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PES University, Bengaluru

(Established under Karnataka Act No. 16 of 2013)

UE17/18/

19MA251

JULY 2021: END SEMESTER ASSESSMENT (ESA) B. TECH IV SEMESTER

UE17/18/19MA251 - Linear Algebra and Its Applications

Time: 3 Hrs	Answer All Questions	Max Marks: 100

Scheme and Solution

1 a) Determine the values of a and b for which the system of equations x+y+az=2b, x+3y+(2+2a)z=7b, 3x+y +(3+3a)z=11b will have (i) trivial solution (ii) unique non-trivial solution (iii) no solution (iv) infinity of solutions.

$$[A:b]=\left[\begin{array}{ccc|c}1&1&a&2b\\1&3&2+2a&7b\\3&1&3+3a&11b\end{array}\right]\xrightarrow[R_3-3R_1]{R_2-R_1}\left[\begin{array}{ccc|c}1&1&a&2b\\0&2&2+a&5b\\0&-2&3&5b\end{array}\right]-(2)\xrightarrow{R_3+R_2}\left[\begin{array}{ccc|c}1&1&a&2b\\0&2&2+a&5b\\0&0&5+a&10b\end{array}\right]-(1)$$

System will have (i) trivial solution if $a\neq-5$ and $b=0$ —(1)

(ii) unique non-trivial solution if $a\neq-5$ and any $b\neq0$ —(1)

(iii) no solution if $a=-5$ and $b\neq0$ —(1)

(iv) infinity of solutions if $a=-5$ and $b=0$ —(1)

b) Factor A=LU and A=LDU for $A=\begin{pmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{pmatrix}$

$$A=\begin{bmatrix}a&a&0\\a&a+b&b\\a&b&b+c\end{bmatrix}\xrightarrow{R_2-R_1}\begin{bmatrix}a&a&0\\0&b&b\\0&b&b+c\end{bmatrix}-(1)\xrightarrow{R_3-R_2}\begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix}=U-(1)$$

$$A=LU=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}-(2)\begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix}$$

$$A=LDU=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a&&\\&b&\\&&c\end{bmatrix}-(1)\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}-(1)$$

c) Use Gauss-Jordan elimination on [A:I] to solve $AA^{-1}=I$

$$\begin{pmatrix}1&1&1\\0&1&2\\1&2&4\end{pmatrix}\begin{pmatrix}x_1&x_2&x_3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$[A:I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&2&0&1&0\\1&2&4&0&0&1\end{array}\right]\xrightarrow{R_3-R_1}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&2&0&1&0\\0&1&3&-1&0&1\end{array}\right]-(1)\xrightarrow{R_3-R_2}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&2&0&1&0\\0&0&1&-1&-1&1\end{array}\right]-(1)$$

$$\xrightarrow[R_2-2R_3]{R_1-R_3}\left[\begin{array}{ccc|ccc}1&1&0&2&1&-1\\0&1&0&2&3&-2\\0&0&1&-1&-1&1\end{array}\right]-(1)\xrightarrow{R_1-R_2}\left[\begin{array}{ccc|ccc}1&0&0&0&-2&1\\0&1&0&2&3&-2\\0&0&1&-1&-1&1\end{array}\right]-(1)$$

$$x_1=\begin{pmatrix}0\\2\\-1\end{pmatrix}-(1)\quad x_2=\begin{pmatrix}-2\\3\\-1\end{pmatrix}-(1)\quad x_3=\begin{pmatrix}1\\-2\\1\end{pmatrix}-(1)$$

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2	a)	Let $A=\begin{pmatrix} 4 & -2 & 0 & 2 \\ -2 & 0 & -5 & -2 \\ 0 & 3 & 1 & 3 \end{pmatrix}$ (a) If column space of A is a subspace of R^k, find k? (b) Find a non-zero vector in nullspace of A. Also find l of R^l such that nullspace of A is a subspace of R^l.

