum_{n=1}^{\infty}\frac{2n(-1)^n\cdot\sin nx}{\pi(\mu^2+n^2)}\right] \quad \ldots (6)$$

Adding (5) and (6) we get

$$e^{\mu x}+e^{-\mu x} = \frac{2\sin h\mu\pi}{\mu\pi} + 2\sum_{n=1}^{\infty}\frac{\sinh\mu\pi}{\pi}\cdot\frac{2\mu(-1)^n\cos nx}{n^2+\mu^2}$$

$$\therefore \cos h\mu x = \frac{e^{\mu x}+e^{-\mu x}}{2} = \frac{\sin h\mu\pi}{\mu\pi} + \frac{\sinh\mu\pi}{\pi}\sum_{n=1}^{\infty}\frac{2\mu(-1)^n\cos nx}{n^2+\mu^2}$$

$$= \frac{\sin h\mu\pi}{\pi}\left[\frac{1}{\mu} + 2\sum_{n=1}^{\infty}\frac{\mu(-1)^n\cos nx}{n^2+\mu^2}\right]$$





## EXERCISES

1. Find the Fourier Series for $f(x) = (\pi - x)^2$ in $(-\pi, \pi)$
$f(x) = \frac{4\pi^2}{3} + \left(\frac{4}{n^2}\right)(-1)^n\cos nx + \frac{4\pi(-1)^n}{n}\sin nx$ [Apr. 87, Civil]

2. Prove that in the interval $-\pi < x < \pi$. [Nov. 87, Civil]
$\cos x = \frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n^2-1)}\sin nx$

3. Obtain the Fourier Series of period $2\pi$ for the function $\alpha x(\pi - x)$ in $0 \le x \le 2\pi$, where $\alpha$ is a constant. [Apr. 88]
[**Ans.** $f(x) = -\frac{\alpha\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\alpha}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{2\pi\alpha}{n}\sin nx$]

4. Obtain the Fourier Series expansion of $\frac{1}{2}(\pi - x)$ in $(0, 2\pi)$.
[Nov. 86, Mech., Nov. 88, Mech.] [**Ans.** $f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}$]

5. Find the Fourier Series of the following function which is assumed to have the period $f(x) = \frac{x^2}{4}$, $-\pi < x < \pi$. [Apr. 87, ECE]
Hence deduce $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$.
[**Ans.** $\frac{x^2}{4} = \frac{\pi^2}{12} - \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right)$]

6. Show that the Fourier Series for the function
$f(x) = \begin{cases} x+1, & 0 < x < \pi \\ x-1, & -\pi < x < 0 \end{cases}$ is $\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n(1+\pi)}{n}\right]\sin nx$.
[Anna Univ. Apr. 2001]

7. Expand the Fourier Series to represent $f(x) = \frac{1}{4}(\pi - x)^2$, $0 < x < 2\pi$.
[**Ans.** $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$]

8. Find a Fourier Series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$.
[**Ans.** $x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$]

9. Obtain the Fourier Series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

$$[Ans.\ e^{-x} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\left\{\frac{1}{2} + \sum_{1}^{\infty} \frac{\cos nx}{1+n^2} + \sum_{1}^{\infty} \frac{n \sin nx}{1+n^2}\right.$$

10. Find the Fourier Series to represent $e^{ax}$ in the interval $-\pi < x < \pi$.

$$[Ans.\ e^{ax} = \frac{\sin h\, a\pi}{\pi}\left\{\frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \cos nx}{a^2+n^2} + \sum_{n=1}^{\infty} \frac{2n(-1)^n \sin nx}{a^2+n^2}\right.$$

11. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval, $0 < x < 2\pi$, show that

$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} = \frac{\pi^2}{12}$

12. Prove that the Fourier Series expansion of

$\frac{1}{2} x$ is $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$

13. Obtain the Fourier Series expansion of $e^x$ in $(0, 2\pi)$.

$$[Ans.\ e^x = \frac{e^{2\pi}-1}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2+1} - \sum_{n=1}^{\infty} \frac{n}{n^2+1} \sin nx\right\}$$

14. Find the Fourier Series expansion of $x \cos x$ in the interval $-\pi < x < \pi$.

$$[Ans.\ x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{(-1)^n \sin nx}{n^2-1}\ ]$$

15. Find the Fourier Series of the function $f(x)$ defined by

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

$$[Ans.\ f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

16. A function is defined as follows :

$$f(x) = \begin{cases} -x \text{ in } -\pi < x < 0 \\ x \text{ in } 0 < x < \pi \end{cases} \text{ show that}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

17. Obtain the Fourier Series of $(1 - \cos x)^{\frac{1}{2}}$ in $0 \le x \le 2\pi$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

$$[Ans.\ \frac{4\sqrt{2}}{\pi}\left\{\frac{1}{2} - \sum_{1}^{\infty} \frac{1}{4n^2-1} \cos nx\right\}\ ]$$





18. Find the Fourier Series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le \pi \\ 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$$

$$[Ans.\ f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right\}\ ]$$

19. Find the Fourier Series for the function

$$f(x) = \begin{cases} -k \text{ in } -\pi < x < 0 \\ k \text{ in } 0 < x < \pi \end{cases}.$$

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$[Ans.\ f(x) = \frac{4k}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n}\ ]$$

20. Obtain the Fourier Series of the function

$$f(x) = \begin{cases} 0 \text{ when } -\pi < x < 0 \\ 1 \text{ when } 0 < x < \pi \end{cases}$$

$$[Ans.\ f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)\ ]$$

21. Obtain the Fourier Series expansion of $f(x)$ if

$$f(x) = \begin{cases} 1 \text{ for } 0 < x < \pi \\ 2 \text{ for } \pi < x < 2\pi \end{cases}.$$

$$[Ans.\ f(x) = \frac{3}{2} - \frac{2}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n}\ ]$$

22. An alternating current after passing through a rectifier has the form

$$i = \begin{cases} i_0 \sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{for } \pi \le x \le 2\pi \end{cases}$$

where $i_0$ is the maximum current and the period is $2\pi$. Express '$i$' in a Fourier Series.

$$[Ans.\ i = \frac{i_0}{\pi} + \frac{i_0}{2}\sin x - \frac{2i_0}{\pi} \sum_{n=2,4,6}^{\infty} \frac{\cos nx}{n^2-1}\ ]$$

23. Find the Fourier Series of the function $f(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$

$$[Ans.\ f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots]$$

24. Obtain the Fourier Series of the following functions in the given interval.

(i) $f(x) = \begin{cases} 1 \text{ in } (0, \pi) \\ 0 \text{ in } (\pi, 2\pi) \end{cases}$

(ii) $f(x) = \begin{cases} 1 \text{ in } -\pi < x < 0 \\ 2 \text{ in } 0 < x < \pi \end{cases}$

(iii) $f(x) = \begin{cases} a \text{ in } 0 < x < \pi \\ -a \text{ in } \pi < x < 2\pi \end{cases}$

(iv) $f(x) = \begin{cases} -1 \text{ in } -\pi < x < 0 \\ 1 \text{ in } 0 < x < \pi \end{cases}$

(v) $f(x) = \begin{cases} 0 \text{ for } -\pi < x < 0 \\ \pi \text{ for } 0 < x < \pi \end{cases}$

(vi) $f(x) = \begin{cases} -x \text{ for } -\pi < x < 0 \\ 0 \text{ for } 0 < x < \pi \end{cases}$

(vii) $f(x) = \begin{cases} x \text{ in } -\pi \le x \le 0 \\ 0 \text{ in } 0 \le x \le \pi \end{cases}$

(viii) $f(x) = \begin{cases} x + \dfrac{\pi}{2} & -\pi \le x \le 0 \\ \dfrac{\pi}{2} - x & 0 \le x \le \pi \end{cases}$

(ix) $f(x) = \begin{cases} x & \text{in } -\pi < x < 0 \\ 0 & \text{in } 0 < x < \dfrac{\pi}{2} \\ x - \dfrac{\pi}{2} & \text{in } \dfrac{\pi}{2} < x < \pi \end{cases}$

*Answers :*

(i) $\dfrac{1}{2} + \dfrac{2}{\pi}\sum_{1}^{\infty}\dfrac{1}{2n-1}\sin(2n-1)x$

(ii) $\dfrac{3}{2} + \dfrac{2}{\pi}\sum_{1}^{\infty}\dfrac{1}{2n-1}\sin(2n-1)x$

(iii) $\dfrac{4a}{\pi}\left[\sum_{n=1,3,5}^{\infty}\dfrac{\sin nx}{n}\right]$

(iv) $\dfrac{4}{\pi}\left[\sum_{n=1,3,5}^{\infty}\dfrac{\sin nx}{n}\right]$

(v) $\dfrac{\pi}{2} + 2\left[\sum_{n=1,3,5}^{\infty}\dfrac{\sin nx}{n}\right]$

(vi) $\dfrac{\pi}{4} - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2} + \sum_{1}^{\infty}\dfrac{(-1)^n \sin nx}{n}$

(vii) $-\dfrac{\pi}{4} + \dfrac{2}{\pi}\sum_{1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2} + \sum_{1}^{\infty}\dfrac{(-1)^{n+1}\sin nx}{n}$

(viii) $\dfrac{4}{\pi}\sum_{n=1,3,5}^{\infty}\dfrac{\cos nx}{n^2}$

(ix) $-\dfrac{3\pi}{16} + \dfrac{1}{\pi}\sum_{n=1,3,5}^{\infty}\dfrac{\cos nx}{n^2} + \dfrac{2}{\pi}\sum_{n=2,6,10}^{\infty}\dfrac{\cos nx}{n^2}$
$- \dfrac{3}{2}\sum_{n=1}^{\infty}\dfrac{(-1)^n \sin nx}{n} - \dfrac{1}{\pi}\left[\dfrac{\sin x}{1^2} - \dfrac{\sin 3x}{3^2} + \dfrac{\sin 5x}{5^2} - \dots\right]$





## CHANGE OF INTERVAL

So far we have discussed that a given function $f(x)$ can be expanded in a Fourier Series in the interval of length $2\pi$. But in many Engineering problems, it is desired to expand a function in a Fourier Series in the interval of length $2l$ and not $2\pi$. To change the length $2l$ to $2\pi$ we put

$\dfrac{x}{l} = \dfrac{z}{\pi}$

*i.e.*, $z = \dfrac{\pi x}{l}$, so that

$z = \dfrac{\pi c}{l} = d$ (say)

when $x = c$,

when $x = c + 2l$, $z = \dfrac{\pi(c + 2l)}{l} = \dfrac{\pi c}{l} + 2\pi = d + 2\pi$

$\therefore$ The function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the function $f\left(\dfrac{lz}{\pi}\right) = F(z)$, say, of period $2\pi$ in $(d, d + 2\pi)$ and the latter function can be expressed in Fourier Series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad \dots (1)$$

where $$a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\,dz \quad \dots (2)$$

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz\,dz \quad \dots (3)$$

and $$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin nz\,dz \quad \dots (4)$$

Now by applying the inverse substitution

$z = \dfrac{\pi x}{l}$, $dz = \dfrac{\pi}{l}dx$

when $z = d$, $x = c$

and when $z = d + 2\pi$, $x = c + 2l$

$\therefore$ Equation (1) becomes

$$F(z) = F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \dots (5)$$

Where $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \quad \text{[by (2)]}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{[by (3)]} \quad \dots (6)$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx \quad [b\ (\ )]$$

$\therefore$ The Fourier Series for $f(x)$ in the interval $c < x < c + 2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Where $a_0$, $a_n$ and $b_n$ are given by (5), (6) and (7).

*NOTE 1* : **If we put $c = 0$ the interval becomes $0 < x < 2l$, then the above equations are reduced to**

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

and $$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

*NOTE 2* : **If we put $c = -l$, then the interval becomes $-l < x < l$ and the above results are reduced to**

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

*NOTE 3* : **If $f(x)$ is an even function then we have**

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx,$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

and $$b_n = 0$$

NOTE 4 : **If $f(x)$ is an odd function then we have $a_0 = 0, a_n = 0$ and**

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$





**EXAMPLE 1**

***Find the Fourier Series expansion of period 2l for the function $f(x) = (l-x)^2$ in the range $(0, 2l)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.***

[Nov.91]

**Solution**

The Fourier Series of $f(x)$ in $(0, 2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \ldots (1)$$

Now $$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{2l} (l-x)^2\,dx$$

$$= \left[\frac{(l-x)^3}{-3l}\right]_0^{2l} = \frac{l^3 + l^3}{3l} = \frac{2l^3}{3l} = \frac{2}{3}l^2 \quad \ldots (2)$$

i.e., $$\boxed{a_0 = \frac{2}{3}l^2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{2l} (l-x)^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)^2\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 2(l-x)(-1)\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\} + 2\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{2l\cos 2n\pi}{\frac{n^2\pi^2}{l^2}} + \frac{2l}{\frac{n^2\pi^2}{l^2}}\right] = \frac{4l^2}{n^2\pi^2} \qquad [\because \sin 2n\pi = 0, \cos 2n\pi = 1] \quad \ldots (3)$$

i.e., $$\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{2l} (l-x)^2\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)^2\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 2(l-x)(-1)\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\} + 2\left\{\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{-l^2\cos 2n\pi}{\frac{n\pi}{l}} + \frac{2\cos 2n\pi}{\frac{n^3\pi^3}{l^3}} + \frac{l^2}{\frac{n\pi}{l}} - \frac{2}{\frac{n^3\pi^3}{l^3}}\right]$$

$$= \frac{1}{l}\left[\frac{-l^2}{\frac{n\pi}{l}} + \frac{l^2}{\frac{n\pi}{l}}\right] = 0 \qquad [\because \cos 2n\pi \ldots]$$

i.e., $\boxed{b_n = 0}$

Substituting (2), (3) and (4) in (1), we get,

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos \frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2} \qquad \ldots (5)$$

Here 0 is a point of discontinuity which is an end point of the given interval $0 < x < 2l$. Therefore, the sum of Fourier series (5) at $x = 0$ is the average of value of $f(x)$ at the end points *i.e.*, at $x = 0$ and at $x = 2l$.

Putting $x = 0$ in (5), we get,

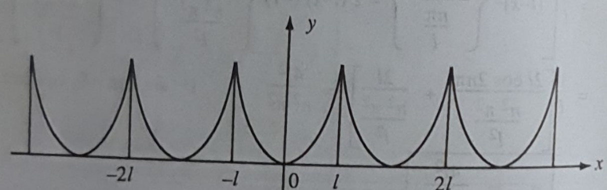
$$\frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{f(0)+f(2l)}{2} = \frac{l^2+l^2}{2} = l^2$$

$$\therefore \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 - \frac{1}{3} = \frac{2}{3}; \qquad \therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

**EXAMPLE 2**

***Obtain the Fourier Series to represent $x^2$ from $x = -l$ to $x = l$.*** *[Nov. 87, Civil]*

**Solution**




The given function $f(x) = x^2$ is an even function in $(-l, l)$.

The Fourier Series for $f(x) = x^2$ in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \qquad \ldots (1)$$

Now $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} x^2\,dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{l^3}{3}+\frac{l^3}{3}\right] = \frac{2l^3}{3l} = \frac{2}{3}l^2$$

i.e., $\boxed{a_0 = \frac{2}{3}l^2}$ $\qquad \ldots (2)$





$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x^2\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 2x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\} + 2\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{2l\cos n\pi}{\frac{n^2\pi^2}{l^2}} + \frac{2l\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right] = \frac{4l^2\cos n\pi}{n^2\pi^2} \qquad [\because \sin n\pi = 0] \qquad \ldots (3)$$

i.e., $\boxed{a_n = \frac{4l^2\cos n\pi}{n^2\pi^2}}$

Substituting (2), (3) in (1) we get

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

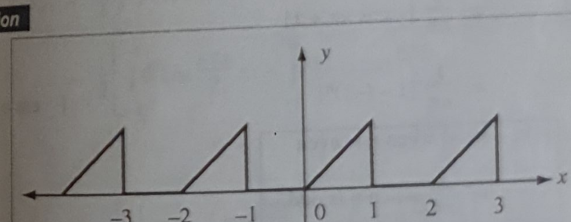
$$x^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\frac{n\pi x}{l}}{n^2}$$

**EXAMPLE 3**

***Find the Fourier Series for f(x) given***

$$f(x) = \begin{cases} 0 \text{ in } -1 < x < 0 \\ 1 \text{ in } 0 < x < 1 \end{cases} \text{ and } f(x+2) = f(x) \text{ for all } x. \qquad \textit{[Apr. 86, Civil]}$$

**Solution**




The Fourier series of $f(x)$ in $-1 < x < 1$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots (1)$$

Now $$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx$$

$$= \frac{1}{1}\left[\int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx\right] = \frac{1}{1}\,[x]_0^1 = 1$$

i.e., $\boxed{a_0 = 1}$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx \qquad \ldots (2)$$

$$= \frac{1}{1}\left[\int_{-1}^{0} 0\cdot\cos\frac{n\pi x}{1}\,dx + \int_{0}^{1} 1\cos\frac{n\pi x}{1}\,dx\right]$$

$$= \frac{1}{1}\left[\frac{\sin\frac{n\pi x}{1}}{\frac{n\pi}{1}}\right]_0^1 \qquad [\because \sin n\pi = 0]$$

i.e., $\boxed{a_n = 0}$ ... (3)

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx$$

$$= \frac{1}{1}\left[\int_{-1}^{0} 0\cdot\sin\frac{n\pi x}{1}\,dx + \int_{0}^{1} 1\cdot\sin\frac{n\pi x}{1}\,dx\right]$$

$$= \frac{1}{1}\left[\frac{-\cos\frac{n\pi x}{1}}{\frac{n\pi}{1}}\right]_0^1$$

$$= \frac{1}{n\pi}\,[-\cos n\pi + 1]$$

$$= \frac{1}{n\pi}\,[1-(-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

$$\boxed{\begin{aligned} b_n &= 0 \text{ when } n \text{ is even} \\ &= \frac{2}{n\pi} \text{ when } n \text{ is odd.}\end{aligned}} \qquad \ldots (4)$$

Substituting (2), (3) and (4) in (1) we get

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi x}{1}$$





$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin n\pi x$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{2n-1}$$

■ EXAMPLE 4 ■

*The function $f(x)$ is defined as follows in the interval $(-2, 2)$.*

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

*Show that* $f(x) = \frac{1}{4} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{n\pi}{4}\cos\frac{n\pi x}{2}$. [Nov. 87, Mech.]

