Linear Algebra & Its Applications

UE20MA251 Scheme & Solution

ESA - May 2022

1a) $$(A:b)=\left(\begin{array}{ccc|c}1&0&1&4\\2&1&3&5\\-3&-3&a^2-5a&b-8\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&1&4\\0&1&1&-3\\0&-3&a^2-5a+3&b+4\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&1&4\\0&1&1&-3\\0&0&a^2-5a+6&b-\end{array}\right)$$ —(2) —(1

(i) If $a^2-5a+6=0 \Rightarrow a=2,3$ —(1) & $b\neq 5$ —(1) $\Rightarrow$ system has no solution

(ii) If $a\neq 2,3$ & any b $\Rightarrow$ system has a unique solution —(1)

(iii) If $a=2,3$ & $b=5$ $\Rightarrow$ system has infinite no. of solutions —(

1b) $$A=\begin{pmatrix}1&1&1\\2&4&5\\0&4&0\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&1\\0&2&3\\0&4&0\end{pmatrix}\text{—(1)}\longrightarrow\begin{pmatrix}1&1&1\\0&2&3\\0&0&-6\end{pmatrix}=U \text{ —(1)}$$

$$E_{21}=\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\text{—(1)}\quad E_{32}=\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\text{—(1)}\quad E_{21}^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\ \&\ E_{32}^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\text{—(1)}$$

$$E_{21}^{-1}E_{32}^{-1}=L=\begin{pmatrix}1&0&0\\2&1&0\\0&2&1\end{pmatrix}\text{(1)}\qquad \therefore A=LU=\begin{pmatrix}1&0&0\\2&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}1&1&1\\0&2&3\\0&0&-6\end{pmatrix}\text{—(1)}$$

1(c) $$(A:I)=\left(\begin{array}{ccc|ccc}1&0&-2&1&0&0\\-3&1&4&0&1&0\\2&-3&4&0&0&1\end{array}\right)\longrightarrow\left(\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&-2&3&1&0\\0&-3&8&-2&0&1\end{array}\right)\text{—(2)}\longrightarrow\left(\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&-2&3&1&0\\0&0&2&7&3&1\end{array}\right)\text{—(1)}$$

$$\longrightarrow\left(\begin{array}{ccc|ccc}1&0&0&8&3&1\\0&1&0&10&4&1\\0&0&2&7&3&1\end{array}\right)\text{—(2)}\longrightarrow\left(\begin{array}{ccc|ccc}1&0&0&8&3&1\\0&1&0&10&4&1\\0&0&1&7/2&3/2&1/2\end{array}\right)\text{—(1)}\quad \therefore A^{-1}=\begin{pmatrix}8&3&1\\10&4&1\\7/2&3/2&1/2\end{pmatrix}=B$$

$\therefore$ 3rd column of $B=A^{-1}$ is $(1,1,1/2)$ —(1)

2a) $$A=\begin{pmatrix}1&1&2&2\\2&2&4&4\\1&c&2&2\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&2&2\\0&0&0&0\\0&c-1&0&0\end{pmatrix}=U \text{ —(1)}$$

(i) If $c=1$ (1) $\Rightarrow U=\begin{pmatrix}1&1&2&2\\0&0&0&0\\0&0&0&0\end{pmatrix}=R \Rightarrow x+y+2z+2t=0 \Rightarrow \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ (1) $\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}$ (1) $\begin{pmatrix}-2\\0\\0\\1\end{pmatrix}$ (1) are special solutions

(ii) If $c\neq 1$ (1) $\Rightarrow U\Rightarrow\begin{pmatrix}1&0&2&2\\0&1&0&0\\0&0&0&0\end{pmatrix}=R \Rightarrow \left.\begin{array}{l}x+2z+2t=0\\y=0\end{array}\right\}\Rightarrow \begin{pmatrix}-2\\0\\1\\0\end{pmatrix}\begin{pmatrix}-2\\0\\0\\1\end{pmatrix}$ —(1)

2b)

$$A=\begin{pmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\-3&2&1&-8&-6\\2&-3&6&7&9\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\0&2&-8&-5&0\\0&-3&12&5&5\end{pmatrix}\text{—(2)}\longrightarrow\begin{pmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\0&0&0&1&-2\\0&0&0&-4&8\end{pmatrix}\text{—(2)}\longrightarrow$$

$$\begin{pmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\0&0&0&1&-2\\0&0&0&0&0\end{pmatrix}\text{—(1)}$$

Basis of $A=\{v_1, v_2, v_4\}$
They span 3-d space in $\mathbb{R}^4$ } —(1)

