Q1) Let S consist of the following vectors $u_1 = (1, 1, 0, -1)$, $u_2 = (1, -2, -1, -1)$ $u_3 = (1, 1, -3, 2)$, $u_4 = (4, -1, 3, 3)$ in R^4. Is S orthogonal, if not make it an orthogonal matrix. Does S form a basis of R^4.

Solution:

$$u_1^T u_2 = \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \\ -1 \end{bmatrix} = 0$$

similarly, $u_1^T u_3 = 0$ and $u_1^T u_4 = 0$

$u_2^T u_3 = 0$ and $u_3^T u_4 = 0$

All vectors are mutually orthogonal, however magnitude of vectors $\neq 1$

⇒ Hence, S is not orthogonal (as not orthonormal, columns)

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}{\sqrt{1^2+1^2+0^2+(-1)^2}} = \begin{bmatrix} 1/\sqrt{3} \\ +1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{bmatrix}$$

Similarly, $q_2 = \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ -2 \\ -1 \\ -1 \end{bmatrix}$; $q_3 = \frac{1}{\sqrt{15}} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$; $q_4 = \frac{1}{\sqrt{35}} \begin{bmatrix} 4 \\ -1 \\ 3 \\ 3 \end{bmatrix}$

$$S' = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{7} & 1/\sqrt{15} & 4/\sqrt{35} \\ 1/\sqrt{3} & -2/\sqrt{7} & 1/\sqrt{15} & -1/\sqrt{35} \\ 0 & -1/\sqrt{7} & -3/\sqrt{15} & 3/\sqrt{35} \\ -1/\sqrt{3} & -1/\sqrt{7} & 2/\sqrt{15} & 3/\sqrt{35} \end{bmatrix} \leftarrow \text{orthogonal matrix}$$

To check if it forms a basis in R^4, we must also ensure they are linearly independent. Make S' into echelon form and check.

$$S': \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{7} & 1/\sqrt{15} & 4/\sqrt{35} \\ 1/\sqrt{3} & -2/\sqrt{7} & 1/\sqrt{15} & -1/\sqrt{35} \\ 0 & -1/\sqrt{7} & -3/\sqrt{15} & 3/\sqrt{35} \\ -1/\sqrt{3} & -1/\sqrt{7} & 2/\sqrt{15} & 3/\sqrt{35} \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{7} & 1/\sqrt{15} & 4/\sqrt{35} \\ 0 & -3/\sqrt{7} & 0 & -5/\sqrt{35} \\ 0 & -1/\sqrt{7} & -3/\sqrt{15} & 3/\sqrt{35} \\ 0 & 0 & 3/\sqrt{15} & 7/\sqrt{35} \end{bmatrix}$$

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$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{7} & 1/\sqrt{15} & 4/\sqrt{35} \\ 0 & -3/\sqrt{7} & 0 & -5/\sqrt{35} \\ 0 & -1/\sqrt{7} & -3/\sqrt{15} & 3/\sqrt{35} \\ 0 & 0 & 3/\sqrt{15} & 7/\sqrt{35} \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3}R_2} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{7} & 1/\sqrt{15} & 4/\sqrt{35} \\ 0 & -3/\sqrt{7} & 0 & -5/\sqrt{35} \\ 0 & 0 & -3/\sqrt{15} & 4/(3\sqrt{35}) \\ 0 & 0 & 3/\sqrt{15} & 7/\sqrt{35} \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + R_3} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{7} & 1/\sqrt{15} & 4/\sqrt{35} \\ 0 & -3/\sqrt{7} & 0 & -5/\sqrt{35} \\ 0 & 0 & -3/\sqrt{15} & 4/(3\sqrt{35}) \\ 0 & 0 & 0 & \frac{25}{3\sqrt{35}} \end{bmatrix}$$

As $r = n = 4$, linearly independent.

S' forms a basis of R^4.

Q2) If $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$

a) Determine (i) Rows of A are orthogonal or not

(ii) Columns of A are orthogonal or not

(iii) A is an orthogonal matrix or not

b) Find a matrix B having orthonormal rows of A.

c) Is B orthogonal?

d) Are columns of B orthogonal.

