UNIT-1

Overview

(1) Formulation of Partial differential equation by eliminating
(a) arbitary constants
(b) arbitary functions

(2) Solution of standard type PDE

(I) Type (I) $f(p,q)=0$
(II) Clairaut's form, $z = px + qy + f(p,q)$
(III) $f(z,p,q) = 0$ or $f(x,p,q)=0$ or $f(y,p,q)=0$
(IV) Seperable equations $f(x,p) = g(y,q)$

(3) Lagrange's equation $Pp + Qq = R$
Auxilary eq^n $\frac{dx}{P} + \frac{dy}{Q} + \frac{dz}{R} =$

(I) Grouping method
(II) Method of multiplier

(4) Solution of PDE of higher order
$Z = CF + PI$
$CF = f_1(y+m_1x) + f_2(y+m_2x) + \ldots$

for PI -
Case I - $g(x,y) = e^{ax+by}$
Case II - $g(x,y) = \sin(ax+by)$ or $\cos(ax+by)$
Case III - $g(x,y) = x^m y^n$
Case IV - $g(x,y) = e^{ax+by}\ \phi(x,y)$
Case V - $\sin ax \sin bx$
Case VI - $\cos ax \cos bx$

General Rule: $PI = \frac{1}{(D-mD')}\phi(x,y) = \int \phi(x, a-mx)\, dx$

If no. of arbitary constants = no. of indipent variables
can 1st order

$\frac{\partial z}{\partial x} = p$ | $\frac{\partial z}{\partial y} = q$ | $\frac{\partial^2 z}{\partial x^2} = r$ | $\frac{\partial^2 z}{\partial x \partial y} = s$ |

$$\frac{\partial^2 z}{\partial y^2} = t$$

Q. form the

$z = (x^2+a)(y^2+b)$ — ① a, b - arbitary constant

differentiating ① partially w.r.t x & y, we get

$\frac{\partial z}{\partial x} = 2x(y^2+b)$ $\Rightarrow \frac{p}{2x} = y^2+b$ — ②

& $\frac{\partial z}{\partial y} = 2y(x^2+a)$, $\frac{q}{2y} = x^2+a$ — ③

using ② & ③ in ①

$z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right)$

$4xyz = pq$

Q. form partial differential equation by eliminating the arbitary constants a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — ①

ellipsoid

differentiating ① partially w.r.t x & y, we get

$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$ — ②

$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$ — ③

differentiating ② partially w.r.t x & y again,

w.r.t x, $\frac{2}{a^2} + \frac{2}{c^2}\left[\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}\right] = 0$ — ④

w.r.t y, & $\frac{2}{c^2}\left[\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + z\frac{\partial^2 z}{\partial y \partial x}\right] = 0$ — ⑤

diff. ③ w.r.t y partially

$\frac{2}{b^2} + \frac{2}{c^2}\left[\left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2}\right] = 0$ — ⑥

from (5)

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + \frac{z\,\partial^2 z}{\partial x \partial y} = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

⇒ To form partial differential equations by eliminating the arbitary functions -

if there are only one function then obtained eqn will be of 1st order.

if there are more than one function than obtained eqn will be higher order.

Q. Form the P.D.E by elimintating the arbitary function from
$z = f(x^2+y^2)$ —— (1)

diff (1) partially w.r.t x & y

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x \quad \text{—— (2)}$$

& $p = 2x f'(x^2+y^2)$

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y$$

$$q = 2y f'(x^2+y^2) \quad \text{—— (3)}$$

Q Obtain the partial differential equation by eliminating

$f(x^2+y^2+z^2,\ x+y+z)=0$ —— (1)

The eq^n (1) can be rework as

$x^2+y^2+z^2 = g(x+y+z)$ —— (2)

diff (2) partially w.r.t x & y

$2x + 2z\frac{\partial z}{\partial x} = g'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$

$2x + 2zp = g'(x+y+z)(1+p)$ —— (3)

&

$2y + 2z\frac{\partial z}{\partial y} = g'(x+y+z)(1+\partial z/\partial y)$

$2y + 2zq = g'(x+y+z)\ (1+q)$ —— (4)

dividing (2) by (3)

$\frac{x+zp}{y+zq} = \frac{1+p}{1+q}$

$x + zp + xq + zpq = y + zq + yp + zpq$

$(z-y)p + (x-z)q = y - x$

$z = xy + f(x+y)$

Q. Obtain the partial differential eq^n by eliminating the arbitary functions from

$z = f(2x+y) + g(3x-y)$ —— (1)

differentiating (1) partidlly w.r.t x & y

$\frac{\partial z}{\partial x} = 2f'(2x+y) + 3g'(3x-y)$ —— (2)

