CHAPTER II

PARTIAL DIFFERENTIAL EQUATIONS

2·1. Introduction

You have already studied ordinary differential equations in the previous semesters. Now we will proceed to the study of partial differential equations. A partial differential equation is one which involves partial derivatives. The *order* of a partial differential equation is the order of the highest derivative occuring in it.

In what follows, z will be taken as the dependent variable and x, y the independent variables so that $z = f(x, y)$. We will use the following notations hereafter:

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x\, \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

2·2. Formation of differential equations

Partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

2·3. By elimination of arbitrary constants

Let us take the function

$$f(x, y, z, a, b) = 0 \qquad \text{...(1)}$$

where a and b are arbitrary constants.

Now we have to eliminate a and b while forming the differential equation.

Differentiating the equation (1) partially with respect to the independent variables x, y we get,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} + 0$$

i.e.,
$$\boxed{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0} \qquad \text{...(2)}$$

Similarly,
$$\boxed{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0} \qquad \text{...(3)}$$

Now, eliminating the two arbitrary constants a and b from (1), (2) and (3), we get a partial differential equation of the first order of the form

$$\phi(x, y, z, p, q) = 0. \qquad ...(4)$$

Note. Equation (1) is said to be the *primitive* or *complete solution* of the first order differential equation (4). If the number of constants to be eliminated is equal to the number of independent variables, the differential equation got after elimination of arbitrary constants will be of the first order. On the other hand, if the number of constants to be eliminated is more than the number of independent variables, the resulting partial differential equation will be of the second or higher orders. This is an important difference between the partial differential equation and the ordinary differential equation where the order of the differential equation equals the number of arbitrary constants eliminated.

Example 1. *Form the partial differential equation by eliminating the arbitrary constants a and b from* $z = (x^2 + a)(y^2 + b)$. **(A.U. '65 B.E.)**

Now, $z = (x^2 + a)(y^2 + b)$ ...(1)

Differentiating partially w.r.t. x and y in turn,

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b),$$

and
$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a).$$

Therefore, $x^2 + a = \dfrac{q}{2y}$

and
$$y^2 + b = \frac{p}{2x}$$

Substituting these in (1), we get,

$$z = \frac{p}{2y} \cdot \frac{q}{2x}$$

i.e.
$$4xyz = pq.$$

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Example 2. *Find the partial differential equation of all planes cutting equal intercepts from the x and y axes.*

Let a, c be the intercepts on x and z axes respectively. why not y axis.

Hence, the equation of the plane is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \qquad ...(1)$$

Differentiating partially w.r.t. x and y in turn,

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial x} = 0 \qquad ...(2)$$

Example 5. *Obtain the partial differential equation by eliminating a, b, c from* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Since the number of arbitrary constants is more than the number of independent variables, we will get the partial differential equation of order greater than 1.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Differentiate partially w.r.t. x and y. (z is dependent)

$$\frac{2x}{a^2}+\frac{2z}{c^2}p=0 \qquad (i)$$

$$\frac{2y}{b^2}+\frac{2z}{c^2}q=0 \qquad (ii)$$

Differentiate (i) w.r.t. x again.

$$\frac{2}{a^2}+\frac{2}{c^2}\left[z\cdot\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right]=0 \qquad (iii)$$

Differentiate (ii) w.r.t. y again.

$$\frac{2}{b^2}+\frac{2}{c^2}\left[z\cdot\frac{\partial^2 z}{\partial y^2}+\left(\frac{\partial z}{\partial y}\right)^2\right]=0 \qquad (iv)$$

Differentiate (i) w.r.t. y

$$0+\frac{2}{c^2}\left[z\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]=0 \qquad (v)$$

From (v), we get

$$z\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}=0 \quad \text{(required equation)}$$

[**Note.** We may also get different partial differential equations. The answer is not unique].

2·4. By elimination of arbitrary. functions

Let u and v be any two given functions of x, y and z. Let u and v be connected by an arbitrary function φ by the relation

$$\varphi(u, v)=0. \qquad (1)$$

Now, we want to eliminate φ.

Differentiating partially, w.r.t. x and y, we get, and $$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0 \qquad ...(3)$$

From (2) and (3), we get $\frac{1}{c}(p-q)=0$

Therefore, the equation is $p-q=0$.

Example 3. *Form the partial differential equation by elimination a and b from* $\log(az-1)=x+ay+b$ **(MS. 1991. Ap.)**

Differentiating the given function partially w.r.t. x,

$$\frac{1}{az-1}ap=1 \qquad ...(1)$$

Differentiating w.r.t y partially,

$$\frac{1}{az-1}\cdot aq=a \qquad ...(2)$$

Dividing (2) by (1), we get

$$a=\frac{q}{p};\ i.e.\ ap=q$$

Putting in (1)

$$q=az-1$$

$$q=\frac{q}{p}z-1$$

$p(q+1)=qz$ is the required equation.

Example 4. *Obtain the partial differential equation of all spheres whose centres lie on $Z=0$ and whose radius is constant and equal to r.*

(MS. 1991. Ap.)

The centre of sphere is $(a, b, 0)$

Hence its equation is $(x-a)^2+(y-b)^2+z^2=r^2$ ...(1)

Differentiate w.r.t. x and y partially in order

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$

$$2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$\therefore$ $$x-a=-pz$$

$$y-b=-qz$$

Substituting in (1),

$$p^2z^2+q^2z^2+z^2=r^2$$

i.e., $z^2(p^2+q^2+1)=r^2$ is the required equation.

$$\frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right)+\frac{\partial \varphi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p\right)=0 \qquad \ldots(2)$$

and
$$\frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right)=0 \qquad \ldots(3)$$

Eliminating $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ from (2) and (3), we obtain,

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$$

which simplifies to

$$Pp+Qq=R \qquad \ldots(4)$$

where
$$P=\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\equiv\frac{\partial (u, v)}{\partial (y, z)}$$

$$Q=\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\equiv\frac{\partial (u, v)}{\partial (z, x)}$$

$$R=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\equiv\frac{\partial (u, v)}{\partial (x, y)}$$

Equation (4) is the required one which is called Lagrange's linear equation.

The relation $\varphi(u, v)=0$ is a solution of (4), whatever may the arbitrary function φ be.

Example 6. *Form the partial differential equation by eliminating the arbitrary functions from (i)* $z=f(x^2+y^2)$ and

$$(ii)\ z = f(x+ct)+\phi(x-ct)$$

(i) Now $z=f(x^2+y^2)$

Differentiating w.r.t x and y in turn,

$$\frac{\partial z}{\partial x}=p=f'(x^2+y^2)\,.\,2x$$

$$\frac{\partial z}{\partial y}=q=f'(x^2+y^2)\,.\,2y$$

Dividing, $\frac{p}{q}=\frac{x}{y}$

$\therefore\ py-qx=0$ is the required one.

(ii) $z=f(x+ct)+\phi(x-ct)$

$$\therefore \qquad \frac{\partial z}{\partial x}=p=f'(x+ct)+\phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \quad \ldots(i)$$

$$\frac{\partial z}{\partial t} = cf'(x+ct) - c\phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct)$$

i.e., $$\frac{\partial^2 z}{\partial t^2} = c^2 \{f''(x+ct) + \phi''(x-ct)\}$$

Now using (i), $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$.

Therefore, the required differential equation is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Example 7. *Form the partial differential equation by eliminating the arbitrary function from the relation* $\phi(x^2+y^2+z^2, lx+my+nz) = 0$.

Now the given relation is of the form $\phi(u, v) = 0$,

where $u = x^2 + y^2 + z^2$,

and $v = lx + my + nz$.

Hence the partial differential equation is

$Pp + Qq = R$, where

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} = 2y \cdot n - 2z \cdot m,$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} = 2z \cdot l - 2x \cdot n,$$

and $$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 2x \cdot m - 2y \cdot l$$

Therefore the required equation is,

$$2(ny - mz)p + 2(lz - nx)q = 2(mx - ly).$$

i.e., $$(ny - mz)p + (lz - nx)q = (mx - ly).$$

Aliter: The equation $\phi(x^2+y^2+z^2, lx+my+nz) = 0$ may be written as $x^2+y^2+z^2 = f(lx+my+nz)$ where f is an arbitrary function. Now differentiate this equation w.r.t x and y partially, treating z as dependent variable.

$$2x + 2zp = f'(lx+my+nz) \times (l+np) \quad \ldots(1)$$

$$2y + 2zq = f'(lx+my+nz) \times (m+nq) \quad \ldots(2)$$

Divide (1) and (2) to eliminate f'

$$\frac{x+pz}{y+qz} = \frac{l+np}{m+nq}$$

$\therefore$ $mx + nqx + pmz + pqnz = ly + lqz + npy + pqnz.$

i.e. $(ny - mz)\, p + (lz - nx)\, q = mx - ly.$

Example 8. *Form the partial differential equation by eliminating f from*

$z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$ **(MS. 1988. Ap.)**

Let $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$ ...(1)

Differentiate w.r.t. x and y.

$$\frac{\partial z}{\partial x} = p = 2x + 2f'\left(\frac{1}{y} + \log x\right) \times \left(\frac{1}{x}\right) \qquad ...(2)$$

$$\frac{\partial z}{\partial y} = q = 2f'\left(\frac{1}{y} + \log x\right) \times \left(-\frac{1}{y^2}\right) \qquad ...(3)$$

Eliminate f' from (2) and (3).

$$\therefore \quad \frac{p-2x}{q} = \frac{-1}{x} \times y^2$$

$$px - 2x^2 = -qy^2$$

$$px + qy^2 = 2x^2$$

Example 9. *Form the partial differential equation by eliminating f and g from*

$$z = f(ax + by) + g(\alpha x + \beta y).$$

Differentiate w.r.t. x and y partially,

$$\frac{\partial z}{\partial x} = p = f'(ax+by)\,.\,a + g'(\alpha x + \beta y)\,.\,\alpha$$

$$\frac{\partial^2 z}{\partial x^2} = f''(ax+by)\,a^2 + g''(\alpha x + \beta y)\,\alpha^2 \qquad ...(i)$$

$$\frac{\partial^2 z}{\partial y \partial x} = f''(ax+by)\,ab + g''(\alpha x + \beta y)\,\alpha\beta \qquad ...(ii)$$

$$\frac{\partial z}{\partial y} = f'(ax+by)\,.\,b + g'(\alpha x + \beta y)\,.\,\beta$$

$$\frac{\partial^2 z}{\partial y^2} = f''(ax+by)\,b^2 + g''(\alpha x + \beta y)\,\beta^2 \qquad (iii)$$

(*i*) $\times\, b\beta - (a\beta + b\alpha)$ (*ii*) $+ a\alpha$ (*iii*) gives

$$b\beta \frac{\partial^2 z}{\partial x^2} - (a\beta + b\alpha)\frac{\partial^2 z}{\partial y \partial x} + a\,\alpha \frac{\partial^2 y}{\partial y^2}$$

$$= \left[a^2 b\beta - (a\beta + b\alpha)\, ab + a\alpha \,.\, b^2 \right] f''(ax + by)$$

$$+ \left[\alpha^2\, b\beta - (a\beta + b\alpha)\, \alpha\beta + a\alpha\beta^2 \right] g''(\alpha x + \beta y)$$

$$= [0]\, f'' + [0]\, g''$$

$$= 0$$

Hence, $b\beta \frac{\partial^2 z}{\partial x^2} - (a\,\beta + b\alpha)\frac{\partial^2 z}{\partial y \partial x} + a\alpha \frac{\partial^2 z}{\partial y^2} = 0.$

Example 10. *Form the partial differential equation by eliminating f from* $z = xy + f(x^2 + y^2 + z^2)$

Let $z = xy + f(x^2 + y^2 + z^2)$.

We will keep f on RHS and all other terms in the LHS.

i.e., $$z - xy = f(x^2 + y^2 + z^2) \quad ...(1)$$

Differentiate w.r.t x and y.

$$p - y = f'(x^2 + y^2 + z^2) \times (2x + 2zp) \quad ...(2)$$

$$q - x = f'(x^2 + y^2 + z^2) \times (2y + 2zq) \quad ...(3)$$

Divide (2) and (3)

$$\frac{p - y}{q - x} = \frac{x + pz}{y + qz}$$

$$pqz + py - y^2 - qyz = qx + pqz - x^2 - pxz$$

$$(y + xz)\, p - (yz + x)\, q = y^2 - x^2$$

Example 11. *Form the partial differential equation by eliminating f from*

$$f(x^2 + y^2 + z^2, x + y + z) = 0$$

Rewriting the given equation as

$$x^2 + y^2 + z^2 = f(x + y + z) \quad ...(1)$$

differentiate w.r.t. x and y partially.

$$2x + 2zp = f'(x + y + z) \times (1 + p) \quad ...(2)$$

$$2y + 2zq = f'(x + y + z) \times (1 + q) \quad ...(3)$$

Divide (2) and (3)

$$\frac{x + pz}{y + qz} = \frac{1 + p}{1 + q}$$

$$x + qx + pz + pqz = y + py + qz + pqz$$

$$(z - y)\, p + (x - z)\, q = y - x.$$

Note. In all problems, where $f(u, v) = 0$ is given, write as $u = \phi(v)$ or $v = \psi(u)$ and proceed so that the elimination of arbitrary function is very easy.

Example 12. *Form the partial differential equation by eliminating f and ϕ from*

$$z = x f\left(\frac{y}{x}\right) + y\,\phi(x) \qquad \text{(MS. 1987)}$$

Since there are two arbitrary functions f and ϕ, we will get a second order partial differential equation.

Differentiate w.r.t. x and y.

$$p = x f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\,\phi'(x)$$

$$p = -\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\,\phi'(x) \qquad \text{...(1)}$$

$$q = x f'\left(\frac{y}{x}\right) \times \frac{1}{x} + \phi(x)$$

$$q = f'\left(\frac{y}{x}\right) + \phi(x) \qquad \text{...(2)}$$

Differentiate once again w.r.t. x and y.

$$r = \frac{\partial^2 z}{\partial x^2} = \cdots\cdots$$

Differentiate (2) w.r.t. x

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = f''\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) + \phi'(x) \qquad \text{...(3)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right) \times \frac{1}{x} \qquad \text{...(4)}$$

$(1) \times x + (2) \times y$ gives

$$px + qy = -y f'\left(\frac{y}{x}\right) + x f\left(\frac{y}{x}\right) + xy\,\phi'(x) + y f'\left(\frac{y}{x}\right) + y\,\varphi(x)$$

$$= xy\,\phi'(x) + x f\left(\frac{y}{x}\right) + y\,\phi(x)$$

$$px + qy = xy\,\phi'(x) + z \qquad \text{...(5)}$$

use (4) in (3)

$$s = -\frac{y}{x} \times t + \phi'(x)$$

$$\frac{xs + yt}{x} = \varphi'(x). \text{ Use this } \varphi'(x) \text{ in } (5)$$

$$px + qy = z + xy\left[\frac{xs + yt}{x}\right]$$

$$= z + xys + y^2t$$

$\therefore\ z = px + qy - xys - y^2t$ is the required equation.

Exercises 2(a)

1. Form the partial differential equations by eliminating the arbitrary constants a and b from the following equations.

   (i) $z = ax^3 + by^3$.

   (ii) $z = ax + by + ab$.

   (iii) $z = a(x + y) + b$.

   (iv) $(x - a)^2 + (y - b)^2 + z^2 = 1$.

   (v) $ax^2 + by^2 + z^2 = 1$.

   (vi) $2z = (ax + y)^2 + b$.

   (vii) $z = (x + a)(y + b)$.

2. Form the partial differential equation by eliminating the arbitrary constants.

   (i) $z = ax^n + by^n$ (*Anna Ap. 2005*)

   (ii) $(x - a)^2 + (y - b)^2 + z^2 = a^2 + b^2$

   (iii) $z = a(x + \log y) - \frac{x^2}{2} - b$

   (iv) $z = a \log\left[\frac{b(y - 1)}{1 - x}\right]$

   (v) $z = \frac{1}{2}[\sqrt{x + a} + \sqrt{y - a}] + b$.

   (vi) $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$, α known constant

   (vii) $z = ax + by + \frac{a}{b} - b$

3. Obtain partial differential equations by eliminating the arbitrary functions:

   (i) $z = F(x^2 + y^2)$. **(M.U. 64 B.E.)**

   (ii) $z = f(x^2 + y^2 + z^2)$. **(M.U. 64 B.E.)**

   (iii) $xyz = \phi(x + y + z)$. **(M.U. 64 B.E.)**

   (iv) $z = y f(x) + xg(y)$. **(S.V.U. 66 B.E.)**

   (v) $z = f(2x + y) + g(3x - y)$. **(M.U. 63 B.E.)**

   (vi) $z = xy + f(x^2 + y^2)$. **(M.U. 65 B.E.)**

(vii) $z = f\left(\frac{y}{x}\right)$ (M.U. 64 B.E.)

(viii) $z = f\left(\frac{xy}{z}\right)$ (M.U. 65 B.E.)