Solution : a) Let r_1, r_2, r_3 be rows of A.

(i) $r_1^T r_2 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = 0$ $\qquad r_1^T r_3 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} = 0$

$r_2^T r_3 = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} = 0$

Hence,
(i) Rows of A are orthogonal.

Let c_1, c_2, c_3 be columns of A.

(ii) $c_1^T c_2 = \begin{bmatrix} 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = -31 \neq 0$

Hence, columns of A are not orthogonal.

(iii) $\because$ columns of A are not orthogonal, A is not orthogonal.

b) each row of B = $\dfrac{\text{row}_i \text{ of } A}{\|\text{row}_i \text{ of } A\|}$

$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{26} & 3/\sqrt{26} & 4/\sqrt{26} \\ 7/\sqrt{78} & -5/\sqrt{78} & 2/\sqrt{78} \end{bmatrix}$$

c) columns are mutually orthogonal and they have unit magnitude

$\Rightarrow$ B is orthogonal.

d) let c_1, c_2, c_3 be columns of $B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{26} & 3/\sqrt{26} & 4/\sqrt{26} \\ 7/\sqrt{78} & -5/\sqrt{78} & 2/\sqrt{78} \end{bmatrix}$

$$c_1^T c_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 3/\sqrt{26} \\ -5/\sqrt{78} \end{bmatrix} = \frac{1}{3} + \frac{3}{26} - \frac{35}{78} = \underline{0}$$

$$c_1^T c_3 = -\frac{1}{3} + \frac{4}{26} + \frac{14}{78} = \underline{0}$$

$$c_2^T c_3 = -\frac{1}{3} + \frac{12}{26} - \frac{10}{78} = 0$$

and this implies, columns of B are orthogonal.

Q3) If $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & -3/\sqrt{14} \\ - & - & - \end{bmatrix}$ find a third row so that matrix Q^T is orthogonal.

Solution :

Orthogonal matrix ; $QQ^T = I$

Let x, y, z be the missing row vector in Q.

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} \\ x & y & z \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x \\ 1/\sqrt{3} & 2/\sqrt{14} & y \\ 1/\sqrt{3} & -3/\sqrt{14} & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$\begin{bmatrix} 1 & 0 & \frac{x+y+z}{\sqrt{3}} \\ 0 & 1 & \frac{x+2y-3z}{\sqrt{14}} \\ \frac{x+y+z}{\sqrt{3}} & \frac{x+2y+(-3z)}{\sqrt{14}} & x^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \frac{x+y+z}{\sqrt{3}} = 0 \quad ; \quad x+y+z = 0$

$\Rightarrow \frac{x+2y-3z}{\sqrt{14}} = 0 \quad ; \quad x+2y-3z = 0$

$\Rightarrow y = 4z$ and $x = -5z$

$\Rightarrow x^2+y^2+z^2 = 1$

$(-5z)^2 + (4z)^2 + z^2 = 1$

$z^2 = \frac{1}{42} \Rightarrow z = \frac{1}{\sqrt{42}}$

$(x, y, z) = \left(\frac{-5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{1}{\sqrt{42}} \right)$

(Q4) Find an orthonormal set q_1, q_2, q_3 for which q_1 and q_2 span column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$.

Which fundamental subspace contains q_3?
What is the least squares solution of $Ax=b$ if $b=(1,2,7)$? Also find $A=QR$ factorization.

Solution:

$$q_1 = \frac{(1,2,-2)}{\sqrt{1^2+2^2+(-2)^2}} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$B = b - (q_1^T b) q_1$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

q_3 belongs to left null space of A as

$$A^T x = 0$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\downarrow R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 6 \end{bmatrix}$$

↳ Free variable 'z'

let ~~z=t~~

(i) $x + 2y - 2z = 0$

(ii) $-3y + 6z = 0 \Rightarrow y = 2z$

From (i)

$$x + 2(2z) - 2z = 0$$

$$x = -2z$$

$$x^2 + y^2 + z^2 = 1$$

$$4z^2 + 4z^2 + z^2 = 1$$

$$z = \frac{1}{3}$$

$$\Rightarrow q_3 = (x, y, z) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \in N(A^T)$$