$$A=\begin{bmatrix} 4 & -2 & 0 & 2 \\ -2 & 0 & -5 & -2 \\ 0 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{R_2+\frac{1}{2}R_1} \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & -1 & -5 & -1 \\ 0 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{R_3+3R_2} \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & -1 & -5 & -1 \\ 0 & 0 & -14 & 0 \end{bmatrix} \text{ —(1) —(1)}$$

$C(A)$ is a subspace of $\mathbb{R}^3=\mathbb{R}^k \Rightarrow k=3$ —(1)

$N(A)$ is a subspace of $\mathbb{R}^4=\mathbb{R}^l \Rightarrow l=4$ —(1)

$N(A)$ is $Ax=0 \Rightarrow \left.\begin{matrix} 4x-2y+2t=0 \\ -y-5z-t=0 \\ -14z=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} z=0 \\ t=1 \\ y=-1 \\ x=-1 \end{matrix}\right\}$ —(2) $(-1,-1,0,1)$ is a non-zero vector in $N(A)$. —(1)

	b)	Find a basis and the dimension of the subspaces $V=\{(a,b,c,d)/b-2c+d=0\}$, $W=\{(a,b,c,d)/a=d, b=2c\}$ and $V\cap W$ in R^4.

$V=\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} / b-2c+d=0\right\}=\left\{\begin{pmatrix} a \\ 2c-d \\ c \\ d \end{pmatrix}\right\}$ Basis for $V=\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$ Dim of $V=3$ —(2)

$W=\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} / \begin{matrix} a=d \\ b=2c \end{matrix}\right\}=\left\{\begin{pmatrix} a \\ 2c \\ c \\ a \end{pmatrix}\right\}$ Basis for $W=\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right\}$ Dim of $W=2$ —(2)

$V\cap W=\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} / \begin{matrix} b-2c+d=0 \\ a=d \\ b=2c \end{matrix}\right\}=\left\{\begin{pmatrix} 0 \\ 2c \\ c \\ 0 \end{pmatrix}\right\}$ Basis for $V\cap W=\left\{\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right\}$ Dim of $V\cap W=1$ —(2)

	c)	Find a matrix A that has V as its row space, and a matrix B that has V as its nullspace, if V is the subspace spanned by $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}\right\}$

$$V=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \text{ —(1)} \qquad V^T=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 5 & 0 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \text{ —(1)} \xrightarrow{R_3-4R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ —(1)}$$

$A=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$ is the matrix that has V as its Row space —(2)

$V^Tx=0 \Rightarrow \left.\begin{matrix} x+y=0 \\ y=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=0 \\ y=0 \\ z=1 \end{matrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in N(V^T)$ —(1) $\therefore B=[0\ 0\ 1]$ has V as its nullspace —(1)

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3 a) What matrix P projects every point in R^3 onto the line of intersection of the planes x+y+z=0 and x-z=0? Find the nullspace matrix of P. What do the column space and row space of matrix P represent?

$\left.\begin{array}{l} x+y+z=0 \\ x-z=0 \end{array}\right\} \begin{array}{l} x=z=k \\ y=-2k \end{array} \quad \begin{pmatrix} k \\ -2k \\ k \end{pmatrix} \text{—(1)} \quad \therefore \vec{a}=\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is the vector which lies on the line of intersection of the planes. —(1)

$P=\frac{aa^T}{a^Ta}=\frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}=\frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \text{—(1)} \simeq \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$Px=0 \Rightarrow x-2y+z=0 \Rightarrow \begin{array}{l} y=1 \\ z=0 \end{array} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \begin{array}{l} y=0 \\ z=1 \end{array} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad N(P)=\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is nullspace matrix —(2)