$\frac{\partial z}{\partial y} = f'(2x+y) - g'(3x-y)$ —— (3)

diff (2) partially w.r.t x & y and (3) w.r.t y

$$r = \frac{\partial^2 z}{\partial x^2} = \cancel{y}\, 4f''(2x+y) + 9g''(3x-y) \quad \text{—} (4)$$

$$s = \frac{\partial^2 z}{\partial y \partial x} = 2f''(2x+y) - 3g''(3x-y) \quad \text{—} (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(2x+y) + g''(3x-y) \quad \text{—} (6)$$

eq (4) + (5) × 3

$$r + 3s = 10f''(2x+y) \quad \text{—} (7)$$

eq (5) + (6) × 3

$$s + 3t = 5f''(2x+y) \quad \text{—} (8)$$

dividing (7) by (8)

$$\frac{r+3s}{s+3t} = 2$$

$$r + 3s = \cancel{2s}\, 2s + 6t$$

$$r + s - 6t = 0$$

→ Solution of Partial Differential eqⁿ

① **Solution** - a solution of a partial diff. eqⁿ is the relation b/w dependent & independent variable which satisfy partial diff. eqⁿ.

(I) **Complete Integral** - a solution which contain arbitary constant equal to number of independent variable is called complete integral.

(II) **Particular Integral** - a solution which is obtained by complete integral by giving some particular value, to arbitary constant.

(III) **Singular Solution** - (no constant)

Let $f(x,y,z,p,q) = 0$ —— ①
PDE where complete integral is
$\phi(x,y,z,a,b) = 0$ —— ②
diff ② partially w.r.t a and b

$$\frac{\partial \phi}{\partial a} = 0 \text{ —— ③ } \& \frac{\partial \phi}{\partial b} = 0 \text{ —— ④}$$

then singular solution obtained by eliminating a & b from 2, 3 & 4.

I

Solution of PDE
Standard form -

Form (I) - If $f(p, q) = 0$
The solution will be
$z = ax + by + c$ —— ①
where a & b connected by the relation
$f(a, b) = 0$

Q. Solve $p^2 + q^2 = 5$
which is of the type $f(p, q) = 0$
$\therefore$ the solution given by
$z = ax + by + c$ —— ①
where a & b connected by
$a^2 + b^2 = 5$
$\Rightarrow b = \pm\sqrt{5 - a^2}$
using in ①
$z = ax \pm \sqrt{(5-a)^2}\, y + c$ —— ②
diff. ② partialy w.r.t c
$0 = 1$ (which is not valid)
$\therefore$ There is no singular solution

$p = a, \quad q = b$

Q. Solve $p^2 + q^2 = npq$
which is the type $f(p, q) = 0$
the solution given by
$z = ax + by + c$ —— ①
where a & b are connected by -
$a^2 + b^2 = nab \Rightarrow b^2 - nab + a^2 = 0$
$b = \dfrac{na \pm \sqrt{n^2a^2 - 4a^2}}{2} = \dfrac{na \pm a\sqrt{n^2 - 4}}{2}$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$z = ax + \left(\frac{na \pm a\sqrt{n^2-4}}{2}\right)y + c \quad \text{—} (2)$$

$0 = 1$ (which is not valid)

$\therefore$ There is no singular solution.

$\boxed{\text{Type II}}$

objective

Clairaut's form

If $z = px + qy + f(p,q)$

Then replacing p by a & q by b

Then complete integral is

$$z = ax + by + f(a,b)$$

ex. Solve $z = px + qy + p^2 - q^2$

which is of the type $z = px + qy + f(p,q)$

$\therefore$ The complete integral is given by

$$z = ax + by + a^2 - b^2 \quad \text{—} (2)$$

diff. (2) partially w.r.t. a & b

$$0 = x + 2a \Rightarrow a = \frac{-x}{2} \quad \text{—} (3)$$

&

$$0 = y - 2b \Rightarrow b = \frac{y}{2} \quad \text{—} (4)$$

putting (3) & (4) in eq^n (2)

$$z = \left(\frac{-x}{2}\right) \times x + \left(\frac{y}{2}\right) \times y + \left(\frac{-x}{2}\right)^2 + \left(\frac{y}{2}\right)^2$$

$$z = \frac{-x^2}{2} + \frac{y^2}{2} - \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = \frac{-x^2}{4} + \frac{y^2}{4} \quad \text{(singular solution)} \quad \text{—} (5)$$

only for general solution.

putting $b = \phi(a)$ in (2)

$$z = ax + \phi(a)y + a^2 - (\phi(a))^2 \quad \text{—— (6)}$$

diff. partially w.r.t a.

$$0 = x + \phi'(a)y + 2a - 2\phi(a)\phi'(a)$$

Now by eliminating a ~~for~~ from (6) & (7) we get general solution.

Imp Q

Solve $z = px + qy + \sqrt{1+p^2+q^2}$ —— (1)

which is of the form $z = px + qy + f(p, q)$.