2c)

$$A=\begin{pmatrix}2&4&-2&1\\-2&-5&7&3\\3&7&-8&6\end{pmatrix}\longrightarrow\begin{pmatrix}2&4&-2&1\\0&-1&5&4\\0&1&-5&9/2\end{pmatrix}\text{—(2)}\longrightarrow\begin{pmatrix}2&4&-2&1\\0&-1&5&4\\0&0&0&17/2\end{pmatrix}\text{—(1)}$$

$$Ax=0\Rightarrow\left.\begin{array}{r}2x+4y-2z+t=0\\-y+5z+4t=0\\t=0\end{array}\right\}\Rightarrow\begin{array}{l}z=1\\y=5\\x=-9\end{array}\quad\begin{pmatrix}-9\\5\\1\\0\end{pmatrix}\in N(A)\text{ —(1)}$$

$$(A:u)=\begin{pmatrix}2&4&-2&1&3\\-2&-5&7&3&-1\\3&7&-1&6&3\end{pmatrix}\longrightarrow\begin{pmatrix}2&4&-2&1&3\\0&-1&5&4&2\\0&1&-5&9/2&-3/2\end{pmatrix}\longrightarrow\begin{pmatrix}2&4&-2&1&3\\0&-1&5&4&2\\0&0&0&17/2&1/2\end{pmatrix}\text{—(1)}$$

System $Ax=u$ is consistent, hence $u\in C(A)$ —(1)
$u\notin N(A)$ as $N(A)$ is a subspace of $\mathbb{R}^4$ & u is in $\mathbb{R}^3$ —(1)

3a) $w=(-1,1,4,3)$ $v_1=(1,1,0,1)$ $v_2=(0,-1,1,1)$

$$V=\begin{pmatrix}1&0\\1&-1\\0&1\\1&1\end{pmatrix}\quad V^TV=\begin{pmatrix}3&0\\0&3\end{pmatrix}\quad (V^TV)^{-1}=\frac{1}{3}\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{—(1)}\quad V^Tw=\begin{pmatrix}3\\6\end{pmatrix}$$

$$\hat{x}=\begin{pmatrix}1\\2\end{pmatrix}\text{—(1)}\quad p=V\hat{x}=\begin{pmatrix}1\\-1\\2\\3\end{pmatrix}\in C(V)\text{—(1)}\quad V^Tx=0\Rightarrow\left.\begin{array}{r}x+y+t=0\\-y+z+t=0\end{array}\right\}\Rightarrow\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}\begin{pmatrix}-2\\1\\0\\1\end{pmatrix}$$

Basis for $V^{\perp}=\{(-1,1,1,0),(-2,1,0,1)\}$ —(2) $p=v$ —(1)
$\therefore u=w-v=(-1,1,4,3)-(1,-1,2,3)=(-2,2,2,0)\in N(V^T)$ —(1)

3b) $T(x,y,z,t)=(x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$

$$T=\begin{pmatrix}1&-1&1&1\\1&0&2&-1\\1&1&3&-3\end{pmatrix}\text{—(1)}\longrightarrow\begin{pmatrix}1&-1&1&1\\0&1&1&-2\\0&2&2&-4\end{pmatrix}\longrightarrow\begin{pmatrix}1&-1&1&1\\0&1&1&-2\\0&0&0&0\end{pmatrix}\text{—(1)}$$

$$Tx=0\Rightarrow\left.\begin{array}{r}x-y+z+t=0\\y+z-2t=0\end{array}\right\}\Rightarrow\begin{array}{l}z=1\\t=0\\y=-1\\x=-2\end{array}\begin{pmatrix}-2\\-1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\1\\-1\\0\end{pmatrix}\text{—(1)};\ \begin{array}{l}z=0\\t=1\\y=2\\x=1\end{array}\begin{pmatrix}1\\2\\0\\1\end{pmatrix}\text{—(1)}$$

Basis for Range of $T=\{(1,1,1),(-1,0,1)\}$ —(1) Dim=2 } —(1)
Basis for Kernel of $T=\{(2,1,-1,0),(1,2,0,1)\}$ Dim=2

3 c) $b=(1,0,0)$ $a_1=(-1,2,2)$ $a_2=(2,2,-1)$ $a_3=(2,-1,2)$

$$p_1 = \frac{-1}{9}\begin{pmatrix}-1\\2\\2\end{pmatrix} \text{ — (1)} \qquad p_2=\frac{2}{9}\begin{pmatrix}2\\2\\-1\end{pmatrix} \text{ — (1)} \qquad p_3=\frac{2}{9}\begin{pmatrix}2\\-1\\2\end{pmatrix} \text{ — (1)}$$

$$p_1+p_2+p_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} = b \quad \text{as } p \in C(A) \text{ — (1)}$$

$$P_1 = \frac{1}{9}\begin{pmatrix}1&-2&-2\\-2&4&4\\-2&4&4\end{pmatrix} \text{ — (1)} \qquad P_2=\frac{1}{9}\begin{pmatrix}4&4&-2\\4&4&-2\\-2&-2&1\end{pmatrix} \text{ — (1)} \qquad P_1P_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \text{ — (1)}$$