**Solution**

The Fourier Series of the function $f(x)$ in $(-2, 2)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2} \qquad \ldots (1)$$

Now $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\,dx + \int_{-1}^{0}(1+x)\,dx + \int_{0}^{1}(1-x)\,dx + \int_{1}^{2} 0\,dx\right]$$

$$= \frac{1}{2}\left[\left\{\frac{(1+x)^2}{2}\right\}_{-1}^{0} + \left\{\frac{(1-x)^2}{-2}\right\}_{0}^{1}\right] = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right] = \frac{1}{2}$$

i.e., $\boxed{a_0 = \frac{1}{2}}$ ... (2)

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\cdot\cos\frac{n\pi x}{2}\,dx + \int_{-1}^{0}(1+x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{1}(1-x)\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2} 0\cdot\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{(1+x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{-1}^{0} + \left\{(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{n^2\pi^2}{4}} - \frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} - \frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} + \frac{1}{\frac{n^2\pi^2}{4}}\right]$$

i.e.,
$$a_n = \frac{4}{n^2\pi^2}\left[1 - \cos\frac{n\pi}{2}\right] \quad \ldots (3)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\cdot\sin\frac{n\pi x}{2}\,dx + \int_{-1}^{0}(1+x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{1}(1-x)\sin\frac{n\pi x}{2}\,dx + \int_{1}^{2} 0\cdot\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{(1+x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{-1}^{0} + \left\{(1-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{1}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{\frac{n\pi}{2}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} - \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} + \frac{1}{\frac{n\pi}{2}}\right] = 0$$

i.e.,
$$\boxed{b_n = 0} \quad \ldots (4)$$

Substituting (2), (3) and (4) in (1) we get

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\, 2\sin^2\frac{n\pi}{4}\cos\frac{n\pi x}{2} \quad \left[\text{Use } 1 - \cos x = 2\sin^2\frac{x}{2}\right]$$

i.e.,
$$f(x) = \frac{1}{4} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{n\pi}{4}\cos\frac{n\pi x}{2}$$





**EXAMPLE 5**

*Obtain the Fourier Series of period 2l for the function $f(x) = |x|$ in $-l \le x \le l$* **[BDN, Apr. '98]**

**Solution**

Here $f(x) = |x|$ is an even function in $(-l, l)$.

$$\therefore b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} \quad \ldots (1)$$

Now
$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l}|x|\,dx \quad [\because \text{In } (0, l),\ |x| = x]$$

$$= \frac{2}{l}\int_{0}^{l} x\,dx$$

$$= \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l \quad \ldots (2)$$

$$a_0 = l$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0, & \text{when '}n\text{' is even} \\ \dfrac{-4l}{n^2\pi^2} & \text{when '}n\text{' is odd}\end{cases} \quad \ldots (3)$$

Substituting (2) and (3) in (1) we get

$$f(x) = \frac{l}{2} - \sum_{n=1,3,\ldots}^{\infty}\frac{4l}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

**EXAMPLE 6**

*Find the Fourier Series of*

$$f(x) = 0 \text{ in } -2 < x < 0$$
$$= x \text{ in } 0 < x < 2$$

**[MU, Oct. '98]**

**Solution**

The Fourier Series of $f(x)$ in $(-2, 2)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2} \quad \ldots (1)$$

$$\text{where } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[0 + \int_0^2 x\,dx\right] = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$a_0 = 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \quad \ldots (2)$$

$$= \frac{1}{2}\left[\int_{-2}^{0} f(x)\cos\frac{n\pi x}{2}\,dx + \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[0 + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \frac{1}{2}\times\frac{4}{n^2\pi^2}[\cos n\pi - 1]$$

$$a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$a_n = \left.\begin{array}{l} 0 \text{ when '}n\text{' is even} \\ = \frac{-4}{n^2\pi^2} \text{ when '}n\text{' is odd}\end{array}\right\} \quad \ldots (3)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} f(x)\sin\frac{n\pi x}{2}\,dx + \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[0 + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$





$$= \frac{-2}{n\pi}\cos n\pi \quad \ldots (4)$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

Substituting (2), (3) and (4) in(1) we get

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}$$

**EXAMPLE 7**

***Find the Fourier series of $f(x)$ of period 4 given by*** **[BDN, Nov. '95]**

$$f(x) = \begin{cases} 1, & -2 < x < 0 \\ e^{-x}, & 0 < x < 2\end{cases}$$

**Solution**

The Fourier series of $f(x)$in $(-2, 2)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2} \quad \ldots (1)$$

$$\text{where } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{0} f(x)\,dx + \int_0^2 f(x)\,dx\right\}$$

$$= \frac{1}{2}\left\{\int_{-2}^{0} 1\,dx + \int_0^2 e^{-x}\,dx\right\} = \frac{1}{2}\left[(x)_{-2}^{0} + (-e^{-x})_0^2\right]$$

$$= \frac{1}{2}[2 - e^{-2} + 1]$$

$$a_0 = \frac{1}{2}[3 - e^{-2}] \quad \ldots (2)$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}\cos\frac{n\pi x}{2}\,dx + \int_0^2 e^{-x}\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_{-2}^{0} + \left\{\frac{e^{-x}}{1+\frac{n^2\pi^2}{4}}\left(-\cos\frac{n\pi x}{2} + \frac{n\pi}{2}\sin\frac{n\pi x}{2}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\times\frac{4}{4+n^2\pi^2}\{-e^{-2}\cos n\pi + 1\}$$

$$a_n = \frac{2}{4+n^2\pi^2}\{1-e^{-2}(-1)^n\}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\int_{-2}^{0} 1\cdot\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} e^{-x}\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_{-2}^{0} + \left\{\frac{e^{-x}}{1+\frac{n^2\pi^2}{4}}\left(-\sin\frac{n\pi x}{2} - \frac{n\pi}{2}\cos\frac{n\pi x}{2}\right)\right\}_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\left(\frac{-2}{n\pi} + \frac{2}{n\pi}\cos n\pi\right) + \frac{4}{4+n^2\pi^2}\left(\frac{-n\pi}{2}\cos n\pi\cdot e^{-2} + \frac{n\pi}{2}\right)\right] \quad \ldots(4)$$

Substituting (2), (3) and (4) in (1) we get

$$f(x) = \frac{(3-e^{-2})}{4} + \frac{2}{(4+\pi^2)}(1+e^{-2})\cos\frac{\pi x}{2} + \frac{2}{(4+4\pi^2)}(1-e^{-2})\cos\frac{2\pi x}{2} + \ldots$$

$$+ \frac{1}{2}\left[\frac{-4}{\pi} + \frac{4}{4+\pi^2}\left(\frac{\pi}{2}e^{-2} + \frac{\pi}{2}\right)\right]\sin \pi x + \ldots$$

**EXAMPLE 8**

*Find the Fourier Series for* $f(x) = \begin{cases} l-x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$

*Hence deduce the sum to infinity of the series* $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ *[BDN, Apr. '98]*

**Solution**

The Fourier Series of $f(x)$ in $(0, 2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \quad \ldots(1)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{l}(l-x)\,dx = \frac{1}{l}\left\{\frac{(l-x)^2}{-2}\right\}_0^{l}$$

$$a_0 = \frac{l}{2} \quad \ldots(2)$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\cos\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l}$$





$$= \frac{l}{n^2\pi^2}[-\cos n\pi + 1] = \frac{l}{n^2\pi^2}[1-(-1)^n]$$

$$a_n = \begin{cases} 0, & \text{when '}n\text{' is even} \\ \frac{2l}{n^2\pi^2} & \text{when '}n\text{' is odd} \end{cases} \quad \ldots(3)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l}$$

$$b_n = \frac{l}{n\pi}[\cos n\pi] = \frac{(-1)^n l}{n\pi} \quad \ldots(4)$$

Substituting (2), (3) and (4) in (1) we get

$$f(x) = \frac{l}{4} + \sum_{n=1,3,5}^{\infty}\frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{(-1)^n l}{n\pi}\sin\frac{n\pi x}{l}$$

$$f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{n\pi x}{l} + \frac{l}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l} \quad \ldots(5)$$

Here '0' is a point of discontinuity which is an end point of the given interval. Therefore, the value of Fourier series (5) at $x = 0$ is the average value of $f(x)$ at $x = 0$ and $x = 2l$.

Putting $x = 0$ in (5), we get,

$$\frac{l}{4} + \frac{2l}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n+1)^2} = \frac{f(0)+f(2l)}{2} = \frac{l}{2}$$

$$\text{i.e., } \frac{2l}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n+1)^2} = \frac{l}{2} - \frac{l}{4} = \frac{l}{4}$$

$$\therefore \sum_{1}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

## EXERCISES

1. If $f(x) = \begin{cases}\pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2\end{cases}$ show that in the interval (0, 2)

   $$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$ *[Nov. 90, Civil]*

2. Obtain the Fourier Series expansion of $f(x)$, if

   $$f(x) = \begin{cases} 1 \text{ for } 0 \le x \le 1 \\ x \text{ for } 1 < x < 2\end{cases} \text{ and } f(x+2) = f(x).$$ *[Apr. 90]*

3. Develop $f(x)$ in Fourier Series in the interval $(-2, 2)$ if

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$$

[*Ans.* $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} \ldots\right.$ [Apr. 90, C...

4. Find the Fourier Series for $f(x) = x^2$ in $-1 < x < 1$. [Nov. 86, C...

[*Ans.* $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\pi x}{n^2}$

5. In the range $(0, 2l)$, $f(x)$ is defined by the relation

$$f(x) = \begin{cases} 0, & 0 < x < l \\ a, & l < x < 2l \end{cases}$$

[Nov. 88, Mec...

[*Ans.* $f(x) = \frac{a}{2} - \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l}$

6. Find the Fourier Series with period 4 to represent the function $f(x) = x^2 - 2$ in the interval $-2 < x < 2$. [Apr. 90, ECE

[*Ans.* $f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty}\frac{16(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2}$

7. Suppose that $f(x)$ is periodic with period 2L and $f(x) = x$, $-l < x < l$. Determine the Fourier representation of $f(x)$. [Nov. 88, ECE

8. Find the Fourier Series of the function $f(x) = 2x - x^2$ for $0 < x < 3$ and $f(x+3) = f(x)$. [Apr. 89, ECE

[*Ans.* $2x - x^2 = \frac{3}{\pi}\left\{\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \ldots\right\} - \frac{9}{\pi^2}\left\{\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} + \frac{1}{3^2}\cos\frac{6\pi x}{3} + \ldots\right.$

9. If $f(x) = \begin{cases} -1 & \text{for } -2 \le x \le -1 \\ x & \text{for } -1 < x < 1 \\ 1 & \text{for } 1 \le x \le 2 \end{cases}$ [Nov. 89, ECE]

Find the Fourier Series which represents $f(x)$ in the interval $(-2, 2)$.

10. Find the Fourier Series of the function of period 4

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < -1 \\ k & \text{when } -1 < x < 1 \\ 0 & \text{when } 1 < x < 2 \end{cases}$$

[*Ans.* $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left\{\frac{1}{1}\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \ldots\right.$

11. Find the Fourier Series for the function

$f(x) = \begin{cases} kx, & 0 < x < l \\ 0, & l < x < 2l \end{cases}$ $k$ being a constant.

[*Ans.* $f(x) = \frac{kl}{4} + \sum_{n=1,3,}^{\infty}\frac{-2kl}{n^2\pi^2}\cos\frac{n\pi x}{l} - \sum_{n=1}^{\infty}\frac{kl(-1)^n}{n\pi}\sin\frac{n\pi x}{l}$]





12. Find Fourier expansion for the function $f(x) = x - x^2$, $-1 < x < 1$.

[*Ans.* $x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi x$]

13. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.

[*Ans.* $e^{-x} = \frac{\sinh l}{l} + \sum_{n=1}^{\infty}\frac{2l(-1)^n \sinh l}{l^2 + (n\pi)^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{2n\pi(-1)^n \sinh l}{l^2 + (n\pi)^2}\sin\frac{n\pi x}{l}$]

14. Find a Fourier Series for $f(x) = 1 - x^2$, when $-1 \le x \le 1$

$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\left(\cos\pi x - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right)$]

15. Obtain a Fourier Series for $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

[*Ans.* $f(x) = \frac{1}{4} - \frac{2}{\pi^2}\cdot\sum_{n=1,3,5}^{\infty}\frac{\cos n\pi x}{n^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi x}{n}$]

16. Find Fourier Series for $f(x) = \pi x$ from $x = -c$ to $x = c$.

[*Ans.* $\pi x = 2c\left[\sin\left(\frac{\pi x}{c}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{c}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{c}\right) - \ldots\right]$

17. Find Fourier Series for $f(x) = x - x^3$ in $-1 < x < 1$.

[*Ans.* $x - x^3 = \frac{12}{\pi^3}\left(\sin\pi x - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} - \ldots\right)$]

18. Find Fourier Series for the function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$$

[*Ans.* $f(x) = -\frac{4}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right) + \frac{2}{\pi}\left(\sin\pi x + \frac{\sin 3\pi x}{3} + \ldots\right)$]

19. If $f(x) = \begin{cases} 1 \text{ for } 0 < x < 1 \\ 2 \text{ for } 1 < x < 3 \end{cases}$ and $f(x) = \frac{3}{2}$, $f(x+3) = f(x)$ then prove that

$$f(x) = \frac{5}{3} - \frac{\sqrt{3}}{2\pi}\left\{\cos\frac{2\pi x}{3} - \frac{1}{2}\cos\frac{4\pi x}{3} + \frac{1}{4}\cos\frac{8\pi x}{3} - \ldots\right\} - \frac{3}{2\pi}\left\{\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{4}\sin\frac{8\pi x}{3} + \ldots\right\}$$

20. Find the Fourier Series for the following functions in the given interval :

(*i*) $f(x) = x + x^2$ in $-1 < x < 1$.

(*ii*) $f(x) = 1 + \sin x$ in $-1 < x < 1$.

(iii) $f(x) = \begin{cases} 0, & -8 < x < 0 \\ 4, & 0 < x < 4 \\ 0, & 4 < x < 8 \end{cases}$

(iv) $f(x) = \begin{cases} l - x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$

(v) $f(x) = \begin{cases} \pi x, & 0 \le x < 1 \\ 0, & 1 < x \le 2 \end{cases}$

(vi) $f(x) = \begin{cases} \pi x, & 0 < x \le 2 \\ \pi \text{ for } x = 0 \text{ and } x = 2 \end{cases}$

**Answers**

(i) $f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x - \frac{2}{\pi} \sum_{1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x$

(ii) $f(x) = 1 + 2\pi \sin 1 \cdot \sum_{1}^{\infty} \frac{n(-1)^n}{1 - n^2\pi^2} \sin n\pi x$

(iii) $f(x) = 1 + \frac{4}{\pi}\left(\frac{1}{1}\cos\frac{\pi x}{8} - \frac{1}{3}\cos\frac{3\pi x}{8} + \frac{1}{5}\cos\frac{5\pi x}{8} \ldots\right)$

$$+ \frac{4}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} + \ldots\right)$$

$$+ \frac{4}{\pi}\left(\sin\frac{\pi x}{8} + \frac{1}{3}\sin\frac{3\pi x}{8} + \frac{1}{5}\sin\frac{5\pi x}{8} + \ldots\right)$$

(iv) $f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right)$

$$+ \frac{l}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right)$$

(v) $f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi x}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{n}$

(vi) $f(x) = \pi - 2 \sum_{n=1}^{\infty} \sin \frac{n\pi x}{n}$

## HALF RANGE EXPANSIONS

In many Engineering problems it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier Series of period $2\pi$ or in the range $(0, l)$ in a Fourier Series of period $2l$. If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

If we extend the function $f(x)$ by reflecting it in the Y axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.





If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half range series.