(ix) $z = x + y + f(xy)$. (M.U. 65 B.E.)

(x) $z = xf(ax + by) + g(ax + by)$. (M.U. 64 B.E.)

(xi) $z = f(x + iy) + (x + iy)\, g(x - iy)$. (M.U. 66 B.E.)

(xii) $xy + yz + zx = f\left(\frac{z}{x + y}\right)$ (M.U. 65 B.E.)

(xiii) $z = e^y f(x + y)$.

(xiv) $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$.

(xv) $f(x^2 + y^2 + z^2, x + y + z) = 0$.

(xvi) $z = x f\left(\frac{y}{x}\right) + y\,\phi(x)$.

(xvii) $z = f_1(x + y) + x f_2(x + y) + f_3(x - y) + x f_4(x - y)$. (S.V.U. 67 B.E.)

(xviii) $z = y^2 + 2 f\left(\frac{1}{x} + \log y\right)$. (M.U. 72 B.E.)

(xix) $f(x + y + z, xy + z^2) = 0$ (Madurai 79 B.E.)

(xx) $z = f(2r + 3y) + \phi(y + 2x)$.

4. Find the partial differential equation of all spheres whose centres lie on the z-axis.

5. Find the partial differential equation of all spheres of radius k units having their centres on the xy-plane.

6. Show that the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 2u/x \text{ is satisfied by } u = \frac{1}{x}\,\phi(y - x) + \phi'(y - x),$$

where ϕ is an arbitrary function.

7. If $z = f(x + iy) + F(x - iy)$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \text{ where } f, F \text{ are arbitrary.}$$

8. If $u = f(x^2 + y) + F(x^2 - y)$, show that

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\cdot\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial y^2} = 0.$$ (M.U. 64 B.E.)

9. Find the differential equation of all planes which are at constant distance k from the origin.

2·5. Solution of Partial differential equations.

A solution or integral of a partial differential equation is a relation between the independent and the dependent variables which satisfies the given partial differential equation. There are two distinct types of solution for partial differential equations, one type of solution containing arbitrary constants and the other type of solution containing an arbitrary function. Both these types of solutions may be given as solutions of the same partial differential equation.

Consider the equations

$$z = ax + by \quad \ldots(1)$$

and

$$z = x f\left(\frac{x}{y}\right) \quad \ldots(2)$$

where a and b are arbitrary constants and f an arbitrary function. By eliminating the arbitrary constants a and b from (1) and the arbitrary function from (2), we obtain the same partial differential equation $px + qy = z$. Therefore, equations (1) and (2) are both solutions of $px + qy = z$.

A solution which contains as many arbitrary constants as there are independent variables is called a *complete integral.*

A solution got by giving particular values to the arbitrary constants in a complete integral is called a *particular integral.*

2·6. To find the singular integral

Suppose that $f(x, y, z, p, q) = 0$ ...(1)

is the partial differential equation whose complete integral is

$$\phi(x, y, z, a, b) = 0 \quad \ldots(2)$$

where a and b are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain,

$$\frac{\partial \phi}{\partial a} = 0 \quad \ldots(3)$$

and

$$\frac{\partial \phi}{\partial b} = 0 \quad \ldots(4)$$

The eliminant of a and b from the equations (2), (3) and (4), when it exists, is called the singular integral of (1). Geometrically, the singular integral represents the envelope of the surfaces represented by the complete integral (2). The singular integral is not contained in the complete integral whereas the particular integral is obtained from the complete integral.

2·7. To find the general integral

In the complete integral (2), assume that one of the constants is a function of the other. That is, $b = f(a)$

Then (2) becomes,

$$\varphi\,[x, y, z, a, f(a)\,] = 0 \quad \ldots(5)$$

Differentiating (2), partially w.r.t. a,

$$\frac{\partial \varphi}{\partial a} + \frac{\partial \varphi}{\partial b} \cdot f'(a) = 0 \qquad \ldots(6)$$

The eliminant of a between the two equations (5) and (6), when it exists, is called the *general integral* of (1).

2.8. Solution of partial differential equations by direct integration

A partial differential equation can be solved by successive integration in all cases where the dependent variable occurs only in the partial derivatives. We shall study a few examples.

Example 13. *Solve* $\dfrac{\partial^2 z}{\partial x\, \partial y} = \sin x$

Integrating w.r.t. x,

$$\frac{\partial z}{\partial y} = -\cos x + f(y), \text{ where } f \text{ is arbitrary.}$$

Again integration w.r.t. y,

we get, $$z = -y \cos x + F(y) + \varphi(x).$$

Example 14. *Solve* $\dfrac{\partial^2 z}{\partial x^2} = xy.$

Integrating w.r.t. x,

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} y + f(y). \text{ Integrating again w.r.t. } x$$

$$z = \frac{x^3}{6} y + x f(y) + \varphi(y).$$

Example 15. *Solve* $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, *given that* $u = 0$ *when* $t = 0$ *and* $\dfrac{\partial u}{\partial t} = 0$ *when* $x = 0$. *Show also that as* $t \to \infty$, $u \to \sin x$. **(MS. 1989 Nov.)**

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$$

Integrating w.r.t. x,

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t).$$

When $x = 0$, $\dfrac{\partial u}{\partial t} = 0.$

$$\therefore \qquad 0 = f(t).$$

Hence $\dfrac{\partial u}{\partial t} = e^{-t} \sin x.$

Integrating this equation w.r.t. t,

$$u(x, t) = -e^{-t} \sin x + \phi(x)$$

When $t = 0$, $u = 0$.

$$\therefore \quad 0 = -\sin x + \phi(x)$$

Hence $\varphi(x) = \sin x$

$$\therefore \quad u(x, t) = \sin x\,(1 - e^{-t}).$$

Example 16. *Solve* $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$, *given that when* $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ *and* $\dfrac{\partial z}{\partial y} = 0$. **(MS. 1991 N.)**

If z were a function of x alone, solving $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$, we get,

$$z = Ae^{ax} + Be^{-ax}.$$

Since z is a function of x and y, A and B will be functions of y alone.

Hence $z = f(y)e^{ax} + \varphi(y)e^{-ax}$ ...(1)

where $f(y)$ and $\phi(y)$ are functions of y alone.

$$\therefore \quad \frac{\partial z}{\partial x} = a f(y)\, e^{ax} - a\varphi(y)e^{-ax}$$

By hyp., when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$.

$$\therefore \quad a \sin y = a\,[f(y) - \varphi(y)]$$

i.e., $\quad f(y) - \varphi(y) = \sin y$ ...(2)

From (1), $\dfrac{\partial z}{\partial y} = e^{ax} f'(y) + e^{-ax} \varphi'(y)$.

By hyp., when $x = 0$, $\dfrac{\partial z}{\partial y} = 0$.

$\therefore \quad 0 = f'(y) + \varphi'(y)$. Intergrating $f(y) + \phi(y) = k$ (constant) ...(3)

From (2) and (3)

$$f(y) = \tfrac{1}{2}(\sin y + k), \text{ and}$$

$$\varphi(y) = \tfrac{1}{2}(k - \sin y)$$

$$\therefore \quad z = \tfrac{1}{2}(\sin y + k)\, e^{ax} + \tfrac{1}{2}(k - \sin y)e^{-ax}$$

$$z = \sin y \sinh ax + k \cosh ax, \text{ where } k \text{ is any constant.}$$

Example 17. *By changing the independent variables by the relations*

$$\therefore \qquad \frac{\partial}{\partial x} \equiv \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)$$

Hence $$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)^2 y$$

i.e., $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial r^2} + \frac{\partial^2 y}{\partial s^2} + 2\frac{\partial^2 y}{\partial r \partial s} \qquad \ldots(2)$$

a^2

$$\therefore \quad \frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = 0,$$

$-4a^4 \dfrac{\partial^2 y}{\partial r \partial s} = 0$

gives $-4a^2 \dfrac{\partial^2 y}{\partial r \partial s} = 0.$

Hence $\dfrac{\partial^2 y}{\partial r \partial s} = 0.$

Integrating w.r.t. r,

$$\frac{\partial y}{\partial s} = \varphi(s).$$

Integrating this equation w.r.t. s,

$$y = \int \phi(s)\, ds$$

$$= F(s) + f(r)$$

$$\therefore \quad y = F(x - at) + f(x + at),$$

where F and f are arbitrary functions.

Example 18. *Solve* $\dfrac{\partial z}{\partial x} = 6x + 3y$; $\dfrac{\partial z}{\partial y} = 3x - 4y.$

From $\dfrac{\partial z}{\partial x} = 6x + 3y,$

we get $z = 3x^2 + 3xy + \phi(y)$

$$\therefore \quad \frac{\partial z}{\partial y} = 3x + \varphi'(y)$$

Hence $3x + \varphi'(y) = 3x - 4y$, using the hyp.

i.e., $\varphi'(y) = -4y$, giving

$\varphi(y) = -2y^2 + k$

$\therefore \quad z = 3x^2 + 3xy - 2y^2 + k$, where k is *a* constant.

$r = x + at$, $s = x - at$, show that the equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ gets transformed into the equation $\frac{\partial^2 y}{\partial r \partial s} = 0$. Hence find a general solution of the partial differential equation.

Here y is a function of x and t.

Using $\quad r = x + at,$

and $\quad s = x - at,$

y can be expressed as a function of r and s.

Hence y is a function of r and s, where r and s are functions of t and x.

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial y}{\partial s} \cdot \frac{\partial s}{\partial t}$$

$$= \frac{\partial y}{\partial r} \cdot a + \frac{\partial y}{\partial s}(-a).$$

$$= a\left(\frac{\partial y}{\partial r} - \frac{\partial y}{\partial s}\right), \text{ since } \frac{\partial r}{\partial t} = a \text{ and } \frac{\partial s}{\partial t} = -a.$$

$$\frac{\partial}{\partial t}\frac{\partial y}{\partial t} = a\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) y.$$

Thus $\quad \frac{\partial}{\partial t} \equiv a\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t} a\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial}\right) y$$

$$= a\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) \cdot a\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) y$$

$$= a^2\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)^2 y$$

$$\frac{\partial^2 y}{\partial t^2} = a^2\left[\frac{\partial^2 y}{\partial r^2} - \frac{2\partial^2 y}{\partial r \partial s} + \frac{\partial^2 y}{\partial s^2}\right] \qquad \text{...(1)}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial y}{\partial s} \cdot \frac{\partial s}{\partial x}$$

Now $\quad \frac{\partial r}{\partial x} = 1$, and $\frac{\partial s}{\partial x} = 1$

Hence $\quad \frac{\partial y}{\partial x} = \frac{\partial y}{\partial r}(1) + \frac{\partial y}{\partial s}(1)$

$$= \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) y.$$

Exercises 2(b)

Solve the following equations (1–12):

1. $\frac{\partial z}{\partial x} = 0.$ 2. $\frac{\partial^2 z}{\partial y^2} = 0$ 3. $\frac{\partial^2 z}{\partial x^2} = \cos x.$

4. $\frac{\partial^2 z}{\partial x \partial y} + \frac{x}{y} = 6.$ 5. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(ax + by).$

6. $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ 7. $x\frac{\partial z}{\partial x} = 2x + y + 3z.$

8. $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, for which $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$, when y is an odd multiple of $\pi/2$.

9. $\frac{\partial^2 z}{\partial y^2} - 5\frac{\partial z}{\partial y} + 6z = 12y.$ 10. $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2.$

11. $\frac{\partial^2 z}{\partial y^2} = z$ given that $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$ when $y = 0$.

12. $\frac{\partial z}{\partial x} = 3x - y$ and $\frac{\partial z}{\partial y} = -x + \cos y.$

13. By interchanging the independent variables by the relations $z = x + iy$, $\bar{z} = x - iy$, show that the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms into $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$. Hence obtain a general solution of the equation. **(M.U. 67 B.E.)**

14. With the help of the substitution $u = x + \alpha y$, $v = x + \beta y$ where α, β are suitable constants, transform the equation

$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$ to the form $\frac{\partial^2 z}{\partial u \partial v} = 0$ and hence obtain its general solution.

15. Solve $\frac{\partial^2 z}{\partial x^2} = 0, \frac{\partial^2 z}{\partial y^2} = 0.$ **(O.U. 66 B.E.)**

2.9. Methods to solve the first order partial differential equations

The partial differential equation of the first order can be written as $F(x, y, z, p, q) = 0$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. We shall see some standard forms of such equations and solve them by special methods.

2.10. Type I. F (p, q) = 0. *i.e., the equations contain p and q only.* Suppose that $z = ax + by + c$ is a solution of the equation $F(p, q) = 0$.

Then $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$.

Substituting these in the given equation, we get $F(a, b) = 0$.

Hence the complete solution of the given equation is

$$z = ax + by + c, \text{ where } F(a, b) = 0.$$

Solving for b from $F(a, b) = 0$ we get $b = \varphi(a)$, say

Then $z = ax + \varphi(a)\, y + c$ ...(1),

is the complete integral of the given equation since it contains two arbitrary constants.

Singular integral is got by eliminating a and c from

$$z = ax + \varphi(a)\, y + c,$$

$$0 = x + \varphi'(a)\, y,$$

and $0 = 1$.

The last equation being absurd, there is no singular integral for the given partial differential equation.

To find the general integral, put $c = f(a)$, f being arbitrary.

Then $z = ax + y\varphi(a) + f(a)$ ...(2)

Differentiating partially w.r.t. a,

we get $0 = x + y\varphi'(a) + f'(a)$ ...(3)

Eliminating a between (2) and (3), we get the general solution.

Example 19. *Solve* $\sqrt{p} + \sqrt{q} = 1$.

This is of the form $F(p, q) = 0$.

Hence the complete integral is $z = ax + by + c$, where $\sqrt{a} + \sqrt{b} = 1$

i.e., $b = (1 - \sqrt{a})^2$

$\therefore$ the complete solution is

z = ax + by + c

$$z = ax + (1 - \sqrt{a})^2\, y + c \qquad \text{...(1)}$$

Differentiating partially w.r.t. c, we find that there is no singular solution.

Taking $c = f(a)$ where f is arbitrary,

$$z = ax + (1 - \sqrt{a})^2\, y + f(a) \qquad \text{...(2)}$$

Differentiating (2) partially w.r.t. a, we get

$$x + 2y(1 - \sqrt{a})\left(-\frac{1}{2\sqrt{a}}\right) + f'(a) = 0 \qquad \text{...(3)}$$

Eliminating a between (2) and (3), we get the general integral.

Example 20. *Solve* $p^2 + q^2 = npq$

The solution of this equation is $z = ax + by + c$

Subject to $a^2 + b^2 = abn$

Solving for b, we get $b = \dfrac{na \pm \sqrt{a^2n^2 - 4a^2}}{2}$

i.e. $$b = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4} \right]$$

Hence the complete solution is

$$z = ax + \frac{a}{2}[\, n \pm \sqrt{n^2 - 4}\,]\, y + c \quad \text{...(1)}$$

Differentiating partially w.r.t. c, we get $0 = 1$ which is absurd.

Hence, there is no singular integral.

To find general solution, put $c = \phi\,(a)$

$$Z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4} \right] y + \phi\,(a) \quad \text{...(2)}$$

Differentiate partially w.r.t. 'a'.

$$0 = x + \frac{1}{2}\left[n \pm \sqrt{n^2 - 4} \right] y + \phi'\,(a) \quad \text{...(3)}$$

Eliminating 'a' between (2) and (3), we get the general solution of the given equation.

2·11. Type II. Clairaut's form . z = px + qy + f (p, q).

Suppose that the given equation is of the form

$$z = px + qy + f\,(p, q) \quad \text{...(1)}$$

We can easily prove that

$$z = ax + by + f\,(a, b) \quad \text{...(2)}$$

is the complete solution of (1), where a, b are arbitrary constants. Differentiating (2) partially w.r.t a and b, we get

$$x + \frac{\partial f}{\partial a} = 0 \quad \text{...(3)}$$

and $$y + \frac{\partial f}{\partial b} = 0 \quad \text{...(4)}$$

By eliminating a and b from (2), (3), and (4), we get the singular integral of (1).

Taking $b = \varphi\,(a)$, (2) becomes,

$$z = ax + \varphi\,(a)\, y + f\,[a, \varphi\,(a)] \quad \text{...(5)}$$

Differentiating partially w.r.t. a, we get

$$0 = x + y\varphi'(a) + f'(a) \qquad ...(6)$$

Eliminating a between (5) and (6), we get the general integral of (1).

Example 21. *Solve* $z = px + qy + \sqrt{1 + p^2 + q^2}$. **(M.S. 1991 Nov.)**

This is of the form $z = px + qy + f(p, q)$.

Hence the complete integral is $z = ax + by + \sqrt{1 + a^2 + b^2}$, where a and b are arbitrary constants.

Singular solution is found as follows.