For least square solution : $A\hat{x} = b$ where $b = (1, 2, 7)$

$A = QR$

$QR\hat{x} = b$

$R\hat{x} = Q^T b \quad (\because Q^T = Q^{-1})$

$$Q^T b = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$$

putting this in $R\hat{x} = Q^T b$

$$\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$$\left.\begin{matrix} 3x - 3y = -3 \\ 3y = 6 \end{matrix}\right\} \quad (x, y) = (1, 2)$$

Q5) Use the Gram-Schmidt process to find a set of orthonormal vectors q_1, q_2, q_3 from the independent vectors $a_1 = (1,-2,0,1)$

$a_2 = (-1,0,0,-1)$

$a_3 = (1,1,0,0)$

Solution :

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ 0 \\ 1/\sqrt6 \end{bmatrix}$$

$$B = a_2 - (q_1^T a_2)\, q_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt6} & \frac{-2}{\sqrt6} & 0 & \frac{1}{\sqrt6} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ 0 \\ 1/\sqrt6 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \frac{+2}{\sqrt6} \begin{bmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ 0 \\ 1/\sqrt6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$B = \begin{bmatrix} -2/3 \\ -2/3 \\ 0 \\ -2/3 \end{bmatrix} \longrightarrow q_2 = \frac{B}{\|B\|} = \begin{bmatrix} -1/\sqrt3 \\ -1/\sqrt3 \\ 0 \\ -1/\sqrt3 \end{bmatrix}$$

$$C = a_3 - (q_1^T a_3)\, q_1 - (q_2^T a_3)\, q_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt6} & \frac{-2}{\sqrt6} & 0 & \frac{1}{\sqrt6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} q_1 - \begin{bmatrix} \frac{-1}{\sqrt3} & \frac{-1}{\sqrt3} & 0 & \frac{-1}{\sqrt3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} q_2$$

$$C = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt6} \begin{bmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ 0 \\ 1/\sqrt6 \end{bmatrix} + \frac{2}{\sqrt3} \begin{bmatrix} -1/\sqrt3 \\ -1/\sqrt3 \\ 0 \\ -1/\sqrt3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{6} - \frac{2}{3} \\ 1 - \frac{2}{6} - \frac{2}{3} \\ 0 + 0 + 0 \\ 0 + \frac{1}{6} - \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ -1/2 \end{bmatrix}$$

Q6) What multiple of $a_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ should be subtracted from $a_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ to make the result orthogonal to a_1? Factor $A = QR$ with orthonormal vectors in Q.

Solution :

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 4/\sqrt{20} \\ 2/\sqrt{20} \end{bmatrix}$$

$$q_2' = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \left[\frac{4}{\sqrt{20}} \;\; \frac{2}{\sqrt{20}}\right] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 4/\sqrt{20} \\ 2/\sqrt{20} \end{bmatrix}$$

$$q_2' = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \frac{8}{\sqrt{20}} \begin{bmatrix} 4/\sqrt{20} \\ 2/\sqrt{20} \end{bmatrix} = \begin{bmatrix} 2 - \frac{32}{20} \\ 0 - \frac{16}{20} \end{bmatrix}$$

$$q_2' = \begin{bmatrix} -2/5 \\ -8/10 \end{bmatrix} = \begin{bmatrix} +2/5 \\ -4/5 \end{bmatrix}$$

$$q_2 = \frac{q_2'}{\|q_2'\|} = \frac{\begin{bmatrix} +\frac{2}{5} \\ -\frac{4}{5} \end{bmatrix}}{\frac{\sqrt{20}}{5}} = \begin{bmatrix} +2/\sqrt{20} \\ -4/\sqrt{20} \end{bmatrix}$$

$$\begin{bmatrix} +\frac{2}{5} \\ -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - K \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad 2 - 4K = +\frac{2}{5}$$

and $\quad -2K = -\frac{4}{5}$

$$\boxed{K = \frac{2}{5}}$$

$$Q = \begin{bmatrix} 4/\sqrt{20} & +2/\sqrt{20} \\ 2/\sqrt{20} & -4/\sqrt{20} \end{bmatrix}$$

$$A = QR$$

$$R = Q^T A = \begin{bmatrix} 4/\sqrt{20} & 2/\sqrt{20} \\ 2/\sqrt{20} & -4/\sqrt{20} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{20} & 8/\sqrt{20} \\ 0 & 4/\sqrt{20} \end{bmatrix}$$