The half range cosines series in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\text{where } a_0 = \frac{2}{l}\int_0^l f(x)\,dx,$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l}\,dx$$

The half-range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

***NOTE :* (i) The half-range cosine series in $(0, \pi)$ is given by**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$\text{where } a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

***NOTE :* (ii) The half-range sine series in $(0, \pi)$ is given by**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

$$\text{where } b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx.$$

**EXAMPLE 1**

*Obtain the half range sine series of the function $f(x) = kx(x - l)$ in $0 \le x \le l$.* *[Apr. 86, Mech.]*

**Solution**

We know that the half range sine series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \ldots (1)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

Now, $b_n = \frac{2}{l}\int_0^l kx(x-l) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l}\int_0^l (kx^2 - klx) \sin \frac{n\pi x}{l}\, dx$

$$= \frac{2}{l}\left[(kx^2 - klx)\left\{\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - (2kx - kl)\left\{\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\} + 2k\left\{\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]$$

$$= \frac{2}{l}\left[\frac{2k \cos n\pi}{\frac{n^3\pi^3}{l^3}} - \frac{2k}{\frac{n^3\pi^3}{l^3}}\right] = \frac{4kl^2}{n^3\pi^3}[(-1)^n - 1]$$

$$\boxed{\begin{aligned} \therefore b_n &= 0 \text{ when } n \text{ is even} \\ &= \frac{-8kl^2}{n^3\pi^3} \text{ when } n \text{ is odd}\end{aligned}}$$

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{-8kl^2}{n^3\pi^3} \sin \frac{n\pi x}{l}$$

i.e., $kx(x-l) = \sum_{n=1,3,5}^{\infty} \frac{-8kl^2}{n^3\pi^3} \sin \frac{n\pi x}{l} = -\frac{8kl^2}{\pi^3}\sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l}$

$$= -\frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{l}$$

■ **EXAMPLE 2** ■

*Obtain the sine series for the function*

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le \frac{l}{2} \\ l - x & \text{in } \frac{l}{2} \le x \le l \end{cases}$$

[Nov. 86, 91, Mech.]

● **Solution**

The sine series for the function $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots (1)$$





where $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

Now, $b_n = \frac{2}{l}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l}\, dx\right]$

$$= \frac{2}{l}\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + 1\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{\frac{-l}{2}\cos \frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}} + \frac{\frac{l}{2}\cos \frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\, 2\sin \frac{n\pi}{2} = \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} \quad \dots (2)$$

i.e., $\boxed{b_n = \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2}}$

Substituting (2) in (1) we get

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} = \frac{4l}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l}$$

■ **EXAMPLE 3** ■

*Find the half-range sine series of $f(x)$ in $(0, \pi)$, given that*

$$f(x) = \begin{cases} kx, & 0 \le x \le \frac{\pi}{2} \\ k(\pi - x), & \frac{\pi}{2} \le x \le \pi \end{cases}$$

[Nov.90, 89, Civil]

● **Solution**

The half-range sine series of $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \dots (1)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

Now, $b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} kx \sin nx\, dx + \int_{\pi/2}^{\pi} k(\pi - x) \sin nx\, dx\right]$

$$= \frac{2}{\pi}\left[\left\{kx\left(\frac{-\cos nx}{n}\right) - k\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{k(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-k)\left(\frac{-\sin nx}{n^2}\right)\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{-k\left(\frac{\pi}{2}\right)\cos\frac{n\pi}{2}}{n} + \frac{k\sin\frac{n\pi}{2}}{n^2} + \frac{k\left(\frac{\pi}{2}\right)\cos\frac{n\pi}{2}}{n} + \frac{k\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\,\frac{2k\sin\frac{n\pi}{2}}{n^2}$$

$$\boxed{b_n = \frac{4k}{\pi n^2}\sin\frac{n\pi}{2}} \quad \ldots (2)$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\frac{4k}{\pi n^2}\sin\frac{n\pi}{2}\sin nx = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}\sin nx}{n^2}$$

**EXAMPLE 4**

*Find a half-range sine series which represents $f(x) = \sin px$ for p not an integer on the interval $0 < x < \pi$* [Nov. 88, Apr. 92]

**Solution**

The half-range sine series for $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin px \sin nx\, dx$$

(Here note that '$p$' is not an integer but '$n$' is an integer.)

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}[\cos(n-p)x - \cos(n+p)x]\, dx$$

$[\because 2\sin A\sin B = \cos(A - B) - \cos(A + B)]$





$$= \frac{1}{\pi}\left[\frac{\sin(n-p)x}{n-p} - \frac{\sin(n+p)x}{n+p}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-p)\pi}{n-p} - \frac{\sin(n+p)\pi}{n+p}\right] \quad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi\cos p\pi - \cos n\pi\sin p\pi}{n-p} - \frac{\sin n\pi\cos p\pi + \cos n\pi\sin p\pi}{n+p}\right]$$

$$= \frac{-1}{\pi}\left[\frac{\cos n\pi\sin p\pi}{n-p} + \frac{\cos n\pi\sin p\pi}{n+p}\right]$$

$[\because \sin n\pi = 0$ since $n$ is an integer also $\sin p\pi \neq 0$ since $p$ is not an integer$]$

$$= \frac{(-1)^{n+1}\sin p\pi}{\pi}\left[\frac{n+p+n-p}{n^2-p^2}\right]$$

$$= \frac{(-1)^{n+1}\sin p\pi}{\pi}\cdot\frac{2n}{n^2-p^2} \quad \ldots (2)$$

i.e., $b_n = \frac{2n}{\pi}\frac{(-1)^{n+1}\sin p\pi}{n^2-p^2}$

Substituting (2) in (1) we get

$$f(x) = \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}\sin p\pi}{\pi(n^2-p^2)}\sin nx$$

$$= \frac{2\sin p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(n)(-1)^{n+1}\sin nx}{n^2-p^2}$$

**EXAMPLE 5**

*Find the half range sine series of $f(x) = 1 - x$ in $(0, 1)$.*

**Solution**

The half range sine series of $f(x)$ in $(0, 1)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n\sin n\pi x \quad \ldots (1)$$

where $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\, dx = 2\int_0^1(1-x)\sin n\pi x\, dx$

$$= 2\left[(1-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 \quad \ldots (2)$$

$$b_n = \frac{2}{n\pi}$$

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x$$

**EXAMPLE 6**

*Find the half range sine series for $f(x) = x(\pi - x)$ in $(0, \pi)$. Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$.* [MU, Oct ...]

$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{4}$

**Solution**

We know that the half range sine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \dots (1)$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-2\cos n\pi}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{4}{n^3\pi}[1 - (-1)^n]$$

$\therefore b_n = 0, \quad$ when '$n$' is even

$= \frac{8}{n^3\pi}, \quad$ when '$n$' is odd

Substituting this value of $b_n$ in (1) we get,

$$f(x) = \frac{8}{\pi}\sum_{n=1,3,5}^{\infty} \frac{1}{n^3}\sin nx \quad \dots (2)$$

$$f(x) = \frac{8}{\pi}\left[\frac{1}{1^3}\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\right]$$

Put $x = \frac{\pi}{2}$, we get

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right)$$

$$= \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots$$





*NOTE : Here $x = \frac{\pi}{2}$ is a point of continuity. Therefore, the Fourier series converges to $f\left(\frac{\pi}{2}\right)$.*

**EXAMPLE 7**

*Express $f(x)$ as a Fourier sine Series where $f(x)$ is given by*

$$f(x) = \frac{1}{4} - x, \; 0 < x < \frac{1}{2}$$
$$= x - \frac{3}{4}, \; \frac{1}{2} < x < 1$$

[MU. Apr. '95]

**Solution**

The half range Fourier sine series of $f(x)$ in (0,1) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \dots (1)$$

where
$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\, dx$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x\, dx + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\, dx\right]$$

$$= 2\left[\left\{\left(\frac{1}{4} - x\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right\}_0^{1/2}\right.$$
$$\left. + \left\{\left(x - \frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right\}_{1/2}^{1}\right]$$

$$= 2\left[\frac{\cos\frac{n\pi}{2}}{4n\pi} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{1}{4n\pi} - \frac{\cos n\pi}{4n\pi} - \frac{\cos\frac{n\pi}{2}}{4n\pi} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$b_n = 2\left[\frac{-2\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{1 - (-1)^n}{4n\pi}\right]$$

$$\therefore b_1 = \left(\frac{-4}{\pi^2} + \frac{1}{\pi}\right)$$

$$b_2 = 0 \quad \dots (2)$$

$$b_3 = \frac{4}{3^2\pi^2} + \frac{1}{3\pi}$$

Substituting (2) in (1), we get

$$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \dots$$

**EXAMPLE 8**

*Find the half-range sine series for the function $f(x) = e^x$ in $0 < x < 1$* [Mano, Nov.

**Solution**

The half-range sine series of $f(x)$ in (0, 1) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where
$$b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x\, dx = 2\int_0^1 e^x \sin n\pi x\, dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(\sin n\pi x - n\pi \cos n\pi x)\right]_0^1$$

$$= 2\left[\frac{-n\pi \cdot \cos n\pi \cdot e^1}{n^2\pi^2+1} + \frac{n\pi}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}[(-1)^{n+1} \cdot n\pi \cdot e + n\pi] \quad \ldots (2)$$

substituting (2) in (1) we get

$$f(x) = 2\pi \sum_{n=1}^{\infty} \frac{n[1 + e(-1)^{n+1}]}{1 + n^2\pi^2} \cdot \sin n\pi x$$

**EXAMPLE 9**

*Check that for $0 < x < l$,*

$$1 = \frac{4}{\pi}\left(\sin \frac{\pi x}{l} + \frac{1}{3}\sin \frac{3\pi x}{l} + \frac{1}{5}\sin \frac{5\pi x}{l} + \ldots\right)$$ [Mano, Nov. '95]

**Solution**

The half range sine series of $f(x) = 1$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l}\left[\int_0^l \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l = \frac{-2}{n\pi}[\cos n\pi - 1]$$

$$= \frac{-2}{n\pi}[(-1)^n - 1]$$





$$\therefore b_n = 0, \text{ when '}n\text{' is even}$$
$$= \frac{4}{n\pi}, \text{ when '}n\text{' is odd} \quad \ldots (2)$$

Substituting (2) in (1) we get

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} \cdot \sin \frac{n\pi x}{l}$$

$$1 = \frac{4}{\pi}\left[\sin \frac{\pi x}{l} + \frac{1}{3}\sin \frac{3\pi x}{l} + \ldots\right]$$

**EXAMPLE 10**

*Find the half-range sine series for $(x-1)^2$ in (0, 1).* [BDN, Apr. '98]

**Solution**

The sine series of $f(x)$ in (0, 1) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \ldots (1)$$

where
$$b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x\, dx = 2\int_0^1 (x-1)^2 \sin n\pi x\, dx$$

$$= 2\left[(x-1)^2\left(\frac{-\cos n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\frac{2\cos n\pi}{n^3\pi^3} + \frac{1}{n\pi} - \frac{2}{n^3\pi^3}\right] \quad \ldots (2)$$

Substituting (2) in (1) we get

$$f(x) = 2\sum_{n=1}^{\infty}\left[\frac{2\cos n\pi}{n^3\pi^3} + \frac{1}{n\pi} - \frac{2}{n^3\pi^3}\right] \sin n\pi x$$

**EXAMPLE 11**

*Obtain a half-range cosine series of the function*

$$f(x) = \begin{cases} kx, & \text{for } 0 \le x < \frac{l}{2} \\ k(l-x), & \frac{l}{2} \le x \le l \end{cases}$$

**Solution**

We know that a half-range cosine series for $f(x)$ in $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \ldots (1)$$

Now, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$

$$= \frac{2}{l}\left[\left(k\frac{x^2}{2}\right)_0^{l/2} + k\left(lx - \frac{x^2}{2}\right)_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

i.e., $\boxed{a_0 = \frac{kl}{2}}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{kx\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - k\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{k(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + k\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left(\frac{\frac{kl}{2}\sin\frac{n\pi}{2}}{\frac{n\pi}{l}}\right) + \left(k\frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right) - \frac{k}{\frac{n^2\pi^2}{l^2}} - \frac{k\cos n\pi}{\frac{n^2\pi^2}{l^2}} - \frac{\frac{kl}{2}\sin\frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{k\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\left[\frac{2k\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}} - \frac{k}{\frac{n^2\pi^2}{l^2}} - \frac{k\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right]$$

$$\therefore a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] \qquad [\because \cos n\pi = (-1)^n]$$

When '$n$' is odd, $\cos\frac{n\pi}{2} = 0$.

$\therefore a_n = 0$ when n is odd. ... (3)

When $n$ is even

$a_2 = \frac{2kl}{2^2\pi^2}[2\cos\pi - 1 - 1] = -\frac{8kl}{2^2\pi^2}$ ... (4)

$a_4 = \frac{2kl}{4^2\pi^2}[2\cos 2\pi - 1 - 1] = 0$ ... (5)





$a_6 = \frac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - 1]$ ... (6)

$= \frac{-8kl}{6^2\pi^2}$ and so on.

Substituting (2), (3), (4) (5) and (6) in (1) we get

$$\therefore f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\right)$$

■ **EXAMPLE 12** ■

*Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$.*

*[Apr. 91, 87, Mech., Apr. 89, ECE]*

**Solution**

The half-range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \qquad \ldots (1)$$

Now, $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^\pi = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$ ... (2)

i.e., $\boxed{a_0 = \pi}$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$\therefore a_n = 0$ when $n$ is even,

$= \frac{-4}{n^2\pi}$ when $n$ is odd. ... (3)

Substituting (2) and (3) in (1) we get

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty}\frac{-4}{n^2\pi}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

## ■ EXAMPLE 13 ■

***Find a cosine series for the function.***

$$f(x) = \begin{cases} x & \text{in } 0 \le x < \dfrac{\pi}{2} \\ \pi - x & \text{in } \dfrac{\pi}{2} \le x < \pi \end{cases}$$

[Apr.

**● Solution**

The cosine series for the function $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$\boxed{a_0 = \frac{\pi}{2}} \qquad \ldots (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\frac{\pi}{2}\cdot\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} \quad \frac{-\frac{\pi}{2}\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right] = \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

When $n$ is odd $a_n = 0$, *i.e.*, $a_1 = a_3 = a_5 = \ldots 0$

When $n$ is even ... (3)

$$a_2 = \frac{2}{2^2\pi}[2\cos\pi - 1 - 1] = -\frac{2}{\pi\cdot 1^2} \qquad \ldots (4)$$





$$a_4 = \frac{2}{4^2\pi}[2\cos 2\pi - 1 - 1] = 0 \qquad (\because \cos 2\pi = 1) \ldots (5)$$

$$a_6 = \frac{2}{6^2\pi}[2\cos 3\pi - 1 - 1] = \frac{-2}{\pi\cdot 3^2} \qquad \ldots (6)$$

and so on.

Substituting (2), (3), (4), (5) and (6) in (1) we get

$$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right]$$

## ■ EXAMPLE 14 ■

***Using an appropriate Fourier expansion show that in the range $(0, \pi)$, the function $\sin x$ can be expressed as*** [Apr. 93]

$$\frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots - \frac{\cos 2nx}{4n^2 - 1}\right)$$

**● Solution**

Here the range is $(0, \pi)$ and the expansion contains only cosine terms. Therefore we have to expand $\sin x$ in a half range Fourier cosine series in $(0, \pi)$.

We know that the half range cosine series of $f(x)$ in $(0, \pi)$ is given by ... (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}[-\cos\pi + \cos 0]$$

$$= \frac{2}{\pi}[1 - (-1)] \qquad [\because \cos\pi = -1]$$

$$a_0 = \frac{4}{\pi} \qquad \ldots (2)$$

Now

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

When $n$ is odd,

$$a_n = \frac{-1}{\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$= 0 \qquad [\because \text{ when n is odd } \cos(1+n)\pi = 1 = \cos(1-n)\pi]$$

i.e., $a_n = 0$ provided $n \neq 1$ and $n$ is odd.

when $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= -\frac{1}{2\pi}[1-1] = 0$$

i.e., $a_1 = 0$ ...(4)

When $n$ is even, $(1+n)$ and $(1-n)$ is odd

$$a_n = -\frac{1}{\pi}\left[\frac{-1}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$[\because \cos(1+n)\pi = -1;\ \cos(1-n)\pi = -1]$$

$$= \frac{1}{\pi}\left[\frac{2}{1+n} + \frac{2}{1-n}\right] = \frac{2}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{1-n^2}\right] \qquad \text{...(5)}$$

Substituting (2), (3), (4) and (5) in (1), we get

$$f(x) = \frac{2}{\pi} + \sum_{n=2,4,6}^{\infty} \frac{4}{\pi(1-n^2)} \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6}^{\infty} \frac{1}{n^2-1} \cos nx$$

$$= \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2-1} \cos 2nx\right]$$

i.e., $$\sin x = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \dots - \frac{\cos 2nx}{4n^2-1}\right]$$

**■ EXAMPLE 15 ■**

*Expand $f(x) = \cos x,\ 0 < x < \pi$ in a Fourier sine series.*

*[Nov. '86, Apr. '88, Nov. '91]*

**● Solution**

We know that the Fourier sine series of $f(x)$ in $0 < x < \pi$ is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx \qquad \text{...(1)}$$





To find $a_n$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} \frac{1}{2}\{\sin(n+1)x + \sin(n-1)x\} \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[-\left(\frac{\cos n\pi \cos \pi - \sin n\pi \sin \pi}{n+1}\right) - \left(\frac{\cos n\pi \cos \pi + \sin n\pi \sin \pi}{n-1}\right) + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi + (n+1)\cos n\pi + (n-1) + n + 1}{(n+1)(n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n \cos n\pi + 2n}{n^2-1}\right] \qquad [\text{ provided } n \neq 1]$$

$$= \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right]$$

$$\left.\begin{aligned} a_n &= 0, && \text{when '}n\text{' is odd} \\ &= \frac{4}{\pi(n^2-1)} && \text{when '}n\text{' is even} \end{aligned}\right\} \qquad \text{... (2)}$$

$$\text{Now } a_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0 \qquad \text{... (3)}$$

Substituting (2) and (3) in (1), we get

$$f(x) = a_1 \sin x + \sum_{n=2}^{\infty} a_n \sin nx = 0 + \sum_{n=2,4,}^{\infty} \frac{4n}{\pi(n^2-1)} \cdot \sin nx$$

$$= \frac{4}{\pi}\left[\frac{2}{3}\sin 2x + \frac{4}{15}\sin 4x + \frac{6}{35}\sin 6x + \dots\right]$$

$$\cos x = \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \dots\right]$$

**■ EXAMPLE 16 ■**

*Expand* $f(x) = \begin{cases} \sin x, & 0 < x < \dfrac{\pi}{4} \\ \cos x, & \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases}$ *in a series of sines.* [Apr...]

