UNIT IV- Orthogonalization, Eigenvalues and Eigenvectors:

NOTES:

Contents:

- Orthogonal Bases,
- The Gram- Schmidt Orthogonalization,
- Introduction to Eigenvalues and Eigenvectors,
- Properties of Eigenvalues and Eigenvectors,
- Symmetric Matrices, Diagonalization of a Matrix.
- Powers and Products of the matrices.

Orthogonalization and the Gram- Schmidt Process

Definition **:**

In an **orthogonal basis**, every vector is perpendicular to every other vector.

The coordinate axes are mutually orthogonal.
Mutually perpendicular unit vectors are called **Orthonormal** vectors.

Examples :

For the vector space R^2,
1. The set (2, 0) , (0, 2) is an orthogonal basis.
2. The set (1, -2) , (2 , 1) is an orthogonal basis.
3. The set (1, 0), (0, 1) is an orthonormal basis.

Definition :

- A matrix with Orthonormal columns will be called Q.
- A square matrix with Orthonormal columns is called an **Orthogonal matrix** denoted by Q.

Examples:
Rotation matrix, any permutation matrix.

Note : The size of Q has to be square or tall.

Properties of Q:

- If Q (square or rectangular) has orthonormal columns, then $Q^T Q = I$.
- An orthogonal matrix is a square matrix with orthonormal columns. Then Q^T is Q^{-1}.
- If Q is rectangular then Q^T is left inverse of Q.
- Multiplication by any Q preserves length. The norms of x and Qx are equal.
- Also, Q preserves inner products and angles, since

$$(Q x)^T (Qy) = x^TQ^TQy = x^Ty.$$

- Since Q preserves lengths and inner products it preserves angle between two vectors.
- If $q_1, q_2 \ldots. q_n$ are orthonormal basis of R^n then any vector b from R^n can be expressed as
  $b = x_1q_1 + x_2q_2 + \ldots.. + x_n q_n$**-------(1)**

  Multiply both sides by q_1^T.

  Then $x_1 = q_1^T b$.

  Similarly, $x_2 = q_2^T b, \ldots\ldots x_n = q_n^T b$.

Hence, $b = (q_1^T b)q_1 + (q_2^T b)q_2 + \ldots.. + (q_n^T b)q_n$

= sum of one dimensional projections on to q_i's.

The matrix form of equation (1) is $Qx = b$ and the solution of this system of equations is

$$x = Q^{-1}b = Q^Tb$$

Note: The rows of a square matrix are orthonormal whenever the columns are.

Example:

Orthonormal columns
Orthonormal rows

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}.$$

Rectangular Matrices with Orthonormal Columns

- If Q has orthonormal columns, the least-squares problem becomes easy.
- $Q^T Q = Q^T b$ are the normal equations for the best solution -in which $Q^T Q = I$.
- p=Q, the projection of b is $(q_1^T b)q_1 + \ldots. + (q_n^T b)q_n$
- $p = QQ^Tb$, the projection matrix is $P = Q Q^T$.

1) The vectors $q_1 = (1, 0, 0)$, $q_2 = (0, 3/5, 4/5)$ and $q_3 = (0, 4/5, -3/5)$ form an orthonormal basis for R^3. Express the vector $v = (7, -5, 10)$ as a linear combination of the q's.

Solution: $\begin{pmatrix} 7 \\ -5 \\ 10 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix}$

Solving the equations $\begin{matrix} 7 = c_1 + 0 + 0 \cdot \\ -5 = Oc_1 + \frac{3}{5}c_2 + \frac{4}{5}c_3 \\ 10 = 0c_1 + \frac{4}{5}c_2 - \frac{3}{5}c_3 \end{matrix}$, we get $c_1 = 7, c_2 - 5, c_3 = -10$

Therefore $v = 7\,q_1 + 5\,q_2 - 10\,q_3$

2) Find a third column so that the matrix $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & --- \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & --- \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & --- \end{bmatrix}$ is orthogonal.

Solution: Let $Q = \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & x \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & y \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & z \end{bmatrix} \end{array}$ i. e, third columns elements are (x, y, z)

Since Q has be orthogonal then

$$a^T c = 0 \Rightarrow \left(\frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x + y + z = 0 \cdot \cdots (1)$$

$$b^T c = 0 \Rightarrow \left(\frac{1}{\sqrt{14}} \frac{2}{\sqrt{14}} \frac{-3}{\sqrt{14}}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x + 2y - 3z = 0 \cdot \cdots (2)$$

Solving equation (1) and equation (2), by taking $z = 1$ (because z is the free variable) we get, $x = -5 \; and \; y = 4$.