Differentiate $z = ax + by + \sqrt{1 + a^2 + b^2}$ wrt a and b

We get

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}}$$

Try

$$0 = y + \frac{b}{\sqrt{1 + b^2 + a^2}}$$

$$x = -\frac{a}{\sqrt{1 + a^2 + b^2}} \qquad ...(1)$$

$$y = -\frac{b}{\sqrt{1 + a^2 + b^2}} \qquad ...(2)$$

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$$1 - x^2 - y^2 = 1 - \frac{a^2 + b^2}{1 + a^2 + b^2} = \frac{1}{1 + a^2 + b^2}$$

$$\therefore \sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$\therefore$ (1) and (2) become,

$$x = -a\sqrt{1 - x^2 - y^2}$$

$$y = -b\sqrt{1 - x^2 - y^2}$$

$$\therefore \quad a = \frac{-x}{\sqrt{1 - x^2 - y^2}}$$

$$b = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

Substituting in the given equation,

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \sqrt{1-x^2-y^2}$$

$z^2 = 1 - x^2 - y^2$ *i.e.*, $x^2 + y^2 + z^2 = 1$ is the singular integral.

Example 22. *Solve* $z = px + qy + p^2q^2$

This is Clairaut's form

The complete solution is $z = ax + by + a^2 b^2$ ...(1)

Differentiate w.r.t. a and b,

$$0 = x + 2ab^2$$

and

$$0 = y + 2ba^2$$

$\therefore$

$$x = -2ab^2 \quad \text{...(2)}$$

$$y = -2ba^2 \quad \text{...(3)}$$

$$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k} \text{ (say)}$$

$$a = ky \text{ and } b = kx$$

Put in (2),

$$x = -2k^3 yx^2$$

$$k^3 = -\frac{1}{2xy}$$

Put a and b in (1),

$$z = kxy + kxy + k^4 x^2 y^2$$

$$= 2kxy + k x^2 y^2 \left(-\frac{1}{2xy}\right)$$

$$= 2k\, xy - \frac{k}{2} xy$$

$$= \frac{3}{2} k\, xy$$

$$z^3 = \frac{27}{8} k^3 x^3 y^3$$

$\therefore$

$$z^3 = \frac{27}{8} x^3 y^3 \left(-\frac{1}{2xy}\right)$$

$$z^3 = -\frac{27}{16} x^2 y^2$$

$16z^3 + 27x^2y^2 = 0$ is the singular solution

Put $b = \phi(a)$ in (1).

$$z = ax + \phi(a)\,y + a^2[\phi(a)]^2 \quad ...(4)$$

Differentiate (4) w.r.t. a and eliminate a to get the general solution.

Example 23. *Solve:* $z = px + qy + p^2 - q^2$

This is Clairaut's form. Hence complete solution is

$$z = ax + by + a^2 - b^2 \quad ...(1)$$

To get singular solution:

Differentiate (1) w.r.t. a and b.

$$0 = x + 2a$$

$$0 = y - 2b \quad ...(2)$$

$$\therefore \quad a = -\frac{x}{2} \text{ and } b = \frac{y}{2} \quad ...(3)$$

Putting in (1),

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$$

$4z = y^2 - x^2$ is the singular solution.

Put $b = \phi(a)$ in (1).

$$z = ax + \varphi(a)\,y + a^2 - [\varphi(a)]^2 \quad ...(4)$$

Differentiate (4) w.r.t. a.

$$0 = x + y\cdot\varphi'(a) + 2a - 2\varphi(a).\,\varphi'(a) \quad ...(5)$$

Eliminating 'a' between (4) and (5) we get the general solution.

2.12. Type III. (a) **$F(z, p, q) = 0$.** *i.e., equations not containing x and y explicitly*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

Now $\quad z = f(u) = f(x + ay).$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}\cdot 1 = \frac{dz}{du}. \qquad \frac{dz}{dx} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = \frac{dz}{du}\cdot a = a\frac{dz}{du}.$$

Substituting these values of p and q in $F(z, p, q) = 0$, we get $F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$, which is an ordinary differential equation of the first order. Solving for $\frac{dz}{du}$, we obtain $\frac{dz}{du} = \varphi(z, a)$ (say) *i.e.,* $$\frac{dz}{\varphi(z, a)} = du$$

Integrating, $$\int \frac{dz}{\varphi(z, a)} = u + c$$

i.e., $$f(z, a) = u + c$$

i.e., $$f(z, a) = x + ay + c$$ ...(

This is the complete integral.

Singular and general integrals are found out as usual.

(*b*) Suppose that the given equation is of the form

$$\boldsymbol{F(x, p, q) = 0}$$ ...(

Since z is a function of x and y

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy}$$

Assume that $q = a$.

Then the equation becomes $F(x, p, a) = 0$.

Solving for p, we obtain $p = \phi(x, a)$.

$\therefore$ $$dz = \varphi(x, a)\,dx + a\,dy$$

Hence $$z = \int \varphi(x, a)\,dx + ay + c$$

$$z = f(x, a) + ay + c$$ ...(

(2) is the complete integral of (1) since it contains two arbitrary consta
a and c.

(*c*) If the given equation is of the form

$\boldsymbol{F(y, p, q) = 0}$, assume $p = a$ and proceed as before.

The complete integral will be of the form

$$z = ax + \int f(y, a)\,dy + c.$$

Example 24. *Solve* $p(1 + q) = qz.$ **(M.S. 1987 A**

Assume $$u = x + ay$$

Then $$p = \frac{dz}{du} \text{ and } q = \frac{dz}{du} \cdot a$$

Substituting these values in the given equation, we get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du}$$

$\therefore$ $$a\frac{dz}{du} = az - 1$$

$dz = p\,dx + q\,dy$

$$\therefore \quad \frac{a\,dz}{az-1} = du$$

$dz = p\,dx + q\,dy$

Integrating, $$a\int \frac{dz}{az-1} = u + c$$

$$\therefore \quad \log(az-1) = u + c$$

i.e., $\log(az-1) = x + ay + c$, which is the required complete integral.

Singular and general integrals are found out as usual.

Example 25. *Solve* $p = 2qx$.

$p = 2ax$

$\dfrac{\partial z}{\partial} = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$

Let $\quad q = a.$

Then $\quad p = 2ax.$

i) $q = a$

But $\quad dz = p\,dx + q\,dy$

ii) $dz = p\,dx + q\,dy$

$\quad = 2ax\,dx + a\,dy$

iii) $dz = 2ax\,dx + a\,dy$

$$\therefore \quad z = ax^2 + ay + c \qquad \ldots(1)$$

(1) Is the complete integral of the given equation. Differentiating partially w.r.t. c, we get $1 = 0$. Hence there is no singular integral. General integral can be found out in the usual way.

Example 26. *Solve:* $q = px + p^2$ **(MS. ’86 Ap.)**

$p = \dfrac{dz}{dx}$

This is of the form $\phi(x, p, q) = 0$.

$f(p, q, x)$

$\therefore$ Assume $q = a =$ constant

Then $p^2 + px - a = 0.$

$$\therefore \quad p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$$

Since, $\quad dz = p\,dx + q\,dy$

$$= \left(\frac{-x \pm \sqrt{x^2+4a}}{2}\right)dx + a\,dy$$

$y = q$

Integrating,

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\int \sqrt{x^2+4a}\,dx + ay + b$$

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left\{2a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) + \frac{x}{2}\sqrt{x^2+4a}\right\} + ay + b$$

is the complete solution.

Singular integral does not exist; find the general solution as usual.

Example 27. *Solve* $pq = y$.

This is of the form $f(p, q, y) = 0$.

Assume $p = a$ = constent.

Then $aq = y \therefore q = \frac{y}{a}$.

$$dz = pdx + qdy$$

$$= adx + \frac{y}{a} dy$$

Integrating, $z = ax + \frac{y^2}{2a} + b$ is the complete solution.

There is no singular integral since $\frac{\partial \varphi}{\partial b} = 0$ gives $1 = 0$ which is absurd

Put $b = \varphi(a)$

$$z = ax + \frac{y^2}{2a} + \varphi(a) \quad ...(3)$$

Differentiate (3) w.r.t. a

$$0 = x - \frac{y^2}{2a^2} + \varphi'(a) \quad ...(4)$$

Eliminate a between (3) and (4) to get general solution.

Example 28. *Solve* $9(p^2 z + q^2) = 4$.

This is of the form $f(p, q, z) = 0$

Assume $z = f(x + ay)$ and $u = x + ay$.

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}.$$

$$q = \frac{\partial z}{\partial y} = a\frac{dz}{du}$$

Substituting in the given equation,

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$\left(\frac{dz}{du}\right)^2 = \frac{4}{9(z + a^2)}$$

$$\frac{dz}{du} = \frac{2}{3}\frac{1}{\sqrt{z + a^2}}$$

$$3\sqrt{z + a^2}\, dz = 2du$$

Integrating, $3\frac{(z + a^2)^{3/2}}{3/2} = 2u + 2b$

$(z+a^2)^{3/2} = (x+ay)+b$

$(z+a^2)^3 = (x+ay+b)^2$. This is the complete solution.

Singular solution and general solution can be found in the usual manner (exercise to the reader).

Example 29. *Solve:* $z = p^2 + q^2$ **(MS. 1986 A)**

Let $z = f(x+ay)$ where $u = x+ay$.

$\therefore p = \frac{dz}{du}, q = a\frac{dz}{du}$. Substituting in the given equation,

$$z = (1+a^2)\left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$\frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}du$$

Integrating, $2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(x+ay)+b$ is the complete solution.

Example 30. *Solve:* $Ap + Bq + cz = 0$ **(MS. 1986 Nov.)**

Let $z = f(x+ay)$ where $u = x+ay$

Hence $p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$.

The equation becomes, $A\frac{dz}{du} + Ba\frac{dz}{du} + cz = 0$

$$\therefore \quad \frac{dz}{du} = \frac{-cz}{a+Ba}$$

$$\frac{dz}{z} = -\frac{c}{A+Ba}du$$

Integrating, $\log z = -\frac{c}{A+Ba}(x+ay)+b$ is the complete solution

2·12. Type IV. Separable equations

We say that a first order partial differential equation is *separable* if it can be written as $f(x, p) = \phi(y, q)$.

We first put each of these equal expressions equal to an arbitrary constant a, say.

Hence $f(x, p) = \phi(y, q) = a$.

Solving for p and q, we get $p = f_1(x, a)$ and $q = \varphi_1(y, a)$

But $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$.

Hence $dz = p\,dx + q\,dy = f_1(x, a)\,dx + \varphi_1(y, a)\,dy$.

$\therefore \quad z = \int f_1(x, a)\,dx + \int \varphi_1(y, a)\,dy + b$

Now (3) contains two arbitrary constants and hence it is the complete integral. The singular and general integrals are found out as usual.

Example 31. *Solve* $p^2 y(1 + x^2) = qx^2$.

The equation is separable.

$\therefore \quad p^2 \frac{(1 + x^2)}{x^2} = \frac{q}{y} = a$, where a is an arbitrary constant.

Thus $\quad p^2 \frac{1 + x^2}{x^2} = a$

Hence $$p = \frac{x\sqrt{a}}{\sqrt{1 + x^2}};$$

Again $$q = ay.$$

But $$dz = pdx + qdy$$

$$= \frac{x\sqrt{a}}{\sqrt{1 + x^2}} dx + aydy$$

$\therefore$ $$z = \sqrt{a} \int \frac{x}{\sqrt{1 + x^2}} dx + a \int ydy$$

$$z = \sqrt{a(1 + x^2)} + \tfrac{1}{2} ay^2 + b$$

This is the complete integral where a and b are arbitrary constants.

Differentiating partially w.r.t. b, we find that there is no singular integral.

Example 32. *Solve* $p^2 + q^2 = x + y$. ...(1)

The equation is separable.

$\therefore \quad p^2 - x = y - q^2 = a$, say.

Thus $\quad p^2 = x + a$, and hence $p = \sqrt{(x + a)}$,

Again $y - q^2 = a$, and hence $q = \sqrt{(y - a)}$

But $\quad dz = pdx + qdy$

$$= \sqrt{(x + a)}\,dx + \sqrt{(y - a)}\,dy$$

$$\therefore \quad z = \tfrac{2}{3}(x + a)^{\frac{3}{2}} + \tfrac{2}{3}(y - a)^{\frac{3}{2}} + b \quad ...(2)$$

This is the complete integral of the given equation. Differentiating (2) partially w.r.t. b, we find that there is no singular integral since we get $1 = 0$ which is absurd.

Example 33. *Solve* $p^2 + q^2 = x^2 + y^2$.

This is a separable equation.

$$p^2 - x^2 = y^2 - q^2 = a^2 \text{ (say)}$$

$$\therefore \quad p^2 - x^2 = a^2 \text{ and } y^2 - q^2 = a^2$$

Hence $\quad p = \sqrt{x^2 + a^2}$ and $q = \sqrt{y^2 - a^2}$

$$dz = pdx + qdy$$

$$= \sqrt{x^2 + a^2}\, dx + \sqrt{y^2 - a^2}\, dy$$

Integrating, $\int dz = \int \sqrt{x^2 + a^2}\, dx + \int \sqrt{(y^2 - a^2)}\, dy$

$$z = \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 + x^2} + \frac{y}{2} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \frac{y}{a} + b$$

is the complete solution.

There is no singular integral.

Example 34. *Solve:* $p - x^2 = q + y^2$

Let $\quad p - x^2 = q + y^2 = a =$ constants

$$p = a + x^2 \text{ and } q = a - y^2$$

$$dz = p\, dx + q\, dy$$

$$= (a + x^2)\, dx + (a - y^2)\, dy$$

Integrating, $z = ax + \dfrac{x^3}{3} + ay - \dfrac{y^3}{3} + b$ is the complete integral.

There is no singular integral.

put $\quad b = \varphi(a)$

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + \phi(a) \qquad \ldots(1)$$

Differentiate w.r.t. a.

$$0 = x + y + \varphi'(a) \qquad \ldots(2)$$

Eliminate a between (1) and (2) to get the general solution.

2.14. Equations reducible to standard forms

Many non-linear partial differential equations of the first order do not fall under any of the four standard types discussed so far. However, in some cases, it is possible to transform the given partial differential equation into one of the standard types by change of variables. We will see below a few types of equation, reducible in each case to one of the standard types.

Case 1. An equation of the form $F(x^m p, y^n q) = 0$, where m and n are constants can always be transformed into an equation of the *First Type*.

By putting $x^{1-m} = X$ and $y^{1-n} = Y$ where $m \neq 1$ and $n \neq 1$, we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{dX}{dx} = (1-m)\, x^{-m} \frac{\partial z}{\partial X} = (1-m)\, x^{-m} P,$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{dY}{dy} = (1-n)\, y^{-n} \frac{\partial z}{\partial Y} = (1-n)\, y^{-n} Q,$$

where $$P = \frac{\partial z}{\partial X} \text{ and } Q = \frac{\partial z}{\partial Y}.$$

Hence the equation reduces to

$F[(1-m)P, (1-n)Q] = 0$, which is of the form $f(P, Q) = 0$.

Case 2. An equation of the form $F(x^m p, y^n q, z) = 0$ can also be transformed to the standard type $f(P, Q, z) = 0$ by the substitutions $x^{1-m} = X$ and $y^{1-n} = Y$ if $m \neq 1$; $n \neq 1$.

Case 3. In the above two cases, if $m = 1$, put $X = \log x$; and if $n = 1$, put $Y = \log y$; whence we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{dX}{dx} = P \cdot \frac{1}{x} \quad \text{i.e., } xp = P,$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{dY}{dy} = Q \cdot \frac{1}{y} \quad \text{i.e., } yq = Q.$$

Case 4. An equation of the form $F(z^k p, z^k q) = 0$, where k is any constant, can be transformed into the *First Type* by proper substitution.

If $k \neq -1$, put $Z = z^{k+1}$.

Then, $$\frac{\partial Z}{\partial x} = (k+1)\, z^k p,$$

and $$\frac{\partial Z}{\partial y} = (k+1)\, z^k q.$$

Hence the given equation reduces to the form

$$f(P, Q) = 0, \text{ where } \frac{\partial Z}{\partial x} \text{ and } Q = \frac{\partial Z}{\partial y}.$$

If $k = -1$, set $Z = \log z$.

$$\therefore \quad \frac{\partial Z}{\partial x} = \frac{1}{z} p$$

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and $$\frac{\partial Z}{\partial y} = \frac{1}{z} q$$

Hence the given equation again reduces to the form $f(P, Q) = 0$.

Case 5. An equation of the form $F(x^m z^k p, y^n z^k q) = 0$ may be transformed into the standard type $f(P, Q) = 0$ by putting $X = x^{1-m}$, $Y = y^{1-n}$ and $Z = z^{k+1}$ if $m \neq 1, n \neq 1$ and $k \neq -1$ or by putting $X = \log x$, $Y = \log y$, $Z = \log z$ if $m = 1$, $n = 1$ and $k = -1$.

