UNIT-1

PARTIAL DIFFERENTIAL EQ^{n}S (P.D.E.)

(QUICK REVISION NOTES)

Linear v/s Non-Linear

eg-1) $\left(\frac{\partial^3 y}{\partial x^3}\right)^1 + \left(\frac{\partial^2 y}{\partial x^2}\right)^2 + \left(\frac{\partial y}{\partial x}\right) + 3y = 0$ (Non-Linear)

(Order = 3
Degree = 1)

- Order = Highest Derivative
- Degree = Power on Highest Derivative

eg-2) $\left(\frac{\partial^3 y}{\partial x^3}\right)^3 + \left(\frac{\partial^2 y}{\partial x^2}\right)^2 + \left(\frac{\partial y}{\partial x}\right) + 3y = 0$ (Non-Linear)

(Order = 3
Degree = 3)

eg-3) $\left(\frac{\partial^3 y}{\partial x^3}\right) + \left(\frac{\partial^2 y}{\partial x^2}\right) + \underbrace{3y\left(\frac{dy}{dx}\right)}_{\text{mul. of derivative with dep. var.}} = 0$ (Non-Linear)

(Order = 3
Degree = 1)

eg-4) $\left(\frac{\partial^3 y}{\partial x^3}\right) + \left(\frac{\partial^2 y}{\partial x^2}\right) + \underbrace{3x\left(\frac{dy}{dx}\right)}_{\text{Mul. of derivative with indep. var.}} = 0$ (Linear)

(Order = 3
Degree = 1)

$z = f(x, y)$

$\frac{\partial z}{\partial x} = p$	$\frac{\partial^2 z}{\partial x^2} = r$	$\frac{\partial^2 z}{\partial y^2} = t$
$\frac{\partial z}{\partial y} = q$	$\frac{\partial^2 z}{\partial x \partial y} = s$	

Formation of P.D.E. by eliminating →

① Arbitrary constants (a.c.)

~~We may get diff PDE (ans is NOT unique~~

If no. of a.c. ≤ no. of indep. var

↓

we get P.D.E. of first order

If no. of a.c. > no. of indep. var

↓

we get PDE of 2nd or higher order

↓

we may get diff. PDE (ans is NOT unique)

P.T.O. →

② Arbitrary functions (a-fn)

If a-fn = 1	If a-fn > 1
↓	↓
PDE of first order	PDE of 2nd or higher order
	↓
	ans is NOT unique

★ **Elimination of a-fn of type $\phi(u, v) = 0$**

(M-I) $Pp + Qq = R$ (Lagrange eqn)

→ $\left(p = \frac{\partial z}{\partial x}\right)$; $\left(q = \frac{\partial z}{\partial y}\right)$

→ $\left(P = \frac{\partial(u,v)}{\partial(y,z)}\right)$; $\left(Q = \frac{\partial(u,v)}{\partial(z,x)}\right)$; $\left(R = \frac{\partial(u,v)}{\partial(x,y)}\right)$

1st co-ordinate missing

2nd coordinate missing

3rd coordinate missing

$\overrightarrow{1 \quad 2 \quad 3} \Rightarrow \begin{matrix}(2,3)\\(3,1)\\(1,2)\end{matrix}$

order matters in denominator (remove the missing co-ordinate & write the other 2 in order)

(M-II) $\phi(u, v) = 0$

$$\boxed{u = f(v) \quad \text{or} \quad v = f(u)}$$

Then solve normally as before.

Solution of standard types of first order P.D.E.

Types of solⁿs

① Complete solⁿ (or) complete integral

② Singular solⁿ

③ General solⁿ (or) General Integral (or) Particular Integral (PI)

Type - I) $F(p, q) = 0$ —— ①

• For complete solⁿ

Standard eqⁿ: $z = ax + by + c$ —— ②

$$\frac{\partial z}{\partial x} = p = a \quad ; \quad \frac{\partial z}{\partial y} = q = b \quad \text{—— ③}$$

→ Put ③ in ①

$\therefore F(a, b) = 0$

Find $b = \phi(a)$ —— ④

→ Put ④ in ②

$$\therefore \boxed{z = ax + \phi(a)y + c} \quad \text{—— ⑤}$$

- For singular solⁿ

→ Differentiate (5) partially w.r.t. 'a' and 'c'

$\therefore \frac{\partial z}{\partial c} = 0 = 1$ ... (Absurd!)