**● Solution**

We know that sine series for $f(x)$ in $\left(0, \dfrac{\pi}{2}\right)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin 2nx$$

To find $b_n$

$$b_n = \frac{1}{\frac{1}{2}\left(\frac{\pi}{2}\right)} \int_0^{\pi/2} f(x) \cdot \sin 2nx \, dx$$

$$= \frac{4}{\pi}\left[\int_0^{\pi/4} \sin x \sin 2nx \, dx + \int_{\pi/4}^{\pi/2} \cos x \sin 2nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/4} \{\cos(2n-1)x - \cos(2n+1)x\}\, dx + \int_{\pi/4}^{\pi/2} \{\sin(2n+1)x + \sin(2n-1)x\}\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\sin(2n-1)x}{2n-1} - \frac{\sin(2n+1)x}{2n+1}\right\}_0^{\pi/4} + \left\{\frac{-\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n+1}\right\}_{\pi/4}^{\pi/2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{4}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{4}}{2n-1}\right]$$

$$\left[\because \cos(2n \pm 1)\frac{\pi}{2} = 0\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{4} - \cos\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n-1} - \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{4} + \cos\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n+1}\right.$$

$$\left. + \frac{\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n+1} + \frac{\cos\frac{n\pi}{2}\cos\frac{\pi}{4} + \sin\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n-1}\right]$$





When '$n$' is even,

$$= \frac{2}{\pi}\left\{\frac{-\cos\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}}}{2n-1} - \frac{\cos\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}}}{2n+1} + \frac{1}{\sqrt{2}}\cdot\frac{\cos\frac{n\pi}{2}}{2n+1} + \frac{1}{\sqrt{2}}\cdot\frac{\cos\frac{n\pi}{2}}{2n-1}\right\}$$

$$\left[\because n \text{ is even } \sin\frac{n\pi}{2} = 0\right]$$

... (2)

$$b_n = \frac{2}{\pi}[0] = [0]$$

When '$n$' is odd

$$b_n = \frac{2}{\pi} \times \frac{1}{\sqrt{2}}\left[\frac{\sin\frac{n\pi}{2}}{2n-1} - \frac{\sin\frac{n\pi}{2}}{2n+1} + \frac{-\sin\frac{n\pi}{2}}{2n+1} + \frac{\sin\frac{n\pi}{2}}{2n-1}\right]$$

$$\left[n \text{ is odd, } \cos\frac{n\pi}{2} = 0\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-2\sin\frac{n\pi}{2}}{(2n+1)} + \frac{2\sin\frac{n\pi}{2}}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{\pi} 2\sin\frac{n\pi}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\cdot\sin\frac{n\pi}{2}\left[\frac{2n+1-2n+1}{4n^2-1}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\sin\frac{n\pi}{2}\left[\frac{2}{4n^2-1}\right]$$

... (3)

$$b_n = \frac{4\sqrt{2}}{\pi(4n^2-1)}\sin\frac{n\pi}{2} \text{ when '}n\text{' is odd.}$$

Substituting (2) and (3) in (1), we get

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{4\sqrt{2}}{(4n^2-1)\pi}\cdot\sin\frac{n\pi}{2}\cdot\sin 2nx$$

$$= \frac{4\sqrt{2}}{\pi}\left[\frac{\sin 2x}{3} - \frac{1}{35}\sin 6x + \ldots\right]$$

**■ EXAMPLE 17 ■** [Apr. '88]

*Expand $x \sin x$ as a sine series in $0 < x < \pi$*

**● Solution**

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ... (1)

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\sin nx\,dx$$

$$= \frac{-2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} x\,[\cos(n-1)x-\cos(n+1)x]\,dx$$

$$= \frac{-1}{\pi}\left\{\int_0^{\pi} x\cos(n-1)x\,dx-\int_0^{\pi} x\cos(n+1)x\,dx\right\}$$

$$= \frac{-1}{\pi}\left[\left\{x\frac{\sin(n-1)x}{(n-1)}-1\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right\}_0^{\pi}-\left\{x\frac{\sin(n+1)x}{n+1}-1\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right\}_0^{\pi}\right]$$

$$= \frac{-1}{\pi}\left[\frac{\cos(n-1)\pi}{(n-1)^2}-\frac{1}{(n-1)^2}-\frac{\cos(n+1)\pi}{(n+1)^2}+\frac{1}{(n+1)^2}\right]$$

$$= \frac{-1}{\pi}\left[\frac{-\cos n\pi}{(n-1)^2}-\frac{1}{(n-1)^2}+\frac{\cos n\pi}{(n+1)^2}+\frac{1}{(n+1)^2}\right]$$

$$= \frac{-1}{\pi}\left[\frac{-(n+1)^2\cos n\pi-(n+1)^2+(n-1)^2\cos n\pi+(n-1)^2}{(n-1)^2(n+1)^2}\right]$$

$$= \frac{-1}{\pi}\left[\frac{-4n\cos n\pi-4\pi}{(n^2-1^2)^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{4n\cos n\pi+4n}{(n^2-1)^2}\right] \quad [\text{ provided } n\neq 1]$$

$$b_n = 0, \quad \text{when '}n\text{' is odd}$$

$$= \frac{8n}{\pi(n^2-1)^2}, \quad \text{when '}n\text{' is even}$$

When $n = 1$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\sin x\,dx \quad [\text{Putting } n = 1 \text{ in (2)}]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin^2 x\,dx = \frac{2}{\pi}\int_0^{\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx-\int_0^{\pi} x\cos 2x\,dx\right]$$





$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi}-\left\{x\left(\frac{\sin 2x}{2}\right)-1\left(\frac{-\cos 2x}{4}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}-\frac{1}{4}+\frac{1}{4}\right] = \frac{\pi}{2} \quad \ldots (4)$$

Substituting (3) and (4) in (1), we get

$$f(x) = b_1\sin x+\sum_{n=2,4}^{\infty} b_n\sin nx$$

$$= \frac{\pi}{2}\sin x+\sum_{n=2,4}^{\infty} b_n\sin nx$$

$$= \frac{\pi}{2}\sin x+\sum_{n=2,4}^{\infty}\frac{8n}{\pi(n^2-1)^2}\sin nx$$

$$\therefore x\sin x = \frac{\pi}{2}\sin x+\frac{8}{\pi}\sum_{n=2,4}^{\infty}\frac{n\sin nx}{(n^2-1)^2}$$

**EXAMPLE 18**

***Find-half range sine series for the function $f(x)$ defined as***

$$f(x) = \begin{cases} x-1, & 0\le x\le 1 \\ 1-x, & 1\le x\le 2 \end{cases}.$$

**Solution**

The half-range sine series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

Here $l = 2$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2} \quad \ldots (1)$$

**To find $b_n$**

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 f(x)\sin\frac{n\pi x}{2}\,dx+\int_1^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 (x-1)\sin\frac{n\pi x}{2}\,dx+\int_1^2 (1-x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[ \left\{ (x-1)\left( \frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - (1)\left( \frac{-\sin \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}} \right) \right\}_0^1 + \left\{ (1-x)\left( \frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - (-1)\left( \frac{-\sin \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}} \right) \right\} \right.$$

$$= \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} - \frac{1}{\frac{n\pi}{2}} + \frac{\cos n\pi}{\frac{n\pi}{2}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}$$

$$= \frac{8 \sin \frac{n\pi}{2}}{n^2\pi^2} + \frac{2}{n\pi}\left[ (-1)^n - 1 \right]$$

$b_n = 0$, when '$n$' is even.

$$b_n = \frac{8 \sin \frac{n\pi}{2}}{n^2\pi^2} - \frac{4}{n\pi} \quad \text{when '}n\text{' is odd.}$$

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} \left( \frac{8 \sin \frac{n\pi}{2}}{n^2\pi^2} - \frac{4}{n\pi} \right) \cdot \sin \frac{n\pi x}{2}$$

$$f(x) = \left( \frac{8}{\pi^2} - \frac{4}{\pi} \right) \sin \frac{\pi x}{2} + \left( \frac{8}{9\pi^2} - \frac{4}{3\pi} \right) \sin \frac{3\pi x}{2}$$

**EXAMPLE 19**

*Prove that in the interval $0 < x < \pi$,*

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[ \frac{\sin x}{a^2+1} - \frac{2 \sin 2x}{a^2+4} + \frac{3 \sin 3x}{a^2+9} \dots \right].$$

[Apr. '86, Nov. '88]

**Solution**

The half range Fourier sine series of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

**To find $b_n$**

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} \cdot \sin nx \, dx$$





$$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[ \int_0^{\pi} e^{ax} \sin nx \, dx - \int_0^{\pi} e^{-ax} \sin nx \, dx \right]$$

$$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[ \left\{ \frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx) \right\}_0^{\pi} - \left\{ \frac{e^{-ax}}{a^2+n^2}(-a \sin nx - n \cos nx) \right\}_0^{\pi} \right]$$

$$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[ \frac{-e^{a\pi}}{a^2+n^2} \cdot n(-1)^n + \frac{n}{a^2+n^2} + \frac{e^{-a\pi}}{a^2+n^2} \cdot n(-1)^n - \frac{n}{a^2+n^2} \right]$$

$$= \frac{2n(-1)^n}{\pi(e^{a\pi} - e^{-a\pi})}\left[ \frac{e^{-a\pi} - e^{a\pi}}{n^2+a^2} \right] = \frac{2n(-1)^{n+1}}{\pi(e^{a\pi} - e^{-a\pi})} \cdot \frac{e^{a\pi} - e^{-a\pi}}{n^2+a^2} \quad \dots (2)$$

$$= \frac{2n(-1)^{n+1}}{\pi(n^2+a^2)}$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{\pi(n^2+a^2)} \cdot \sin nx$$

i.e., $$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[ \frac{\sin x}{a^2+1} - \frac{2 \sin 2x}{a^2+4} + \frac{3 \sin 3x}{a^2+9} \dots\dots \right]$$

**EXAMPLE 20**

*Show that in $0 \le x \le \pi$,* $x(\pi - x) = \dfrac{\pi^2}{6} - \left( \dfrac{\cos 2x}{1^2} + \dfrac{\cos 4x}{2^2} + \dfrac{\cos 6x}{3^2} + \dots \right)$

[Apr. '88, Nov. '89, Apr. '91]

**Solution**

The half range cosine series of the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots (1)$$

Here $f(x) = x(\pi - x)$ (or) $\pi x - x^2$

**To find $a_0$**

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \, dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \, dx$$

$$= \frac{2}{\pi}\left[ \frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi}\left[ \frac{\pi^3}{2} - \frac{\pi^3}{3} \right]$$

$$= \frac{2}{\pi} \cdot \frac{\pi^3}{6} = \frac{\pi^2}{3} \quad \dots (2)$$

**To find $a_n$**

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n^2} - \frac{\pi}{n^2}\right] = \frac{-2}{n^2}\left[1 + (-1)^n\right]$$

$$a_n = \begin{cases} 0, & \text{when '}n\text{' is odd} \\ \frac{-4}{n^2}, & \text{when '}n\text{' is even} \end{cases}$$

Substituting (2) and (3) in (1), we get

$$f(x) = \frac{\pi^2}{6} + \sum_{n=2,4}^{\infty} \frac{-4}{n^2}\cdot\cos nx$$

$$= \frac{\pi^2}{6} - 4\sum_{n=2,4}^{\infty}\frac{1}{n^2}\cdot\cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - 4\left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \ldots\ldots\right)$$

## EXAMPLE 21

***Obtain cosine series for x sin x in $0 < x < \pi$*** [Nov. '86, Apr. '89]

**Solution**

The half range cosine series for $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \quad \ldots(1)$$

Here $f(x) = x\sin x$ defined in $(0, \pi)$.

**To find $a_0$**

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\,dx$$

$$= \frac{2}{\pi}\left[(x(-\cos x) - 1(-\sin x)\right]_0^\pi = \frac{2}{\pi}[\pi] = 2 \quad \ldots(2)$$

**To find $a_n$**

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx \quad \ldots(3)$$

$$= \frac{1}{\pi}\int_0^\pi x\{\sin(n+1)x - \sin(n-1)x\}\,dx$$





$$= \frac{1}{\pi}\left[\int_0^\pi x\sin(n+1)x\,dx - \int_0^\pi x\sin(n-1)x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos(n+1)x}{n+1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right\}_0^\pi - \left\{x\left(\frac{-\cos(n-1)x}{n-1}\right) - 1\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right\}_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right]$$

$$= \frac{1}{\pi}\cdot\pi\left[-\frac{\cos n\pi\cos\pi}{n+1} + \frac{\cos n\pi\cos\pi}{n-1}\right] \quad [\because \sin n\pi = 0]$$

$$= -(-1)^n\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$= (-1)^{n+1}\left[\frac{n+1-n+1}{n^2-1}\right]$$

$$= \frac{2(-1)^{n+1}}{n^2-1}, \text{ provided } n \neq 1 \quad \ldots(4)$$

When $n = 1$, From (3)

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^\pi x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^\pi$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{2}\right) = -\frac{1}{2} \quad \ldots(5)$$

Substituting (2), (4), (5) in (1), we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{(-1)^n\cos nx}{1-n^2}$$

■ EXAMPLE 22 ■

*Obtain cosine series for*

$$f(x) = \begin{cases} \cos x \text{ in } 0 < x < \frac{\pi}{2} \\ 0 \text{ in } \frac{\pi}{2} < x < \pi \end{cases}$$

[Apr. '889, Nov ...]

● **Solution**

The half range cosine series for $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=2}^{\infty} a_n \cos nx$$

$$\text{Here } f(x) = \begin{cases} \cos x \text{ in } 0 < x < \frac{\pi}{2} \\ 0 \text{ in } \frac{\pi}{2} < x < \pi \end{cases}$$

**To find $a_0$**

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} 0\,dx\right] = \frac{2}{\pi}[\sin x]_0^{\pi/2}$$

$$= \frac{2}{\pi} \quad \ldots (2)$$

**To find $a_n$**

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\cdot\cos nx\,dx + 0\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \{\cos(n+1)x + \cos(n-1)x\}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$





$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}\cdot\sin\frac{\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}\cdot\sin\frac{\pi}{2}}{n-1}\right] \left[\because \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{\cos\frac{n\pi}{2}}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{\cos\frac{n\pi}{2}}{\pi}\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{-2\cos\frac{n\pi}{2}}{\pi(n^2-1)} \text{ provided } n \neq 1. \quad \ldots (3)$$

When $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\cdot\cos x\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\,dx\right] = \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}x + \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{2}{\pi}\left[\frac{\pi}{4}\right] = \frac{1}{2} \quad \ldots (4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x - \sum_{n=2}^{\infty} \frac{2\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cos nx$$

■ EXAMPLE 23 ■

**Obtain the Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$ and hence deduce the value of $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \ldots$** *[Anna Univ. Oct. '01]*

● **Solution**

We know that the Fourier cosine series of $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx \quad \ldots (1)$$

Now $$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\left[x(-\cos x) - (-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}[\pi] = 2 \quad \ldots (2)$$

$$\therefore \boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cdot\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\times\frac{1}{2}\int_0^\pi x\{\sin(1+n)x+\sin(1-n)x\}\,dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi x\sin(1+n)x\,dx+\int_0^\pi x\sin(1-n)x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos(1+n)x}{1+n}\right)-\left(\frac{-\sin(1+n)x}{(1+n)^2}\right)\right\}_0^\pi + \left\{x\left(\frac{-\cos(1-n)x}{1-n}\right)-\left(\frac{-\sin(1-n)x}{(1-n)^2}\right)\right\}_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n}-\frac{\pi\cos(1-n)\pi}{1-n}\right]$$

$$= \left[\frac{\cos n\pi}{1+n}+\frac{\cos n\pi}{1-n}\right] = (-1)^n\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$a_n = \frac{2(-1)^n}{1-n^2} \text{ provided } n\neq 1 \qquad \ldots(3)$$

When $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^\pi x\cdot\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-\left(\frac{-\sin 2x}{4}\right)\right]_0^\pi = \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$a_1 = -\frac{1}{2} \qquad \ldots(4)$$

Substituting (2), (3), and (4) in (1), we get

$$f(x) = 1-\frac{1}{2}\cos x-2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

i.e., $x\sin x = 1-\frac{1}{2}\cos x-2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n+1)(n-1)}\cos nx$

Put $x=\frac{\pi}{2}$, we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1-\frac{1}{2}\cos\frac{\pi}{2}-2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)(n+1)}\cos\frac{n\pi}{2}$$





$$\frac{\pi}{2} = 1-2\sum_{n=2,4}^{\infty}\frac{(-1)^n}{(n-1)(n+1)}\cos\frac{n\pi}{2} \qquad \left[\because \cos\frac{n\pi}{2}=0 \text{ when } n \text{ is odd}\right]$$

i.e., $\frac{\pi}{2} = 1-2\left[-\frac{1}{1\cdot3}+\frac{1}{3\cdot5}-\frac{1}{5\cdot7}+\ldots\ldots\right]$

i.e., $1+2\left[\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\ldots\ldots\right] = \frac{\pi}{2}$

i.e., $\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\ldots\ldots = \frac{\pi-2}{4}$

■ EXAMPLE 24 ■

*Express $f(x)=x$ as a half range cosine series in $(0, l)$ and deduce* $\sum_1^\infty\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$. *[BDN, Nov. '95]*

● Solution

The half range cosine series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \qquad \ldots(1)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left(\frac{x^2}{2}\right)_0^l$

$$a_0 = l \qquad \ldots(2)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\times\frac{l^2}{n^2\pi^2}[\cos n\pi-1]$$

$$a_n = \frac{2l}{n^2\pi^2}[(-1)^n-1]$$

$$\left.\begin{aligned} a_n &= 0, && \text{when '}n\text{' is even}\\ &= \frac{-4l}{n^2\pi^2}, && \text{when '}n\text{' is odd}\end{aligned}\right\} \qquad \ldots(3)$$

Substituting (2) and (3) in (1) we get

$$f(x) = \frac{l}{2} + \sum_{n=1,3,5}^{\infty} -\frac{4l}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$\text{i.e.,} \quad = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}$$

Here $x = 0$ is a point of discontinuity which is an end point of the interval $0 < x < l$. Therefore, the value of F.C.S at $x = 0$ is the average value of $f(x)$ at $x = 0$ and $x = l$.