Example 35. *Solve:* $x^2p^2 + y^2q^2 = z^2$. *(MS. 1988 April)*

This equation is not in any of the four standard types. But this is reducible to one of the standard types by proper substitution of the variables. Rewriting the equation, we get,

$$\left(\frac{xp}{z}\right)^2 + \left(\frac{yq}{z}\right)^2 = 1.$$

This is of the form explained in case (5), where $m = 1$, $n = 1$ and $k = -1$. Hence put

$X = \log x$, $Y = \log y$ and $Z = \log z$.

$\partial X = \frac{1}{x} dx$

Then $$P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X}$$

$$= \frac{1}{z} \cdot p \cdot x = \frac{px}{z}$$

and $$Q = \frac{\partial Z}{\partial Y} = \frac{qy}{z}$$

$\therefore$ The equation reduces to $P^2 + Q^2 = 1$.

$\therefore$ the complete solution is

$Z = aX + bY + c$, where $a^2 + b^2 = 1$.

i.e., $\log z = a \log x + \sqrt{1 - a^2} \log y + c$.

The other solutions can be got in the usual manner.

Example 36. *Solve:* $2x^4p^2 - yzq - 3z^2 = 0$. *(MS. 1987 Nov.)*

Rewriting this equation, we get

$$2\left(\frac{x^2p}{z}\right)^2 - \frac{yq}{z} - 3 = 0.$$

This is of the form explained in case 5 where $m = 2$, $n = 1$, $k = -1$

Hence put $X = x^{-1}$, $y = \log y$ and $Z = \log z$.

$$P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{1}{z} p\,(-x^2) = -\frac{px^2}{z};$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial y} = \frac{1}{z} \cdot q \cdot y = \frac{qy}{z}$$

$\therefore$ The equation becomes,

$$2P^2 - Q - 3 = 0.$$

This is of the standard type.

$\therefore$ The complete integral is

$$Z = aX + bY + c, \text{ where } 2a^2 - b - 3 = 0$$

i.e., $\log z = \frac{a}{x} + (2a^2 - 3) \log y + c.$

Example 37. *Solve:* $z^2\,(p^2 + q^2) = x^2 + y^2$. **(MS. 1988 April)**

Rewriting this equation, we get,

$$(zp)^2 + (zq)^2 = x^2 + y^2$$

Put $Z = z^2$. Then $P = \frac{\partial Z}{\partial x} = 2zp,\ Q = \frac{\partial Z}{\partial y} = 2zq$

$\therefore$ the equation reduces to

$$P^2 + Q^2 = 4(x^2 + y^2)$$

i.e., $$P^2 - 4x^2 = 4y^2 - Q^2 = 4a \text{ (say)}$$

$\therefore$ $P = \sqrt{(4a + 4x^2)}$, and $Q = \sqrt{(4y^2 - 4a)}$.

$$dZ = 2\sqrt{(a + x^2)}\, dx + 2\sqrt{(y^2 - a)}\, dy$$

$\therefore$

$$Z = 2\left[\frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}} + \frac{y}{2}\sqrt{(y^2 - a)} - \frac{a}{2}\cosh^{-1}\frac{x}{\sqrt{a}}\right] + b$$

i.e., $z^2 = x\sqrt{(x^2 + a)} + a \sinh^{-1}\frac{x}{\sqrt{a}} + y\sqrt{(y^2 - a)} - a\cosh^{-1}\frac{y}{\sqrt{a}} + b.$

Example 38. *Solve:* $p^2x^4 + y^2zq = 2z^2$

This can be written as

$$(px^2)^2 + (qy^2)\,z = 2z^2$$

which is of the form $F\,(x^m p, y^n q, z) = 0$ (Case 2)

where $m = 2, n = 2$

Put $X = x^{1-m} = \frac{1}{x}$; $Y = y^{1-n} = \frac{1}{y}$

$$P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial X} = p \cdot (-x^2) = -px^2$$

$$Q = \frac{\partial Z}{\partial Y} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y} = q(-y^2) = -qy^2$$

Substituting in the given equation,

$$P^2 - Qz = 2z^2$$

This of the form $f(p, q, z) = 0$

$\therefore$ Let $Z = f(X + aY)$ where $u = X + aY$

$$P = \frac{dz}{du}, \quad Q = a\frac{dz}{du}.$$

Equation becomes,

$$\left(\frac{dz}{du}\right)^2 - az\frac{dz}{du} - 2z^2 = 0$$

Solving for $\frac{dz}{du}$, we get

$$\frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2 + 8z^2}}{2}$$

$$\frac{dz}{z} = \frac{a \pm \sqrt{a^2 + 8}}{2}\, du$$

$$\log z = \frac{a \pm \sqrt{a^2 + 8}}{2}\,(X + aY) + b$$

$\therefore \quad \log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)\left(\frac{1}{x} + \frac{a}{y}\right) + b$ is the complete solution.

Example 39. *Solve :* $z^2(p^2x^2 + q^2) = 1$ **(*BR. 1995 April*)**

This can be written as $(pxz)^2 + (qy^0z)^2 = 1$.

This is of the form $F(x^m z^k p, y^n z^k q) = 0$

where $m = 1, n = 0, k = 1$.

$\therefore$ Put $\log x = X$, (refer to Case 5)

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot \frac{1}{x}.$$

$px = P$ where $P = \frac{\partial z}{\partial X}$

Hence the given equation reduces to

$$P^2 + q^2 = \frac{1}{z^2}$$

This is of the form $f(P, q, z) = 0$

$\therefore$ Let $Z = f(X + ay)$ where $u = X + ay$

$P = \frac{dz}{du}$, $q = a\frac{dz}{du}$. Hence, the equation becomes,

$$(a^2+1)\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2}$$

$$\sqrt{a^2+1}\ z\frac{dz}{du} = 1$$

$$\int \sqrt{a^2+1}\ z\,dz = \int du$$

$$\sqrt{a^2+1}\cdot\frac{z^2}{2} = u + b$$

$$\sqrt{a^2+1}\cdot\frac{z^2}{2} = X + ay + b$$

$\sqrt{a^2+1}\ \frac{z^2}{2} = \log x + ay + b$ is the complete solution.

Example 40. *Solve:* $p^2 + x^2y^2q^2 = x^2z^2$.

Dividing by x^2, $(px^{-1})^2 + (qy)^2 = z^2$

This is of the form $F(px^m, qy^n, z) = 0$.

where $m = -1, n = 1$

put $X = x^{1-m}$ and $Y = \log y$ (Case 2)

i.e., $X = x^2$ and $Y = \log y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = P\cdot 2x$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = Q\cdot\frac{1}{y}$$

$$\therefore \quad p = 2x\,P,\quad q = \frac{Q}{y}$$

Hence the given equation reduces to

$$4P^2 + Q^2 = z^2 \qquad ...(1)$$

This is of the form $F(p, q, z) = 0$

$\therefore$ Let $z = f(u)$ where $u = X + aY$

$\therefore P = \frac{dz}{du}$, $Q = a\frac{dz}{du}$.

Hence, (1) becomes, $(a^2+4)\left(\frac{dz}{du}\right)^2 = z^2$

$$\sqrt{a^2+4}\ \frac{dz}{z} = du$$

Integrating, $\sqrt{a^2+4}\ \log z = X + aY + b$

$\sqrt{a^2+4}\ \log z = x^2 + a\log y + b$ is the complete solution.

Example 41. *Solve:* $p^2+q^2 = z^2(x^2+y^2)$

This can be rewritten as $\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2+y^2.$

Hence putting $Z = \log z$ *i.e.*, $z = e^Z$ (case 5)

$$p = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}e^Z = e^Z\cdot\frac{\partial Z}{\partial x}$$

$$= zP \text{ where } P = \frac{\partial Z}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}e^Z = e^Z\frac{\partial Z}{\partial y} = z\ Q$$

where $Q = \frac{\partial Z}{\partial y}$

$\therefore\ \frac{p}{z} = P$ and $\frac{q}{z} = Q$

Hence equation reduces to,

$$P^2+Q^2 = x^2+y^2$$

$$P^2-x^2 = y^2-Q^2 = a^2$$

$\therefore\quad P = \sqrt{x^2+a^2}$ and $Q = \sqrt{y^2-a^2}$

$$dZ = P\,dx + Q\,dy$$

$$= \sqrt{x^2+a^2}\ dx + \sqrt{y^2-a^2}\ dy$$

Integrating,

$$\log z = Z = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2}$$

$$+\frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b$$

is the complete solution

Example 42. *Solve:* $(x+pz)^2+(y+qz)^2 = 1$

Put $\quad z^2 = Z$

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = 2zp \,;\ pz = \frac{P}{2}$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = 2zq \,;\ qz = \frac{Q}{2}$$

Substituting in the given equation

$$\left(x + \frac{P}{2}\right)^2 + \left(y + \frac{Q}{2}\right)^2 = 1$$

This is separable equation.

$$\left(x + \frac{P}{2}\right)^2 = 1 - \left(y + \frac{Q}{2}\right)^2 = a^2$$

$$\therefore\ x + \frac{P}{2} = a \qquad \left(y + \frac{Q}{2}\right)^2 = 1 - a^2$$

$$P = 2\,(a - x) \qquad Q = 2\,[\sqrt{1 - a^2} - y]$$

$$\therefore\ dZ = P\,dx + Q\,dy$$

$$dZ = 2\,(a - x)dx + 2\,[\sqrt{1 - a^2} - y]\,dy$$

Integrating,

$$Z = -\,(a - x)^2 + 2\left[\sqrt{1 - a^2} \cdot y - \frac{y^2}{2}\right] + b$$

$$\therefore\ z^2 = -\,(a - x)^2 + 2\sqrt{1 - a^2}\,y - y^2 + b$$

is the complete solution.

Exercises 2(c)

Solve the following equations (1 to 10):

1. $p + q = pq$.
2. $pq = 1$.
3. $p^2 + q^2 = m^2$.
4. $p = q^2$.
5. $2p + 3q = 1$.
6. $p^2 + q^2 = npq$.
7. $q^2 - 3q + p = 2$.
8. $q^2 - p^2 = 9$.
9. $q + \sin p = 0$.
10. $p^2 - 2pq + 3q = 5$.

Obtain the complete solutions of the following equations:

11. $z = xp + yq + p^2 - q^2$.
12. $z = px + qy + 3pq$.
13. $z = px + qy - 4p^2q^2$.
14. $(p + q)\ (z - px - qy) = 1$.
15. $z = px + qy + \sqrt{(p^2 + q^2)}$.
16. $z^2 = pq$.
17. $z = p^2 + q^2$.
18. $p^2z^2 + q^2 = 1$.
19. $p(1 + q^2) = q(z - a)$.
20. $z^4q^2 - z^2p = 1$.
21. $z^2(p^2 + q^2 + 1) = 1$.
22. $q^2 = (1 - p^2)z^2p^2$.
23. $p(1 + q) = qz$.
24. $p^2 + q^2 = x - y$.
25. $\sqrt{p} + \sqrt{q} = \sqrt{x}$.

26. $pq = xy$.
27. $p^2 + q^2 = x + y$.
28. $p + q = x + y$.
29. $p + q = \sin x + \sin y$.
30. $(1 - x^2)yp^2 + x^2q = 0$.
31. $q(p - \sin x) = \cos y$.
32. $px \tan y = q + 1$.
33. $(p^2 - q^2)\, z = x - y$.
34. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.
35. $(x^2 + y^2)(p^2 + q^2) = 1$.
36. $p^2 + x^2y^2q^2 = x^2z^2$.
37. $p^2 + q^2 = z^2 (x^2 + y^2)$.
38. $q = xp + p^2$.
39. $p^2x + q^2y = z$.
40. $(p^2x^2 + q^2)z^2 = 1$ (Put $X = \log x$)
41. $z(p^2 - q^2) = x - y\left(\text{Put } z = \frac{2}{3} z^{3/2}\right)$
42. $pqz = p^2 (qx + p^2) + q^2 (py + q^2)$
43. $\frac{x^2}{p} + \frac{y^2}{q} = z$ (**Hint:** put $x^3 = X, y^3 = Y$ $\left(\textbf{Ans: } \frac{3z^2}{2} = \frac{x^3}{a} + \frac{y^3}{1-a} + b\right)$
44. $z^4 q^2 - z^2 p = 1$. (**Hint:** put $z^3 = Z$ (**Ans:** $z^3 = ax \pm \sqrt{3a + 9}\, y + b$)
45. $q^2y^2 = z(z - px)$ (put $\log x = X$, $\log y = Y$

$$\left(\textbf{Ans: } \log z = \frac{-1 \pm \sqrt{1 + 4a^2}}{2a^2} (X + aY) + b\right)$$

46. $\frac{p}{x^2} + \frac{q}{y^2} = z\left(\text{put } x^3 = X, y^3 = Y;\ \log z = \frac{1}{3(1 + a)} (x^3 + ay^3) + b\right)$
47. $px^2 + qy^2 = z^2$ $\left(\textbf{Ans: } \frac{(1 + a)}{z} = \frac{1}{x} + \frac{a}{y} + b\right)$
48. $z^2 (p^2 - q^2) = 1$ (**Ans:** $z^2 = ax \pm \sqrt{a^2 - 4}\, y + b$)
49. $z^2 \left(\frac{p^2}{x^2} + \frac{q^2}{y^2}\right) = 1$ (**Ans:** $z^4 (1 + a^2) = (x^2 + ay^2)^2$
50. $z^2 (p^2 + q^2) = x + y$ $\left(\textbf{Ans: } \frac{3z^2}{2} = (4x + a)^{3/2} + (4y - a)^{3/2} + b\right)$
51. $4z^2q^2 = y - x + 2zp$. (put $Z = z^2$;

$$\left(\textbf{Ans: } z^2 = \frac{2}{3} (y + a)^{3/2} + \frac{(x + a)^2}{2} + b\right)$$

52. $p^2z^2 \sin^2 x + q^2z^2 \cos^2 y = 1$.

(**Ans:** $z^2 = 2\sqrt{a} \log \tan x/2 + 2\sqrt{1-a} \log (\sec y + \tan y) + c$)

2·15. Lagrange's linear equation. A linear partial differential equation of the first order known as Lagrange's linear equation is of the form

$$Pp + Qq = R \qquad \text{...(1)},$$

where P, Q and R are functions of x, y, z. We have already seen, under article 2·4, that by eliminating the arbitrary function ϕ from the relation

$$\phi (u, v) = 0 \qquad \text{...(2)},$$

where u, v are functions of x, y, z, we get a partial differential equation of the form

$$Pp + Qq = R, \text{ where}$$

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

and $$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

Hence $\phi(u, v) = 0$ is the general solution of (1), ϕ being any arbi... function.

Now suppose $u = a$ and $v = b$ where a, b are constants.

$$\therefore \quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} \cdot dz = 0,$$

and $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

From these equations, we get

$$\frac{dx}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}}$$

or

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The solutions of equations (3) are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the general solution of (1), where $u = a$ and $v = b$ are the solutions of (3).

Thus to solve the equation $Pp + Qq = R$,

(*i*) *form the auxiliary simultaneous equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R};$$

(*ii*) *solve these auxiliary simultaneous equations giving two independent solutions* $u = a$ and $v = b$;

(*iii*) *then write down the solution as* $\phi(u, v) = 0$ *or* $u = f(v)$ *or* $v = F(u)$, *where the function is arbitrary.*

2·16. Solution of the subsidiary equation by the method of multipliers

The subsidiary equations

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be solved as follows.

Using the two sets of multipliers $x, y, z; l, m, n$ each of the ratio in (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x\,(mz - ny) + y\,(nx - lz) + z\,(ly - mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{l\,(mz - ny) + m\,(nx - lz) + n\,(ly - mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

Hence $x\,dx + y\,dy + z\,dz = 0$ and $l\,dx + m\,dy + n\,dz = 0$.

Integrating we get,

$$x^2 + y^2 + z^2 = a \text{ and } lx + my + nz = b.$$

Hence the general integral is

$$\phi\,(x^2 + y^2 + z^2,\ lx + my + nz) = 0.$$

Example 45. *Find the general solution of*

$x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$ **(B.E. 1963, 78; 1976 Madurai)**

The subsidiary equations are

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \qquad \text{....(1)}$$

Taking the two sets of multipliers as x, y, z and

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of ratio in (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{\Sigma x^2(z^2 - y^2)} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{\Sigma\,(z^2 - y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{0}$$

Hence $x\,dx + y\,dy + z\,dz = 0$ and $\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$.

Integrating, $x^2 + y^2 + z^2 = a$ and $\log x + \log y + \log z = k$.

i.e., $x^2 + y^2 + z^2 = a$ and $x\,yz = b$.

Hence the general integral is $\phi\,(x^2 + y^2 + z^2, xyz) = 0$.

Example 46. *Solve:* $\frac{y^2 z}{x}\,p + xzq = y^2.$ **(BR. 1995 April)**

The subsidiary equations are

By algebra, we know,

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}=\frac{l'dx+m'dy+n'dz}{l'P+m'Q+n'R}$$

where the two sets of multipliers $l, m, n; l', m', n'$ may be constants or variables in x, y, z. Choosing l, m, n such that $lP+mQ+nR=0$, we have

$$ldx+mdy+ndz=0 \qquad \text{....(1)}$$

If $ldx+mdy+ndz$ is a perfect differential of some function, say, $u(x, y, z)$ then $du=0$, by (1). Hence integrating (1), we get

$u=a$, as one solution.