$C(P)=\left\{c_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} / c_1 \in \mathbb{R}\right\}$ is a line in $\mathbb{R}^3$ passing through $(1,-2,1)$ $=C(P^T)$
—(1) (1)

b) Find the matrix of the linear transformation $T: R^3 \to R^2$ defined by T(x,y,z)=(x+y, 2z-x) with respect to
(i) the standard basis (1,0,0),(0,1,0),(0,0,1) and
(ii) the basis (1,0,-1),(1,1,1), (1,0,0)

$T(x,y,z)=(x+y,\ 2z-x)$

(i) $T(1,0,0)=(1,-1)$; $T(0,1,0)=(1,0)$; $T(0,0,1)=(0,2)$ $\therefore T=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ —(2)

(ii) $T(1,0,-1)=(1,-3)$; $T(1,1,1)=(2,1)$; $T(1,0,0)=(1,-1)$ —(3)

$\therefore T=\begin{pmatrix} 1 & 2 & 1 \\ -3 & 1 & -1 \end{pmatrix}$ —(5)

c) Find $\|E\|^2=\|Ax-b\|^2$ and solve the normal equations $A^TA\hat{x}=A^Tb$. Find the solution $\hat{x}$ and the projection $p=A\hat{x}$. (Use Least squares method) Given

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, x=\begin{pmatrix} u \\ v \end{pmatrix}, b=\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$Ax=b \Rightarrow \left.\begin{array}{l} u=1 \\ v=3 \\ u+v=4 \end{array}\right\}$ —(1)

$\|E\|^2=\|Ax-b\|^2=(u-1)^2+(v-3)^2+(u+v-4)^2$
$=2u^2+2v^2-10u-14v+2uv+26$ —(1)

Normal equations are $A^TA\hat{x}=A^Tb \Rightarrow \hat{x}=(A^TA)^{-1}A^Tb$

$A^TA=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ —(1) $A^Tb=\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ —(1) $(A^TA)^{-1}=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\therefore \hat{x}=\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ —(2) $p=A\hat{x}=\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}=b$ —(1)

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4	a)	Let W = { (a, b, b) / a , b are real } and let v = (3, 2, 6). (i) Find an orthonormal basis for W. (ii) Find the projection of v onto W , say v_1 (iii) Decompose v into a sum of two vectors $v_1 + v_2$ where v_2 is projection of v onto $W^{\perp}$.	

$W = \left\{ \begin{pmatrix} a \\ b \\ b \end{pmatrix} \right\} \equiv \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ —(1) $\quad \therefore W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$

(a) Orthonormal basis for $W = \left\{ \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ —(1)

(b) $v_1 = p = W\hat{x}$ where $\hat{x} = (W^T W)^{-1} W^T v$

$W^T W = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad (W^T W)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ —(1) ; $W^T v = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$ —(1) $\therefore \hat{x} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ —(1) $\Rightarrow v_1 = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$ —(1)

$v = v_1 + v_2 \Rightarrow v_2 = v - v_1 = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \in W^{\perp}$ —(1)

$\therefore v = v_1 + v_2 = (3,4,4) + (0,-2,2)$

b) Check if the following symmetric matrix $A = \begin{pmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{pmatrix}$ is orthogonally diagonalizable. If so, orthogonally diagonalize it as $A = S\Lambda S^{-1} = Q\Lambda Q^{-1} = Q\Lambda Q^T$ where Q is an orthogonal matrix.

Characteristic equation for A is $|A - \lambda I| = 0 \Rightarrow \lambda = 1, 7, 13$

$\lambda_1 = 1 \Rightarrow x_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $\lambda_2 = 7 \Rightarrow x_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$, $\lambda_3 = 13 \Rightarrow x_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ —(3)

Eigen vectors are orthogonal hence A is orthogonally diagonalizable

$q_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$ $\quad q_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ $\quad q_3 = \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$ $\quad \therefore Q = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}$ —(1)

$A = Q\Lambda Q^T = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & & \\ & 7 & \\ & & 13 \end{pmatrix}}_{(1)} \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix}$ —(1)

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c) Use the Gram – Schmidt process to find a set of orthonormal vectors from the independent vectors $a_1 = (1, -1, 0)$, $a_2 = (0, 1, -1)$ and $a_3 = (1, 0, -1)$. How many non-zero orthonormal vectors are obtained?