Therefore $c = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} OR \quad c = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$.

Let $q_3 = \pm \begin{pmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{pmatrix}$ (After Normalization i.e.,$\sqrt{(-5)^2 + (4)^2 + 1^2} = \sqrt{42}$)

Hence $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{42}} \end{bmatrix}$ OR $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \frac{+5}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{42}} \end{bmatrix}$

3) Let $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$, $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$. Verify that

(i) $Q^T Q = I$ (ii) $\|Qx\| = \|x\|, \|Qy\| = \|y\|$ (iii) $(Qx)^T (Qy) = x^T y$

Solution: To Prove that $Q^T Q = I$:

$$Q^T Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(ii) To prove that $\|Qx\| = \|x\|, \|Qy\| = \|y\|$

Consider $Qx = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$,

$$\|Qx\| = \sqrt{3^8 + (-1)^2 + 1^2} = \sqrt{11}$$

$$\|x\| = \sqrt{\left(\sqrt{2}\right)^2 + (3)^2} = \sqrt{11}$$

Hence $\|Qx\| = \|x\|$

Consider $Qy = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} . = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$,

$$\|Qy\| = \sqrt{1^8 + (-7)^2 + 2^2} = \sqrt{54}$$

$$\|x\| = \sqrt{\left(-3\sqrt{2}\right)^2 + (6)^2} = \sqrt{54}$$

Hence $\|Qy\| = \|y\|$

(iii) $(Qx)^T (Qy) = x^T y$

$Qx = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $Qy = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} . = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$,

$$(Qx)^T = [3,-1,1]\begin{bmatrix}1\\-7\\2\end{bmatrix} = 12$$

4) If W is a subspace spanned by the orthogonal vectors $(\,2,5,-1\,)$ and $(\,-2\,,1,1\,)$ find the point in W that is closest to $(\,1,2,3\,)$.

Solution: Let $\omega = \left\{C_1\begin{pmatrix}2\\5\\-1\end{pmatrix} + C_2\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\}$ and $A = \begin{bmatrix}2 & -2\\-5 & 1\\1 & 1\end{bmatrix}$

Normalizing A we get $A = \begin{bmatrix}\frac{2}{\sqrt{2^2+5^2+(-1)^2}} = \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{[-2]^2+1^2+1^2}} = \frac{-2}{\sqrt{6}}\\ \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}}\end{bmatrix} = Q$

We have $P = A\hat{x}$, $\hat{x} = \frac{Q^Tb}{Q^TQ}$ and $b = (1,2,3)$

$$\hat{x} = \frac{Q^Tb}{Q^TQ} = \frac{\begin{bmatrix}\frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{-1}{\sqrt{30}}\\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}(1,2,3)}{\begin{bmatrix}\frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{-1}{\sqrt{30}}\\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{6}}\\ \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}}\end{bmatrix}} = \begin{bmatrix}\frac{9}{\sqrt{30}}\\ \frac{3}{\sqrt{6}}\end{bmatrix}$$

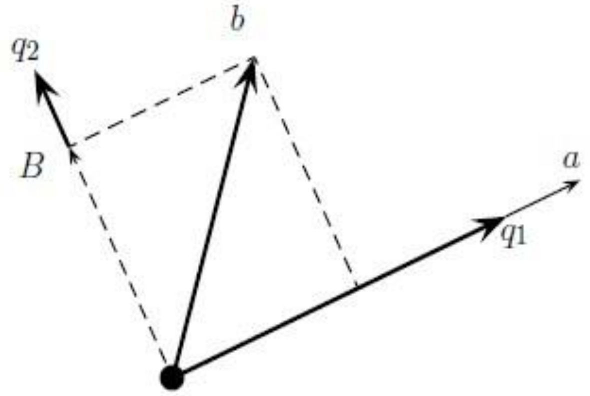
$$P = A\hat{x} = \begin{bmatrix}\frac{2}{\sqrt{30}} & = \frac{-2}{\sqrt{6}}\\ \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{9}{\sqrt{30}}\\ \frac{3}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}\frac{-2}{5}\\ 2\\ \frac{1}{5}\end{bmatrix}$$

