ENGINEERING MATHEMATICS-IV
LINEAR ALGEBRA
MATLAB

Department of Science and Humanities

QR Decomposition with Gram-Schmidt

The QR decomposition (also called the QR factorization) of a matrix is a decomposition of the matrix into an orthogonal matrix and a triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

A = QR,

where Q is an orthogonal matrix (i.e. $Q^T Q = I$) and R is an upper triangular matrix. If A is nonsingular, then this factorization is unique. There are several methods for actually computing the QR decomposition. One of such method is the Gram-Schmidt process.

QR Decomposition with Gram-Schmidt

Gram-Schmidt process:

Consider the GramSchmidt procedure, with the vectors to be considered in the process as columns of the matrix A. That is,

$$A = \left[\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n \right].$$

Then,

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1, \qquad \mathbf{e}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \\
\mathbf{u}_2 &= \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1, \qquad \mathbf{e}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||}. \\
\mathbf{u}_{k+1} &= \mathbf{a}_{k+1} - (\mathbf{a}_{k+1} \cdot \mathbf{e}_1)\mathbf{e}_1 - \cdots - (\mathbf{a}_{k+1} \cdot \mathbf{e}_k)\mathbf{e}_k, \qquad \mathbf{e}_{k+1} = \frac{\mathbf{u}_{k+1}}{||\mathbf{u}_{k+1}||}.
\end{aligned}$$

Note that $|| \cdot ||$ is the L_2 norm.

QR Decomposition with Gram-Schmidt

QR factorization:

The resulting QR factorization is

$$A = \left[\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n \right] = \left[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n \right] \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{e}_1 & \mathbf{a}_2 \cdot \mathbf{e}_1 & \cdots & \mathbf{a}_n \cdot \mathbf{e}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{e}_2 & \cdots & \mathbf{a}_n \cdot \mathbf{e}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{a}_n \cdot \mathbf{e}_n \end{bmatrix} = QR.$$

Note that once we find $\mathbf{e}_1, \ldots, \mathbf{e}_n$, it is not hard to write the QR factorization.

QR Decomposition with Gram-Schmidt

Find QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

Code:

```
A=[1,1,0;1,0,1;0,1,1]
[Q,R]=qr(A)
```

Or

```
[Q,R]=qr([1,1,0;1,0,1;0,1,1])
```

QR Decomposition with Gram-Schmidt

Output:

Q =

$$\begin{matrix} -0.7071 & 0.4082 & -0.5774 \\ -0.7071 & -0.4082 & 0.5774 \\ 0 & 0.8165 & 0.5774 \end{matrix}$$

R =

$$\begin{matrix} -1.4142 & -0.7071 & -0.7071 \\ 0 & 1.2247 & 0.4082 \\ 0 & 0 & 1.1547 \end{matrix}$$

QR Decomposition with Gram-Schmidt

QR Factorization of Pascal Matrix

```
>>A = sym(pascal(3))
```

Output

A =

[1, 1, 1]
[1, 2, 3]
[1, 3, 6]

```
[Q,R] = qr(A)
```

QR Decomposition with Gram-Schmidt

Output

Q =

[3^(1/2)/3, -2^(1/2)/2, 6^(1/2)/6]
[3^(1/2)/3, 0, -6^(1/2)/3]
[3^(1/2)/3, 2^(1/2)/2, 6^(1/2)/6]

R =

[3^(1/2), 2*3^(1/2), (10*3^(1/2))/3]
[0, 2^(1/2), (5*2^(1/2))/2]
[0, 0, 6^(1/2)/6]

QR Decomposition with Gram-Schmidt

```
isAlways(A == Q*R)
```

Output

```
ans =
      3×3 logical array
      1     1     1
      1     1     1
      1     1     1
```

QR Decomposition with Gram-Schmidt

QR Decomposition to Solve Matrix Equation of the form Ax=b

```
A = sym(invhilb(5))
b = sym([1:5]‘)
```

Output:

A =

[25, -300, 1050, -1400, 630]
[-300, 4800, -18900, 26880, -12600]
[1050, -18900, 79380, -117600, 56700]
[-1400, 26880, -117600, 179200, -88200]
[630, -12600, 56700, -88200, 44100]

b =

1; 2; 3; 4 ;5

QR Decomposition with Gram-Schmidt

```
[C,R] = qr(A,b);
X = R\C
```

Output:

```
X =
          5
          71/20
          197/70
         657/280
          1271/630
```

QR Decomposition with Gram-Schmidt

```
isAlways(A*X == b)
```

Output:

```
ans =
        5×1 logical array
        1
        1
        1
        1
        1
```

QR Decomposition with Gram-Schmidt

Example problems:

Find QR Decomposition (Gram Schmidt Method) ...

Using MATLAB command

Example 1

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

QR Decomposition with Gram-Schmidt

Example problems:

Example 2:

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

QR Decomposition with Gram-Schmidt

Example problems:

Example 3:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 9 \\ 5 & 7 & 3 \end{bmatrix}$$

THANK YOU