$\therefore$ The complete integral is given by

$$z = ax + by + \sqrt{1+a^2+b^2} \quad \text{—— (2)}$$

diff (2) partially w.r.t a & b

$$0 = x + \frac{1}{\cancel{2}\sqrt{1+a^2+b^2}}\cancel{2}(a)$$

$$0 = y + \frac{1}{\sqrt{1+a^2+b^2}}(b) \quad \text{—— (4)}$$

Now :

Ans —

$$z^2 = 1 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 1 \quad \text{— singular solution}$$

Type III -

(a) If $f(z, p, q) = 0$
Then let $z = f(U)$ where $U = x + ay$
$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial U}{\partial x} = \frac{\partial z}{\partial U}$

& $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial U}\frac{\partial U}{\partial y} = a\frac{\partial z}{\partial U}$

$p = \frac{\partial z}{\partial U}$
$q = a\frac{\partial z}{\partial U}$

(b) If $f(x, p, q) = 0$
put $q = a$ and find p
and using in
$dz = p\,dx + q\,dy$

(c) If $f(y, p, q) = 0$
put $p = a$ and find q & using in -
$dz = p\,dx + q\,dy$

Q. Solve $p^2 + q^2 = z$.
which is of the form $f(z, p, q) = 0$
Let $z = f(U)$ where $U = x + ay$ — Type (a)
Then $p = \frac{dz}{dU}$ & $q = a\frac{dz}{dU}$

$$\left(\frac{dz}{dU}\right)^2 + \left(a\frac{dz}{dU}\right)^2 = z$$

$$\left(\frac{dz}{dU}\right)^2 (1 + a^2) = z$$

$$\frac{dz}{dU} = \sqrt{\frac{z}{1+a^2}}$$

by variable seperation

$$\frac{dz}{z^{1/2}} = \frac{dU}{\sqrt{1+a^2}}$$

by integration -

$$\frac{dz}{z^{1/2}} = \frac{du}{\sqrt{1+a^2}}$$

$$\frac{z^{-1/2+1}}{-1/2+1} = \frac{u}{\sqrt{1+a^2}} + c$$

$$2z^{1/2} = \frac{1}{\sqrt{1+a^2}}(x+ay)+c \quad \text{——} ②$$

differentiating ② partially w.r.t c which gives $0=1$ (which is not possible).

$\therefore$ Singular solution is not possible.

Q Solve $p = 2qx$

which is the form $f(x,p,q)=0$ — Type (b)

$\therefore$ Let $q = a$ $\therefore$ $p = 2ax$

Now,

$dz = pdx + qdy$

Integrating

$$z = \int 2ax\,dx + \int a\,dy + c$$

$$z = ax^2 + ay + c \quad \text{——} ②$$

diff ② partially w.r.t c which gives $0=1$ (which is not possible).

$\therefore$ Singular solution not possible.

Q. Solve $pq = y$

which is the form $f(y,p,q)=0$ — Type (c)

$\therefore$ Let $p = a$ $\therefore$ $q = \frac{y}{a}$

Page No: ______
Date: / /

Now $dz = p\,dx + q\,dy$

Integrating $z = \int a\,dx + \int \frac{y}{a}\,dy + c$

$= ax + \frac{1}{2a}y^2 + c$

$\boxed{\text{Type IV}}$ –

If $f(x,p) = g(y,q)$

Seperable equation –

Then $f(x,p) = g(y,q) = a$ (constant)

& also

$dz = p\,dx + q\,dy$

Q. Solve $p^2 + q^2 = x + y$

$p^2 - x = y - q^2$

which is of the form $f(x,p) = g(y,q)$

Let $p^2 - x = y - q^2 = a$ (constant)

$\therefore$ $p = \sqrt{x+a}$ &

$q = \sqrt{(y-a)}$

$dz = p\,dx + q\,dy$

Integration

$$z = \int (x+a)^{1/2}\,dx + \int (y-a)^{1/2}\,dy + c$$

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$$

① Solve $p(1+q) = qz$ ③ Solve $p^2y(1+x^2) = qx^2$

② Solve $q = px + p^2$

$\frac{p^2}{x^2}(1+x^2) = \frac{q}{y}$

→ Equations Reducible to Standard form:

① If $F(x^m p, y^n q) = 0$

② If $F(x^m p, y^n q, z) = 0$

③ If $F(z^k p, z^k q) = 0$

④ If $F(x^m z^k p, y^n z^k q) = 0$

Note:

$m \neq 1$ then put $X = x^{1-m}$ if $m = 1$ then $X = \log x$

$n \neq 1$ then put $Y = y^{1-n}$ if $n = 1$ then $Y = \log y$

$k \neq -1$ then put $Z = z^{k+1}$ if $k = -1$ then $Z = \log z$

Q. Solve $x^2p^2 + y^2q^2 = z^2$

we can write-

$$\left(\frac{xp}{z}\right)^2 + \left(\frac{yq}{z}\right)^2 = 1 \quad \text{——} ②$$