Q7) Apply Gram-Schmidt process to find a set of orthonormal vectors q_1, q_2, q_3 from the independent vectors $a_1 = (1,1,1)$, $a_2 = (-1,0,-1)$, $a_3 = (-1,2,3)$. Factor $A = QR$ where $A = (a_1, a_2, a_3)$

Solution :

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$B = a_2 - (q_1^T a_2) q_1$$

$$B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right) q_1$$

$$B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + \frac{2}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \frac{\sqrt{3}}{\sqrt{2}} B$$

$$q_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$C = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$C = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right) q_1 - \left(\begin{bmatrix} \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right) q_2$$

$$C = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix} - \begin{bmatrix} -2/6 \\ 4/6 \\ -2/6 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$q_3 = \frac{C}{\|C\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

8. Find the eigen values and the corresponding eigen vectors of:

a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Characteristic equation :

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0.$$

$$2-\lambda(2-\lambda)^2 - 0(0) + 1(-1)(2-\lambda) = 0.$$

$$(2-\lambda)^3 + \lambda - 2 = 0.$$

$$8 - \lambda^3 - 3(4)(\lambda) + 3(2)(\lambda)^2 + \lambda - 2 = 0.$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda - 6 = 0.$$

$$-(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0.$$

$$-(\lambda-1)(\lambda-2)(\lambda-3) = 0.$$

$$\lambda = 1, 2, 3.$$

(i) $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $z = t$.

$\Rightarrow y = 0$

$x + z = 0$

$\Rightarrow x = -t$

eigen vector $\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

Characteristic equation :

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$4-\lambda\left[(5-\lambda)(2-\lambda)-(-2)\right]-1\left[2(2-\lambda)-(-2)\right]-1\left[2-(5-\lambda)\right]$$

(ii) $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_2]{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \cancel{0} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

we have : $x = 0$.
$z = 0$.
$y = k$.

eigen vector : $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(iii) $\lambda = 3$.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_1 \to -R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

let $z = t$.
$\Rightarrow y = 0$.
$x - z = 0$
$x = z = t$

eigen vector : $\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

Characteristic equation :

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(5-\lambda)(2-\lambda) - (-2)\right] - 1\left[2(2-\lambda) - (-2)\right] - 1\left[2 - (5-\lambda)\right].$$

$$(4-\lambda)\left[12 - 7\lambda + \lambda^2\right] - \left[6 - 2\lambda\right] - \left[-3 + \lambda\right] = 0.$$

$$\cancel{48 - 12\lambda}$$

$$-\lambda^3 + 7\lambda^2 - 12\lambda + 4\lambda^2 - 28\lambda + 48 - 6 + 2\lambda + 3 - \lambda = 0.$$

$$-\lambda^3 + 11\lambda^2 - 39\lambda + 45 = 0.$$

$$-(\lambda - 5)(\lambda^2 - 6\lambda + 9) = 0.$$

$$-(\lambda - 5)(\lambda - 3)^2 = 0.$$

$$\lambda = 3, 3, 5$$

(i) $\lambda = 3$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let $y = k \quad z = t.$

$$x + y - z = 0.$$

$$x = z - y = t - k.$$

e. vector : $\begin{bmatrix} t-k \\ k \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$

(ii) $\lambda = 5$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & +1 & -3 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{R_1 \to -R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & +2 & -4 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & +2 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $z = t$.

$2y - 4z = 0$.

$y = +2z = +2t$.

$x - y + z = 0$.