<u>The Gram-Schmidt Process</u>

- This is a process of converting a set of linearly independent vectors into a set of orthonormal vectors. The number of vectors given is always equal to the number of vectors produced.
- Consider any 3 independent vectors a, b, c. Then the first orthonormal q_1= a/norm(a).
- If ‘b’ is perpendicular to the vector ‘a’ then q_2=b/norm(b) otherwise we subtract the component of b in q_1 direction to get

$$B=b-(q_1^T b)q_1$$

- q_2=B/norm(B).



- The third vector c is not in the plane of a and b (or q_1 and q_2). If ‘c’ is perpendicular to the plane spanned by the vectors a and b then
- q_3=c/norm(c)

  Otherwise

  C=$c-({q_1}^T c)q_1-({q_2}^T c)q_2$

  q_3=C/norm(C).

This is the one idea of the whole Gram-Schmidt process, to subtract from every new vector its components in the directions that are already settled. That idea is used over and over again. When there is a fourth vector, we subtract away its components in the directions of q_1, q_2, q_3.

A = QR factorization

Where Q-Orthonormal columns and R is the upper triangular matrix(square).

The Factorization A=QR

- We started with a matrix A, whose columns were a, b, c.
- We ended with a matrix Q, whose columns are q_1, q_2, q_3.
- A and Q are of order m by n.

To find a relation between A and Q we express a, b, c as linear combinations of q_1, q_2, q_3

If suppose A is the 3x3 matrix then $R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}$

How is this A = QR factorization useful?

It simplifies the least squares problem Ax = b

$$A^T A = R^T Q^T Q R = R^T R.$$

The fundamental equation $A^T A\hat{x} = A^T b$ simplifies to a triangular system:

$$R^T R\hat{x} = R^T Q^T b \qquad \text{or} \qquad R\hat{x} = Q^T b.$$

5) What multiple of $a_1 = (1, 1)$ should be subtracted from $a_2 = (4, 0)$ to make the result orthogonal to a_1? Factorize $A = [a_1, a_2]$ into QR.
Solution: Form the given data, we can consider, $a_2 - C_1 a_1$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} - C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 - C_1 \\ -C_1 \end{pmatrix}$$

Now this must be orthogonal to a_1, therefore $(1, 1) \begin{pmatrix} 4 - C_1 \\ -C_1 \end{pmatrix} = 0$

$$\Rightarrow 4 - C_1 - C_1 = 0$$

$$\Rightarrow C_1 = 2$$

To find $A = QR, A = [a_1, a_2] =.\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$

$$q_1 = \frac{a_1}{||a_1||} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e = a_2 - (q_1^T a_2) q_1$$

$$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (1,1) \begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$q_2 = \frac{e}{||e||} = \frac{\begin{pmatrix} 2 \\ -2 \end{pmatrix}}{\sqrt{8}}$$

$$Q = [q_1 q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \end{bmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{8}} & \frac{4}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{8}} & \frac{4}{\sqrt{2}} \end{bmatrix}$$

6) Find an orthonormal set q_1, q_2, q_3 for which q_1 and q_2 span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. which fundamental subspace contains q_3? What is the least squares solution of $Ax = b$ if $b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$?

Solution: (i) Let $a = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ $and\ b_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{9}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{pmatrix}$$

$$e = b - (q_1^T b) q_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|}, \|e\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{7}$$

$$q_2 = \frac{e}{\|e\|} = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$e_1 = b_1 - (q_1^T b_1) q_1 - (q_2^T b_1) q_2$$

$$= \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{pmatrix} - \left(\begin{bmatrix} \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \right) \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$q_3 = \frac{e_1}{\|e_1\|} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\|e_1\| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{9}$=3

$$\begin{array}{cccc} & q_1 & q_2 & q_3 \\ \text{Therefore } Q = & \left(\begin{array}{c} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{array} \right. & \begin{array}{c} \frac{2}{\sqrt{7}} \\ \frac{1}{\sqrt{7}} \\ \frac{2}{\sqrt{7}} \end{array} & \left. \begin{array}{c} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{array} \right) \end{array}$$