Putting $x = 0$ in (4), we get,

$$\frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{f(0)+f(l)}{2} = \frac{l}{2}$$

put $x = 0$, we get

$$0 = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\text{i.e.,} \sum_{1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

**EXAMPLE 25**

*Expand $f(x) = x(l-x)$ over the interval $(0, l)$ as a Fourier cosine series of period '$l$'.* [BU, Apr. '97]

**Solution**

The cosine series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \dots (1)$$

$$\text{where } a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right] = \frac{2}{l}\left[\frac{3l^3 - 2l^3}{6}\right]$$

$$a_0 = \frac{l^2}{3} \quad \dots (2)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx\right]$$





$$= \frac{2}{l}\left[\left[(lx - x^2)\right]\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\times l\left[\frac{-\cos n\pi}{\frac{n^2\pi^2}{l^2}} - \frac{1}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{-2l^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$\left.\begin{aligned} a_n &= 0, && \text{when '}n\text{' is odd} \\ &= \frac{-4l^2}{n^2\pi^2}, && \text{when '}n\text{' is even}\end{aligned}\right\} \quad \dots (3)$$

Substituting (2) and (3) in (1), we get

$$f(x) = \frac{l^2}{6} + \sum_{2,4,6}^{\infty}\frac{-4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} = \frac{l^2}{6} - \frac{4}{\pi^2}\sum_{2,4,6}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}$$

**EXAMPLE 26**

*Expanding $x(\pi - x)$ as a sine series in $(0, \pi)$ show that* [BU, Oct. '95]

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$$

**Solution**

The half-range Fourier sine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \dots (1)$$

$$\text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{n^3\pi}\left[-2\cos n\pi + 2\right] = \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

$$b_n = 0, \quad \text{when '}n\text{' is even}$$
$$= \frac{8}{n^3\pi}, \quad \text{when '}n\text{' is odd}$$

Substituting (2) in (1), we get

$$f(x) = \sum_{1,3,5}^{\infty} \frac{8}{n^3\pi} \sin nx$$

$$f(x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\right]$$

put $x = \frac{\pi}{2}$ in (3), we get

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \dots\right]$$

$$\text{But } f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi}{4}$$

substituting (5) in (4) we get

$$\therefore \frac{\pi}{4} = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \dots\right]$$

i.e., $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \dots = \frac{\pi^2}{32}$

**EXAMPLE 27**

***Find the sine series of f (x) = x in (0, l)*** *[MKU, Apr. '95]*

**Solution**

The Fourier sine series of $f(x)$ in $(0, l)$ is given by,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad \dots (1)$$

$$\text{where } a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l}\, dx$$





$$= \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l\cos n\pi}{\frac{n\pi}{l}}\right] = \frac{2l}{n\pi}(-1)^{n+1} \quad \dots (2)$$

substituting (2) in (1) we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l} = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$

**EXAMPLE 28**

***Develope in Fourier expansion of sin $\left(\frac{\pi x}{l}\right)$ in a cosine series for $0 < x < l$.*** *[BDN, Nov. '97]*

**Solution**

The half range cosine series of $f(x)$ in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \dots (1)$$

$$\text{where } a_0 = \frac{2}{l}\int_0^l f(x)\, dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\, dx \quad \left[\because f(x) = \sin\frac{\pi x}{l}\right]$$

$$= \frac{2}{l}\left[\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right]_0^l$$

$$= \frac{2}{\pi}[1+1] = \frac{4}{\pi} \quad [\because \cos\pi = -1] \quad \dots (2)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\, dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\int_0^l \left\{\left[\sin\left(\frac{\pi}{l} + \frac{n\pi}{l}\right)x + \sin\left(\frac{\pi}{l} - \frac{n\pi}{l}\right)x\right] dx\right\}$$

$$= \frac{1}{l}\left[\frac{-\cos\frac{(1+n)\pi x}{l}}{\frac{(1+n)\pi}{l}} - \frac{\cos\frac{(1-n)\pi x}{l}}{\frac{(1-n)\pi}{l}}\right]_0^l$$

$$= \frac{-\cos(1+n)\pi}{(1+n)\pi} - \frac{\cos(1-n)\pi}{(1-n)\pi} + \frac{1}{(1+n)\pi} + \frac{1}{(1-n)\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1-n)\cos n\pi + (1+n)\cos n\pi + (1-n) + (1+n)}{1-n^2}\right]$$

$$a_n = \frac{1}{\pi(1-n^2)}[2\cos n\pi + 2] = \frac{2}{\pi(1-n^2)}[(-1)^n + 1]$$

$$\left.\begin{aligned} a_n &= 0, && \text{'}n\text{' is odd} \\ &= \frac{4}{\pi(1-n^2)}, && \text{'}n\text{' is even}\end{aligned}\right\}$$

Substituting (2) and (3) in (1) we get

$$f(x) = \frac{2}{\pi} + \sum_{n=2,4,\ldots}^{\infty} \frac{4}{\pi(1-n^2)} \cdot \cos\frac{n\pi x}{l}$$

## • EXERCISES

1. Obtain cosine series expansion of $f(x) = (x-1)^3$ in (0, 1) and hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$ [Apr. 90]

   [Ans. $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$]

2. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$ and hence find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ [Nov. 89, Mech.]

   [Ans. $f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right)$]

3. Show that $\log\left(2\sin\frac{x}{2}\right) = -\sum_{n=1}^{\infty}\frac{\cos nx}{n}$ if $0 < x < \pi$. [Nov. 87, ECE]

4. Find the half-range cosine series of $f(x) = x(2-x)$ in $0 \le x \le 2$. Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ [Nov. 90]

5. Obtain the cosine series of

$$f(x) = \begin{cases} \frac{\pi}{2}x & \text{in } \left(0, \frac{\pi}{2}\right) \\ \frac{\pi}{2}(\pi - x) & \text{in } \left(\frac{\pi}{2}, \pi\right)\end{cases}$$ [Apr. 90, ECE]

   [Ans. $f(x) = \frac{\pi^2}{8} + \sum_{n=1}^{\infty}\left(\frac{2\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right)\cos nx$]





6. Obtain the cosine series of the function [Nov. 87, ECE]

$$f(x) = \begin{cases} 0 \text{ for } 0 < x < 1 \\ 1 \text{ for } 1 < x < 2\end{cases}$$

   [Ans. $f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}$]

7. Find the half-range sine series for the function [Nov. 90, Civil]

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ (\pi - x), & \frac{\pi}{2} < x < \pi\end{cases}$$

   [Ans. $f(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right)$]

8. In the range (0, 2) the function $f(x)$ is defined as follows [Apr. 87, Mech.]

$$f(x) = \begin{cases} \pi x, & 0 \le x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x \le 2\end{cases}$$

   Show that $f(x) = 2\left(\sin \pi x - \frac{\sin 2\pi x}{3} + \frac{\sin 3\pi x}{3} + \ldots\right)$

9. Show that the half-range sine series of the function $lx - x^2$ in $(0, l)$ is $\frac{8l^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{l}$ [Apr. 89, 90, Mech., Nov. 91, Civil]

10. Expand $f(x) = x^2$ in sine series valid for the interval $0 < x < \pi$. [Nov. 89, ECE, Nov. 91, Civil]

   [Ans. $x^2 = \frac{2}{\pi}\left\{\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{4}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin x - \frac{\pi^2}{2}\sin 4x + \ldots\right\}$]

11. Obtain the sine series for unity in $0 < x < \pi$. Hence show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ [Nov. 88, Mech.]

   [Ans. $f(x) = 1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$]

12. Find the half-range sine series of the following functions :
   (i) $x^3$ in $0 \le x \le \pi$ [Apr. 91]
   (ii) $x^2$ in $0 < x < 1$ [Apr. 87]
   (iii) $x$ in $0 < x < 2$ [Apr. 88, ECE]

(iv) $f(t)=\begin{cases}\dfrac{2kt}{l} & \text{when } 0<t<\dfrac{l}{2}\\[2mm] \dfrac{2k}{t}(l-t) & \text{when } \dfrac{l}{2}<t<l\end{cases}$ [Apr. 87,

**Answers**

(i) $f(x)=2\sum_{n=1}^{\infty}(-1)^n\left[\dfrac{6}{n^3}-\dfrac{\pi^2}{n}\right]\sin nx.$

(ii) $f(x)=2\left[\left(\dfrac{\pi^2-4}{\pi^3}\right)\sin \pi x-\dfrac{1}{2\pi}\sin 2x+\left(\dfrac{3^2\pi^2-4}{3^3\pi^3}\right)\sin 3x-\dfrac{1}{4x}\sin 4x+\right.$

(iii) $f(x)=\dfrac{-4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\cdot\sin\dfrac{n\pi x}{2}$

(iv) $f(t)=\dfrac{8k}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin\dfrac{n\pi t}{l}.$

13. Find the Fourier sine series for the function

$$f(x)=\begin{cases}1, & 0\le x\le \dfrac{l}{2}\\[2mm] 0, & \dfrac{l}{2}\le x\le l\end{cases}$$

[Ans. $f(x)=\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin^2\dfrac{n\pi}{4}\sin\dfrac{n\pi x}{l}$

14. Show that the Fourier sine series for $e^x$ in $0<x<1$ is

$$\pi\sum_{n=1}^{\infty}\frac{n[1-e(-1)^n]}{1+n^2\pi^2}\sin n\pi x.$$

15. Show that the Fourier cosine series for the function $x\sin x$ in the interval $(0,\pi)$ is

$$1-\frac{1}{2}\cos x-2\left(\frac{\cos 2x}{1\cdot 3}-\frac{\cos 3x}{2\cdot 4}+\frac{\cos 4x}{3\cdot 5}-\dots\right)$$

## COMPLEX OR EXPONENTIAL FORM OF FOURIER SERIES

We know that the Fourier Series of the function $f(x)$ in the interval $c<x<c+2l$ is given by

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+\dots+a_n\cos\frac{n\pi x}{l}+\dots$$
$$+b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}+\dots+b_n\sin\frac{n\pi x}{l}+\dots \quad \dots(1)$$

Also we know that

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}\qquad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$







i.e. $\cos\dfrac{\pi x}{l}=\dfrac{e^{\frac{i\pi x}{l}}+e^{\frac{-i\pi x}{l}}}{2}$, $\cos\dfrac{2\pi x}{l}=\dfrac{e^{\frac{i2\pi x}{l}}+e^{\frac{-i2\pi x}{l}}}{2}$, ..., $\cos\dfrac{n\pi x}{l}=\dfrac{e^{\frac{in\pi x}{l}}+e^{\frac{-in\pi x}{l}}}{2}$

and $\sin\dfrac{\pi x}{l}=\dfrac{e^{\frac{i\pi x}{l}}-e^{\frac{-i\pi x}{l}}}{2i}$, $\sin\dfrac{2\pi x}{l}=\dfrac{e^{\frac{i2\pi x}{l}}-e^{\frac{-i2\pi x}{l}}}{2i}$, ..., $\sin\dfrac{n\pi x}{l}=\dfrac{e^{\frac{in\pi x}{l}}-e^{\frac{-in\pi x}{l}}}{2i}$

Substituting these values in (1), we get

$$f(x)=\frac{a_0}{2}+a_1\left(\frac{e^{\frac{i\pi x}{l}}+e^{\frac{-i\pi x}{l}}}{2}\right)+\dots+a_n\left(\frac{e^{\frac{in\pi x}{l}}+e^{\frac{-in\pi x}{l}}}{2}\right)+\dots$$
$$+b_1\left(\frac{e^{\frac{i\pi x}{l}}-e^{\frac{-i\pi x}{l}}}{2i}\right)+\dots+b_n\left(\frac{e^{\frac{in\pi x}{l}}-e^{\frac{-in\pi x}{l}}}{2i}\right)+\dots$$
$$=\frac{a_0}{2}+\left(\frac{a_1e^{\frac{i\pi x}{l}}}{2}+\frac{b_1e^{\frac{i\pi x}{l}}}{2i}\right)+\dots+\left(\frac{a_ne^{\frac{in\pi x}{l}}}{2}+\frac{b_ne^{\frac{in\pi x}{l}}}{2i}\right)+\dots$$
$$+\left(\frac{a_1e^{\frac{-i\pi x}{l}}}{2}-\frac{b_1e^{\frac{-i\pi x}{l}}}{2i}\right)+\dots+\left(\frac{a_ne^{\frac{-in\pi x}{l}}}{2}-\frac{b_ne^{\frac{-in\pi x}{l}}}{2i}\right)$$
$$=\frac{a_0}{2}+\left(\frac{a_1-ib_1}{2}\right)e^{\frac{i\pi x}{l}}+\dots+\left(\frac{a_n-ib_n}{2}\right)e^{\frac{in\pi x}{l}}+\dots$$
$$+\left(\frac{a_1+ib_1}{2}\right)e^{\frac{-i\pi x}{l}}+\dots+\left(\frac{a_n+ib_n}{2}\right)e^{\frac{-in\pi x}{l}}\dots \quad \dots(2)$$

$\left[\because \dfrac{1}{i}=-i\right]$

Let $c_0=\dfrac{a_0}{2}$, $c_n=\dfrac{a_n-ib_n}{2}$, $c_{-n}=\dfrac{a_n+ib_n}{2}$

Now the series (2) becomes

$$f(x)=c_0+c_1e^{\frac{i\pi x}{l}}+c_2e^{\frac{i2\pi x}{l}}+\dots+c_ne^{\frac{in\pi x}{l}}+\dots$$
$$+c_{-1}e^{\frac{-i\pi x}{l}}+c_{-2}e^{\frac{-i2\pi x}{l}}+\dots+c_{-n}e^{\frac{-in\pi x}{l}}+\dots$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

This form of the Fourier series is called **complex or exponential form of Fourier Series**

The coefficient $c_n$ is given by

$$c_n = \frac{a_n - ib_n}{2} = \frac{a_n}{2} - \frac{i}{2} b_n$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{2l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left[\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right] dx$$

$$c_n = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx \quad \ldots (4)$$

$[\because \cos\theta - i\sin\theta = e^{-i\theta}]$

$$\therefore c_{-n} = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{\frac{in\pi x}{l}}\, dx$$

[ (replacing $n$ by $-n$ in (4) ]... (5)

$$c_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\, dx$$

[putting $n = 0$ in (4)]

**NOTE 1 : In the interval $-\pi < x < \pi$,**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

**NOTE 2 : In the interval $0 < x < 2\pi$, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$,**

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\, dx.$$

**NOTE 3 : In the interval $-l < x < l$,**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}},\quad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$$

**NOTE 4 : In the interval $0 < x < 2l$,**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}},\quad c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$$





**EXAMPLE 1** *Find the complex form of Fourier Series of $f(x) = e^{ax}$ $(-\pi < x < \pi)$ in the form.*

$$e^{ax} = \frac{\sin h\, a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}\, e^{inx}$$

**And hence prove that** $\dfrac{\pi}{a \sin h\, a\pi} = \sum_{-\infty}^{\infty}\dfrac{(-1)^n}{n^2+a^2}$.

*[Apr. 91, Nov. 89, Civil, Nov. 89, Mech., Nov. 91]*

**• Solution**

The complex form of Fourier Series of $f(x)$ in the interval $(-\pi, \pi)$ is given by,

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \ldots (1) \quad \text{[Refer to Note (1)]}$$

where $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\, dx = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)\pi} - e^{-(a-in)\pi}}{a-in}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(\cos n\pi - i\sin n\pi) - e^{-a\pi}(\cos n\pi + i\sin n\pi)\right]$$

$[\because e^{-in\pi} = \cos n\pi - i\sin n\pi]$

$$= \frac{1}{2\pi(a-in)}\left[(e^{a\pi} - e^{-a\pi})\cos n\pi\right] \quad [\because \sin n\pi = 0]$$

$$c_n = \frac{1}{\pi(a-in)}\sin h\, a\pi\,(-1)^n \quad \left[\because \sin hx = \frac{e^x - e^{-x}}{2}\right]$$

Substituting this value of $c_n$ in (1) we get,

$$f(x) = \frac{\sin h\, a\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{1}{a-in}\, e^{inx}$$

$$e^{ax} = \frac{\sin h\, a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}\, e^{inx} \quad \ldots (2)$$