$\therefore$ No singular solⁿ.

- For general solⁿ (PI)

Put $c = \psi(a)$ in (5)

$$\therefore z = ax + \phi(a)y + \psi(a) \quad \text{— (6)}$$

$$\frac{\partial z}{\partial a} = 0 = x + \phi'(a)y + \psi'(a) \quad \text{— (7)}$$

→ Eliminating 'a' b/w (6) & (7); we will get general solⁿ.

★ Sri Dharacharya Rule (Quadratic formula)

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \leftarrow \left(\begin{array}{c} ax^2 + bx + c = 0 \\ \text{(quad. in } x\text{)} \end{array}\right)$$

$b = \phi(a)$ निकालने में कभी- कभी iska use kiya jaaeya

(उस case में quad. eqⁿ को 'b' की form में rkhna (mtlb → quad. in 'b'))

Type-2) CLAIRANT'S FORM

$$z = px + qy + f(p,q) \quad \text{--- (1)}$$

(i) For complete solⁿ

[Standard eqⁿ]: $z = ax + by + c$

$$\frac{\partial z}{\partial x} = p = a; \quad \frac{\partial z}{\partial y} = q = b \quad \text{--- (2)}$$

→ Put (2) in (1)

$$\therefore \boxed{z = ax + by + f(a,b)} \quad \text{--- (3)} \; //$$

Type-1 में hmari eqⁿ 3 var's (a, b, c) में थी तो isiliye 'b' को 'a' की form में लिखकर reduce करना पड़ रहा था

→ BUT yha already 2 var (a, b) में bn rhi hai, तो yehi complete solⁿ hai.

(ii) For singular solⁿ

→ Differentiate (3) partially w.r.t. 'a' and 'b'

& $\frac{\partial z}{\partial a} = 0$; $\frac{\partial z}{\partial b} = 0$

→ Find values of 'a' and 'b' in terms of x and y.

→ substitute above found values of 'a' and 'b' in (3) to get singular solⁿ.

◉ For general solⁿ (PI).

Put $b = \phi(a)$ in ③

$$\therefore \quad z = ax + \phi(a)y + f(a, \phi(a)) \quad \text{—④}$$

$$\frac{\partial z}{\partial a} = 0 = x + \phi'(a)y + \frac{\partial}{\partial a}\left(f(a, \phi(a))\right) \quad \text{—⑤}$$

→ eliminating 'a' b/w ④ & ⑤, we will get general solⁿ.

Type-3)

(A) $F(z, p, q) = 0.$ —①

◉ For complete solⁿ

Let $z = f(u)$ where $u = x + ay$.

$$\Rightarrow p = \frac{\partial z}{\partial x} = \left(\frac{dz}{du}\right)\left(\frac{\partial u}{\partial x}\right) = \left(\frac{dz}{du}\right) \quad \left[\text{as } \left(\frac{\partial u}{\partial x}\right) = 1\right]$$

$$\therefore \quad \boxed{p = \frac{dz}{du}} \quad \text{—②}$$

$$\Rightarrow q = \frac{\partial z}{\partial y} = \left(\frac{dz}{du}\right)\left(\frac{\partial u}{\partial y}\right) = a\frac{dz}{du} \quad \left[\text{as } \left(\frac{\partial u}{\partial y}\right) = a\right]$$

$$\therefore \quad \boxed{q = a\frac{dz}{du}} \quad \text{—③}$$

→ Substitute ② & ③ in ①

→ seperate the variables (z and u) & then integrate

→ At last, put $u = x + ay$ to get complete sol^n.

$\boxed{}$ — (4)

(i) For singular sol^n

→ Differentiate (4) partially w.r.t. 'a' and 'c' (constant of integration)

$\frac{\partial z}{\partial c} = 0 = 1$ ... (Absurd!)

∴ No singular sol^n!

(ii) For general sol^n (PI)

Put $c = \phi(a)$ in (4)

$\boxed{}$ — (5)

— (6)

$\frac{\partial z}{\partial a} = 0 =$ ....

→ Eliminating 'a' b/w (5) & (6), we will get general sol^n.

(B) $F(x, p, q) = 0$ — ①

① For complete sol^n

Put $\boxed{q = a}$ in ①

$\therefore F(x, p, a) = 0$

Solve for 'p' and get $\boxed{p = f_1(x, a)}$

Now, $\boxed{dz = p\,dx + q\,dy}$ — ②

→ Substitute values of 'p' and 'q' in ②

→ then ~~variable separation~~ integrate to get complete sol^n — ③

① For singular sol^n

→ Differentiate ③ partially w.r.t. 'a' and 'c'

$\frac{\partial z}{\partial c} = 0 = 1$ ...... (Absurd!)