Similarly, the other set of multipliers l', m', n' can be found out so that $l'P+m'Q+n'R=0$.

Hence $l'\,dx+m'\,dy+n'\,dz=0$.

This yields another solution $v=b$.

Therefore the general solution is

$$\phi(u, v)=0, \text{ or } u=f(v).$$

Here, the set of multipliers l, m, n and l', m', n' are called Lagrangian multipliers.

Example 43. *Find the general integral of $px+qy=z$.*

Comparing the equation with $Pp+Qq=R$, we get

$$P=x, Q=y \text{ and } R=z.$$

The subsidiary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}.$$

In these, the variables are separated.

From $\frac{dx}{x}=\frac{dy}{y}$, we get $\log x=\log y+\log a$.

i.e., $\frac{x}{y}=a.$

Similarly from $\frac{dy}{y}=\frac{dz}{z}$, we get $\frac{y}{z}=b$.

Hence the general integral is $\phi\left(\frac{x}{y}, \frac{y}{z}\right)=0$.

Example 44. *Solve:* $(mz-ny)\,p+(nx-lz)\,q=ly-mx$.

Lagrange's subsidiary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx} \qquad \text{...(1)}$$

$$\frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

From the first equality, $x^2dx = y^2dy$, giving $x^3 - y^3 = c$.

From the first and the last ratios,

$$x\,dx = z\,dz, \text{ giving } x^2 - z^2 = k.$$

Hence the general integral is $F(x^3 - y^3, x^2 - z^2) = 0$.

Example 47. *Find the general solution of* $(y+z)p + (z+x)q = x + y$.

The auxiliary equation are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

each is equal to $$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} \quad \text{....(1)}$$

Taking the first two ratios,

$$\frac{d\,(x+y+z)}{2\,(x+y+z)} = \frac{-d\,(x-y)}{(x-y)}$$

Integrating,

$$\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log c.$$

$$\therefore \log(x+y+z) = \log(x-y)^{-2} + \log k.$$

$$\therefore (x+y+z) = k\,(x-y)^{-2}$$

i.e., $$(x+y+z)(x-y)^2 = k \quad \text{....(2)}$$

Taking the last two ratios of equations (1),

$$\frac{d\,(x-y)}{x-y} = \frac{d\,(y-z)}{y-z}$$

Integrating, $\log(x-y) = \log(y-z) + \log b$

$$\therefore \frac{x-y}{y-z} = b \quad \text{....(3)}$$

Solutions given by (2) and (3) are independent.

Hence, the general solution is

$$(x+y+z)(x-y)^2 = f\left(\frac{x-y}{y-z}\right)$$

Example 48. *Find the general solution of* $p\tan x + q\tan y = \tan z$.

This is Lagrange's equation.

The auxiliary equations are

i.e., $x+y+z=a$ ...(1)

Taking the Lagrangian multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$, we have

$$\frac{\frac{1}{x}dx}{y-z}=\frac{\frac{1}{y}dy}{z-x}=\frac{\frac{1}{z}dz}{x-y}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{\Sigma(y-z)}$$

$$=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

Hence, $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

Integrating, $\log x+\log y+\log z=\log b$

$\therefore \quad xyz=b$ ...(2)

Hence, the general solution is $\phi\,(xyz, x+y+z)=0$.

Example 51. *Solve* : $(y-xz)p+(yz-x)q=(x+y)(x-y)$ **(Anna 2002 Nov.)**

Solution. The auxiliary equations are

$$\frac{dx}{y-xz}=\frac{dy}{yz-x}=\frac{dz}{x^2-y^2}$$

Each is equal to $\frac{y\,dx+x\,dy+dz}{y(y-xz)+x(yz-x)+(x^2-y^2)}$

$$=\frac{d(xy+z)}{0}$$

$\therefore\ d(xy+z)=0$, Hence $xy+z=a$ ....(1)

Again each ratio is equal to $\frac{x\,dx+y\,dy}{xy-x^2z+y^2z-xy}$

Therefore, $$\frac{\frac{1}{2}d(x^2+y^2)}{z(y^2-x^2)}=\frac{dz}{x^2-y^2}$$

i.e. $$d(x^2+y^2)=-2z\,dz$$

i.e. $$d(x^2+y^2+z^2)=0$$

$\therefore\ x^2+y^2+z^2=b$ ...(2)

Using (1) and (2), the general solution is

$$\phi\,(x^2+y^2+z^2,\ xy+z)=0$$

Example 52. *Solve:* $pz+qy=x$.

The auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \cot x\, dx = \int \cot y\, dy = \int \cot z\, dz$$

Taking the first two ratios,

$\log \sin x = \log \sin y + \log a$

$$\frac{\sin x}{\sin y} = a \quad \text{...(1)}$$

Similarly, taking the last two ratios,

we get $$\frac{\sin y}{\sin z} = b \quad \text{...(2)}$$

Hence, the general solution is

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0, f \text{ being arbitrary}$$

Example 49. *Solve :* $(y - z)\, p + (z - x)\, q = x - y$

The auxiliary equations are

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}.$$

Each is equal to $$\frac{d\,(x + y + z)}{y - z + z - x + x - y} = \frac{d\,(x + y + z)}{0}$$

Since the denominator is zero, $d\,(x + y + z) = 0$

$$\therefore \quad x + y + z = a \quad \text{...(1)}$$

Also, taking Lagrange's multipliers x, y, z.

each radio is $$= \frac{x\,dx + y\,dy + z\,dz}{x\,(y - z) + y\,(z - x) + z\,(x - y)} = \frac{\frac{1}{2}\, d\,(x^2 + y^2 + z^2)}{0}$$

Hence, $d\,(x^2 + y^2 + z^2) = 0$

i.e., $x^2 + y^2 + z^2 = b$ ...(2)

$\therefore$ The general solution is $\phi\,(x + y + z, x^2 + y^2 + z^2) = 0$

Example 50. *Solve:* $x\,(y - z)\, p + y\,(z - x)\, q = z\,(x - y)$

The auxiliary eqations are

$$\frac{dx}{x\,(y - z)} = \frac{dy}{y\,(z - x)} = \frac{dz}{z\,(x - y)}.$$

Each is equal to $$\frac{dx + dy + dz}{\Sigma\, x\,(y - z)} = \frac{d\,(x + y + z)}{0}$$

Hence, $d\,(x + y + z) = 0$

Hence the general solution is $\phi\left(\frac{1}{x}+\frac{1}{y}, \frac{z}{x+y}\right)=0$

Example 54. *Solve* $p-q=\log(x+y)$

The auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{\log(x+y)}$$

Hence, $\frac{dx}{1}=\frac{dy}{-1}$ gives, on integration

$$x=-y+a \quad \therefore \quad x+y=a \qquad \text{....(1)}$$

$$\frac{dx}{1}=\frac{dz}{\log(x+y)}=\frac{dz}{\log a}$$

Hence $x=\frac{1}{\log a}z+b$, on integration

$$x-\frac{z}{\log(x+y)}=b \qquad \text{....(2)}$$

The general solutions is $\phi\left(x+y,\ x-\frac{z}{\log(x+y)}\right)=0$

Example 55. $(2z-y)\,p+(x+z)\,q+2x+y=0.$

This is Lagrange's equation. Hence, the auxiliary *equations are*

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-2x-y}$$

Each equal to $\frac{dz+2dy-dx}{-2x-y+2x+2z-2z+y}$

$$=\frac{d(z+2y-x)}{0}$$

Hence $d(z+2y-x)=0 \quad \therefore \quad z+2y-x=a$ ....(1)

Also, each ratio is equal to

$$=\frac{x\,dx+y\,dy+z\,dz}{2\,xz-xy+xy+yz-2\,xz-yz}$$

$$=\frac{\frac{1}{2}d(x^2+y^2+z^2)}{0}$$

$$\therefore \quad d(x^2+y^2+z^2)=0$$

$$x^2+y^2+z^2=b \qquad \text{....(2)}$$

Therefore, the general solution is

$$\phi(x^2+y^2+z^2, z+2y-x)=0$$

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

Taking the first and the last ratios, we get,

$$x\,dx = zdz$$

Integrating $\frac{x^2}{2} = \frac{z^2}{2} + c$

$\therefore \quad x^2 - z^2 = a$ ....(2)

each ratio of (1) is equal to $= \frac{dx + dy + dz}{x + y + z}$

Hence $\frac{dy}{y} = \frac{d\,(x + y + z)}{x + y + z}$

Integrating, $\log y = \log\,(x + y + z) + \log b$

$\therefore \quad \frac{y}{x + y + z} = b$ ....(2)

Hence, the general solution is $\phi\left(x^2 - z^2, \frac{y}{x + y + z}\right) = 0$

Example 53. *Solve:* $z\,(x - y) = px^2 - qy^2$ **(MS. 1988 Nov.)**

This is Lagrange's equation.

The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z\,(x - y)}$$

Each equal to $\frac{dx + dy}{x^2 - y^2}$

Hence $\frac{dx}{x^2} = -\frac{dy}{y^2}$. Integrating.

$$-\frac{1}{x} = \frac{1}{y} + k \quad \therefore \quad \frac{1}{x} + \frac{1}{y} = a \qquad ....(1)$$

Also $\frac{dz}{z\,(x - y)} = \frac{d\,(x + y)}{x^2 - y^2}$

i.e., $\frac{dz}{z} = \frac{d\,(x + y)}{x + y}$

Integrating, $\log z = \log\,(x + y) + \log b$

$$\frac{z}{x + y} = b$$

Both solutions are independent

The general solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ ...I

We have to find that function ϕ satisfying (I)

and also $x^2 + y^2 + z^2 = 4$ ....(3)

and $x + y + z = 2$ ....(4)

Hence, we will eliminate x, y, z from (1), (2), (3), (4).

From (2), $y = bz$ From (1), $x = ay = abz$

using these values in (3) and (4)

$$a^2 b^2 z^2 + b^2 z^2 + z^2 = 4$$

i.e., $$z^2 (1 + b^2 + a^2 b^2) = 4 \quad ....(5)$$

Also, $$abz + bz + z = 2$$

$$(1 + b + ab)\, z = 2. \quad ....(6)$$

Eliminate z between (5) and (6)

Squaring (6),

$$(1 + b + ab)^2 z^2 = 4 \quad ....(7)$$

From (5) and (7), $1 + b^2 + a^2b^2 = (1 + b + ab)^2$

Simplifying, $b + ab^2 + ab = 0.$

i.e., $$1 + ab + a = 0 \quad ....(8)$$

using $$a = \frac{x}{y}, b = \frac{y}{z} \text{ in } (8)$$

$$1 + \frac{xy}{yz} + \frac{x}{y} = 0 \quad \therefore \quad 1 + \frac{x}{z} + \frac{x}{y} = 0$$

i.e., $yz + xy + xz = 0$ is the required surface.

Exercises 2(d)

Find the general solution of the following partial differential equations (1 to 18):

1. $px^2 + qy^2 = (x + y)\, z.$ **2.** $px - qy = xz.$ **3.** $(p - q)\, z = z^2 + (x + y)^2.$

4. $p\, zx + q\, zy = xy.$ **5.** $px^2 + qy^2 = z^2.$ **6.** $p + q = 1.$

7. $(a - x)\, p + (b - y)\, q = c - z.$ **8.** $p\sqrt{x} + q\sqrt{y} = \sqrt{z}.$ **(M.U. 64 B.E.)**

9. $px + qy = x.$ **10.** $px + qy = nz.$ **(M.U. 86 Ap.)**

11. $py^2 z + qx^2 z = xy^2.$ **(M.U. 71 B.E.)**

12. $px\,(y^2 + z) - qy\,(x^2 + z) = z\,(x^2 - y^2).$ **(MS. 1991 Ap.)**

13. $px^2 - qy^2 = z^2.$ **14.** $py^2 - xyq = x\,(z - 2y)$

solutions for $\frac{x-y}{y-z}+1$ gives $\frac{x-z}{y-z}$ which is the reciprocal of the second solution.

Therefore solution given by (4) and (5) are not independent. Hence we have to search for another independent solution.

using multipliers x, y, z in ...(1)

each ratio is $$= \frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz}$$

using multipliers 1, 1, 1.

each ratio is $$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\therefore \quad \frac{x\,dx + y\,dy + dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\frac{\frac{1}{2}d\,(x^2 + y^2 + z^2)}{(x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{d\,(x + y + z)}{x^2 + y^2 + z^2 - xy - yz - zx}$$

Hence $\quad \frac{1}{2}d\,(x^2 + y^2 + z^2) = (x + y + z)\,d\,(x + y + z)$

Integrating $\quad \frac{1}{2}(x^2 + y^2 + z^2) = \frac{(x + y + z)^2}{2} + k$

$$\therefore \quad (x^2 + y^2 + z^2) = (x + y + z)^2 + 2k$$

$\therefore \quad xy + yz + zx = b$ on simplification.

$\therefore \quad$ The general solution

$$\phi\left(xy + yz + zx\,,\ \frac{x-y}{y-z}\right) = 0.$$

Example 59. *Find the equation of the curve satisfying $px + qy = z$ and passing through the circle*

$$x^2 + y^2 + z^2 = 4, \ \ x + y + z = 2.$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking the first two ratios, $\log x = \log y + \log a$

$$\therefore \quad \frac{x}{y} = a$$

Similarly, $\frac{y}{z} = b$

$$= \frac{\frac{1}{2} d\,(x^2+y^2+z^2)}{0}$$

Hence $d\,(x^2+y^2+z^2) = 0 \quad \therefore \quad x^2+y^2+z^2 = a$ ...(2)

using multipliers 2, 3, 4,

each of equation (1) is

$$= \frac{2\,dx + 3\,dy + 4\,dz}{6z - 8y + 12x - 6z + 8y - 12x}$$

$$= \frac{2dx + 3\,dy + 4dz}{0}$$

$\therefore \quad 2dx + 3dy + 4dz = 0$

Hence $\quad 2x + 3y + 4z = b$ ....(3)

General solution is $\phi\,(x^2+y^2+z^2,\ 2x+3y+4z) = 0$

Example 58. *Solve:* $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$ (*Anna Ap. 2005*)

The subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad ....(1)$$

$$\text{each} = \frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dx - dz}{x^2 - yz - z^2 + xy}$$

i.e., $$\frac{d\,(x-y)}{(x^2 - y^2) + z\,(x-y)} = \frac{d\,(y-z)}{y^2 - z^2 + x\,(y-z)} = \frac{d\,(x-z)}{x^2 - z^2 + y\,(x-z)}$$

i.e., $$\frac{d\,(x-y)}{(x-y)\,(x+y+z)} = \frac{d\,(y-z)}{(y-z)\,(x+y+z)} = \frac{d\,(x-z)}{(x-z)\,(x+y+z)} \qquad ...(2)$$

i.e., $$\frac{d\,(x-y)}{(x-y)} = \frac{d\,(y-z)}{y-z} = \frac{d\,(x-z)}{x-z} \qquad ...(3)$$

Taking the first two ratios, and integrating

$\log\,(x-y) = \log\,(y-z) + \log a$

$\therefore \quad \dfrac{x-y}{y-z} = a$ ...(4)

Similarly taking the last two ratios of (3)

we get $\dfrac{y-z}{x-z} = b$ ...(5)

But $\dfrac{x-y}{y-z}$, and $\dfrac{y-z}{x-z}$ *are not independent*

Example 56. *Solve:* $(y^2 + z^2 - x^2)\, p - 2\, xyq + 2\, xz = 0.$

(MS. 1988 Nov.)

The auxiliary equations are

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2\, xy} = \frac{dz}{-2\, xz}$$

From the last two ratios,

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, $\log y = \log z + \log a$

$$\frac{y}{z} = a \qquad \text{....(1)}$$

Taking Lagrangian multipliers x, y, z we get

each ratio is equal to $\dfrac{x\, dx + y\, dy + z\, dz}{-x\,(x^2 + y^2 + z^2)}$

$$\therefore \quad \frac{dy}{-2\, xy} = \frac{x\, dx + y\, dy + z\, dz}{-x\,(x^2 + y^2 + z^2)}$$

$$\therefore \quad \frac{dy}{y} = \frac{d\,(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

$$\log y = \log\,(x^2 + y^2 + z^2) + \log b.$$

$$\therefore \quad \frac{y}{x^2 + y^2 + z^2} = b \qquad \text{....(2)}$$

Hence the general solution is

$$\phi\left(\frac{y}{z}, \frac{y}{x^2 + y^2 + z^2}\right) = 0$$

Example 57. *Solve:* $(3z - 4y)\, p + (4x - 2z)\, q = 2y - 3x.$

(MS. 19

The auxiliary equations are

$$\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}.$$

use Lagrangian multipliers $x, y, z.$ we get

each ratio is $\dfrac{x\, dx + y\, dy + z\, dz}{3\, xz - 4\, xy + 4\, xy - 2\, yz + 2\, yz - 3\, xz}$

15. $p \cot x + q \cot y = \cot z$. (Os. 64 B.E.)

16. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$. (Os. U. 65 B.E.)

17. $(1+y)p + (1+x)q = z$

18. $(x+y)zp + (x-y)zq = x^2 + y^2$.

19. Show that the surface which satisfies the differential equation $(x^2 - a^2)p + (xy - az\tan\alpha)q = xz - ay\cot\alpha$ and passes through the curve $x^2 + y^2 = a^2$, $z = 0$ is $x^2 + y^2 - a^2 = z^2 \tan^2\alpha$.

20. Show that the integral surface of the linear partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0$, $z = 1$ is $x^2 + y^2 + 2xyz - 2z + 2 = 0$.

21. Show that the integral surface of the equation $2y(z-3)p + (2x - z)q = y(2x - 3)$ which passes through the circle $x^2 + y^2 = 2x$, $z = 0$ is $x^2 + y^2 - \ldots - 2x + 4z = 0$. (M.U. 70 B.E.)

22. If the expression $(y^2 + z)\,dx + (x^2 + z)\,dy$ is an exact differential in x and y, show that $z = 2xy + f(x+y)$, where f is arbitrary. Find f if $z = 2y + 1$ when $x = 0$. (M.U. 65 B.E.)

23. $pz + qy = x$

24. $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

25. $pz - qz = z^2 + (x+y)^2$

2.17. Partial Differential Equation of Higher order

A thorough treatment of the subject of partial differential equations of order higher than the first is too vast to study here. We shall study only linear partial differential equations of higher order with *constant coefficients*. We can divide this study into two groups, *viz*. (*i*) homogeneous linear and (*ii*) non-homogeneous linear equations.

For example, the equation,

$$2\frac{\partial^3 z}{\partial x^3} + 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} + 5\frac{\partial^3 z}{\partial y^3} = x^2 + y \quad \ldots(i)$$

is an equation in which the partial derivatives occurring are all of the same order and the coefficients are constants whereas, the equation

$$\frac{\partial^3 z}{\partial x^3} + 2\frac{\partial^2 z}{\partial y^2} - 4\frac{\partial z}{\partial x} + z = x^2 + y^2 \quad \ldots(ii)$$

possesses derivatives which are not all of the same order but with constant coefficients. (*i*) is called a homogeneous linear equation with constant coefficients whereas (*ii*) is called a non-homogenous linear equation with constant coefficients.

We shall use the differential operators D and D' to denote $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$ and hence (*i*) can be written symbolically as

$$(2D^3 + 3D^2D' + 4DD'^2 + 5D'^3)\, z = x^2 + y.$$

(*ii*) becomes $(D^3 + 2D'^2 - 4D + 1)\, z = x^2 + y^2$.

2·18. Homogeneous linear equation

A homogeneous linear partial differential equation of n^{th} order with constant coefficients is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \ldots\ldots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \quad \ldots(i)$$

where a's are constants and F is a known function of x, y. Writing symbolically, (*i*) can be written as

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n)\, z = F(x, y) \quad \ldots(ii)$$

or $$f(D, D')\, z = F(x, y) \quad \ldots(iii)$$

where $f(D, D')$ stands for the polynomial expression

$$a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n.$$

The method of solving (*iii*) is anologous to that of solving ordinary linear differential equation with constant coefficients. The complete solution of (*iii*) consists of two parts namely the *complementary function* and the *particular integral*. The complementary function of (*iii*) is the solution of

$$f(D, D')\, z = 0 \quad \ldots(iv)$$

and the particular integral of (*iii*) is a particular solution of (*iii*) given symbolically by

$$\frac{1}{f(D, D')} F(x, y).$$

Hence the complete solution = complementary function + particular integral

= C.F. + P.I.

Note. If $z = z_1, z = z_2, \ldots\ldots, z = z_n$ are solutions of (*iv*), then $z = \lambda_1 z_1 + \lambda_2 z_2 + \ldots\ldots + \lambda_n z_n$ is also a solution of (*iv*), where λ's' are constants arbitrarily chosen.

2·19. Complementary function of homogeneous linear equations, with constant coefficients

Let $(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n)\, z = F(x, y) \quad \ldots(1)$

or $f(D, D')\, z = F(x, y) \quad \ldots(2)$

be the given homogeneous linear equation with constant coefficients. Now the complementary function of (2) is the solution of

$$f(D, D')\, z = 0 \quad \ldots(3)$$

Since $f(D, D')$ is a polynomial which is homogeneous of degree n in D and

D', we can factorise $f(D, D')$ into linear factors, and hence (3) can be written as

$$(D - m_1 D')(D - m_2 D')......(D - m_n D')\, z = 0 \quad ...(4)$$

where $m_1, m_2, ...m_n$ are the roots of

$$(m - m_1)(m - m_2)\ ......\ (m - m_n) = 0 \quad ...(5)$$

i.e., of $\quad f(m, 1) = 0.$

Equation (3) will be satisfied by the solution of each of the component differential equations,

$$(D - m_1 D')\, z = 0,\ (D - m_2 D')\, z = 0, ... (D - m_n D')z = 0 \quad ...(6)$$

$(D - m_r D')\, z = 0$ is Lagrange's equation.

Hence the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}.$$

Therefore, $y + m_r x = c$ and $z = k$.

Hence the general solution of $(D - m_r D')\, z = 0$ is

$$z = \phi_r\, (y + m_r x).$$

Putting $r = 1, 2, ..., n$ we get the general solutions of the component equations (6). Hence the most general solution of (3) is $z = \phi_1\, (y + m_1 x) + \phi_2\, (y + m_2 x) + \cdots + \phi_n\, (y + m_n x)$, where $\phi_1, \phi_2, ...\phi_u$ are arbitrary functions.

That is, the complementary function of (1) is

$$z = \phi_1\, (y + m_1 x) + \phi_2\, (y + m_2 x) + \cdots + \phi_n\, (y + m_n x),$$

where $m_1, m_2, ..., m_n$ are the roots of the auxiliary equation $f(m, 1) = 0$ which is got by replacing D by m and D' by 1 in $f(D, D')$ and equating it to zero.

Note. The above argument is valid only if $m_1 \neq m_2 \neq .... \neq m_n$.

2·20. Auxiliary equation with repeated roots

Suppose the auxiliary equation $f(m, 1) = 0$ possesses two equal roots, say $m_1 = m_2$. Then the above method will give a complementary function with $(n-1)$ arbitrary functions only which will not be the complete solution of the given equation.

If $m_1 = m_2$, we come across the component equation

$$(D - m_1 D')^2\, z = 0$$

Let $\quad (D - m_1 D')\, z = u$

Then (7) becomes

$$(D - m_1 D')\, u = 0.$$

$\therefore \quad u = \phi_1 (y + m_1 x)$

Substituting this value of u in (8), we get

$$(D - m_1 D') z = \phi_1 (y + m_1 x) \qquad ...(9)$$

This is Lagrange's equation whose subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi_1 (y + m_1 x)}$$

Solving these, we get,

$$y + m_1 x = c \text{ and } z - x\phi_1 (y + m_1 x) = k.$$

Hence the solution of (7) is

$$z - x\phi_1 (y + m_1 x) = \phi_2 (y + m_1 x)$$

i.e.,

$$z = x\phi_1 (y + m_1 x) + \phi_2 (y + m_1 x) \qquad ...(10)$$

(10) is the solution of (7).

Note. If the auxiliary equation $f(m, 1) = 0$ has r equal roots $m_1 = m_2 = m_3 = \cdots = m_r$, then the corresponding part in the complementary function is

$$z = \phi_1 (y + m_1 x) + x\phi_2 (y + m_1 x) + \cdots + x^{r-1} \phi_r (y + m_1 x).$$

2.21. The Particular integral

Evaluation of the particular integral in P.D.E., is analogous to that of the P.I. in an ordinary linear diferential equation. There are short methods to evaluate the particular integrals of the homogeneous linear equation with constant coefficients. The proofs are simple and the students can prove them by themselves. The methods are given below.

Type 1. $\dfrac{1}{f(D, D')} e^{ax+by} = \dfrac{1}{f(a, b)} e^{ax+by}$, if $f(a, b) \neq 0$.

Type 2. $\dfrac{1}{f(D, D')} x^r y^s = [f(D, D')]^{-1} x^r y^s$, where

$[f(D, D')]^{-1}$ is to be expanded in powers of D, D'.

Type 3. $\dfrac{1}{f(D^2, DD', D'^2)} \begin{matrix}\sin \\ \cos\end{matrix} \left(ax + by\right)$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \begin{matrix}\sin \\ \cos\end{matrix} \left(ax + by\right)$$

Type 4. $\dfrac{1}{f(D, D')} e^{ax+by} \phi(x, y) = e^{ax+by} \dfrac{1}{f(D + a, D' + b)} \phi(x, y)$.

Type 5. $\dfrac{\sin ax \sin by}{f(D^2, D'^2)} = \dfrac{\sin ax \sin by}{f(-a^2, -b^2)}$ if denominator $\neq 0$

Type 6. $\dfrac{\cos ax \cos by}{f(D^2, D'^2)} = \dfrac{\cos ax \cos by}{f(-a^2 - b^2)}$ if denominator $\neq 0$

General rule to find $\dfrac{1}{D - mD'} F(x, y)$.

First change y to $y - mx$ in $F(x, y)$, integrate it w.r.t. x treating y as a constant and then in the resulting integral change y to $y + mx$. The result thus got is the value of $\dfrac{1}{D - mD'} F(x, y)$.

OR

Integrate $F(x, a - mx)$ w.r.t. x and after integration replace 'a' by $y + mx$

Example 60. $(D^2 - 4DD' + 4D'^2)\, z = 0$.

The auxiliary equation is

$m^2 - 4m + 4 = 0$ {Replace D by m and D' by 1 in $f(D, D')$ and equate to zero}

Solving $m = 2, 2$ (equal roots)

Since R.H.S. is zero, there is no Particular integral.

Hence $z = C.F.$ alone

i.e. $$z = \phi_1(y + 2x) + x\, \phi_2(y + 2x)$$

Example 61. *Solve* $(D^3 - 3D^2 D' + 2DD'^2)\, Z = 0$

The auxiliary equations $m^3 - 3m^2 + 2m = 0$

i.e. $$m\,(m^2 - 3m + 2) = 0$$

i.e. $$m\,(m - 1)(m - 2) = 0$$

$\therefore\ m = 0, 1, 2$

General solution is

$$\therefore\ Z = f_1(y + 0\cdot x) + f_2(y + x) + f_3(y + 2x)$$

$$Z = f_1(y) + f_2(y + x) + f_3(y + 2x)$$

Example 62. $(D^3 + DD'^2 - D^2 D' - D'^3)\, Z = 0$

Auxiliary equation is

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m - 1) + (m - 1) = 0$$

$$(m - 1)(m^2 + 1) = 0$$

$$\therefore\quad m = 1, i, -i$$

General Solution is

$$Z = \phi(y + x) + f(y + ix) + F(y - ix)$$

Example 63. Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = 0.$ **(M.U. 63 B.E.)**

Writing this equation symbolically, we get,

$$(D^2 + DD' - 2D'^2)\, z = 0 \qquad \ldots(1)$$

The auxiliary equation is $m^2 + m - 2 = 0$,

giving $m = 1, -2$.

Hence the C.F. of (1) given by

$$z = \phi_1(y + x) + \phi_2(y - 2x). \qquad \ldots(2)$$

R.H.S. of (1) is zero. Hence the complete solution of (1) is the C.F. itself.

Thus the complete solution is (2). P.I1 P.I2

Example 64. *Solve* $(D^3 - 7DD'^2 - 6D'^3)\, z = x^2 y + \sin(x + 2y)$. **(MS. 1988 Nov.)**

The auxiliary equation is $m^3 - 7m - 6 = 0$

i.e., $(m + 1)(m + 2)(m - 3) = 0$

$\therefore\ m = -1, -2, 3$

The C.F. of the given P.D.E. is

$$z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x)$$

$$(\text{P.I.})_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 y)$$

$$= \frac{1}{D^3}\left[1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right]^{-1}(x^2 y)$$

$$= \frac{1}{D^3}\left[1 + \frac{7D'^2}{D^2} + \frac{6D'^3}{D^3} + \ldots\right](x^2 y).$$

$$= \frac{1}{D^3}(x^2 y)$$

$$= \frac{1}{60}x^5 y.$$

$$(\text{P.I.})_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x + 2y)$$

$$= \frac{1}{-D + 28D + 24D'}\sin(x + 2y)$$

$$= \frac{1}{3} \cdot \frac{1}{9D + 8D'} \sin(x + 2y)$$

$$= \frac{1}{3} \cdot \frac{9D - 8D'}{81D^2 - 64D'^2} \sin(x + 2y)$$

$$= \frac{1}{3} \frac{9D - 8D'}{-81 + 256} \sin(x + 2y)$$

$$= \frac{1}{525} [(-7 \cos(x + 2y)]$$

$$= -\frac{1}{75} \cos(x + 2y)$$

Hence the complete solution of the given equation is

$$z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) + \frac{x^5 y}{60} - \frac{1}{75} \cos(x + 2y).$$

Example 65. *Solve* $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x + 2y}$.

Writing in the symbolic form, the equation is

$$(D^3 - 3D^2D' + 4D'^3) z = e^{x + 2y}$$

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$

i.e., $(m + 1)(m - 2)^2 = 0$

i.e., $m = -1, 2, 2$

Hence the C.F. is $\phi_1(y - x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2D' + 4D'^3} \cdot e^{x + 2y}$$

$$= \frac{e^{x + 2y}}{1 - 6 + 32}$$

$$= \frac{e^{x + 2y}}{27}$$

The complete solution is

$$z = \phi_1(y - x) + \phi_2(y + 2x) + x\phi_3(y + 2x) + \frac{1}{27} e^{x + 2y}.$$

Example 66. *Solve* $(D^3 + D^2D' - DD'^2 - D'^3) z = e^x \cos 2y$.

The auxiliary equation is

$$m^3 + m^2 - m - 1 = 0$$

i.e., $(m + 1)^2 (m - 1) = 0$

$\therefore$ $m = 1, -1, -1$

$\therefore$ C.F. is $\phi_1(y+x)+\phi_2(y-x)+x\,\phi_3(y-x)$

$$\text{P.I.} = \frac{1}{D^3+D^2D'-DD'^2-D'^3}\, e^x \cos 2y$$

$$= e^x \frac{\cos 2y}{(D+1)^3+(D+1)^2 D'-(D+1)D'^2-D'^3}$$

$= e^x$. Real part of

$$\frac{e^{i2y}}{(D+1)^3+(D+1)^2 D'-(D+1)D'^2-D'^3}$$

$$= e^x \text{ R.P. of } \frac{e^{i2y}}{1+2i+4+8i}$$

$$= \frac{e^x}{5} \text{. R.P. of } \frac{1-2i}{(1+2i)(1-2i)} (\cos 2y + i \sin 2y)$$

$$= \tfrac{1}{5} e^x . \tfrac{1}{5} (\cos 2y + 2 \sin 2y)$$

$$= \frac{e^x}{25} (\cos 2y + 2 \sin 2y)$$

The complete solution is

$$Z = \phi_1(y+x)+\phi_2(y-x)+x\phi_3(y-x)+\frac{e^x}{25}(\cos 2y + 2\sin 2y).$$

Example 67. *Solve:* $(D^3-7DD'^2-6D'^3)\,z = e^{2x+y}$ **(BR. 1995 Ap.)**

The auxiliary equation is

$$m^3-7m-6=0$$

Evidently $m=-1$ is a root.