$(ii) q_3\ \epsilon N(A^T),\ Since\ N(A^T)\ \perp C(A)\ \ and\ q_3 \perp C(A)$

(iii) We have A=QR, $R = Q^T A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$

Also we have $R\hat{x} = Q^T b_1$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

This is in matrix form $A\hat{x} = b$, let $\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

$$\Rightarrow 3x_1 - 3x_2 = -3$$

$$0x_1 + 3x_2 = 6$$

Solving $x_1 = 1, x_2 = 2,$

Therefore, the least squares solution $\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

7) Use the Gram – Schmidt process to find a set of orthonormal vectors from the independent vectors $a_1 = (1,1,1)$, $a_2 = (0,1,1)$ and $a_3 = (0,0,1)$. Also find the $A = QR$ factorization where $A = [\, a_1\, a_2\, a_3]$.

Solution: Let $a = \begin{pmatrix}1\\1\\1\end{pmatrix}, b = \begin{pmatrix}0\\1\\1\end{pmatrix}$ $and\ c = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$$

$$e = b - (q_1^T b)q_1 = \begin{pmatrix}0\\1\\1\end{pmatrix} - \left(\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\right)\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix} = \begin{pmatrix}\frac{-2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|}, \|e\| = \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}$$

$$q_2 = \frac{e}{\|e\|} = \frac{3}{\sqrt{6}}\begin{pmatrix}\frac{-2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix} = \begin{pmatrix}\frac{-2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$$

$$e_1 = c - (q_1^T c)q_1 - (q_2^T c)q_2$$

$$= \begin{pmatrix}0\\0\\1\end{pmatrix} - \left(\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} - \left(\begin{bmatrix}\frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\begin{pmatrix}\frac{-2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$$

$$= \begin{pmatrix}0\\\frac{-1}{2}\\\frac{1}{2}\end{pmatrix}$$

$$q_3 = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \|e_1\| = \sqrt{(0)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\text{Therefore } Q = \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{array}$$

TO find R:

We have A=QR,

$$R = Q^T A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$OR\ R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Eigen values and Eigen vectors

Definition :

Let A be a square matrix of order n. If there exists a real or complex number λ and a non zero vector x such that Ax = b = λ x then x is called the Eigenvector of A and λ is its corresponding Eigen value.

Procedure to find eigenvalues and eigenvectors of A

1. Find the characteristic equation, that is determinant of A - λ I = 0.
2. This gives an equation of degree n. It starts with $(-\lambda)^n$.
3. Find the roots of this equation. The n roots are the eigenvalues of A.

4. For each eigenvalue λ, solve the equation (A - λ I)x = 0. Since the determinant of A - λ I is zero, there are solutions other than x = 0. Those are the eigenvectors.

Note :<u>Corresponding to 'n' distinct Eigen values we get 'n' independent Eigen vectors. But when 2 or more eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to repeated roots.</u>

<u>Properties of Eigen Values and Eigen vectors</u>

- If λ is an Eigen value of A with x as the corresponding Eigen vector then λ^2 is an Eigen value of A^2 with the same Eigen vector x.
- For a given Eigen vector x, there corresponds only one Eigen value λ.
- For a given Eigen value there corresponds infinitely many Eigen vectors.
- $\lambda = 0$ is an Eigen value of A , if and only if A is singular i.e det(A)=0.
- If λ is an Eigen value of A with x as the Eigen vector then $1/\lambda$ is an Eigen value of A^{-1} provided A^{-1} exists.
- A and its transpose A^T have the same Eigen values.
- The Eigen values of a diagonal matrix are just the diagonal elements of the matrix.
- The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.
- The product of the Eigen values of a matrix A is equal to its determinant.

<u>The Cayley-Hamilton Theorem</u>

Statement:

Every square matrix satisfies its own characteristic equation .
Example: Let the characteristic equation is:

$\det(A-tI) = t^2-4t+2 = 0$ and hence it can be verified that

$A^2 - 4A + 2I = 0$

Note: If a matrix is invertible then we can find its inverse using

Cayley- Hamilton Theorem.

Example: For the Matrix above Cayley Hamilton theorem, $A^2-4A+2I= 0$.

Therefore $A^{-1} = (4I-A)/2$.

8) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, shift A to $A - 7\,I$ what are the eigenvalues and eigenvectors and how are they related to those of A ?