$x = y - z = +t$

eigen vector : $\begin{bmatrix} +t \\ +2t \\ t \end{bmatrix} = \begin{bmatrix} +1 \\ +2 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

Characteristic equation :

$$\begin{vmatrix} 0-\lambda & 0 & 3 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0.$$

$-\lambda[-\lambda(3-\lambda)+1] - 0[1(3-\lambda)] + 3\cancel{[1-(-\lambda)0]}\ \cancel{3[1(3-\lambda)]}$.

$-\lambda[\lambda^2 - 3\lambda + 1] + 3 \mp \lambda = 0$.

$-\lambda^3 + 3\lambda^2 \mp \lambda + 3 = 0$.

$= -(\lambda - 3)(\lambda^2 + 1)$

$= -(\lambda - 3)(\lambda - i)(\lambda + i) = 0$.

$\lambda = 3, i, -i$

(i) $\lambda = 3$.

$$\begin{bmatrix} -3 & 0 & 3 \\ 1 & -3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -1 \\ -3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & -3 & -1 \\ 0 & -9 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2/9} \begin{bmatrix} 1 & -3 & -1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let $z = t$.

$\Rightarrow y = 0$.

$x - 3y - z = 0$

$x = z = t$

eigen vector: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(ii) $\lambda = -i$

$$\begin{bmatrix} i & 0 & 3 \\ 1 & i & -1 \\ 0 & 1 & 3+i \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1/i} \begin{bmatrix} i & 0 & 3 \\ 0 & i & \cancel{1}+3i \\ 0 & 1 & 3+i \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2/i} \begin{bmatrix} i & 0 \\ 0 & i & 3i \\ 0 & 0 \end{bmatrix}$$

We have : $z = k$.

$\cancel{3}iy + (3i-1)z = 0$.

$\cancel{i\{3y}$ $\cancel{3}iy = -(3i-1)z$.

$y = \dfrac{-(3i-1)z}{\cancel{3}i}$

$= (-3-i)k$.

$iz + 3z = 0$

$x = \dfrac{-3z}{i} = \dfrac{-3z\,i}{-1}$

$= 3ik$.

eigen vector: $\begin{bmatrix} 3i \\ -3-i \\ 1 \end{bmatrix}$.

(iii) $\lambda = i$

$$\begin{bmatrix} -i & 0 & 3 \\ 1 & -i & -1 \\ 0 & 1 & 3-i \end{bmatrix} \xrightarrow[\cancel{R_1/2}]{R_1 \to R_1/(-i)} \begin{bmatrix} 1 & 0 & 3i \\ 1 & -1 & -1 \\ 0 & 1 & 3-i \end{bmatrix} \xrightarrow[R_2 \to R_2/(-i)]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 3i \\ 0 & 1 & 3-i \\ 0 & 1 & 3-i \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 3i \\ 0 & 1 & 3-i \\ 0 & 0 & 0 \end{bmatrix}.$$

$z = t$

$\Rightarrow y = -(3-i)z$.

$= (-3+i)z$.

$x = -3iz$

eigen vector: $\begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix}$.

9. Verify Cayley Hamilton Theorem and hence find its inverse given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$|A - \lambda I| = 0$.

$\lambda^3 - \text{trace}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0$.

$\text{trace}(A) = 1 - 1 + 1 = 1$

$M_{11} = -1 - 4 = -5$

$M_{22} = 1 - 9 = -8$.

$M_{33} = -1 - 4 = -5$

$\det(A) = 1(-1-4) - 2(2-12) + 3(2+3)$.

$= -5 + 20 + 15 = 30$.

$\lambda^3 - \lambda^2 + (-18)\lambda - 30 = 0$.

$\lambda^3 - \lambda^2 - 18\lambda - 30 = 0$

$A^3 - A^2 - 18A - 30 = 0$.

Multiplying by A^{-1}

$A^2 - A - 18I - 30A^{-1} = 0$.

$30A^{-1} = A^2 - A - 18I$.