$\therefore$ No singular sol^n !

① For general sol^n (P I)

→ Put $c = \phi(a)$ in ③

— ④

$\frac{\partial z}{\partial a} = 0 =$ - - - - - — ⑤

→ Eliminating 'a' b/w ④ & ⑤ ; we will get general sol^n.

(c) $F(y, p, q) = 0$ —— (1)

- For complete sol^n

put $\boxed{p = a}$ in (1)

$\therefore F(y, a, q) = 0$

solve for 'q' and get $\boxed{q = f(a, y)}$.

Now, $\boxed{dz = p\,dx + q\,dy}$ —— (2)

→ Substitute values of 'p' and 'q' in (2)

→ then integrate to get complete sol^n

$\boxed{}$ —— (3)

- For singular sol^n

→ Differentiate (3) partially w.r.t. 'a' and 'c'

$\frac{\partial z}{\partial c} = 0 = 1$ ... (Absurd!)

$\therefore$ NO singular sol^n!

- For general sol^n

→ Put $c = \phi(a)$ in (3)

$\boxed{}$ —— (4)

$\therefore$ ------------ —— (5)

$\frac{\partial z}{\partial a} = 0 =$ ------

→ Eliminating 'a' b/w (4) & (5), we will get general sol^n.

Type-4) SEPERABLE eq^n

$$f(x, p) = \phi(y, q) = a \quad \text{--- (1)}$$

- **For complete sol^n**

$p = f_1(x, a)$; $q = \phi_1(y, a)$

$$\boxed{dz = p\,dx + q\,dy} \quad \text{--- (2)}$$

→ Integrate to get complete sol^n

$$\boxed{} \quad \text{--- (3)}$$

- **For singular sol^n**

→ Differentiate (3) partially w.r.t. 'a' and 'c'.

$\frac{\partial z}{\partial c} = 0 = 1$ ..... (Absurd!)

∴ No singular sol^n!

- **For general sol^n**

→ Put $c = \phi(a)$ in (3)

∴ $\boxed{}$ --- (4)

$\frac{\partial z}{\partial a} = 0 = \ldots\ldots$ --- (5)

→ Eliminating 'a' b/w (4) & (5), we will get general sol^n.

Lagrange's Linear Eq^n

① grouping method

② Method of multipliers.

→ sirf general sol^n निकालना है यहाँ और वो direct निकलकर आएगा।

$$\boxed{Pp + Qq = R}$$ Lagrange's Linear Eq^n

$$\left(p = \frac{\partial z}{\partial x} \;;\; q = \frac{\partial z}{\partial y}\right)$$

P, Q, R direct eq^n से पता chlenge.

p का coeff = P
q का coeff = Q
constant = R

Working Rule ....

★ Auxilliary Eq^n

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

(Put P, Q, R in this)

① Grouping Method

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

→ Take any 2 pairs to solve the auxilliary eqn that are easy to integrate

$\therefore$ $(u = c_1)$; $(v = c_2)$

general solⁿ :→ $\boxed{f(u, v) = 0}$ //.

② Method of multipliers

Let (l, m, n) be 1 set of multipliers

[(l, m, n) may be $f(x, y, z)$ or constants]

choose them in the way that

$d^n = 0$ (denominator $= 0$)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{\boxed{lP + mQ + nR} = 0}$$

$P, Q, R \neq 0$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$

→ Integrating to get $(u = c_1)$

llly, choose another set of multipliers so that we get $(v = c_2)$

Now,

general solⁿ ⇒ $\boxed{f(u, v) = 0}$ //

commonly - used multipliers

1. x, y, z
2. $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$
3. x^2, y^2, z^2
4. $1, 1, 1$
5. $l, m, n.$

Linear P.D.E of 2nd & higher order with const coeffs of homogenous types.

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots\cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

$$\frac{\partial}{\partial x} = D \quad \& \quad \frac{\partial}{\partial y} = D'$$

$$\therefore \left(a_0 D^n + a_1 D^{n-1} D' + \cdots\cdots + a_n D'^n\right) z = F(x, y)$$

$$\boxed{f(D, D') z = F(x, y)}$$

general solⁿ → $\boxed{z = \underbrace{CF}_{\text{complementary fn}} + PI}$ → Particular Integral

To find C.F.

