

Unit – 5

Singular Value Decomposition

1. Consider an $n \times n$ symmetric matrix, A then any two eigen vectors from different eigen spaces are orthogonal
2. A symmetric matrix, A is orthogonally diagonalizable, (i.e) the eigen vector matrix, S is a orthogonal matrix then $S^{-1} = S^T$.

Then Diagonal matrix, $\Lambda = S^{-1} A S$

and Factorization of matrix , $A = S \Lambda S^{-1}$.

Then it is possible to find powers of A as $A^n = S \Lambda^n S^{-1}$

3. The set of all eigen values of A is called Spectrum of A

- 4 .Quadratic Forms, $Q(x)$: $Q(x)$ has only quadratic terms in the form purely x^2 , or Cross products, xy , xz or zx

A $Q(x)$ is represented as $Q(x) = x^T A x$, where A is called matrix of the quadratic form and A is a symmetric matrix

$Q(x) = x^2 + y^2 + z^2$, then A is exactly diagonal matrix

5. Classification of QF :

- a) Positive definite : if $Q(x) > 0$, Eigen values of $A > 0$
- b) Negative definite : if $Q(x) < 0$, Eigen values of $A < 0$
- c) Indefinite : If $Q(x)$ assumes both positive & negative & Eigen values of A are both positive & negative
- d) Positive Semi definite : if $Q(x) \geq 0$
- e) Negative Semi definite : if $Q(x) \leq 0$

Tests for positive definiteness :

- a) $x^T A x > 0$
- b) All the eigen values of $A > 0$
- c) All the upper sub matrices of A must have positive determinant
- d) All the pivots > 0

Tests for positive semi definite:

a) $x^T A x \geq 0$

b) All the eigen values of $A \geq 0$

c) All the upper sub matrices of A must have non negative determinant

d) All the pivots ≥ 0

Singular value decomposition, SVD:

SVD is used to factorize $m \times n$ matrix, A by using the fact that $A^T A$ is symmetric matrix

Factorization of $A = U \Sigma V^T$ where,

$U = m \times m$ orthogonal matrix

$V = n \times n$ orthogonal matrix

$\Sigma =$ diagonal matrix

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

Singular values of A , $\sigma =$ square root of $\lambda =$ length of the vector AX

1. Compute quadratic form for $A = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}$

Sol: $x = (x, y)$

$$QF = x^T A x = 3x^2 - 4xy + 7y^2$$

2. Check the positive definiteness of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

Sol: Eigen values = 1, 1, 7 > 0

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 2/3 & 2/5 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5/3 & 0 \\ 0 & 0 & 7/5 \end{pmatrix}$$

Pivots = 3, 5/3, 7/5 > 0

$$|A| = 7$$

A is positive definite

3. Find SVD of $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$

$$\text{Sol: } A^T A = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

Eigen values of $A = 0, 18$

Factorization of $A = U \Sigma V^T$ where,

$U = m \times m$ orthogonal matrix

$V = n \times n$ orthogonal matrix

$\Sigma = \text{diagonal matrix}$

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

Singular values = $\sqrt{0}, \sqrt{18} = 0, 3\sqrt{2}$

$$v^T = (1/\sqrt{2}) \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\varepsilon = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 1/\sqrt{45} \end{pmatrix}$$