Solution: To find the eigen values of A:

Consider the characteristic equation $|A - \lambda I| = 0$

i.e., $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (4-\lambda)(2-\lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda = 1{,}5$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3x+3y=0, $x + y = 0 \Rightarrow x = -y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = k_1$ be the free variable, then $x = -k_1$,

Therefore, the eigen vector is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} k_1$$

Case 2: When $\lambda = 5$.

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-x+3y=0, $x - 3y = 0 \Rightarrow x = 3y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = k_2$ be the free variable, then $x = 3k_2$,

Therefore, the eigen vector is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} k_2$$

Trace 0f A=sum of the elements of principal diagonal of A=4+2=6.

Sum of the eigen values=1+5=6.

Trace of A=Sum of the eigen values.

Determinant of A=$\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$ and Product of the eigen values =1x5=5.

Therefore, Determinant of A=Product of the eigen values.

Now we shift A to A-7I i.e., $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & -5 \end{bmatrix} = A - 7I = B(say)$

The eigen values of B:

Consider the characteristic equation $|B - \lambda I| = 0$

i.e., $|B - \lambda I| = \begin{vmatrix} -3-\lambda & 3 \\ 1 & -5-\lambda \end{vmatrix} = 0$

$$\Rightarrow (-3-\lambda)(-5-\lambda) - 3 = 0$$

$$\lambda = -2, -6$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} -3-\lambda & 3 \\ 1 & -5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = -2$.

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-x+3y=0, $x - 3y = 0 \Rightarrow x = 3y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = k_1$ be the free variable, then $x = 3k_1$,

Therefore, the eigen vector is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} k_1$$

Case 2: When $\lambda = -6$.

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3x+3y=0, $x + y = 0 \Rightarrow x = -y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = k_2$ be the free variable, then $x = -k_2$,

Therefore, the eigen vector is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} k_2$$

Therefore, we can say that Eigen vectors are same for A and B=A- λI and eigen values are been shifted $\lambda\ to\ \lambda - 7$

9) Find the eigenvalues of A , A^2 , A^{-1} and $A + 4I$ if

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

i.e., $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(2-\lambda) - 2 = 0$$

$$\lambda = 1,3$$

Eigen values of A are 1,3.

We know that the eigen values of A^{-1} is $\frac{1}{\lambda}$, if eigen values of A is λ.

Hence the eigen values of $A^{-1} = 1,\ \frac{1}{3}$

The eigen values of A^2 is $1^2, 3^2 = 1{,}9$.

Eigen values of $A + 4I$: Here we need to shift $\lambda\ to\ \lambda + 4$

Eigen value of $A + 4I$: $1 + 4 = 4, 3 + 4 = 7$.

10) Write three different 2 x 2 matrices for which the eigenvalues are 4, 5 and determinant is 20.

Solution: $\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}$(For upper and lower triangular matrices the principal diagonal given the eigen values).

11) Find the eigenvalues and the corresponding eigenvectors of

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & -3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\lambda^3 - (sum\ of\ element\ of\ diagonal\ of\ A)\lambda^2 + (sum\ of\ the\ mionors\ of\ A)\lambda - Determinant\ of\ A$$

Solving we get $\lambda = -2, 3, 6$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Case 1: When $\lambda = -2$.

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$3x + y + 3z = 0$$

$$x + 7y + z = 0$$

$$3x + y + 3z = 0$$

Taking any two different equations say

$$3x + y + 3z = 0$$

$$x + 7y + z = 0$$

($ax + by + cz = 0$ and $dx + ey + fz = 0$, then $\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$)

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20}$$

$$X_1(Eigen\ vector) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Case 2: When $\lambda = 3$.

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + 3z = 0$$

$$x + 2y + z = 0$$

$$3x + y - 2z = 0$$

Taking any two different equations say

$$-2x + y + 3z = 0$$

$$x + 2y + z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x}{-5} = -\frac{y}{-5} = \frac{z}{-5}$$

$$X_2(Eigen\ vector) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 6$.