Factorizing,

$$(m+1)(m^2-m-6)=0$$

$$(m+1)(m-3)(m+2)=0$$

$\therefore$ $m=-1,-2,3$

$\therefore$ C.F. is $\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$.

$$\text{P.I.} = \frac{e^{2x+y}}{D^3-7DD'^2-6D'^3}$$

$$= \frac{e^{2x+y}}{8-7(2)(1)-6(1)}$$

$$= -\frac{1}{12} e^{2x+y}$$

The complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - \frac{1}{12}e^{2x + y}.$$

Example 68. $(D^2 - 2DD' + D'^2)z = e^{x + 2y}$

Auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\therefore \quad m = 1, 1$$

C.F. is $\phi_1(y + x) + x\,\phi_2(y + x)$

$$\text{P.I.} = \frac{e^{x + 2y}}{(D - D')^2}$$

$$= \frac{e^x + 2y}{(1 - 2)^2} = e^{x + 2y}$$

Hence, the complete solution is $z = \phi_1(y + x) + x\,\phi_2(y + x) + e^{x+2y}$

Example 69. *Solve* $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = e^{x + 2y}$

Writing in the symbolic form, we have

$$(D^3 - 2D^2D')Z = e^{x + 2y}$$

Auxiliary equation is $m^3 - 2m^2 = 0$

$$m^2(m - 2) = 0$$

$$\therefore \quad m = 0, 0, 2.$$

C.F. is $\phi_1(y) + x\,\phi_2(y) + \phi_3(y + 2x)$

$$\text{P.I.} \quad \frac{e^{x + 2y}}{D^3 - 2D^2D'}$$

$$= \frac{e^{x + 2y}}{1 - 2(2)}$$

$$= \frac{-1}{3}e^{x + 2y}$$

The complete solution is

$$z = \phi_1(y) + x\,\phi_2(y) + \phi_3(y + 2x) - \frac{1}{3}e^{x + 2y}.$$

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Example 70. *Solve* $(D^4 - D'^4)\, z = e^{x+y}$

The auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$\therefore \; m = 1, -1, i, -i$$

C.F. is $\phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix)$

$$\text{P.I.} = \frac{e^{x+y}}{D^4 - D'^4}$$

(Now replacing D by 1, D' by 1, the denominator becomes zero.

$$= \frac{e^{x+y}}{(D - D')(D + D')(D^2 + D'^2)}$$

$$= \frac{e^{x+y}}{(D - D')(1 + 1)(1 + 1)}$$ (Replace D by 1, D' by 1 in these factors which do not vanish.)

$$= \frac{1}{4} \cdot \frac{e^{x+y}}{D - D'} \qquad ...(1)$$

To evaluate $\dfrac{e^{x+y}}{D - D'}$ we remember the rule $\dfrac{F(x, y)}{D - mD'}$.

Change y as y-x and integrate w.r.t. x treating y as a constant and then change y into $y + x$.

Here, $F(x, y) = e^{x+y}$

Changing y as $y - x$, it becomes e^y

Integrating w.r.t x, we get xe^y

Changing y as $y + x$, we get xe^{x+y}

$$\therefore \; \text{P.I.} = \frac{1}{4} x \,.\, e^{x+y}.$$

$\therefore$ The complete solution is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix) + \frac{1}{4} x\, e^{x+y}.$$

Note 1. $$\frac{e^{ax+by}}{D - \frac{a}{b} D'} = x\, e^{ax+by}$$ (where Denominator vanishes when D is replaced by a and D' by b)

Note 2. $\dfrac{e^{ax+by}}{\left(D-\dfrac{a}{b}D'\right)^2}$

$= \dfrac{x^2}{2!}e^{ax+by}$

Note 3. using the result of Note 1 in the above Proble
P.I. $= \dfrac{1}{4}xe^{x+y}$.

Example 71. *Solve* $(D^2 - 4DD' + 4D'^2)\,z = e^{2x+y}$.

Auxiliary equation is $m^2 - 4m + 4 = 0$

Solving, $\therefore\ m = 2, 2.$

O.F. is $\phi_1\,(y + 2x) + x\,\phi_2\,(y + 2x)$.

$$\text{P.I.} = \frac{e^{2x+y}}{(D-2D')^2}$$

Replacing D by 2, D' by 1, the denominator is zero.

using the *Note 2* under the previous problem, P.I. $= \dfrac{x^2}{2}e^{2x+y}$

The complete solution is

$$z = \phi_1\,(y + 2x) + x\,\phi_2\,(y + 2x) + \frac{x^2}{2}e^{2x+y}.$$

Example 72. *Solve* $(D^2 + 2DD' + D'^2)\,z = \text{Sinh}\,(x + y) + e^{x+2y}$

The auxiliary equation is

$m^2 + 2m + 1 = 0$

$\therefore\ m = -1, -1$

C.F. is $\phi_1\,(y - x) + x\,\phi_2\,(y - x)$

$$(\text{P.I.})_1 = \frac{Sinh\,(x+y)}{(D+D')^2} = \frac{\frac{1}{2}\left(e^{x+y} - e^{-(x+y)}\right)}{(D+D')^2}$$

$$= \frac{1}{2}\frac{e^{x+y}}{(D+D')^2} - \frac{1}{2}\frac{e^{-x-y}}{(D+D')^2}$$

$$= \frac{1}{2}\frac{e^{x+y}}{4} - \frac{1}{2}\frac{e^{-x-y}}{4}$$

$$= \frac{1}{4}\,\text{Sinh}\,(x+y)$$

$$(PI)_2 = \frac{e^{x+2y}}{(D+D^1)^2}$$

$$= \frac{e^{x+2y}}{9}$$

$\therefore$ The complete solution is

$$z = C.F. + P.I.$$

$$z = \phi_1(y-x) + x\,\phi_2(y-x) + \frac{1}{4}\text{Sinh}(x+y) + \frac{1}{9}e^{x+2y}$$

Example 73. *Solve:* $(D^3 - 2D^2D')z = \sin(x+2y) + 3x^2y.$

The auxiliary equation is $m^3 - 2m^2 = 0$

$$\therefore \phi\, m = 0, 0, 2$$

C.F. is $\phi_1(y) + x\,\phi_2(y) + \phi_3(y+2x)$

$$(PI)_1 = \frac{\sin(x+2y)}{D^3 - 2D^2D'}$$

$$= \frac{\sin(x+2y)}{D(-1) - 2D(-2)}, \text{ replace } D^2 \text{ by } -1 \text{ and } DD' \text{ by } -2$$

$$= \frac{\sin(x+2y)}{3D}$$

$$= \frac{-1}{3}\cos(x+2y)$$

$$(PI)_2 = \frac{3x^2y}{D^3 - 2D^2D'}$$

$$= \frac{1}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1}(3x^2y)$$

$$= \frac{1}{D^3}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \cdots\right](3x^2y)$$

$$= \frac{1}{D^3}\left[3x^2y + \frac{6x^2}{D}\right]$$

$$= 3y\,.\,\frac{x^5}{3\times4\times5} + 6\,.\,\frac{x^6}{3\times4\times5\times6} = \frac{x^5y}{20} + \frac{x^6}{60}$$

Complete solution is Z = C.F. + P.I.

i.e.,

$$z = \phi_1(y) + x\,\phi_2(y) + \phi_3(y+2x) - \frac{1}{3}\cos(x+2y) + \frac{x^5y}{20} + \frac{x^6}{60}.$$

Example 74. *Solve:* $(D^2 - 2DD')z = x^3y + e^{2x}$ (MS. 1989 Nov)

The auxiliary equation is $m^2 - 2m = 0$

$\therefore \quad m = 0, 2$

C.F. is $\phi_1(y) + \phi_2(y + 2x)$

$$(\text{P.I.}) = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1}(x^3y)$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \cdots\right](x^3y)$$

$$= \frac{1}{D^2}\left[x^3y + \frac{2x^3}{D}\right]$$

$$= \frac{x^5y}{20} + 2 \cdot \frac{x^6}{4 \times 5 \times 6}$$

$$= \frac{x^5y}{20} + \frac{x^6}{60}$$

$$(\text{PI})_2 = \frac{e^{2x}}{D^2 - 2DD'}$$

$$= \frac{e^{2x}}{4} \text{ replace } D \text{ by 2 and } D' \text{ by 0.}$$

$\therefore$ The complete solution is

$$Z = \phi(y) + \phi_2(y + 2x) + \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$$

Example 75. *Solve:* $(D^2 + 3DD' + 2D'^2)z = x + y.$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

C.F. is $\phi_1(y - x) + \phi_2(y - 2x)$.

$$\text{P.I.} = \frac{x + y}{D^2 + 3DD' + 2D'^2}$$

$$= \frac{1}{D^2}\left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^{-1}(x + y)$$

$$\text{P.I.} = \frac{\sin x \cos y}{D^2 - 3DD' + 2D'^2}$$

$$= \frac{1}{2} \cdot \frac{\sin(x+y) + \sin(x-y)}{D^2 - 3DD' + 2D'^2}$$

$$= \frac{1}{2} \cdot \frac{\sin(x+y)}{D^2 - 3DD' + 2D'^2} + \frac{1}{2} \frac{\sin(x-y)}{(-1-3-2)}$$

$$= \frac{1}{2} I - \frac{1}{12} \sin(x-y)$$

$$I = \frac{\sin(x+y)}{D^2 - 3DD' + 2D'^2}$$

$$= \frac{\sin(x+y)}{(D-2D')(D-D')}$$

$$= \frac{1}{D-2D'} \cdot \frac{\sin(x+y)}{D-D'}$$

$$= \frac{1}{D-2D'} \cdot x \cdot \sin(x+y) \quad \text{use general rule}$$

$$= \int x \sin(a-x)\, dx \quad \text{where } a = y + 2x$$

$$= x[+\cos(a-x)] - \sin(a-x) \quad \text{where } a = y + 2x$$

$$= x\cos(x+y) - \sin(x+y)$$

Hence $z = \phi_1(x+y) + \phi_2(y+2x)$

$$+ \frac{1}{2} x \cos(x+y) - \frac{1}{2} \sin(x+y) - \frac{1}{12} \sin(x-y)$$

Example 80. *Solve* $(D^2 + DD' - 6D'^2) z = y \cos x$ ***(MS. 1990 Ap.)***

Auxiliary equation is $m^2 + m - 6 = 0$

$$m = 2, -3$$

C.F. is $\phi_1(y+2x) + \phi_2(y-3x)$

$$\text{PI} = \frac{y \cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{1}{(D-2D')} \frac{y \cos x}{(D+3D')}$$

$$= \frac{1}{(D-2D')} \left[\int (a+3x) \cos x\, dx \right] \quad \text{where } a + 3x = y$$

$$= \frac{1}{D-2D'} \left[(a+3x) \sin x - 3 \int \sin x\, dx \right] \quad \text{where } a + 3x = y$$

$$m^2 - 4 = 0$$
$$m = \pm 2$$

C.F. is $\phi_1 (y + 2x) + \phi_2 (y - 2x)$

$$\text{P.I.} = \frac{\cos 2x \cos 3y}{D^2 - 4D'^2}$$
$$= \frac{\cos 2x \cos 3y}{-4 - 4(-9)}$$
$$= \frac{1}{32} \cos 2x \cos 3y$$

The general solution is

$$Z = \phi_1 (y + 2x) + \phi_2 (y - 2x) + \frac{1}{32} \cos 2x \cos 3y.$$

Example 78. *Solve* $(D^2 - 2DD' + D'^2) Z = \cos (x - 3y)$

(MS. 1990)

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$
$$m = 1, 1$$

C.F. is $\phi_1 (y + x) + x \phi_2 (y + x)$

$$\text{P.I.} = \frac{\cos (x - 3y)}{D^2 - 2DD' + D'^2}$$
$$= \frac{\cos (x - 3y)}{-1 - 2(3) - 9}$$
$$= -\frac{1}{16} \cos (x - 3y)$$

$\therefore$ The complete solution is,

$$z = \phi_1 (y + x) + x \phi_2 (y + x) - \frac{1}{16} \cos (x - 3y)$$

Example 79. *Solve:* $(D^2 - 3DD' + 2D'^2) z = \sin x \cos y$

(MS. 1989)

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$(m - 1)(m - 2) = 0$$
$$\therefore \; m = 1, 2$$

C.F. is $\phi_1 (y + x) + \phi_2 (y + 2x)$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^2 - \cdots\right](x+y)$$

$$= \frac{1}{D^2}\left[x + y - \frac{3}{D}\right] = \frac{1}{D^2}x + \frac{1}{D^2}y - \frac{3}{D^3}$$

$$= \frac{x^3}{6} + \frac{x^2y}{2} - \frac{x^3}{2}$$

$$= \frac{x^2y}{2} - \frac{1}{3}x^3$$

Hence, the complete solution of the equation is

$$z = \phi_1(y - x) + \phi_2(y - 2x) + \frac{x^2y}{2} - \frac{x^3}{3}.$$

Example 76. *Solve:* $(D^2 - DD')\,z = \sin x \sin 2y$

The auxiliary equation is $m^2 - m = 0$

Solving, $m = 0, 1$

C.F. is $\phi_1(y) + \phi_2(y + x)$.

$$\text{P.I.} = \frac{\sin x \sin 2y}{D^2 - DD'}$$

$$= \frac{D(D + D')\sin x \sin 2y}{D(D - D').D(D + D')}$$

$$= D(D + D')\frac{\sin x \sin 2y}{D^2(D^2 - D'^2)}$$

$$= D(D + D')\frac{\sin x \sin 2y}{(-1)[(-1) - (-4)]}$$

$$= -\frac{1}{3}.D(D + D')\sin x \sin 2y$$

$$= -\frac{1}{3}(D^2 + DD')\sin x \sin 2y$$

$$= -\frac{1}{3}[-\sin x \sin 2y + 2\cos x \cos 2y]$$

Hence, the general solution is

$$z = \phi_1(y) + \phi_2(y + x) - \frac{1}{3}[2\cos x \cos 2y - \sin x \sin 2y]$$

Example 77. *Solve* $(D^2 - 4D'^2)\,z = \cos 2x \cos 3y$

The auxiliary equation is

$$= \frac{1}{D-2D'} [\, y \sin x + 3 \cos x \,]$$

$$= \int [(a-2x) \sin x + 3 \cos x]\, dx \text{ where } a - 2x = y$$

$$= [\, (a-2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x \,] \text{ where } a - 2x = y$$

$$= -y \cos x + \sin x$$

$$\therefore \quad z = \phi_1 (y + 2x) + \phi_2 (y - 3x) + \sin x - y \cos x.$$

Example 81. *Solve* $(D^2 + 2DD' + D'^2)\, z = 2 \cos y - x \sin y.$

Auxiliary equation is $m^2 + 2m + 1 = 0$

$\therefore \quad m = -1, -1$

C.F. is $\phi_1 (y - x) + x\, \phi_2 (y - x)$

$$\text{PI} = \frac{2 \cos y - x \sin y}{(D + D')(D + D')}$$

$$= \frac{1}{D + D^1} \int [2 \cos (a + x) - x \sin (a + x)\,]\, dx \text{ where } y = x + a$$

$$= \frac{1}{D + D'} [2 \sin (a + x) + x \cos (a + x) - \sin (x + a)\,]$$

where $y = x + a$

$$= \frac{1}{D + D'} [\, x \cos (a + x) + \sin (a + x)\,]$$

$$= \frac{1}{D + D'} [\, x \cos y + \sin y\,]$$

$$= \int [x \cos (a + x) + \sin (a + x)]\, dx \text{ where } y = a + x$$

$$= x \sin (a + x) + \cos (a + x) - \cos (a + x) \text{ where } y = a + x$$

$$= x \sin y.$$

The general solution is

$$z = \phi_1 (y - x) + x\, \phi_2 (y - x) + x \sin y.$$

Example 82. *Solve* $(D^2 + 4DD' - 5D'^2)\, z = x + y^2 + \pi$

The auxiliary equation is

$$m^2 + 4m - 5 = 0$$

$$(m + 5)(m - 1) = 0$$

$$\therefore \quad m = 1 \text{ or } -5$$

C.F. is $\phi_1 (y + x) + \phi_2 (y - 5x)$.

$$\text{P.I.} = \frac{x + y^2 + \pi}{D^2 + 4DD' - 5D'^2}$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)\right]^{-1} (x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right) + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)^2 - \cdots\right](x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \frac{4D'}{D} + \frac{5D'^2}{D^2} + \frac{16D'^2}{D^2} + \cdots\right](x + y^2 + \pi)$$

(collect only terms upto D'^2 in the numerator as we have y^2 only in the operand.)

$$= \frac{1}{D^2}\left[x + y^2 + \pi - \frac{4}{D}(2y) + \frac{21}{D^2}(2)\right]$$

$$= \frac{x^3}{6} + (y^2 + \pi)\frac{x^2}{2} - 8\left(\frac{yx^3}{6}\right) + 42 \cdot \frac{x^4}{24}$$

$$= \frac{x^3}{6} + \frac{x^2}{2}(y^2 + \pi) - \frac{4}{3}x^3 y + \frac{7}{4}x^4.$$

The general solution is

$$z = \phi_1 (x + y) + \phi_2 (y - 5x) + \frac{x^3}{6} + \frac{x^2}{2}(y^2 + \pi)$$

$$\frac{-4}{3}x^3 y + \frac{7}{4}x^4$$

Example 83. *Solve* $(D^2 - 3DD' + 2D'^2)\, z = (2 + 4x)\, e^{x + 2y}$

The auxiliary equation is $m^2 - 3m + 2 = 0$

$\therefore\ m = 1, 2$

C.F. is $\phi_1 (y + x) + \phi_2 (y + 2x)$

$$\text{P.I.} = \frac{e^{x+2y} \times (2 + 4x)}{D^2 - 3DD' + 2D'^2}$$

$$= \frac{e^{x+2y} \times (2 + 4x)}{(D - 2D')(D - D')}$$

$$= e^{x+2y} \times \frac{(2 + 4x)}{[(D + 1) - 2(D' + 2)]\,[(D + 1) - (D' + 2)]}$$

$$= e^{x+2y}\, \frac{2 + 4x}{(D - 2D' - 3)(D - D' - 1)}$$

$$= e^{x+2y} \frac{2+4x}{3\left[1 - \frac{D-2D'}{3}\right][1-(D-D')]}$$

$$= \frac{1}{3} e^{x+2y} \left[1 - \frac{D-2D'}{3}\right]^{-1} [1-(D-D')]^{-1} (2+4x)$$

$$= \frac{1}{3} e^{x+2y} \left[1 + \frac{D-2D'}{3} + \cdots\right] [1+(D-D') + \ldots] (2+4x)$$

$$= \frac{1}{3} e^{x+2y} \left[1 + \frac{4D-5D'}{3} + \cdots\right] (2+4x)$$

$$= \frac{1}{3} e^{x+2y} \left[2 + 4x + \frac{1}{3}(16)\right]$$

$$= \frac{1}{3} e^{x+2y} \left[\frac{22}{3} + 4x\right]$$

Hence the general solution is

$$Z = \phi_1 (y+x) + \phi_2 (y+2x) + \frac{2}{9} e^{x+2y} \times (11+6x)$$

Exercises 2 (e)

