.Unit 4: Orthogonalization , Eigen Values and Eigen Vectors

Orthogonalization -The Gram-Schmidt Orthogonalization, Introduction to Eigenvalues and Eigenvectors, Properties of Eigenvalues and Eigenvectors, Cayley-Hamilton theorem(Statement only), Symmetric Matrices, Diagonalization of a Matrix.

Class No.	Portions to be covered
42	Orthogonalization - Orthogonal Matrices, Properties
43	Rectangular Matrices with orthonormal columns
44	The Gram- Schmidt Orthogonalization
45	A = QR Factorization
46	**Matlab Class Number 7- Projection by Least Squares**
47	Introduction to Eigen values and Eigenvectors
48	Properties of eigenvalues and eigenvectors, Cayley-Hamilton theorem
49	Problems on Properties of Eigen values and Eigen vectors
50-51	**Matlab Class Number 8 & 9- The Gram- Schmidt process, A=QR Factorization**
52	Symmetric Matrices, Diagonalization of a Matrix
53-54	Problems on Diagonalization of a Matrix
55	Powers and Products of Matrices
56	**Applications**

Classwork problems:

1.	Let S consist of the following vectors $u_1=(1, 1, 0, -1)$, $u_2=(1, -2, -1, -1)$, $u_3=(1, 1, -3, 2)$, $u_4=(4, -1, 3, 3)$ in R^4. Is S orthogonal, if not make it an orthogonal matrix. Does S form a basis of R^4.
2.	If $A=\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ (a)Determine whether or not (i)Rows of A are orthogonal (ii)Columns of A are orthogonal (iii) A is an orthogonal matrix. (b)Find a matrix B having orthonormal rows of A. (c)Is B an orthogonal matrix? (d)Are the columns of B orthogonal? **Answer:** A and columns of A are not orthogonal.
3.	If $Q=\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & -3/\sqrt{14} \\ -- & -- & -- \end{pmatrix}$ find a third row so that the matrix Q^T is orthogonal. **Answer :** $\left(-5/\sqrt{42}, 4/\sqrt{42}, 1/\sqrt{42}\right)$

4.	Find an orthonormal set q_1, q_2, q_3 for which q_1 and q_2 span the column space of $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$ Which fundamental subspace contains q_3 ? What is the least squares solution of Ax = b if b = (1, 2, 7) ? Also find A = QR factorization. **Answer :** $q_2 = \left(2/3, 1/3, 2/3\right)$ $q_3 = \left(-2/3, 2/3, 1/3\right)$. x = (1, 2)
5.	Use the Gram – Schmidt process to find a set of orthonormal vectors q_1, q_2, q_3 from the independent vectors a_1 = (1, -2, 0, 1), a_2 =(-1, 0, 0, -1) a_3 = (1, 1, 0, 0). **Answer :** $\left(1/\sqrt{6}, -2/\sqrt{6}, 0, 1/\sqrt{6}\right)$, $\left(-1/\sqrt{3}, -1/\sqrt{3}, 0, -1/\sqrt{3}\right)$, $\left(1/\sqrt{2}, 0, 0, -1/\sqrt{2}\right)$
6.	What multiple of $a_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ should be subtracted from $a_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to make the result orthogonal to a_1? Factor A = QR with orthonormal vectors in Q. Answer : 1/2
7.	Apply Gram – Schmidt process to find a set of orthonormal vectors q_1, q_2, q_3 from the independent vectors a_1 = (1, 1, 1), a_2 =(-1, 0, -1) a_3 = (-1, 2, 3).Factor A=QR where A=(a_1, a_2, a_3). **Answer :** $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$ $\left(-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\right)$, $\left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right)$
8.	Find the eigenvalues and the corresponding eigenvectors of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ Answer :Eigen values are (i) 1, 2, 3 (ii) 3, 3, 5 (iii) 3, i, -i
9.	Verify Cayley Hamilton theorem for A and hence find its inverse given A= $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix}$
10.	The eigen vectors of a 3x3 matrix A corresponding to the eigen values -2. 3. 6 are (1, 0, -1), (1, -1, 1) and (1, 2, 1). Find the matrix A.
11.	If 3 and 6 are two eigen values of find eigen values of A^2, A^{-1} and A + 5I. (Use Property)
12.	Factor $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ into $S\Delta S^{-1}$ and hence compute A^{55}. Answer: Eigen values are 1, 5 and Eigen vectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

13.	Find the matrices S and S^{-1} to diagonalize $A = \begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix}$ What are limits of A^k and $S\Lambda^k S^{-1}$ as k → ∞ ? Answer : eigenvalues of A are 0.9 and 0.3 with eigenvectors (3 , 1) , (-3 , 1).
14.	If λ is an eigen value of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and μ is an eigen value of $B = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$, then is λμ an eigen value of the product AB.
15.	Check if the following matrices are orthogonally diagonalizable. If not, then orthogonally diagonalize them as $A = S\Delta S^{-1} = Q\Delta Q^{-1} = Q\Delta Q^T$ where Q is an orthogonal matrix. (i) A= $\begin{pmatrix} 4 & 1 & -1 \\ 2 & 6 & -2 \\ 1 & 1 & 2 \end{pmatrix}$ (ii) A= $\begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}$