Auxilliary eq^n

In LHS → $D = m,\ D' = 1,\ z = 1$

RHS = 0

solve eq^n in 'm'.

$m_1 \neq m_2$

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$m_1 = m_2 = m$

$$CF = f_1(y + mx) + x f_2(y + mx)$$

$m_1 = m_2 = m_3 = m$

$$CF = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx)$$

⋮

To find P.I. $= \dfrac{RHS}{LHS}$

Type-1)

$$\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by} \quad (\text{if } f(a, b) \neq 0)$$

$D \to a$

$D' \to b$.

Type-2) $\dfrac{1}{f(D^2, DD', D'^2)} \sin/\cos(ax+by)$.

$\Downarrow$

$\dfrac{1}{f(-a^2, -ab, -b^2)} \sin/\cos(ax+by)$.

$D^2 = -a^2$
$D'^2 = -b^2$
$DD' = -ab$

Type-3) $\dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$

$\downarrow$

converting in Binomial form

→ In d^n, take common the highest power term in 'D'.

$$\boxed{(1+x)^{-1} = 1 - x + x^2 - x^3 + - \cdots}$$

Type-4) $\dfrac{1}{f(D, D')} e^{ax+by} \phi(x, y) = e^{ax+by} \dfrac{1}{f(D+a, D'+b)} \phi(x, y)$

This is done

Now do it acc. to $\phi(x, y)$

if $\phi(x, y) = \sin/\cos(ax+by)$

then use → Imag./Real part of $e^{i\theta}$ concept!

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Type-5) $\dfrac{\sin(ax)\sin(by) / \cos(ax)\cos(by)}{f(D^2, D'^2)}$

$\Downarrow$

$$\frac{(\quad\quad)}{f(-a^2, -b^2)}.$$

→ If $d^n = 0$ while solving PI then multiply numerator with 'x' & d^n को 'D' के respect से differentiate karo & continue doing this until you stop getting 0 in d^n.

★ General Rule to find PI (all types)

(P.S → with some condⁿs)

$$P.I = \frac{F(x,y)}{\boxed{f(D,D')}}$$

→ only applicable if d^n can be factorised.

$$PI = \frac{1}{D - mD'} F(x,y) = \int F(x, \underbrace{c - mx}_{\Downarrow\, y})\, dx$$

(where 'c' is replaced by $y + mx$ after integration.)

TIP. Easy way to solve integral of 2 fns than ILATE formula

Q $\int \underbrace{x^3}_{I} \underbrace{\sin x}_{II} \, dx$

$+(\quad) - (\quad) + (\quad) - (\quad) \ldots$

aise - aise chlna hai jb तक terms 0 ना आ जाए

⊙ for 1st term

$\left(I \int II \, dx \right)$

⊙ for rest of the terms, do →
consider I and II of just previous term
& do → $\left[\frac{d}{dx}(I) * \int II \, dx \right]$

Sol^n $+ \Big[\underbrace{x^3}_{I'} \underbrace{(-\cos x)}_{II'} \Big] - \Big[\underbrace{3x^2}_{I''} \underbrace{(-\sin x)}_{II''} \Big] + \Big[\underbrace{6x}_{I'''} \underbrace{(\cos x)}_{II'''} \Big]$

$- \Big[\underbrace{6}_{I''''} \underbrace{(\sin x)}_{II''''} \Big] + \Big[0 \Big]$

$I \int II$

$\frac{d}{dx}(I') * \int II' \, dx$

$\frac{d}{dx}(I'') * \int II'' \, dx$

$\frac{d}{dx}(I''') * \int II''' \, dx$

stop!

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$ //.

Hyperbolic functions

sin ht ; cos ht } Hyperbolic fns.

$$e^{ix} = \cos x + i\sin x \quad -(A)$$
$$e^{-ix} = \cos x - i\sin x \quad -(B)$$

$$e^{x} = \cos hx + \sin hx \quad -(C)$$
$$e^{-x} = \cos hx - \sin hx \quad -(D)$$

(A + B) ; $$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

(A − B) ; $$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

(C + D) ; $$\boxed{\cos hx = \frac{e^{x} + e^{-x}}{2}}$$

(C − D) ; $$\boxed{\sin hx = \frac{e^{x} - e^{-x}}{2}}$$