Comparing with $F(x^m z^k p, y^n z^k q) = 0$

here

$m = 1$ ∴ Let $X = \log x$

$n = 1$ $Y = \log y$

$k = -1$ $Z = \log z$

∴ $\frac{\partial X}{\partial x} = \frac{1}{x}$, $\frac{\partial Y}{\partial y} = \frac{1}{y}$, $\frac{\partial Z}{\partial z} = \frac{1}{z}$

Now, $P = \frac{\partial z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X}$

$= \frac{1}{Z} px = \frac{px}{Z}$

also $Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y}$

$= \frac{1}{Z} \cdot q \cdot y = \frac{yq}{Z}$

using in ②

$P^2 + Q^2 = 1$ —— ③ —— Type ①

which is in the form of $(P, Q) = 0$

$\therefore Z = aX + bY + c$ —— ④

where $a^2 + b^2 = 1$

$\Rightarrow b = \pm\sqrt{1-b^2}$

$\therefore \log z = a \log x \pm \sqrt{(1-x^2)} \log y + c$

Q. Solve $2x^4p^2 - yzp - 3z^2 = 0$

we can write -

$2\left(\frac{x^2p}{z}\right)^2 - \left(\frac{yq}{z}\right) - 3 = 0$ —— ②

Comparing with $F(x^m z^k p, y^n z^k q) = 0$

then

$m = 2$ Let $X = x^{1-2} = x^{-1}$

$n = 1$ $Y = \log y$

$k = -1$ $Z = \log z$

$\therefore \frac{\partial X}{\partial x} = -x^{-2}$

$\frac{\partial Y}{\partial y} = \frac{1}{y}$, $\frac{\partial Z}{\partial z} = \frac{1}{z}$

$P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \mathfrak{z}} \cdot \frac{\partial \mathfrak{z}}{\partial x} \cdot \frac{\partial x}{\partial X}$

$= \frac{1}{z} \cdot p(-x^2) = -\left(\frac{x^2 p}{z}\right)$

Also $Q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \cdot \frac{\partial \mathfrak{z}}{\partial y} \cdot \frac{\partial y}{\partial Y} =$

$\frac{1}{z} \cdot q \cdot y = \frac{yq}{\mathfrak{z}}$

using in (2)

$2P^2 - Q - 3 = 0$

which is in the form $f(P, Q) = 0$

$\therefore Z = aX + bY + C$ — (4)

where

$2a^2 - b - 3 = 0$

$\Rightarrow b = 2a^2 - 3$

$z = aX + (2a^2 - 3)Y + C$

$\log z = ax^{-1} + (2a^2 - 3)\log y + C$

ex- Solve $z^2(p^2+q^2) = x^2+y^2$

we have,

$(zp)^2 + (zq)^2 = x^2+y^2$ —— ①

Comparing with $f(z^k p, z^k q) = 0$

here $k=1$

so, let $Z = z^{k+1} = z^2$

$\frac{\partial Z}{\partial z} = 2z$

Now $P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = 2zp \Rightarrow zp = \frac{P}{2}$

$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = 2zq \Rightarrow zq = \frac{Q}{2}$

using in ①

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2+y^2$$

$$P^2 + Q^2 = 4x^2+4y^2$$

$$P^2 - 4x^2 = 4y^2 - Q^2 \text{ — ②}$$

which is of the form $f(x,P) = g(y,Q)$

$$z^2 = 2\left[\frac{x}{2}\sqrt{(x^2+a)} + \frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}}\right] +$$

$$2\left[\frac{y}{2}\sqrt{(y^2-a)} - \frac{a}{2}\cosh^{-1}\frac{y}{\sqrt{a}}\right] + c$$

Q. Solve $p^2x^4 + y^2zq = 2z^2$

we have

$(x^2p)^2 + (y^2q)z = 2z^2$

comparing with $f(x^m p, y^n q, z) = 0$

Here $\therefore X = x^{1-n} = x^{-1}$

$m = 2$ $\quad Y = y^{1-n} = y^{-1}$

$n = 2$

$$\frac{\partial X}{\partial x} = -x^{-2} \quad \& \quad \frac{\partial Y}{\partial y} = -y^{-2}$$

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = p(-x^{-2}) = -x^2 p$$

$\therefore x^2 p = -P$

$$Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y} = q(-y^2) = -qy^2$$

$\Rightarrow y^2 q = -Q$

Using in (1)