$a_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix} \quad q_1 = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix}$ —(1)

$(q_1^T a_2)q_1 = \frac{-1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\0\end{pmatrix} = \hat{x}q_1$ —(1) ; $e = a_2 - \hat{x}q_1 = \begin{pmatrix}1/2\\1/2\\-1\end{pmatrix}$; $q_2 = \frac{e}{\|e\|} = \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}$ —(1)

$\hat{x}q_1 = (q_1^T a_3)q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}$ —(1) ; $\hat{y}q_2 = (q_2^T a_3)q_2 = \frac{3}{\sqrt{6}}\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\-1\end{pmatrix}$ —(1)

$e = a_3 - (\hat{x}q_1 + \hat{y}q_2) = \begin{pmatrix}1\\0\\-1\end{pmatrix} - \left[\begin{pmatrix}1/2\\-1/2\\0\end{pmatrix} + \begin{pmatrix}1/2\\1/2\\-1\end{pmatrix}\right] = \begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ —(1)

There are only 2 non-zero orthonormal vectors. —(1)

5 a) Write the quadratic form of the matrix $\begin{pmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{pmatrix}$ and express it as a sum of squares using A=LDU factorization.

$A = \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix} \xrightarrow[R_3+\frac{1}{2}R_1]{R_2+\frac{1}{2}R_1} \begin{bmatrix}2 & -1 & -1\\0 & 3/2 & -3/2\\0 & -3/2 & 3/2\end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix}2 & -1 & -1\\0 & 3/2 & -3/2\\0 & 0 & 0\end{bmatrix} = U$ —(1)

$L = \begin{bmatrix}1 & 0 & 0\\-1/2 & 1 & 0\\-1/2 & -1 & 0\end{bmatrix}$ —(1) $\quad L^T x = \begin{pmatrix}1 & -1/2 & -1/2\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x - y/2 - z/2\\y - z\\0\end{pmatrix}$ —(2) $\quad D = \begin{pmatrix}2 & & \\ & 3/2 & \\ & & 0\end{pmatrix}$ —(1)

Quadratic form of A is $x^TAx = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$ —(1)

$x^TAx = x^T(LDU)x = x^T L D L^T x = (L^T x)^T D (L^T x)$

$= 2(x - y/2 - z/2)^2 + 3/2(y - z)^2$ —(1) which is the sum of squares.

b) Test if A^TA is positive definite or positive semi-definite given $A = \begin{pmatrix}1 & 1\\1 & 2\\2 & 1\end{pmatrix}$

$A^TA = \begin{pmatrix}6 & 5\\5 & 6\end{pmatrix}$ —(1) $\longrightarrow \begin{pmatrix}6 & 5\\0 & 11/6\end{pmatrix}$ —(1)

A^TA has positive pivots (i.e. $d_i > 0$) —(1) hence A^TA is positive definite —(1)

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c) Find SVD of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$

$AA^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ — (1) $A^TA = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ — (1) Eigen values are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0$

$\sigma_1 = \sqrt{3}, \sigma_2 = \sqrt{2}, \sigma_3 = 0$

Eigen vectors are $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $x_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore u_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ — (1) $u_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ — (1) $u_3 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$ — (1) $\therefore U = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$

$v_1 = \dfrac{u_1^T A}{\sigma_1} = \dfrac{(1/\sqrt{3} \;\; 1/\sqrt{3} \;\; 1/\sqrt{3})}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ — (1) $\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$ — (1)

$v_2 = \dfrac{u_2^T A}{\sigma_2} = \dfrac{(-1/\sqrt{2} \;\; 0 \;\; 1/\sqrt{2})}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ — (1) $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$A = U \Sigma V^T = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ — (1)

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