Putting $x = 0$ in (2) we get

$$1 = \frac{\sin h\, a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}$$

i.e., $$\frac{\pi}{\sinh a\pi} = \sum_{-\infty}^{\infty} (-1)^n \frac{a+in}{a^2+n^2}$$

Equating the real part we get

$$\frac{\pi}{a \sinh a\pi} = \sum_{-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

**EXAMPLE 2**

*Find the complex form of the Fourier Series $f(x) = \cos ax$ in $-\pi < x < \pi$*

**Solution**

The complex form of Fourier Series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \ldots (1)$$

Now, $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{i^2n^2+a^2}(-in\cos ax + a\sin ax)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\cos a\pi + a\sin a\pi) - \frac{e^{in\pi}}{a^2-n^2}(-in\cos a\pi - a\sin a\pi)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[-in\cos a\pi\,(e^{-in\pi} - e^{in\pi}) + a\sin a\pi\,(e^{in\pi} + e^{-in\pi})\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[2in\cos a\pi \sin n\pi + 2a\sin a\pi\cos n\pi\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[2a\sin a\pi\cos n\pi\right] \quad [\because \sin n\pi = 0]$$

$$= \frac{(-1)^n \cdot a\sin a\pi}{\pi(a^2-n^2)} \quad [\because a \text{ is not an integer}] \quad \ldots (2)$$

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin a\pi \cdot a}{\pi(a^2-n^2)} \cdot e^{inx}$$

i.e., $$\cos ax = \frac{a\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n e^{inx}}{a^2-n^2}$$





**EXAMPLE 3**

*Find the complex form of the Fourier series of $f(x) = e^{ax}$ in $(-l, l)$.* [MKU, Apr. '95]

**Solution**

The complex form of Fourier series of $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \quad \ldots (1)$$

where $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{\frac{-in\pi x}{l}}\,dx = \frac{1}{2l}\int_{-l}^{l} e^{ax}\cdot e^{\frac{-in\pi x}{l}}\,dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(a-\frac{in\pi}{l}\right)x}\,dx = \frac{1}{2l}\left[\frac{e^{\left(a-\frac{in\pi}{l}\right)x}}{a-\frac{in\pi}{l}}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\,\frac{l}{al-in\pi}\left[e^{\left(a-\frac{in\pi}{l}\right)l} - e^{-\left(a-\frac{in\pi}{l}\right)l}\right]$$

$$= \frac{1}{2(al-in\pi)}\,2\cdot\sinh\left(a-\frac{in\pi}{l}\right)l$$

$$C_n = \frac{\sinh(al-in\pi)}{(al-in\pi)} \quad \ldots (2)$$

Substituting (2) in (1) we get

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\sinh(al-in\pi)}{(al-in\pi)}\cdot e^{\frac{in\pi x}{l}}$$

**EXAMPLE 4**

*Find the complex form of Fourier Series of the periodic function $f(x) = \sin(kx)$ in $-\pi < x < \pi$* [BDN, Nov. '97]

**Solution**

The complex form of the Fourier Series of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \ldots (1)$$

Now $$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin kx\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2+k^2}(-in\sin kx - k\cos kx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{k^2 - n^2} \left[e^{-in\pi}(-in \sin k\pi - k \cos k\pi) - e^{in\pi}(in \sin k\pi - k \cos k\pi)\right]$$

$$= \frac{1}{2\pi(k^2 - n^2)} \left[-in \sin k\pi \left(e^{in\pi} + e^{-in\pi}\right) + k \cos k\pi \left(e^{in\pi} - e^{-in\pi}\right)\right]$$

$$= \frac{1}{2\pi(k^2 - n^2)} \left[-in \sin k\pi \cdot 2 \cos n\pi + k \cos k\pi \cdot 2i \sin n\pi\right]$$

Using $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ ; $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$= \frac{-in}{\pi(k^2 - n^2)} [\sin k\pi \cos n\pi] \qquad [\because \sin n\pi = 0]$$

$$C_n = \frac{in(-1)^{n+1}}{\pi(k^2 - n^2)} \cdot \sin k\pi \quad \dots (2) \qquad [\because \cos n\pi = (-1)^n]$$

Substituting (2) in (1) we get

$$f(x) = \frac{\sin k\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{in(-1)^{n+1}}{(k^2 - n^2)} e^{inx}$$

**EXAMPLE 5**

*Find the complex form of the Fourier Series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.* *[MU, Apr. '91]*

**Solution**

The complex form of Fourier Series of $f(x)$ in $(-1, 1)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad \dots (1)$$

where

$$c_n = \frac{1}{2} \int_{-1}^{1} e^{-x} \cdot e^{-in\pi x} \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1 + in\pi)x} \, dx = \frac{1}{2}\left[\frac{e^{-(1 + in\pi)x}}{-(1 + in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{-2(1 + in\pi)} \left[e^{-(1 + in\pi)} - e^{(1 + in\pi)}\right]$$

$$= \frac{-1}{2(1 + in\pi)} \left[e^{-1}(\cos n\pi - i \sin n\pi) - e^{1}(\cos n\pi + i \sin n\pi)\right]$$

$$= -\frac{\cos n\pi}{2(1 + in\pi)} \left[e^{-1} - e^{1}\right]$$





$$= \frac{\cos n\pi}{2(1 + in\pi)} 2 \sin h\, 1 \qquad \left[\because \sin hx = \frac{e^x - e^{-x}}{2}\right]$$

$$c_n = \frac{(-1)^n \sin h\, 1}{1 + in\pi} \quad \dots (2)$$

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin h\, 1}{1 + in\pi} e^{in\pi x}$$

$$= \sin h\, 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} e^{in\pi x}$$

**EXAMPLE 6**

*Find the complex form of the Fourier series of the function* *[Mano, Nov. '95]*

$$f(x) = e^{k+x} \text{ when } -\pi < x < \pi$$

**Solution**

The complex form of the Fourier series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{inx} \quad \dots (1)$$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{k+x} \cdot e^{-inx} \, dx$

$$= \frac{e^k}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x} \, dx = \frac{e^k}{2\pi} \left[\frac{e^{(1-in)x}}{1 - in}\right]_{-\pi}^{\pi}$$

$$= \frac{e^k}{2\pi(1-in)} \left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right]$$

$$= \frac{e^k}{2\pi(1-in)} \left[e^{\pi}(\cos n\pi - i \sin n\pi) - e^{-\pi}(\cos n\pi + i \sin n\pi)\right]$$

$$[\because e^{ix} = \cos x + i \sin x]$$

$$= \frac{e^k (-1)^n}{2\pi(1-in)} \left[e^{\pi} - e^{-\pi}\right] \qquad [\because \sin n\pi = 0]$$

$$C_n = \frac{e^k (-1)^n}{\pi(1 - in)} \cdot \sin h\pi \quad \dots (2)$$

Substituting (2) in (1) we get

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{e^k (-1)^n \sin h\pi}{\pi(1 - in)} \cdot e^{inx}$$

$$e^{k+\pi} = \frac{e^k \sin h\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cdot e^{inx}}{(1 - in)}$$

■ EXAMPLE 7 ■

$$f(x) = \begin{cases} x, -\pi < x < 0 \\ x^2, 0 < x < \pi \end{cases}$$ *expand f (x) in complex Fourier series.*

[BU, Apr.]

**Solution**

The complex form of the Fourier series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{inx}$$

where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} x\cdot e^{-inx}\,dx + \int_0^{\pi} x^2\cdot e^{-inx}\,dx\right]$

$$= \frac{1}{2\pi}\left[\left\{x\left(\frac{e^{-inx}}{-in}\right) - (1)\left(\frac{e^{-inx}}{i^2 n^2}\right)\right\}_{-\pi}^{0}\right.$$

$$\left. + \left\{x^2\left(\frac{e^{-inx}}{-in}\right) - 2x\left(\frac{e^{-inx}}{i^2 n^2}\right) + 2\left(\frac{e^{-inx}}{-i^3 n^3}\right)\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n^2} - \frac{\pi e^{in\pi}}{in} - \frac{e^{in\pi}}{n^2} - \frac{\pi^2 e^{-in\pi}}{in} + \frac{2\pi e^{-in\pi}}{n^2} + \frac{2e^{-in\pi}}{in^3} - \frac{2}{in^3}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n^2} + \frac{i\pi\cos n\pi}{n} - \frac{1}{n^2}\cos n\pi + \right.$$

$$\left.\frac{i\pi^2\cos n\pi}{n} + \frac{2\pi}{n^2}\cos n\pi - \frac{2i\cos n\pi}{n^3} + \frac{2i}{n^3}\right]$$

$$\left[\because e^{in\pi} = \cos n\pi + i\sin n\pi \text{ and } \sin n\pi = 0\right]$$

$$C_n = \frac{1}{2\pi}\left[\frac{1}{n^2} + \frac{2i}{n^3} + \cos n\pi\left(\frac{i\pi}{n} - \frac{1}{n^2} + \frac{i\pi^2}{n} + \frac{2\pi}{n^2} - \frac{2i}{n^3}\right)\right]$$

Substituting this value of $C_n$ in (1) we get the required Fourier series.

■ EXAMPLE 8 ■

*Find the complex form of the Fourier Series of the periodic function $f(x) = \sin x$ in $0 < x < \pi$* [Mano, Nov, '97]

**Solution**

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \dots (1)$$

where $C_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cdot e^{-inx}\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\cdot e^{-inx}\,dx$





$$= \frac{1}{\pi}\left[\frac{e^{-inx}}{(-in)^2+1}(-in\sin x - \cos x)\right]_0^{\pi}$$

$$= \frac{1}{(1-n^2)\pi}\left[e^{-in\pi} + 1\right]$$

$$= \frac{1}{(1-n^2)\pi}\left[\cos n\pi - i\sin n\pi + 1\right]$$

$$C_n = \frac{1}{(1-n^2)\pi}\left[(-1)^n + 1\right]$$

$$C_n = \begin{cases}\frac{2}{(1-n^2)\pi} & \text{when '}n\text{' is even} \\ 0, & \text{when '}n\text{' is odd}\end{cases} \quad \dots (2)$$

Substituting (2) in (1) we get

$$f(x) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{1-n^2} = \frac{2}{\pi}\sum_{n=-4,-2,0}^{n=0,2,4}\frac{e^{inx}}{(1+n)(1-n)}$$

$$\sin x = \frac{2}{\pi}\left[1 - \frac{e^{2ix}+e^{-2ix}}{1.3} - \frac{e^{4ix}+e^{-4ix}}{3.5}\dots\right]$$

[By putting $n = 0, -2, -4, \dots$ $n = 0, 2, 4, \dots$]

**EXERCISES**

1. Show that the complex form of the Fourier Series of the function $f(x) = e^x, -\pi < x < \pi$, is [Nov. 91, Civil]

$$f(x) = \frac{\sin h\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}.$$

2. Prove that the complex form of the Fourier series of the function $f(x) = e^{-x}, -1 < x < 1$, is

$$f(x) = \sum_{n=-\infty}^{\infty}(-1)^n\frac{1-in\pi}{1+n^2\pi^2}\sin h\,1\cdot e^{in\pi x}$$

## ROOT MEAN SQUARE VALUE (R.M.S. VALUE)

**Definition**

The root mean square value (or R.M.S. Value) of $f(x)$ over the interval $(a, b)$ is defined as

$$\text{R.M.S.} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

R.M.S. Value is also called effective value and this term is usually applied to periodic functions only and plays an important role in the calculations of electrical circuits.

If the interval is $(-l, l)$ then

$$\text{R.M.S.} = \sqrt{\frac{\int_{-l}^{l}[f(x)]^2\,dx}{2l}}$$

If we denote the R.M.S. Value by $\overline{y}$, then

$$\overline{y} = \sqrt{\frac{\int_{-l}^{l}[f(x)]^2\,dx}{2l}}$$

i.e., $\overline{y}^2 = \frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx$

**NOTE :1. The RMS value of $f(x)$ in $(0, 2l)$ is given by,**

$$\overline{y}^2 = \frac{1}{2l}\int_{0}^{2l}[f(x)]^2\,dx$$

**2. The RMS value of $f(x)$ in $(-\pi, \pi)$ is given by,**

$$\overline{y}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$$

**3. The RMS value of $f(x)$ in $(0, 2\pi)$ is given by,**

$$\overline{y}^2 = \frac{1}{2\pi}\int_{0}^{2\pi}[f(x)]^2\,dx$$

**4. The RMS value of $f(x)$ in $(0, l)$ is given by,**

$$\overline{y}^2 = \frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx$$

**5. The RMS value of $f(x)$ in $(0, \pi)$ is given by**

$$\overline{y}^2 = \frac{2}{\pi}\int_{0}^{\pi}[f(x)]^2\,dx$$





## PARSEVALS IDENTITY (OR THEOREM)

**If the Fourier Series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$, then**

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

*PROOF :* We know that the Fourier Series corresponding to $f(x)$ in $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \quad \text{... (1)}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l}f(x)\,dx$ (or) $a_0 l = \int_{-l}^{l}f(x)\,dx$ ... (2)

$$a_n = \frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx \text{ (or)}$$

$$a_n l = \int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{... (3)}$$

$$b_n = \frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx \text{ (or)}$$

$$b_n l = \int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx \quad \text{... (4)}$$

Now multiplying both sides of (1) by $f(x)$ and integrating term by term from $-l$ to $l$, we get

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l}f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx + b_n\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx\right] \quad \text{... (5)}$$

Substituting (2), (3), (4) in (5) we get

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0}{2}\cdot a_0 l + \sum_{n=1}^{\infty}[a_n\cdot a_n l + b_n\cdot b_n l] = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$$

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \text{... (A)}$$

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

i.e., R.M.S. value of $f(x)$ is

$$\overline{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

*COROLLARY (i)* : If $f(x)$ is an even function, then the Fourier coefficients $b_n = 0$ in (A) and L.H.S. of (A) can be written as $\frac{2}{l}\int_0^l [f(x)]^2 dx$.

i.e., $$\boxed{\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2} \qquad (\because b_n = 0)$$

*COROLLARY (ii)* : If $f(x)$ is an odd function then the Fourier coefficients $a_0 = 0$ and $a_n = 0$ in (A). Since $f(x)$ is odd $|f(x)|^2$ is an even function and therefore L.H.S. of (A) can be written as $\frac{2}{l}\int_0^l [f(x)]^2 dx$.

i.e., $$\frac{2}{l}\int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$$

[$\because f(x)$ is an odd $[f(x)]^2$ is an even function $\therefore a_0 = 0, a_n = 0$]

*COROLLARY (iii)* : If the half-range cosine series in $(0, l)$ for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \text{ then}$$

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

*COROLLARY (iv)* : If the half range cosine series in $(0, \pi)$ for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ then}$$

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

*COROLLARY (v)* : If the half-range sine series in $(0, l)$ for $f(x)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ then $\frac{2}{l}\int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$

*COROLLARY (vi)* : If the half range sine series in $(0, \pi)$ for $f(x)$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{ then}$$

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$$





............ (2)

**EXAMPLE 1**

*Prove that in the interval $0 < x < l$,*

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \dots\right) \text{ and deduce that}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

**Solution**

In the half-range cosine expansion of $x$ in $0 < x < l$, we can easily find that $a_0 = l$, $a_1 = \frac{-4l}{\pi^2}$, $a_3 = \frac{-4l}{3^2\pi^2}$, ... and $a_2 = a_4 = \dots = 0$.

i.e., $$a_n = \frac{-4l}{n^2\pi^2} \text{ where '}n\text{' is odd.}$$

By using corollary (iii) we have

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \Sigma a_n^2$$

i.e., $$\frac{2}{l}\int_0^l (x)^2 dx = \frac{a_0^2}{2} + \Sigma a_n^2$$

i.e., $$\frac{2}{l}\int_0^l x^2 dx = \frac{l^2}{2} + \sum_{n=0}^{\infty} \frac{16l^2}{n^4\pi^4}$$

i.e., $$\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{l^2}{2} + \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

i.e., $$\frac{2}{3}l^2 - \frac{l^2}{2} = \frac{16\,l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

or $$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

**EXAMPLE 2**

*Find the Fourier Series for $f(x) = x^2$ in $-\pi < x < \pi$. Hence show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

**Solution**

We can easily show that the Fourier Series of $f(x)$ is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

The co-efficients $a_0$, $a_n$, $b_n$ are

$$_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0$$

is

$$(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\therefore \int_{-\pi}^{\pi}[x^2]^2\,dx = 2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

i.e., $$\left(\frac{x^5}{5}\right)_{-\pi}^{\pi} = 2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

i.e., $$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{18\pi^5 - 10\pi^5}{45} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{8\pi^4}{45} = \sum_{n=1}^{\infty}\frac{16}{n^4}$$

i.e., $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

i.e., $$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$$

**■ EXAMPLE 3 ■**

*If for $0 < x < l$, the function $f(x)$ has the expansion*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \text{ show that}$$

$$\int_0^l [f(x)]^2\,dx = \frac{l}{2}[b_1^2 + b_2^2 + \ldots]$$

*[Mano, Apr. '98]*

**● Solution**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (1)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$





i.e., $$b_n\, l = 2\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad \ldots (2)$$

multiplying both sides of (1) by $f(x)$ and integrating from 0 to $l$, we get

$$\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n\left[\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right] \qquad \ldots (3)$$

substituting (2) in (3) we get

$$\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n \cdot \frac{b_n l}{2} = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2$$

$$\int_0^l [f(x)]^2\,dx = \frac{l}{2}[b_1^2 + b_2^2 + \ldots]$$

**■ EXAMPLE 4 ■**

*Expand $f(x) = x - x^2$ as a Fourier series in $-l < x < l$ and using this series find the r.m.s value of $f(x)$ in the interval.* *[BU, Apr. '95]*