$(-P)^2 + (-Q)z = 2z^2$

$P^2 - zQ - 2z^2 = 0$ —— (2)

which is of the form $f(z, P, Q) = 0$

$\therefore$ Let $z = f(U)$ where, where $U = X + aY$

and $P = \frac{dz}{dU}$ & $Q = a\frac{dz}{dU}$

using (2)

$$\left(\frac{dz}{dU}\right)^2 - az\left(\frac{dz}{dU}\right) - 2z^2 = 0 \quad \text{—— (3)}$$

using

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \frac{dz}{dU} = \frac{-(-az) \pm \sqrt{(-az)^2 - 4(1)(-2z^2)}}{2}$$

$$\frac{dz}{du} = z\left[\frac{a \pm \sqrt{a^2+8}}{2}\right]$$

by variable seperable.

$$\frac{dz}{z} = \left[\frac{a \pm \sqrt{a^2+8}}{2}\right] dU$$

Integrating,

$$\log z = \frac{1}{2}\left[a \pm \sqrt{a^2+8}\right] U + c$$

$$= \frac{1}{2}\left[a \pm \sqrt{a^2 \pm 8}\right] (x + aY) + C$$

$$\log z = \frac{1}{2}\left[a \pm \sqrt{a^2+8}\right] (x^{-1} + ay^{-1}) + C$$

Lagrange's equation-

An equation of the form-

$$Pp + Qq = R$$

where,

P, Q, R function of x, y, z.

The Lagrange Auxilary equation are-

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(1) Grouping method (2) Method of multipliers.

Q. Find the general Solution of $px + qy = z$

Imp

Comparing with

$$Pp + Qq = R$$

Here $P = x$, $Q = y$, $R = z$

Lagrange auxilary equation are-

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Z}$$

Grouping method

ie. $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

(I) (II) (III)

From (I) & (II)

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrate, $\log x = \log y + \log C_1$

$\log x = \log C_1 y$

$x = C_1 y$

$\Rightarrow$ $\frac{x}{y} = C_1$ —(1)

from (I) & (III)

$$\frac{dx}{x} = \frac{dz}{z}$$

Integrate, $\log x = \log z + C_2$

$\log x = \log z C_2$

$x = C_2 z$

$\frac{x}{z} = C_2$ —— (2)

Then solution is –

$$f\left(\frac{x}{y}, \frac{x}{z}\right) = 0 \quad \text{or} \quad \frac{x}{y} = g\left(\frac{x}{z}\right) \quad \text{or} \quad \frac{x}{z} = h\left(\frac{x}{y}\right)$$

Imp

Method of Multipliers

Q. Solve $(mz - ny)p + (nx - lz)q = ly - mx$

$P = (mz - ny)$, $Q = (nx - lz)$, $R = (ly - mx)$

Lagrange auxilary equation are –

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

(I) (II) (III)

(grouping not possible)

Taking l, m, n as multipliers then each ratio equal to.

$$\frac{l\,dx + m\,dy + n\,dz}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

i.e. $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mz} = \frac{ldx+mdy+ndz}{0}$

$\therefore$ from (I) & last ratios:

$ldx + mdy + ndz = 0$

Integrating: $lx + my + nz = C_1$ —— ①

Again taking x, y, z as multiplier, then each ratio equal to –

i.e $\frac{xdx + ydy + zdz}{0}$

i.e. $xdx + ydy + zdz = 0$

Integrating:

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_2}{2}$

$\Rightarrow x^2 + y^2 + z^2 = C_2$ —— ②

Solution is

$f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$

Q. Solve $\frac{y^2 z}{x} p + xzq = y^2$

Comparing with

$Pp + Qq = R$

Here $P = \frac{y^2 z}{x}$, $Q = xz$, $R = y^2$

Lagrange auxilary equation are –

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e. $\frac{xdx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$

(I) (II) (III)

from (I) & (II)

$$\frac{x\,dx}{y^2\cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$x^2 dx = y^2 dy$$

Integrating:

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{C_1}{3}$$

$$x^3 - y^3 = C_1 \quad \text{--- (1)}$$

from (1) & (III) ratio:

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2} \Rightarrow x\,dx = z\,dz$$

Integration:

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + \frac{C_2}{2}$$

$$\Rightarrow x^2 - z^2 = C_2 \quad \text{--- (2)}$$

So,

$$f(x^3 - y^3,\ x^2 - z^2) = 0$$

NOTE -

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking 1, 1, 1 ; 1, 1, 0 ; 1, -1, 1

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \Rightarrow \frac{dx+dy+dz}{x+y+z},\ \frac{dx+dy}{x+y},\ \frac{dx-dy+dz}{x-y+z}$$

Q. Find the general solution of -

$$x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$$

Comparing with $Pp + Qq = R$

where,

$$P = x(z^2-y^2),\ Q = y(x^2-z^2),\ R = z(y^2-x^2)$$

∴ Lagrange's subsidiary equations are -

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

x, y, z & $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

∴ the ratio

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

from (I) & (II) ratio

$$x\,dx + y\,dy + z\,dz = 0$$

Integrating:

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_1}{2}$$

$$\Rightarrow x^2 + y^2 + z^2 = C_1 \quad \text{—(1)}$$

from (I) & (III) ratio -

Integrating

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\log x + \log y + \log z = \log C_2$