$A^{-1} = \cancel{\frac{1}{30}}(A^2 - A - 18I)\frac{1}{30}$

$$= \frac{1}{30}\left(\begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right).$$

$$= \frac{1}{30}\begin{bmatrix} -5 & 1 & 11 \\ 10 & -8 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

10. The eigen vectors of a 3×3 matrix A corresponding to the eigen values $-2, 3, 6$ are $(1,0,-1)$, $(1,-1,1)$, and $(1,2,1)$. Find the matrix A.

$$A = S \Lambda S^{-1}$$

$$\Lambda = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \qquad S^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 2 \\ 3 & 0 & -3 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & -1/3 & 1/3 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & -2 \\ 12 & -3 & 0 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & -1/3 & 1/3 \\ 1/2 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

11. If 3 and 6 are 2 eigen values of A matrix, find the eigen values for A^2, A^{-1} and $A + 5I$

Eigen values for $A^2 = \lambda^2 = 3^2, 6^2 = 9, 36$.

Eigen values for $A^{-1} = 1/\lambda = 1/3, 1/6$

Eigen values for $A + 5I$:

$A \rightarrow A + kI$ then $\lambda_k = k + \lambda$.

$\lambda = 5+3, 5+6 = 8, 11$.

12 Factor $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ into $S \Lambda S^{-1}$ and hence compute A^{55}

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

$(4-\lambda)(2-\lambda) - 3 = 0.$

$8 - 6\lambda + \lambda^2 - 3 = 0.$ $\qquad \lambda = 1, 5$

$\lambda^2 - 6\lambda + 5 = 0$

$\lambda^2 - \lambda - 5\lambda + 5 = 0.$

$\lambda(\lambda-1) - 5(\lambda-1) = 0$

$(\lambda-5)(\lambda-1) = 0$

(i) $\lambda = 1$

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_1 \to R_1/3} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let $y = k.$

$x + y = 0.$

$x = -y = -k$

eigen vector : $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(ii) $\lambda = 5$

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = k.$

$-x + 3y = 0.$

$x = 3y = 3k.$

eigen vector : $\begin{bmatrix} 3 \\ 1 \end{bmatrix}.$

$S = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ $\qquad S^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$

$$A = S \Lambda S^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$A^{55} = S \Lambda^{55} S^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5^{55} & 0 \\ 0 & 1^{55} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 \cdot 5^{55} & -1 \\ 5^{55} & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 \cdot 5^{55} + 1 & 3 \cdot 5^{55} - 3 \\ 5^{55} - 1 & 5^{55} + 3 \end{bmatrix}.$$

13. Find the matrices S and S^{-1} to diagonalize $A = \begin{bmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}$ What are limits A^k and $S \Lambda^k S^{-1}$ as $k \to \infty$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0.6-\lambda & 0.9 \\ 0.1 & 0.6-\lambda \end{vmatrix} = 0$$

$(0.6-\lambda)^2 - 0.09 = 0.$

$\lambda^2 - \cancel{0}1.2\lambda + 0.36 - 0.09 = 0.$

$\lambda^2 - 1.2\lambda + 0.27 = 0.$ $\quad \frac{100}{100}\left(\lambda - \frac{3}{10}\right)\left(\lambda - \frac{9}{10}\right) = 0.$

$\lambda = \frac{3}{10}, \frac{9}{10}$

a) $\lambda = 3/10.$

$$\begin{bmatrix} 0.3 & 0.9 \\ 0.1 & 0.3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1/3} \begin{bmatrix} 0.3 & 0.9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let $y = k.$

$3x + 9y = 0.$ $\quad$ eigen vector : $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$x = -3y = -3k$

b) $\lambda = 9/10$.

$$\begin{bmatrix} -3/10 & 9/10 \\ 1/10 & -3/10 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1/3} \begin{bmatrix} -0.3 & 0.9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let $y = k$.

$-x + 3y = 0$

$x = 3y = 3k$

eigen vector : $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix} \qquad S^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 9/10 & 0 \\ 0 & 3/10 \end{bmatrix}$$

$$A = S\Lambda S^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}$$

$$A^k = S\Lambda^k S^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.9^k & 0 \\ 0 & 0.3^k \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}$$

as $k \to \infty$ $\quad 0.9^k \to 0 \quad 0.3^k \to 0$

and $A^k \to 0$.

$$A^k \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

14. If λ is an eigen value of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and μ is an eigent value of $B = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, then is $\lambda\mu$ is an eigen value of AB?