**● Solution**

The Fourier series of $f(x)$ in $(-1, 1)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots (1)$$

where $$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-1}^{1} = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right)$$

$$a_0 = \frac{-2}{3} \qquad \ldots (2)$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{1}(x - x^2)\cos n\pi x\,dx$$

$$= \left[(x - x^2)\left(\frac{\sin n\pi x}{n}\right) - (1 - 2x)\left(\frac{-\cos n\pi x}{n^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3}\right)\right]_{-1}^{1}$$

$$= \frac{-\cos n\pi}{n^2} - \frac{3\cos n\pi}{n^2} \qquad \ldots (3)$$

$$a_n = -\frac{4\cos n\pi}{n^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{1}(x - x^2)\sin n\pi x\,dx$$

$$= \left[(x - x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1-2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]$$

$$= \frac{-2\cos n\pi}{n^3\pi^3} - \frac{2\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3}$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi}$$

Substituting (2), (3), (4) in (1) we get

$$f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2}\cos n\pi x + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi x$$

we know that r.m.s. value of $f(x)$ in $(-l, l)$ is

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \dots (5)$$

From (2) we get

$$a_0 = \frac{-2}{3} \Rightarrow a_0^2 = \frac{4}{9} \quad \dots (6)$$

From (3) we get,

$$a_n = \frac{4(-1)^{n+1}}{n^2} \Rightarrow a_n^2 = \frac{16}{n^4} \quad \dots (7)$$

From (4) we get

$$b_n = \frac{2(-1)^{n+1}}{n\pi} \Rightarrow b_n^2 = \frac{4}{n^2\pi^2} \quad \dots (8)$$

Substituting (6), (7) and (8) in (5) we get

$$\bar{y}^2 = \frac{1}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{16}{n^4} + \frac{4}{n^2\pi^2}\right)$$

## EXAMPLE 5

***Show that for $0 < x < l$***

$$x = \frac{2l}{p}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} \dots\right)$$

***Using root mean square value of x, deduce the value of***

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

*[Mano, Nov. '97]*

**Solution**

The sine series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \dots (1)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$





$$= \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$b_n = \frac{-2l\cos n\pi}{n\pi} = \frac{2l}{n\pi}(-1)^{n+1} \quad \dots (2)$$

Substituting (2) in (1) we get

$$f(x) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$

$$x = \frac{2l}{\pi}\left[\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \dots\right]$$

We know that if,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \text{ then}$$

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

$$\therefore \frac{2}{l}\int_0^l x^2\,dx = \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}(-1)^{2n+2}$$

$$\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

[$\because (-1)^{2n+2} = 1$ whether '$n$' is odd or even]

$$\frac{2l^3}{3l} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

i.e., $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

## EXERCISES

1. If $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ in (0, L) show that $\int_0^L [f(x)]^2\,dx = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2$.

   Choosing $f(x) = 1$ and using this result prove that

   $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$. *[Nov. 89, Mech.]*

2. If $f(x)$ has the half-range Fourier expansion

   $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$ in $0 < x < l$, then prove that

   $\int_0^l [f(x)]^2\,dx = \frac{l}{2}\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2\right]$ *[Apr., Nov. 86, Mech.]*

3. If $f(x)$ has the Fourier Series expansion

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ in } a \le x \le a+2l, \text{ prove that}$$

$$\int_a^{a+2l} [f(x)]^2\, dx = 2l\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

[Apr. 88, Civil, E

4. If $f(x) = b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + \ldots$ in $(0, L)$ show that

$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$. Hence check that in $(0, L)$

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \ldots\right]$$

[Apr. 91, Ci

5. Find the Fourier cosine series for $x(\pi - x)$ in $0 < x < \pi$ and show t

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

## ❑ HARMONIC ANALYSIS

When $f(x)$ is a given function defined analytically in $(0, 2\pi)$ then we kno how to expand $f(x)$ in a Fourier Series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

In other words if $f(x)$ is defined analytically in $(0, 2\pi)$ we know how to fin the Fourier coefficients $a_0$, $a_n$ and $b_n$ analytically.

But in practical Engineering problems, the function $f(x)$ to be expanded i Fourier Series is not defined by analytical expression. Instead, we know a fe values of the function or its graph. In such cases, the Fourier coefficients $a_0$, a and $b_n$ in Fourier Series are calculated by means of approximate integration. Th term harmonic analysis is usually applied to numerical methods used for thi purpose. *i.e.*,

**The process of finding the Fourier Series for a function given by numerical value is known as harmonic analysis. In harmonic analysis the Fourier coefficients $a_0$, $a_n$ and $b_n$ of the function $y = f(x)$ in $(0, 2\pi)$ are given by**

$$\left.\begin{aligned} a_0 &= 2\,[\text{mean value of } y \text{ in } (0, 2\pi)] \\ a_n &= 2\,[\text{mean value of } y \cos nx \text{ in } (0, 2\pi)] \\ b_n &= 2\,[\text{mean value of } y \sin nx \text{ in } (0, 2\pi)] \end{aligned}\right\} \text{For reason refer to Note 3}$$

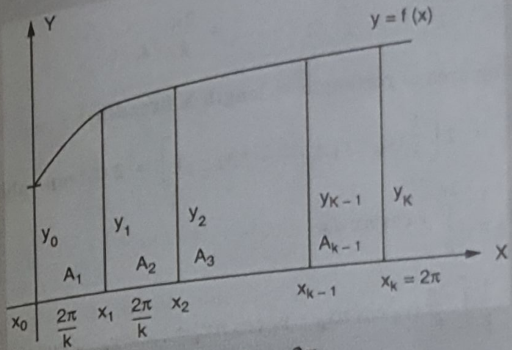




*NOTE 1 : The term $(a_1 \cos x + b_1 \sin x)$ is called the fundamental or first harmonic, the term $(a_2 \cos 2x + b_2 \sin 2x)$ is called the second harmonic and so on.*

***NOTE 2 : The number of ordinates used should not be less than twice the number of highest harmonic to be found.***

***NOTE 3 : Let the range $(0, 2\pi)$ be divided into 'k' equal parts as shown in fig.***




**Each subinterval is of length $\frac{2\pi - 0}{k} = \frac{2\pi}{k}$**

**Let the values of the function $y = f(x)$ at the points**

$$x_0 = 0, x_1, x_2, \ldots\ldots, x_{k-1}, x_k = 2\pi$$

**be denoted by $y_0, y_1, y_2, \ldots\ldots, y_{k-1}, y_k$**

**We know that the Fourier series of $f(x)$ in $(0, 2\pi)$ is**

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots\ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots \quad \ldots (1)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_0^{2\pi} y\, dx \quad [\because y = f(x)] \quad \ldots (2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y \cos nx\, dx \quad \ldots (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\, dx \quad \ldots (4)$$

By numerical integration (rectangular rule) we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} y\,dx = \frac{1}{\pi}\left[\frac{2\pi}{k}(y_0 + y_1 + \ldots\ldots + y_{k-1})\right]$$

[Area enclosed between the curve $y = f(x)$, X-axis, and the ordinates and $y_k$ is given by

$$\text{Area} = A_1 + A_2 + \ldots\ldots\ldots + A_{k-1}$$

$$= \frac{2\pi}{k}\cdot y_0 + \frac{2\pi}{k}y_1 + \ldots\ldots + \frac{2\pi}{k}y_{k-1}$$

(using area of rectangle = length X breadth)]

$$= 2\left[\frac{1}{k}(y_0 + y_1 + \ldots\ldots + y_{k-1})\right] = 2\,[\text{Mean value of } y \text{ in } (0, 2\pi)]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{k}(y_0\cos nx_0 + y_1\cos nx_1 + \ldots + y_{k-1}\cos nx_{k-1})\right]$$

$$= 2\left[\frac{1}{k}(y_0\cos nx_0 + \ldots\ldots + y_{k-1}\cos nx_{k-1})\right]$$

$$= 2\,[\text{mean value of } y\cos nx \text{ in } (0, 2\pi)]$$

Similarly,

$$b_n = 2\,[\text{mean value of } y\sin nx \text{ in } (0, 2\pi)]$$

NOTE 1 : Suppose the function $f(x)$ is defined in the interval $(0, 2l)$, then its Fourier series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

and now, $a_0 = 2\,[\text{Mean value of } y \text{ in } (0, 2l)]$

$$a_n = 2\left[\text{Mean value of } y\cos\frac{n\pi x}{l} \text{ in } (0, 2l)\right]$$

$$b_n = 2\left[\text{Mean value of } y\sin\frac{n\pi x}{l} \text{ in } (0, 2l)\right]$$

2. If the half range Fourier sine series of $f(x)$ in $(0, l)$ is,





$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}, \text{ then}$$

$$b_n = 2\left[\text{Mean value of } y\sin\frac{n\pi x}{l}, \text{ in } (0, l)\right]$$

3. If the half range Fourier sine series of $f(x)$ in $(0, \pi)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n\sin nx$$

then, $b_n = 2\,[\text{Mean value of } y\sin nx \text{ in } (0, \pi)]$

4. If the half range Fourier cosine series of $f(x)$ in $(0, l)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

then, $a_0 = 2\,[\text{Mean value of } y \text{ in } (0, l)]$

$$a_n = 2\left[\text{Mean value of } y\cos\frac{n\pi x}{l} \text{ in } (0, l)\right]$$

5. If the half range Fourier cosine series of $f(x)$ in $(0, \pi)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx, \text{ then,}$$

$$a_0 = 2\,[\text{Mean value of } y \text{ in } (0, \pi)]$$

$$a_n = 2\,[\text{Mean value of } y\cos nx \text{ in } (0, \pi)].$$

**EXAMPLE 1**

*Find the Fourier Series expansion of period $2\pi$ for the function $y = f(x)$ which is defined in $(0, 2\pi)$ by means of the table of values given below. Find the series upto the third harmonic.* [Apr. 2000]

| $x$ | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $y$ | 1.0 | 1.4 | 1.9 | 1.7 | 1.5 | 1.2 | 1.0 |

**Solution**

Since the last value of y is a repetition of the first, only the first six values will be used.

The values of $y\cos x$, $y\cos 2x$, $y\cos 3x$, $y\sin x$, $y\sin 2x$, $y\sin 3x$ are tabulated in next page.

| $x$ | $y$ | $\cos x$ | $\sin x$ | $\cos 2x$ | $\sin 2x$ | $\cos 3x$ | $\sin 3x$ | $y \cos x$ | $y \sin x$ | $y \cos 2x$ | $y \sin 2x$ | $y \cos 3x$ | $y \sin 3x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\frac{\pi}{3}$ | 1.4 | 0.5 | 0.866 | −0.5 | 0.866 | −1 | 0 | 0.7 | 1.212 | −0.7 | 1.212 | −1.4 | 0 |
| $\frac{2\pi}{3}$ | 1.9 | −0.5 | 0.866 | −0.5 | −0.866 | 1 | 0 | −0.95 | 1.65 | −0.95 | −1.645 | 1.9 | 0 |
| $\pi$ | 1.7 | −1 | 0 | 1 | 0 | −1 | 0 | −1.7 | 0 | 1.7 | 0 | −1.7 | 0 |
| $\frac{4\pi}{3}$ | 1.5 | −0.5 | −0.866 | −0.5 | 0.866 | 1 | 0 | −0.75 | −1.299 | −0.75 | 1.299 | 1.5 | 0 |
| $\frac{5\pi}{3}$ | 1.2 | 0.5 | −0.866 | −0.5 | −0.866 | −1 | 0 | 0.6 | −1.039 | −0.6 | −1.039 | −1.2 | 0 |
| | 8.7 | | | | | | | −1.1 | 0.5196 | −0.3 | −0.1732 | 0.1 | 0 |
| | $\Sigma y$ | | | | | | | $\Sigma y \cos x$ | $\Sigma y \sin x$ | $\Sigma y \cos 2x$ | $\Sigma y \sin 2x$ | $\Sigma y \cos 3x$ | $\Sigma y \sin 3x$ |





$$\therefore a_0 = \frac{2\Sigma y}{6} = 1.5, \; a_1 = \frac{2}{6}\Sigma y \cos\theta = 0.37$$

$$b_1 = \frac{2}{6}\Sigma y \sin\theta = 1.00456$$

Substituting these values of $a_0$, $a_1$ and $b_1$ in (2), we get

$$\therefore f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$$

**EXAMPLE 3**

*Find the Fourier Series upto the third harmonic for the function $y = f(x)$ defined in $(0, \pi)$ from the table.* *[MU, Oct. '99]*

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{2\pi}{6}$ | $\frac{3\pi}{6}$ | $\frac{4\pi}{6}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2.34 | 2.2 | 1.6 | 0.83 | 0.51 | 0.88 | 1.19 |

**Solution**

We can express the given data in a half range Fourier sine series.

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \quad \text{... (1)}$$

| $x$ | $y = f(0)$ | $\sin x$ | $\sin 2x$ | $\sin 3x$ | $y \sin x$ | $y \sin 2x$ | $y \sin 3x$ |
|---|---|---|---|---|---|---|---|
| 0 | 2.34 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | 2.2 | 0.5 | 0.87 | 1 | 1.1 | 1.91 | 2.2 |
| 60 | 1.6 | 0.87 | 0.87 | 0 | 1.392 | 1.392 | 0 |
| 90 | 0.83 | 1 | 0 | −1 | 0.83 | 0 | −0.83 |
| 120 | 0.51 | 0.87 | −0.87 | 0 | 0.44 | −0.44 | 0 |
| 150 | 0.88 | 0.5 | −0.87 | 1 | 0.44 | 0.76 | 0.88 |
| 180 | 1.19 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | 4.202 | 3.622 | 2.25 |

Now

$$b_1 = 2\left[\frac{\Sigma y \sin x}{6}\right] = \frac{1}{3}[4.202] = 1.40$$

$$b_2 = 2\left[\frac{\Sigma y \sin 2x}{6}\right] = \frac{1}{3}[3.622] = 1.207$$

$$b_3 = 2\left[\frac{\Sigma y \sin 3x}{6}\right] = \frac{1}{3}(2.25) = 0.75$$

Substituting $b_1$, $b_2$, $b_3$ in (1) we get

$$f(x) = 1.4\sin x + 1.21\sin 2x + 0.75\sin 3x$$

## EXAMPLE 4

*Compute the first two harmonics of the Fourier series for f(x) from the following data [Nov. 2000]*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 0 | 30 | 60 | 90 | 120 | 150 | 180 |
| $f(x)$ : | 0 | 5224 | 8097 | 7850 | 5499 | 2626 | 0 |

**Solution**

Here the length of the interval is $\pi$. $\therefore$ we can express the given data in a half range Fourier sine series

i.e., $f(x) = b_1 \sin x + b_2 \sin 2x$ ... (1)

| $x$ | $y$ | $\sin x$ | $\sin 2x$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 30 | 5224 | .5 | 0.87 |
| 60 | 8097 | 0.87 | 0.87 |
| 90 | 7850 | 1 | 0 |
| 120 | 5499 | 0.87 | −0.87 |
| 150 | 2626 | 0.5 | −0.87 |

Now $b_1 = 2\left(\dfrac{\Sigma y \sin x}{6}\right) = 7867.84$

$b_2 = 2\left(\dfrac{\Sigma y \sin 2x}{6}\right) = 1506.84$

$\therefore f(x) = 7867.84 \sin x + 1506{\cdot}84{\cdot}\sin 2x$

## EXAMPLE 5

*Find the Fourier series as far as the second harmonic to represent the function given in the following data.*

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 3 | 4 | 5 |
| $f(x)$ : | 9 | 18 | 24 | 28 | 26 | 20 |

[BU, Apr. '97]

**Solution**

Here the length of the interval is 6 (not $2\pi$)

i.e., $2l = 6$ (or) $l = 3$

$\therefore$ The Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3} \quad \text{... (1)}$$





| $x$ | $\frac{\pi x}{3}$ | $\frac{2\pi x}{3}$ | $y$ | $y\cos\frac{\pi x}{3}$ | $y\sin\frac{\pi x}{3}$ | $y\cos\frac{2\pi x}{3}$ | $y\sin\frac{2\pi x}{3}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 9 | 9 | 0 | 9 | 0 |
| 1 | $\pi/3$ | $2\pi/3$ | 18 | 9 | 15.7 | −9 | 15.6 |
| 2 | $2\pi/3$ | $4\pi/3$ | 24 | −12 | 20.9 | −24 | 0 |
| 3 | $\pi$ | $2\pi$ | 28 | −28 | 0 | 28 | 0 |
| 4 | $4\pi/3$ | $8\pi/3$ | 26 | −13 | −22.6 | −13 | 22.6 |
| 5 | $5\pi/3$ | $10\pi/3$ | 20 | 10 | −17.4 | −10 | −17.4 |
| | | | 125 | −25 | −3.4 | −19 | 20.8 |

Now $a_0 = 2\left(\dfrac{\Sigma y}{6}\right) = \dfrac{2(125)}{6} = 41.66$

$a_1 = 2\left[\dfrac{\Sigma y \cos\frac{\pi x}{3}}{6}\right] = -8.33$

$b_1 = 2\left[\dfrac{\Sigma y \sin\frac{\pi x}{3}}{6}\right] = -1.13$

$a_2 = 2\left[\dfrac{\Sigma y \cos\frac{2\pi x}{3}}{6}\right] = -6.33$

$b_2 = 2\left[\dfrac{\Sigma y \sin\frac{2\pi x}{3}}{6}\right] = 6.9$

Substituting these values of $a_0$, $a_1$, $b_1$, $a_2$ and $b_2$ in (1), we get

$$f(x) = \frac{41.66}{2} - 8.33\cos\frac{\pi x}{3} - 6.33\cos\frac{2\pi x}{3} - 1.13\sin\frac{\pi x}{3} + 6.9\sin\frac{2\pi x}{3}$$

## EXERCISES

1. Compute the first two harmonics of the Fourier Series of $f(x)$ from the following table :

| $x$ | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 2·34 | 3·01 | 3·68 | 4·15 | 3·69 | 2·2 | 0·83 | 0·51 | 0·88 | 1·09 | 1·19 | 1·64 |

[Nov. 89]

[*Ans.* $y = 2{\cdot}10 - 0{\cdot}283\cos x - 0{\cdot}18\cos 2x + 1{\cdot}6\sin x - 0{\cdot}49\sin 2x$]

2. Using the 6 ordinate scheme, analyse harmonically the data to 2 harmonics [Nov. 88, Nov. 87]

| $x$ | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $y$ | 10 | 12 | 15 | 20 | 17 | 11 | 10 |

3. Find numerically the Fourier Series, upto the third harmonic, for the function $f(x)$, using the table of values of $f(x)$ given below

| $x$ | $0$ | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.0 | 1.21 | 1.27 | 1.3 | 1.27 | 1.21 | 1.0 |

4. The displacement $y$ of a part of a mechanism are tabulated with corresponding angular movement $x$ of the crank. Express $y$ as a Fourier Series neglecting the harmonics above the third, given

| $x°$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1.8 | 1.1 | 0.3 | 0.16 | 0.5 | 1.3 | 2.16 | 1.25 | 1.3 | 1.52 | 1.76 | 2.0 |

[Ans. $y = 1.26 + 0.04 \cos x + 0.53 \cos 2x - 0.1 \cos 3x$
$- 0.63 \sin x - 0.23 \sin 2x + 0.085 \sin 3x$]





## Two Marks Q & A

1. Pick out the even functions : $x^2, \sin x, 1 + x, e^x$. [Apr. 95]

Ans : $x^2$ is an even function. $[\because f(x) = x^2 = f(-x)]$

$\sin x$ is odd functions $e^x$ and $1 + x$ is neither even nor odd function.