$\Rightarrow \log(xyz) = \log C_2$

$xyz = C_2$ —— ②

Solution is –

$f(x, y, z, x^2+y^2+z^2) = 0$

Q. Find the general solution of $(y+z)p + (z+x)q = x+y$

Comparing with $Pp + Qq = R$

where

$P = y+z$, $Q = z+x$, $R = x+y$

$\therefore$ Lagrange's substituting equations are –

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Taking 1, 1, 1 ; 1, −1, 0 ; 1, 0, −1 as multipliers then the each ratio equal to –

$$\underset{(I)}{\frac{dx+dy+dz}{2(x+y+z)}} = \underset{(II)}{\frac{dx-dy}{y-x}} = \underset{(III)}{\frac{dx-dz}{z-x}}$$

Solve (I) & (II)

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x}$$

$$\frac{dx+dy+dz}{x+y+z} = \frac{-2\,dx\,dy}{x-y}$$

Integrating:

$$\log(x+y+z) = -2\log(x-y) + \log c_1$$

$$\log(x+y+z) + 2\log(x-y) = \log c_1$$

$$\log(x+y+z)(x-y)^2 = \log c_1$$

$$\Rightarrow (x-y)^2(x+y+z) = c_1 \quad \text{--- (1)}$$

from (II) & (I)

$$\frac{dx-dy}{y-x} = \frac{dx-dz}{z-x}$$

$$\frac{dx-dy}{x-y} = \frac{dx-dz}{x-z}$$

Integrating $\log(x-y) = \log(x-z) + \log c_2$

$$\frac{x-y}{x-z} = c_2 \quad \text{--- (2)}$$

$\therefore$ Solution:

$$f\left(\frac{x-y}{x-z}, (x-y)^2(x+y+z)\right) = 0$$

Q Solve $p - q = \log(x+y)$

Comparing with $P_p + Q_q = R$

then:

$$P = 1, \quad Q = -1, \quad R = \log(x+y)$$

Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

from (I) & (II)

$$dx = -dy$$
$$dx + dy = 0$$

Integrating:

$$x + y = c_1 \quad \text{---} \; ①$$

from (I) & (III)

$$\frac{dx}{1} = \frac{dz}{\log c_1} \qquad \text{from (1)}$$

Integrating -

$$x = \frac{1}{\log c_1} z + c_2$$

$$\therefore x - \frac{z}{\log(x+y)} = c_2 \quad \text{---} \; ②$$

$\therefore$ Solution,

$$f\left(x+y,\; x - \frac{z}{\log(x+y)}\right) = 0$$

Q. Solve $p \log x + q \log y = \tan z$

Q. Solve $(y - xz)p + (yz - x)q = (x+y)(x-y)$

Q. Solve $(3z - \quad)p + (4x - 2z)q = 2y - 3x$

Homogeneous Linear Equation -

Q. homogeneous linear partial differential equation of nth order with constant coefficient is of the form -

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

Replacing $\frac{\partial}{\partial x} = D$ & $\frac{\partial}{\partial y} = D'$

$$a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D^2$$

$$f(D, D') = f(x,y)$$

$$a_0, a_1, a_2 \dots a_n$$

Complementary function (CF)

The solution of f $f(D, D') = 0$

Replacing D by m & $D' = 1$

The auxiliary eqn is given by -

$f(m,1) = 0$ solving and find the value of m

Case I → If all roots are distinct.

Let m, m_1, m_2, m_3 ...

Then

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots$$

Case II - If all roots are ~~distinct~~ repeated

Let m, m_1, m_2, m_3 ...

Then

$$CF = f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_2 x)$$

Q. Solve $(D^2 - 4DD' + 4D'^2)z = 0$

The auxiliary equation is given by

$m^2 - 4m + 4 = 0$

$m = 2, 2$

$\therefore CF = f_1(y+2x) + x f_2(y+2x)$

Here RHS = 0

$\therefore PI = 0$

$\therefore z = CF + PI$

$= f_1(y+2x) + x f_2(y+2x)$

Q. Solve $(D^3 - 3D^2D' + 2DD'^2)z = 0$

The auxiliary equation is $m^3 - 3m^2 + 2m = 0$

$m(m^2 - 3m + 2) = 0 \Rightarrow m = 0, 1, 2$

$C.F. = f_1(y + 0x) + f_2(y+x) + f_3(y+2x)$

$R.H.S = 0 \therefore PI = 0$

$z = CF + PI$

$= f_1(y) + f_2(y+x) + f_3(y+2x)$

Particular Integral (PI)

we have $f(D, D')z = g(x,y)$

$$PI = \frac{1}{f(D,D')} = g(x,y)$$

Type I - If $g(x,y) = e^{ax+by}$

Then $PI = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$, if $f(a,b) = 0$