No $\lambda\mu$ does not have to be an eigen value of AB.

ex: $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, both having eigen values ± 1. So if $\lambda = +1$, $\mu = -1$

$\lambda\mu = -1$

But $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, having eigen values : $\frac{-\sqrt{5}+3}{2}$, $\frac{\sqrt{5}+3}{2}$.

and -1 is not an eigen value

15 Check ~~if~~ if the following matrices are orthogonally diagonaliza
If not, then orthogonally diagonalize them as $A = S\Lambda S^{-1} = Q\Lambda$
$= Q\Lambda Q^T$ where Q is the orthogonal matrix.

(i) $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

Characteristic equation:

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 6-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(6-\lambda)(2-\lambda)+2\right] - 1\left[2(2-\lambda)+2\right] - 1\left[2-(6-\lambda)\right] = 0.$$
$$(4-\lambda)\left[\lambda^2-8\lambda+14\right] - \left[4-2\lambda\right] - \left[-4+\lambda\right] = 0.$$
$$-\lambda^3 + 8\lambda^2 \mp 14\lambda + 4\lambda^2 - 32\lambda + 56 - 4 + 2\lambda + 4 - \lambda = 0.$$
$$-\lambda^3 + 12\lambda^2 - 45\lambda + 54 = 0$$
$$-(\lambda-6)(\lambda^2-6\lambda+9) = 0.$$
$$-(\lambda-6)(\lambda-3)^2 = 0$$
$$\lambda = 6, 3$$

i) $\lambda = 6$

$$\begin{bmatrix} -2 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2+R_1 \\ R_3 \to R_3 + R_1/2}]{} \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 3/2 & -9/2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{3R_2}{2}} \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

let $z = t$
$y = 3z = 3t$
$-2x = z - y = t - 3t$
$= -2t$
$x = +t$

eigen vector: $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

ii) $\lambda = 3$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = t$.

$y = 0$.

$x + y = z$

$x = z = t$

eigen vectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

As there are only 2 eigen values, and hence 2 eigen vectors, it is not diagonalizable.

Since algebraic multiplicity of 3 $(=2)$ is not equal to the geometric multiplicity of 3 $(=1)$

It is not diagonalizable.

[Algebraic Multiplicity : # repetitions of e. value λ
Geometric Multiplicity : # e. vectors obtained for e. value λ]

15) Check if the following matrices are orthogonally diagonalizable. If not, then orthogonally diagonalize them as $A = S\Lambda S^{-1} = Q\Lambda Q^{-1} = Q\Lambda Q^T$ where Q is an orthogonal matrix.

(ii) $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

Solution:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-5-\lambda)(2-\lambda)+6\right] + 1\left[7(2-\lambda)-6\right] + 1\left[-42-6(-5-\lambda)\right] = 0$$

$$(3-\lambda)\left[\lambda^2+3\lambda-4\right] + (8-7\lambda) + (-12+6\lambda) = 0$$

$$-\lambda^3 + 12\lambda - 16 = 0$$

$$\lambda_1 = -4$$

$$\lambda_2 = 2, \lambda_3 = 2$$

$\lambda = -4$:

$$\begin{bmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - \frac{6}{7}R_1$

$$\begin{bmatrix} 7 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & \frac{-36}{7} & \frac{36}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y = z$ & $7x - y + z = 0$ eigenvector $\lambda = -4$: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$x = 0$

For $\lambda = 2$

$$|A - 2I|x = \begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$6x - 6y = 0$

$x = y = k$

and $x - y + z = 0$

$\Rightarrow z = 0$

eigenvector for $\lambda = 2$; $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\cancel{S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & \end{bmatrix}}$$

Algebraic multiplicity of 2 = 2.

Geometric multiplicity of 2 = 1

Since the algebraic multiplicity is not equal to the geometric multiplicity of $\lambda = 2$, the matrix is not diagonalizable