4a). $a_1=(2,0,1)$ $a_2=(4,-1,2)$

$$q_1=\begin{pmatrix}2/\sqrt5\\0\\1/\sqrt5\end{pmatrix} \text{ — (1)} \qquad p_1=\frac{10}{\sqrt5}\begin{pmatrix}2/\sqrt5\\0\\1/\sqrt5\end{pmatrix}=\begin{pmatrix}4\\0\\2\end{pmatrix} \text{ — (1)} \qquad e=a_2-p_1=\begin{pmatrix}4\\-1\\2\end{pmatrix}-\begin{pmatrix}4\\0\\2\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}=q_2 \text{ — (1)}$$

$$\begin{pmatrix}2&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{matrix}2x+z=0\\ y=0\end{matrix} \Rightarrow \begin{matrix}z=1\\ x=-1/2\end{matrix} \quad \begin{pmatrix}-1/2\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\2\end{pmatrix}=a_3 \ \therefore q_3=\begin{pmatrix}-1/\sqrt5\\0\\2/\sqrt5\end{pmatrix} \text{ — (1)}$$

$q_1^Ta_1=\sqrt5$, $q_1^Ta_2=2\sqrt5$, $q_1^Ta_3=0$ — (1)

$q_2^Ta_2=1$, $q_2^Ta_3=0$; $q_3^Ta_3=\sqrt5$ — (1)

$$A=(a_1\ a_2\ a_3)=QR=\begin{pmatrix}2/\sqrt5&0&-1/\sqrt5\\0&-1&0\\1/\sqrt5&0&2/\sqrt5\end{pmatrix}\begin{pmatrix}\sqrt5&2\sqrt5&0\\&1&0\\&&\sqrt5\end{pmatrix} \text{ — (1)}$$

4b)

$$\Lambda=\begin{pmatrix}-2&&\\&3&\\&&6\end{pmatrix} \text{ — (1)} \qquad x_1=\begin{pmatrix}1\\0\\-1\end{pmatrix} \quad x_2=\begin{pmatrix}1\\-1\\1\end{pmatrix} \quad x_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$S=\begin{pmatrix}1&1&1\\0&-1&2\\-1&1&1\end{pmatrix} \text{ — (1)} \qquad S^{-1}=\frac{1}{6}\begin{pmatrix}3&0&-3\\2&-2&2\\1&2&1\end{pmatrix} \text{ — (2)}$$

$$S\Lambda S^{-1}=\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix} \text{ — (2)}$$

<u>Note</u>: $S\Lambda$ — (1) $S\Lambda S^{-1}$ — (1)

4c) Characteristic eq^{ty} of A given $\lambda^3 - 3\lambda^2 = 0$

$\Rightarrow \lambda = 0, 0, 3$ are eigen values — (1)

$\lambda_1 = \lambda_2 = 0 \Rightarrow x + y + z = 0 \Rightarrow x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ — (1) $\quad x_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ — (1) $\quad x_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ — (1)

$\lambda_3 = 3 \Rightarrow \left.\begin{aligned} -2x + y + z &= 0 \\ x - 2y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}\right\} \quad x_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ — (1) $\quad S_1 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ — (1) $\quad S_2 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ — (1)

5a) $x^T A x = 4(x - y/2)^2 + 2(y + z/2)^2 + z^2/2$

$= 4x^2 + 3y^2 + z^2 - 4xy + 2yz$ — (1)

Pivots are $4, 2, 1/2$ — (1)

Multipliers are $-1/2, 1/2$ — (1)

$LDU = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 4 & & \\ & 2 & \\ & & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} = A$ — (1) — (1) — (1)

5b) $x_1 = \begin{pmatrix} 120 \\ 61 \end{pmatrix}$ $x_2 = \begin{pmatrix} 125 \\ 60 \end{pmatrix}$ $x_3 = \begin{pmatrix} 125 \\ 64 \end{pmatrix}$ $x_4 = \begin{pmatrix} 135 \\ 68 \end{pmatrix}$ $x_5 = \begin{pmatrix} 145 \\ 72 \end{pmatrix}$ — (1)

Sample Mean $M = \frac{\sum x_i}{n} = \frac{1}{5}\begin{pmatrix} 650 \\ 325 \end{pmatrix} = \begin{pmatrix} 130 \\ 65 \end{pmatrix}$ — (1)

Mean deviations from mean $\hat{x}_1 = \begin{pmatrix} -10 \\ -4 \end{pmatrix}$ $\hat{x}_2 = \begin{pmatrix} -5 \\ -5 \end{pmatrix}$ $\hat{x}_3 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$ $\hat{x}_4 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ $\hat{x}_5 = \begin{pmatrix} 15 \\ 7 \end{pmatrix}$ — (1)

$B = \begin{pmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{pmatrix}$ — (1)

Covariance Matrix is $S = \frac{1}{n-1} BB^T = \frac{1}{4}\begin{pmatrix} 400 & 190 \\ 190 & 100 \end{pmatrix} = \begin{pmatrix} 100 & 47.5 \\ 47.5 & 25 \end{pmatrix}$ — (1)

5c) $A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ — (1) $\quad \lambda = 3, 1 ; \sigma_1 = \sqrt{3}, \sigma_2 = 1$ — (1)

$\lambda_1 = 3 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 1 \Rightarrow x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ — (1) $\quad v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ — (1) $\quad V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$u_1 = \frac{Av_1}{\sigma_1} = \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$ — (1) $\quad u_2 = \frac{Av_2}{\sigma_2} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ — (1) $\quad a_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\quad u_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$ — (1)

SVD for $A = U\Sigma V^T = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ (1) (1)

X —