Type II - If $g(x) =$ if $\sin(ax+by)$ or $\cos(ax+by)$

$$PI = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \text{ y } f(-a^2, -ab, -b^2) \neq 0$$

Type III - If $g(x,y) = x^m y^n$

$$PI = \frac{1}{f(D,D')}(x^m y^n)$$

$$= [f(D,D')]^{-1}(x^m y^n)$$

{using binomial theorem}

Type IV - If $g(x,y) = e^{ax+by} \phi(x,y)$

$$P.I = \frac{1}{f(D,D')} e^{ax+by} \phi(x,y)$$

$$= e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x,y)$$

Type VI & VII

If $g(x,y) = \sin ax \sin by$ or $\cos ax \cos by$

$$\therefore PI = \frac{1}{f(D^2, D'^2)} \sin ax \sin by \text{ or } \cos ax \cos by$$

$$= \frac{1}{f(-a^2, -b^2)} \sin ax \sin by \text{ or } \cos ax \cos by$$

General equation - rule

$$PI = \frac{1}{D - mD'} g(x,y) = \int g(x, a-mx)\,dx$$

i.e Replace y by $a - mx$ & integration w.r.t x.

NOTE:

→ If denominator is 0 in any type then differentiate partially dinominator w.r.t D. & multiplying by x in numerator.

Q. Solve $(D^3 - 7DD'^2 - 6D'^3)z = x^2y + \sin(x+2y) + e^{x+2y}$

Sol- The auxilary equation is given by -

$m^3 - 7m - 6 = 0$

Clearly $m = -1$ satisfy above equation

So, $(m+1)$ will be factor of this eqn

$m^2(m+1) \quad - m(m+1) \quad - 6(m+1)$

$m^3 + \cancel{m^2} - \cancel{m^2} - m - 6m - 6$

$\Rightarrow m^2 - m - 6$

$(m+1)(m^2 - m - 6) = 0$

$(m+1)(m^2 - 3m + 2m - 6) = 0$

$(m+1)[m(m-3) + 2(m-3)] = 0$

$(m+1)(m+2)(m-3) = 0 \Rightarrow$

$m = -1, -2, 3$

$CF = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$

Now $(PI)_1 = \dfrac{1}{(D^3 - 7DD'^2 - 6D'^3)} x^2y$

$= \dfrac{1}{D^3\left[1 - \left(7\dfrac{D'^2}{D^2} + 6\dfrac{D'^3}{D^3}\right)\right]} x^2y$

$= \dfrac{1}{D^3}\left[1 - \left(\dfrac{7D'^2}{D^2} + \dfrac{6D'^3}{D^3}\right)\right]^{-1} x^2y$

NOTE: (1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

(2) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

(3) $(1-x)$

$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$= \frac{1}{D^2}\left[x^2y + 0 + 0 + \cdots\right]$

$= \frac{1}{D^2}\left[\frac{x^3}{3}y\right] = \frac{1}{D}\left[\frac{x^4}{3\cdot 4}y\right]$

$= \frac{x^5}{3\cdot 4\cdot 5}y = \frac{1}{60}x^5y$

$(PI)_2 = \frac{1}{(D^2 - 7DD'^2 - 6D'^3)} \sin(x+2y)$

$= \frac{1}{(-1^2)D - 7D(-2^2) - 6(-2^2)D'} \sin(x+2y)$

$= \frac{1}{3(9D+8D')} \sin(x+2y)$

$= \frac{1}{3}\frac{(9D-8D')}{(81D^2 - 64D'^2)} \sin(x+2y)$

$= \frac{1}{3}\frac{(9D-8D')}{81(-1^2)-64(-2^2)} \sin(x+2y)$

$= \frac{1}{3}\frac{9D\sin(x+2y) - 8D'\sin(x+2y)}{-81+256}$

$= \frac{1}{3}\frac{9\cos(x+2y) - 8\times 2\cos(x+2y)}{175}$

$= \frac{-\cancel{7}^{1}}{3\times\cancel{175}_{25}}\cos(x+2y) = -\frac{1}{75}\cos(x+2y)$

$(PI)_3 = \frac{1}{(D^3 - 7DD'^2 - 6D'^3)} e^{x+2y}$

$= \frac{1}{(1)^3 - 7(1)(2)^2 - 6(2)^3} e^{x+2y} = -\frac{1}{75} e^{x+2y}$

$\therefore PI = (PI)_1 + (PI)_2 + (PI)_3$

$= \frac{1}{60} x^5 y - \frac{1}{75} \cos(x+2y) - \frac{1}{75} e^{x+2y}$

$\therefore z = CF + PI$

$= f_1(y-x) + f_2(y-2x) + f_3(y+3x) + \frac{1}{60} x^5 y - \frac{1}{75} \cos(x+2y) - \frac{1}{75} e^{x+2y}$