Solve the equations given below where

$$\frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

1. $(2D^2 + 5DD' + 2D'^2)\, z = 0.$
2. $(D^2 + DD' - 2D'^2)\, z = 0.$
3. $(D^2 + 6DD' + 9D'^2)\, z = 0.$
4. $\frac{\partial^3 z}{\partial x^3} + 3\frac{\partial^3 z}{\partial x^2 \partial y} + 3\frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} = 0$
5. $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial x^2 \partial y} - 8\frac{\partial^3 z}{\partial x \partial y^2} + 12\frac{\partial^3 z}{\partial y^3} = 0.$
6. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$
7. $(2D^2 + DD' - D'^2)\, z = 1.$
8. $(D^3 - 4D^2 D' + 5DD'^2 - 2D'^3)\, z = e^{x+y} + e^{y-2x} + e^{y+2x}.$
9. $4r + 12s + 9t = e^{3x-2y}.$
10. $(D^2 - D'^2)z = e^{x+2y}.$ (S.V.U.
11. $(D^2 - 2DD' + D'^2)\, z = e^{x+2y}.$
12. $(D^2 - 2DD')\, z = e^{2x} + x^2 y.$
13. $(D^3 - 2D^2 D')\, z = 2e^{2x} + 3x^2 y.$
14. $(D^2 + 4DD' - 5D'^2)\, z = x + y^2.$ (S.V.U.
15. $(D^2 + 2DD' + D'^2)z = x^2 y.$ (S.V.U.
16. $(D^2 - DD' - 6D'^2)\, z = xy.$
17. $(D^2 + 3DD' + 2D'^2)z = x + y.$ (M.U. 64

18. $(D^3 - 7DD'^2 - 6D'^3) z = x^2 + xy^2 + y^3 + \cos(x - y)$.

19. $(D^3 + D^2D' - DD'^2 - D'^3) z = e^{2x+y} + \cos(x + y)$. (Anna Ap. 2005)

20. $(D^2 - DD')z = \cos x \cos 2y$. **21.** $(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$.

22. $(D^2 - 6DD' + 9D'^2) z = 6x + 2y$ **23.** $r + s - 6t = \cos(2x + y)$.

24. $r - 2s + t = \sin x$. **25.** $(D^2 - DD') z = \sin x \cos 2y$.

26. $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3) z = \sin(2x+ y)$.

27. $(D^2 - DD' - 2D'^2) z = (y - 1) e^x$ **28.** $(D^2 + 4DD' - 5D'^2)z = x + y$.

29. $(D^2 - 4D'^2)z = \sin(2x + y)$ **30.** $(D^2 - D'^2) z = e^{x+y}$

31. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{y}{x^2 + y^2}$ ***(B.E. 1966)***

32. $(D^3 + D^2D' - DD'^2 - D'^3) z = 3 \sin(x + y)$ ***(MS. 1986 Nov.)***

33. $(D^3 - 7DD'^2 - 6D'^3) z = \cos(x - y) + x^2 + xy^2 + y^3$

34. $(D - D')^2 z = 2e^{x+y} \cos^2\left(\dfrac{x+y}{2}\right)$

2·22. Non-homogenous linear equation

In the P.D. Equation,

$$f(D, D') z = F(x, y) \qquad ...(1)$$

if the polynomial expression $f(D, D')$ in D, D' is not homogenous in D, D', then the equation (1) is called non-homogenous linear equation. Here also, the complete solution = C.F. + P.I. The particular integral is found by the same methods as in the case of homogeneous linear equations. To obtain the C.F., we have to find the solution of

$$f(D, D') z = 0 \qquad ...(2)$$

Assume a trial solution

$$z = ce^{hx+ky} \qquad ...(3)$$

where c, h, k are all arbitrary constants.

Substituting (3) in (2), we get

$$cf(h, k)\, e^{hx+ky} = 0. \qquad ...(4)$$

Hence $\qquad f(h, k) = 0$

If $f(D, D')$ is of degree r in D', then $f(h, k) = 0$ will be of r^{th} degree in k. Solving for k from (4) in terms of h, we get

$$K_s = f_s(h) \text{ where } s = 1, 2, \ldots r.$$

Hence $z = C_s e^{hx+f_s(h)y}$ where $s = 1, 2, \ldots r$ are separate solutions of (2). By giving all possible arbitrary values to C_s and h we get

$z = \Sigma C_1 e^{hx+f_1(h)y}$, $\Sigma C_2 e^{hx+f_2(h)y}$, ... to be the solutions of (2).

Hence the most general solution of (2) is

$z = \Sigma C_1 e^{hx + f_1(h) y} + \Sigma C_2 e^{hx + f_2(h) y} + \ldots + \Sigma C_r e^{hx + f_r(h) y}$.

Note. Suppose k is linear in h, say $k_1 = f_1(h) = \alpha h + \beta$ where α, β are constants.

The corresponding part of the C.F. is

$$\Sigma C_1 e^{hx + (\alpha h + \beta) y}$$
$$= \Sigma C_1 e^{h(x + \alpha y)} \cdot e^{\beta y}$$
$$= e^{\beta y} \Sigma C_1 e^{h(x + \alpha y)}$$
$$= e^{\beta y} \phi(x + \alpha y), \quad \text{where } \phi \text{ is arbitrary function.}$$

Example 84. *Solve* $(D^2 + DD' + D' - 1) z = 5e^x$.

Assume $z = Ce^{hx + ky}$ to be a trial solution.

of $(D^2 + DD' + D' - 1) z = 0$.

Then we get,

$$h^2 + kh + k - 1 = 0$$

i.e., $(h + 1)(h + k - 1) = 0$

i.e., $h = -1$ or $h = 1 - k$.

Then the C.F. of the given equation is

$$\Sigma C_1 e^{-x + ky} + \Sigma C_2 e^{(1-k)x + ky}$$
$$= e^{-x} \Sigma C_1 e^{ky} + e^x \Sigma C_2 e^{k(y - x)}$$
$$= e^{-x} \phi_1(y) + e^x \phi_2(y - x).$$

P.I. $$= \frac{5e^x}{D^2 + DD' + D' - 1}$$
$$= \frac{5e^x}{D^2 - 1}$$
$$= \frac{5e^x}{(D+1)(D+1)}$$
$$= \frac{5}{2} \frac{1}{D - 1} e^x = \frac{5}{2} xe^x.$$

The complete solution is $z = e^{-x} \phi_1(y) + e^x \phi_2(y - x) + \frac{5}{2} xe^x$.

Example 85. *Solve.* $(2D^4 - 3D^2 D' + D'^2) z = e^{2x + y}$

$f(D, D')$ on the L.H.S is not homogeneous.

The equation can be written as

$$(2D^2 - D')(D^2 - D')z = e^{2x+y}$$

Let $z = ce^{hx+ky}$ be a solution of $f(D, D')z =$

Then $(2h^2 - k)(h^2 - k) = 0$

$\therefore k = 2h^2$ or $k = h^2$

Hence, C.F. is $\sum C_1 e^{hx+2h^2y} + \sum C_2 e^{h_1x+h_1^2y}$

$$\text{P.I.} = \frac{e^{2x+y}}{2D^4 - 3D^2D' + D'^2}$$

$$= \frac{e^{2x+y}}{2(2)^4 - 3(2)^2(1) + (1)^2} \text{ replacing } D \text{ by 2 and } D' \text{ by 1}$$

$$= \frac{e^{2x+y}}{21}$$

The complete solution is

$$z = \sum C_1 e^{hx+2h^2y} + \sum C_2 e^{h_1x+h_1^2y} + \frac{1}{21}e^{2x+y}.$$

2.23. None homogeneous linear equation $f(D, D')z = F(x, y)$ where $f(D, D')$ is factorisable into linear factors

Case 1. Consider $(D - mD' - c)z = 0$

Rewriting the equation as $p - mq = cz$

This is Lagrange's equation.

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \qquad ...(1)$$

$dy = -mdx$ gives

$$y + mx = k_1 = \text{constant} \qquad ...(2)$$

Taking $dx = \frac{dz}{cz}$, and integrating,

$$\log z = cx + k_2$$

i.e., $z = e^{cx+k_2} = e^{cx} \cdot e^{k_2} = e^{cx} \cdot k_3$

$$\therefore \quad \frac{z}{e^{cx}} = k_3 = \text{constant} \qquad ...(3)$$

From (2) and (3), the complete solution is

$$\frac{z}{e^{cx}} = f(y + mx)$$

$z = e^{cx} f(y + mx)$, where f is arbitrary

Case 2. If $(D - m_1 D' - c_1)(D - m_2 D' - c_2) \cdots (D - m_n D' - c_n)$ ... the partial differential equation, then the general solution of this equat...

$$z = e^{c_1 x} \phi_1 (y + m_1 x) + e^{c_2 x} \phi_2 (y + m_2 x) + \cdots e^{c_n x} \phi_n (y + m_n x)$$

In case of repeated factors, namely,

Case 3. $(D - mD' - c)^r z = 0$ has the complete solution.

$$z = e^{cx} \phi_1 (y + mx) + x e^{cx} \phi_2 (y + mx) + x^2 e^{cx} \phi_3 (y + mx) + \cdots + x^{r-1} e^{cx} \phi_r (y + mx)$$

Example 86. *Solve:* $(D - D' - 1)(D - D' - 2) z = e^{2x - y}$

Referring to case (2), $m_1 = 1$, $c_1 = 1$, $m_2 = 1$, $c_2 = 2$,

Hence, the C.F. is $z = e^x \phi_1 (y + x) + e^{2x} \phi_2 (y + x)$

$$\text{P.I.} = \frac{e^{2x-y}}{(D - D' - 1)(D - D' - 2)}$$

$$= \frac{e^{2x-y}}{(2 + 1 - 1)(2 + 1 - 2)} \text{ replacing } D \text{ by 2 and } D' \ldots$$

$$= \frac{1}{2} e^{2x-y}$$

Hence, the complete solution is

$$z = e^x \phi_1 (y + x) + e^{2x} \phi_2 (y + x) + \frac{1}{2} e^{2x-y}$$

Example 87. *Solve* $(D + D' - 1)(D + 2D' - 3) z = 4 + 3x + 6y$.

(MI...

Comparing with case 2, $m_1 = -1$, $c_1 = 1$, $m_2 = -2$, $c_2 = 3$

Hence the C.F. is $e^x \phi_1 (y - x) + e^{3x} \phi_2 (y - 2x)$

$$\text{P.I.} = \frac{4 + 3x + 6y}{(D + D' - 1)(D + 2D' - 3)}$$

$$= \frac{4 + 3x + 6y}{3[1 - (D + D')]\left[1 - \frac{D + D'}{3}\right]}$$

$$= \frac{1}{3} [1 - (D + D')]^{-1} \left[1 - \frac{D + D'}{3}\right]^{-1} (4 + 3x + 6y)$$

$$= \frac{1}{3}[\,1 + (D + D') + (D + D')^2 + \cdots\,]$$

$$\times \left[\,1 + \frac{1}{3}(D + 2D') + \frac{1}{9}(D + 2D')^2 + \cdots\,\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[\,1 + \frac{4}{3}D + \frac{5}{3}D' + \cdots\,\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[\,4 + 3x + 6y + \frac{4}{3}(3) + \frac{5}{3}(6)\,\right]$$

$$= x + 2y + 6.$$

The general solution is

$$z = e^x\phi_1(y - x) + e^{3x}\phi_2(y - 2x) + x + 2y + 6.$$

Example 88. *Solve:* $(D^2 - D'^2 - 3D + 3D')z = xy + 7$ *(Anna. Ap 2005)*

The equation can be rewritten as

$$(D - D')(D + D' - 3)\,z = xy + 7$$

Hence, $m_1 = 1, c_1 = 0, m_2 = -1, c_2 = 3$

$\therefore$ C.F. is $e^{ox}\,\phi(y + x) + e^{3x} f(y - x)$

i.e., $\phi(y + x) + e^{3x} f(y - x)$

$$\text{P.I.} = \frac{xy + 7}{(D - D')(D + D' - 3)}$$

$$= \frac{1}{D}\frac{xy + 7}{\left(1 - \dfrac{D'}{D}\right)(-3)\left(1 - \dfrac{D + D'}{3}\right)}$$

$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D + D'}{3}\right)^{-1}(xy + 7)$$

$$= -\frac{1}{3D}\left[\,1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \cdots\,\right]$$

$$\times\left[\,1 + \frac{D + D'}{3} + \frac{1}{9}(D + D')^2 + \cdots\,\right](xy + 7)$$

$$= -\frac{1}{3D}\left[\,1 + \frac{D}{3} + \frac{D'}{3} + \frac{D^2}{9} + \frac{2DD'}{9} + \frac{DD'}{3D} + \cdots\,\right](xy + 7)$$

$$= -\frac{1}{3}\left[\frac{1}{D} + \frac{1}{3} + \frac{2D'}{3D} + \frac{D}{9} + \frac{D'}{3} + \frac{D'}{D^2} + \frac{4DD'}{27}\right](xy + 7)$$

$$= -\frac{1}{3}\left[\frac{x^2}{2}y + 7x + \frac{xy}{3} + \frac{67}{27} + \frac{x^2}{3} + \frac{x}{3} + \frac{x^3}{6} + \frac{y}{9}\right]$$

$$= -\frac{1}{3}\left[\frac{x^2y}{2} + \frac{xy}{3} + 7x + \frac{x^2}{3} + \frac{x}{3} + \frac{y}{9} + \frac{x^3}{6} + y + \frac{67}{27}\right]$$

Hence, the general solution is

$$z = \phi(y + x) + e^{3x} f(y - x) - \frac{1}{3}\left[\frac{x^2y}{2} + \frac{xy}{3} + 7x + \frac{x^2}{3} + \frac{x}{3} + \frac{y}{9} + \frac{x^3}{6} + \ldots\right.$$

Example 89. *Solve:* $(D - D' - 1)(D - D' - 2)\, z = e^{2x + y}$

Here $m_1 = 1, c_1 = 1, c_2 = 2, m_2 = 1$

C.F. is $e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$

$$\text{P.I.} = \frac{e^{2x+y}}{(D - D' - 1)(D - D' - 2)}$$

$$= \frac{1}{(D - D' - 1)} \cdot \frac{e^{2x+y}}{(2 - 1 - 2)}$$

$$= -\frac{e^{2x+y}}{D - D' - 1}$$ replacing D by 2 and D' by 1 in second factor other factor vanishes.

$$= -x\, e^{2x+y}$$

$\therefore$ The general solutions is

$$z = e^x \phi_1(y + x) + e^{2x}\phi_2(y + x) - x\, e^{2x+y}$$

Exercises 2(f)

Solve the equations (1 to 7):

1. $(D^2 + DD' + D' - 1)\, z = e^{-x}$.
2. $(D^2 + 2DD' + D'^2 - 2D - 2D')\, z = e^{x-y} + x^2y$.
3. $D(D + D' - 1)(D + 3D' - 2)\, z = xy + e^{2x+3y}$. (S.V.U. 64 B.E.)
4. $(2DD' + D'^2 - 3D')\, z = 3\cos(3x - 2y)$. (S.V.U. 64 B.E.)
5. $(2D^2 - DD' - D'^2 + 6D + 3D')\, z = 0$.
6. $(D^2 - DD' - 2D'^2 + 2D + 2D')\, z = e^{2x+3y} + \sin(2x + y) + xy$.
7. $(D^2 - DD' + D')\, z = z + e^y + \cos(x + 2y)$.
8. Substituting $u = \log x$, $v = \log y$, solve $(x^2D^2 - y^2D'^2 - yD' + xD)\, z = 0$.
9. $(r - s + p) = x^2 + y^2$ 10. $r - s + q = z + e^v + \cos(x + 2y)$
11. $(D - D' - 1)(D - D' - 2)\, z = e^{-2x+y}$
12. $(D^2 - DD' + D' - 1)\, z = e^{y-x} + \cos(x + 2y)$
13. $(D^2 - DD' + D' - 1)\, z = \cos(x + 2y)$ (B.R. 1995)