2. Explain periodic function with two examples. [Apr. 95]

Ans : A function $f(x)$ is said to have a period $T$ if for all $x$, $f(x + T) = f(x)$, where $T$ is a positive constant. The least value of $T > 0$ is called the period of $f(x)$.

Example : (a)
$$f(x) = \sin x$$
$$f(x + 2\pi) = \sin(x + 2\pi) = \sin x$$
$$\text{Here } f(x) = f(x + 2\pi)$$

$\therefore \sin x$ is a periodic function with period $2\pi$.

(b)
$$f(x) = \tan x$$
$$f(x + \pi) = \tan(\pi + x) = \tan x$$
$$\text{since } f(x + \pi) = f(x)$$

$\therefore \tan x$ is a periodic function with period $\pi$.

3. State Dirichlet's conditions for a function to be expanded as a Fourier series. [Nov. 95, Apr. 95]

Ans : A function $f(x)$ which satisfies the following condition can be expanded as a Fourier series in $(-\pi, \pi)$.

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(-\pi, \pi)$.

(ii) $f(x)$ is periodic with period $2\pi$.

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-\pi, \pi)$.

4. Write the formulae for Fourier constants for $f(x)$ in the interval $(-\pi, \pi)$. [Oct. 95]

Ans :
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$$

5. What is the value of the Fourier constant $a_n$ when odd function $f(x)$ is expanded in $(-\pi, \pi)$. [Oct. 95]

Ans : $a_n = 0$.

6. ***Find the constant $a_0$ of the Fourier series for function $f(x) = x$, $0 \le x \le 2\pi$*** [Nov. 95]

**Ans :** $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$

$$= \frac{1}{\pi}\left[\frac{4\pi^2}{2}\right] = 2\pi$$

7. ***The Fourier series of $f(x)$ in $(0, l)$ is***

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, ***then $a_0 = \ldots$ and $a_n = \ldots$*** [Apr. 95]

**Ans :** $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

8. ***Choose the correct answer.***

***The Fourier series expansion of an even function contains***

(a) *Sine terms only*

(b) *Cosine terms only*

(c) *Both sine and cosine terms*

(d) *Neither sine nor cosine terms* [Nov. 95]

**Ans :** (c) Cosine terms only

9. ***When an even function $f(x)$ is expanded in a Fourier series in the interval from $-\pi$ to $\pi$ show that $b_n = 0$.*** [Nov. 95]

**Ans :** $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Since $f(x)$ is even and $\sin nx$ is odd the product $f(x)\sin nx$ is an odd function.

$\therefore$ By property of definite integrals

$$b_n = 0$$

[Property : $\int_{-\pi}^{\pi} f(x)\,dx = 0$ if $f(x)$ is odd funciton]

10. ***Find $b_n$ in the expansion of $x^2$ as a Fourier series in $(-\pi, \pi)$.*** [Apr. 95]

**Ans :** Since $f(x) = x^2$ is an even function the value of $b_n = 0$.





[Apr. 95]

11. ***Write the Fourier sine series of k in $(0, \pi)$.***

**Ans :** $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} k\sin nx\,dx = \frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{2k}{n\pi}\left[1-(-1)^n\right]$$

i.e., $b_n = 0$, $n$ is even

$= \frac{4k}{n\pi}$, $n$ is odd

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} \frac{4k}{n\pi}\sin nx = \frac{4k}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)x$$

12. ***What is the sum of the Fourier series at a point $x = x_0$ where the function $f(x)$ has a finite discontinuity.*** [Apr. 95]

**Ans :** $f(x) = \frac{f(x+x_0) + f(x-x_0)}{2}$ [Oct. 95]

13. ***Obtain the sine series for unity in $(0, \pi)$.***

**Ans :** Here $f(x) = 1$; $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin nx\,dx = \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right]$$

i.e., $b_n = 0$, $n$ is even

$= \frac{4}{n\pi}$, $n$ is odd

$$\therefore f(x) = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin nx$$

14. ***Write Parseval's theorem on Fourier constants.*** [Apr. 95, Oct. 95]

**Ans :** If the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$ then

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

**15. Define Root Mean Square value of a function.** [Apr. 9...]

*Ans :* The root mean square value of $f(x)$ over the interval $(a, b)$ is de... as

$$\text{R.M.S} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

**16. Find the constant $a_0$ of the Fourier series for the function $f(x) = \ldots$ $0 < x < 2\pi$** [Apr. 9...]

*Ans :* $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} k\,dx = \frac{k}{\pi}[x]_0^{2\pi} = \frac{2\pi k}{\pi}$

$a_0 = 2k$

**17. If $f(x) = x^2$ in $-\pi \le x \le \pi$, then the R.M.S value of $f(x)$ is $\frac{\pi^4}{5}$ say True o... False.** [Apr. 9...]

*Ans :* False.

Since $\text{R.M.S} = \sqrt{\frac{\int_{-\pi}^{\pi} [f(x)]^2\,dx}{\pi-(-\pi)}} = \sqrt{\frac{\int_{-\pi}^{\pi} x^4\,dx}{2\pi}}$

$$= \sqrt{\frac{2\int_0^{\pi} x^4\,dx}{2\pi}}$$

$$= \sqrt{\frac{\left[\frac{x^5}{5}\right]_0^{\pi}}{\pi}}$$

$$= \sqrt{\frac{\pi^5}{5\pi}} = \frac{\pi^2}{\sqrt{5}}$$

**18. Write the Fourier series in complex form for $f(x)$ defined in the interval c to $c + 2\pi$** [Nov. 95]

*Ans :* $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$

where $C_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\, e^{-inx}\,dx$





**Choose the correct answer :**

**19. If the complex form of the Fourier series of $f(x)$ is $\sum_{n=-\infty}^{\infty} C_n e^{inx}$ in $(0, 2\pi)$ then $C_n$ is** [Apr. 95]

(a) $\frac{1}{\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$ (b) $\frac{2}{\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$

(c) $\frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$ (d) $\frac{2}{\pi}\int_0^{\pi} f(x)\, e^{-inx}\,dx$

*Ans :* (c) $\frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$

**20. Write the complex form of Fourier series for $f(x)$ defined in the interval $c, c+2l$.** [Oct. 95]

*Ans :* $f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi x}{l}}$,

where $C_n = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{\frac{-in\pi x}{l}}\,dx$

**21. Write the formulas of Fourier constants for $f(x)$ in $c, c+2l$.** [Apr. 96]

*Ans :* $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$

**22. Find the Fourier constants $b_n$ for $x \sin x$ in $(-\pi, \pi)$.** [Apr. 96]

*Ans :* $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin nx\,dx$

$= 0$ [$\because x\sin x\sin nx$ is an odd function]

**23. Find the cosine series for**

$f(x) = 1$ for $0 \le x \le \frac{a}{2}$

$= -1$ for $\frac{a}{2} < x < a$ [Apr. 96]

*Ans :* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a}$

$$a_0 = \frac{2}{a}\int_0^a f(x)\,dx = \frac{2}{a}\left[\int_0^{a/2} 1\,dx + \int_{a/2}^{a} -1\,dx\right]$$

$$= \frac{2}{a}\left[[x]_0^{a/2} + [-x]_{a/2}^{a}\right] = \frac{2}{a}\left[\frac{a}{2} - a + \frac{a}{2}\right]$$

$$= 0$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[\int_0^{a/2}\cos\frac{n\pi x}{a}\,dx + \int_{a/2}^{a} -\cos\frac{n\pi x}{a}\,dx\right]$$

$$= \frac{2}{a}\left[\left\{\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right\}_0^{a/2} - \left\{\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right\}_{a/2}^{a}\right]$$

$$= \frac{2}{a}\left[\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{a}} + \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{a}}\right] = 4\,\frac{\sin\frac{n\pi}{2}}{n\pi}$$

$$\therefore f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cdot\cos\frac{n\pi x}{a}$$

**24. *In the expansion of $f(x) = \sin hx$ in $(-\pi, \pi)$ as a Fourier series, find the coefficient of $a_n$.* [Apr. 96]**

*Ans :* $f(x) = \sin hx = \sin h(-x) = -\sin hx = -f(x)$

Since $f(x) = \sin hx$ is an odd function, the Fourier coefficient $a_n = 0$

**25. *State Parseval's identity for the full range expansion of $f(x)$ as a Fourier series in $(0, 2l)$.* [Apr. 96]**

*Ans :* Parseval's identity in $(0, 2l)$ is

$$\frac{1}{l}\int_0^{2l}|f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

**26. *Find the R.M.S value of the function $f(x) = x$ in $(0, l)$.* [Apr. 96]**

*Ans :* $$\text{R.M.S} = \sqrt{\frac{\int_0^l x^2\,dx}{l}} = \sqrt{\frac{\left(\frac{x^3}{3}\right)_0^l}{l}} = \sqrt{\frac{l^3}{3l}}$$

$$= \sqrt{\frac{l^3}{3l}} = \frac{l}{\sqrt{3}}$$





**27. *Fourier series of period 2 for $|x|$ in $(0, 2)$ contains only cosine terms. Say true or false.* [Apr. 96]**

*Ans :* True. Since $f(x) = |x|$ is an even function and therefore $b_n = 0$.

**28. *Find $a_n$ in the expansion of $x$ as a Fourier series in $(-\pi, \pi)$.* [Apr. 96]**

*Ans :* $a_n = 0$

Since $f(x) = x$ is an odd function

**29. *Write the formula for finding Euler's constants of a Fourier series in $(0, 2\pi)$.***

*Ans :* $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \;;\; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

**30. *If $f(x)$ is a function defined in $(-2 \le x \le 2)$, what is the value of $b_n$?* [Apr. 96]**

*Ans :* $$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

**31. *Find the value of $a_n$ in the cosine series expansion of $f(x) = k$ in $(0, 10)$.* [Apr. 96]**

*Ans :* $$a_n = \frac{1}{5}\int_0^{10} k\cdot\cos\frac{n\pi x}{10}\,dx = \frac{k}{5}\left[\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]_0^{10} = 0$$

**32. *State Parseval's identity for the half-range cosine expansion of $f(x)$ in $(0, 1)$.* [Nov. 96]**

*Ans :* $$\frac{2}{1}\int_0^1 |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

**33. *If $a_n$, $b_n$ are the Fourier coefficients in the real expansion of $f(x)$ over $(0, 2l)$ and $c_n$ the coefficients in the complex expansion, what is the connection between them?* [Nov. 96]**

*Ans :* $$c_n = \frac{a_n - i\,b_n}{2}$$

**34. *Find $b_n$ in the expansion of $x^2$ as Fourier series in $(-l, l)$.* [Nov. 96]**

*Ans :* $f(x) = x^2$ in an even function

$b_n = 0$

**35. *Pick out the odd functions : $\sin x$, $x\cos x$, $\cos hx$, $\cos x$.* [Nov. 96]**

*Ans :* $\sin x$, $x\cos x$

36. ***Find the constant term in the Fourier series corresponding to $f(x) = x - x^3$ in $(-\pi, \pi)$.*** [Oct.

**Ans :** Given :

$$f(x) = x - x^3$$

$$f(-x) = -x + x^3 = -(x - x^3) = -f(x)$$

i.e.,

$$f(-x) = -f(x)$$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$.

Hence $a_0 = 0$

37. ***Without evaluating any integral, write the half-range series with terms for $f(x) = \sin^3 x$ in $(0, \pi)$.*** [Oct.

**Ans :**

$$f(x) = \sin^3 x = \frac{1}{4}[3 \sin x - \sin 3x]$$

38. ***If $f(x) = |x|$ expanded as a Fourier series in $-\pi < x < \pi$ Find $a_0$.*** [Apr.

**Ans :** $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx \qquad [\because |x| \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \pi$$

39. ***The Fourier series expansion of an odd function contains ......... only.***

**Ans :** Sine terms

40. ***To what value, the Fourier series corresponding to $f(x) = x^2$ in $(0, 2\pi)$ converges at $x = 0$*** [Apr. 97]

**Ans :** The Fourier series converges to $\dfrac{f(0) + f(2\pi)}{2} = 2\pi^2$

41. ***One of the Dirichlet's conditions for convergence is that a function may have finite number of maxima or minima in any one period. Say true or false.***

**Ans :** True.

42. ***If a periodic function $f(x)$ is even in $(-l, l)$ then $a_0$ is ......*** [Apr. 97]

**Ans :** $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

43. ***Fourier series of period 2 for $|x|$ in $(0, 2)$ contains only cosine terms. Say true or false.*** [Apr. 97]

**Ans :** False. It contains both cosine and sine terms.

[Here $|x|$ is an even function. An even function contains only cosine terms either in the interval $(-l, l)$ or $(-\pi, \pi)$ but not in $(0, 2l)$ or $(0, 2\pi)$.]





44. ***For the half-range sine series in the interval $0 < x < l$ the coefficient $b_n$ is given by .........*** [Apr. 97]

**Ans :** $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$$

45. ***Examine whether the function $f(x) = \dfrac{1}{1-x}$ can be expanded in a Fourier series in any interval including $x = 1$.***

**Ans :** At $x = 1$, the function

$$f(x) = \frac{1}{1-x} \text{ is not continuous}$$

By Dirichlets condition, we cannot expand $f(x)$ as a Fourier series.

46. ***Write $a_0$, $a_n$ in the expression $x + x^3$ as a Fourier series in $(-\pi, \pi)$.*** [Apr. 97]

**Ans :** Here $f(x)$ is an odd function.

$$[\because f(-x) = -x - x^3 = -(x + x^3) = -f(x)]$$

So, $a_0$ and $a_n$ are zero.

47. ***What are the values of $a_0$, $a_n$ and $b_n$ for the Fourier series of $f(x)$ in $(c, c + 2\pi)$ ?*** [Apr. 97]

**Ans :** $$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx\,dx$$

48. ***If $f(x)$ is an odd function defined in $(-1, 1)$ what are the values of $a_0$ and $a_n$*** [Apr. 97]

**Ans :** $a_0 = 0$, $a_n = 0$

49. ***In the Fourier series expansion of $f(x) = |\sin x|$ in $(-\pi, \pi)$ what is the value of $b_n$?*** [Oct. 98]

**Ans :** Since $f(x) = |\sin x|$ is an even function

$$b_n = 0$$

51. *Is the function* $f(x) = \begin{cases} 1+\dfrac{2x}{\pi}, & -\pi<x<0 \\ 1-\dfrac{2x}{\pi}, & 0<x<\pi \end{cases}$ *is odd. Say true or false*

[Nov. 97]

*Ans :* False – Even function

52. *Euler's formulae for the Fourier coefficients in the half range cosine series of* $f(x)$ *in* $(0, l)$ *are ........* [Nov. 97]

*Ans :*

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

53. *Find the constant* $a_0$ *of the Fourier series for the function* $f(x) = x\cos x$ *in* $-\pi<x<\pi$. [Apr. 98]

*Ans :*

$$f(x) = x\cos x$$

$$f(-x) = -x\cos x = -f(x)$$

$\therefore$ $f(x)$ is an odd function.

$$a_0 = 0$$

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