Q. Solve $(D^4 - D'^4) z = e^{x+y}$

Ans- The A.E. is given by

$m^4 - 1 = 0$

$(m^2-1)(m^2+1) = 0$

$m = \pm 1, \pm i$

Again ∴

$PI = \frac{1}{(D^4 - D'^4)} e^{x+y}$ Here $f(1,1) = 0$

$= \frac{1}{(D-D')(D+D')(D^2+D'^2)} e^{x+y}$

$= \frac{1}{(D-D')(1+1)(1^2+1^2)} e^{x+y}$

$= \frac{1}{4(D-D')} e^{x+y}$

$P.I. = \frac{1}{4} \frac{x}{(1-0)} e^{x+y} = \frac{1}{4} x e^{x+y}$

$\therefore z = CF + PI$

$\Rightarrow f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix) + \frac{1}{4} x e^{x+y}$

Q. Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$

The auxiliary equation is

$$m^3 + m^2 - m - 1 = 0$$

$$m^2(m+1) - 1(m+1) = 0$$

$$(m+1)(m^2-1) = 0$$

$$(m+1)(m+1)(m-1) = 0$$

$$\Rightarrow m = -1, -1, 1$$

$$\therefore CF = f_1(y-x) + x f_2(y-x) + f_3(y+x)$$

$$PI = \frac{1}{(D^3 + D^2D' - DD'^2 - D'^3)} e^x \cos 2y$$

$$= e^x \frac{1}{[(D+1)^3 + (D+1)^2 D' - (D+1)D'^2 - D'^3)} \cos 2y$$

$$= e^x \text{ The real part of } \frac{1}{[(D+1)^3 + (D+1)^2 D' - (D+1)D'^2 - D'^3]} e^{i(2y)}$$

$e^{i\theta} = \cos\theta + i\sin\theta$ Euler's form

$$= e^x \text{ The real part of } \frac{1}{(0+1)^3 + (0+1)^2(2i) - (0+1)(2i)^2 - (2i)^3}$$

$$= e^x \text{ The R.P. of } \frac{\cos 2y + i\sin 2y}{1 + 2i + 4 + 8i}$$

$$= e^x \text{ The R.P. of } \frac{\cos 2y + i\sin 2y}{5(1+2i)}$$

$$= \frac{1}{5} e^x \text{ The R.P. of } \frac{(\cos 2y + i\sin 2y)(1-2i)}{(1+4)}$$

$$= \frac{1}{25} e^x \text{ The R.P. of } [(\cos 2y + 2\sin 2y) + i(\sin 2y - 2\cos 2y)]$$

$= \frac{1}{25} e^x (\cos 2y + 2\sin 2y)$

$z = CF + PI$

$= f_1(y-x) + x f_2(y-x) + f_3(y+x) + \frac{1}{25} e^x(\cos 2y + 2\sin 2y)$

Q. Solve $(D^2 - DD')z = \sin x \sin 2y$

The A.E. is   Type (5)

$m^2 - m = 0$

$\Rightarrow m = 0, 1$

$CF = f_1(y) + f_2(y+x)$

$PI = \frac{1}{(D^2 - DD')} (\sin x \sin 2y)$

$= \frac{(D^2 + DD')}{(D^2)^2 - D^2 D'^2} (\sin x \sin 2y)$

$= \frac{(D^2 + DD')}{(-1)^2 - (-1)(-2^2)} \sin x \sin 2y$

$= \frac{-1}{3} \left[D^2(\sin x \sin 2y) + DD'(\sin x \sin 2y) \right]$

$= \frac{-1}{3} \left[-\sin x \sin 2y + 2\cos x \cos 2y \right]$

$\therefore z = CF + PI$

$= f_1(y) + f_2(y+x) - \frac{1}{3} \left[-\sin x \sin 2y + 2\cos x \cos 2y \right]$

Q. Solve $(D^2+DD'-6D'^2)z = y\cos x$

A.E. is

$m^2+m-6=0$

$\Rightarrow m = 2, -3$

$CF = f_1(y+2x) + f_2(y-3x)$

$$PI = \frac{1}{D^2+DD'-6D'^2} y\cos x$$

$$= \frac{1}{(D-2D')(D+3D')} y\cos x$$

$$= \frac{1}{(D-2D')}\int (a+3x)\cos x\,dx \qquad [\text{put } y = a+3x]$$

$$= \frac{1}{(D-2D')}\left[(a+3x)\sin x - (3)(-\cos x)\right]$$

$$= \frac{1}{(D-2D')}\left[y\sin x + 3\cos x\right]$$

$$= \int \left[(b-2x)\sin x + 3\cos x\right]dx \qquad (\text{put } y = b-2x)$$

$$= (b-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x$$

$$= -y\cos x + \sin x$$

$\therefore Z = CF + PI$

$$= f_1(y+2x) + f_2(y-3x) - y\cos x + \sin x$$