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-5x + y + 3z = 0$$

$$x - y + z = 0$$

$$3x + y - 5z = 0$$

Taking any two different equations say

$$-5x + y + 3z = 0$$

$$x - y + z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{4} = -\frac{y}{-8} = \frac{z}{4}$$

$$X_3(Eigen\ vector) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

12) Use the Cayley – Hamilton’s theorem to find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

Put $\lambda = A,$ we get $A^3 - A\lambda^2 - A + 9 = 0$

Multiply by $A^{-1}, we\ get\ A^2 - 3A - I + 9A^{-1} = 0,$

$(AA^{-1} = I\ and\ IA^{-1} = A^{-1})$

$$A^{-1} = \frac{1}{9}[3A + I - A^2]$$

$$\frac{1}{9}\left[3\begin{bmatrix}1 & 0 & 3\\2 & 1 & -1\\1 & -1 & 1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0 & 3\\2 & 1 & -1\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\2 & 1 & -1\\1 & -1 & 1\end{bmatrix}\right]$$

$$= \begin{bmatrix}0 & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{9} & \frac{-7}{9}\\ \frac{1}{3} & \frac{-1}{9} & \frac{-1}{9}\end{bmatrix}$$

Symmetric Matrices:

A symmetric matrix is a matrix A such that $A^T = A$, such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries in pairs- on opposite sides of the main diagonal

For eg: $A = \begin{bmatrix}6 & -2 & -1\\-2 & 6 & -1\\-1 & -1 & 5\end{bmatrix}$

NOTE: (i) If A is symmetric, then any two eigen vectors from different eigen values are orthogonal

(ii) An n x n matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

Diagonalization of a Matrix

Key Idea : The eigenvectors diagonalize a matrix.

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then S^{-1} AS is a ***diagonal matrix*** . The eigenvalues of A are on the diagonal of Λ.

The matrix S is called an ***eigenvector matrix***.

- If the matrix A has no repeated eigenvalues then its n eigenvectors are automatically independent .
- Therefore any matrix with distinct Eigen values can be diagonalized.
- The diagonalizing matrix S is not unique. An eigenvector x can be multiplied by a constant, and remains an eigenvector.
- Diagonalizability of A depends on enough eigenvectors.
- Invertibility of A depends on non zero eigen values.

Powers and Products

If A is diagonalizable then A = S Λ S^{-1}. So A^K = $S\Lambda S^{-1}$

13) If possible, diagonalize the $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 17\lambda^2 - 90\lambda + 144 = 0$$

Solving we get $\lambda = 8, 3, 6$

Consider $(A - \lambda I)X = 0$

$$\begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Case 1: When $\lambda = 8$.

The eigen vector $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Case 2: When $\lambda = 6$.

The eigen vector $v_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

Case 1: When $\lambda = 3$.

The eigen vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Normalizing we get

$$u_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix},$$

Let $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Then $A = PDP^{-1}$

Since P is the square $P^{-1} = P^T$

i.e., $A = PDP^T$

14) Factor $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and hence compute A^{100}.

Solution: To find the eigen values of A:

Consider the characteristic equation $|A - \lambda I| = 0$

i.e., $|A - \lambda I| = \begin{vmatrix} 4 - \lambda & 3 \\ 1 & 2 - \lambda \end{vmatrix} = 0$

$$\Rightarrow (4-\lambda)(2-\lambda)-3=0$$

$$\lambda^2-6\lambda+5=0$$

$$\lambda=1,5$$

Consider $(A-\lambda I)X=0$

$$\begin{bmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda=1$.

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3x+3y=0, $x+y=0 \Rightarrow x=-y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y=1$ be the free variable, then $x=-1$,

Therefore, the eigen vector is

$$X_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda=5$.

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-x+3y=0, $x-3y=0 \Rightarrow x=3y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y=1$ be the free variable, then $x=3$,

Therefore, the eigen vector is

$$X_2=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Therefore $S=[X_2,X_1]=\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, $\wedge=\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, $S^{-1}=\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

We have $A=S\wedge S^{-1}$ and $A^{100}=S\wedge^{100}S^{-1}$

$$=\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}^{100}\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5^{100} & 0 \\ 0 & 1^{100} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

15) Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots, a matrix R such that $R^2 = A$. How many such square root matrices are there?

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

i.e., $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$$\Rightarrow (5-\lambda)(5-\lambda) - 16 = 0$$

$$(\lambda - 9)(\lambda - 1) = 0$$

$$\lambda = 1,9$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4x+4y=0, $4x + 4y = 0 \Rightarrow x = -y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = 1$ be the free variable, then $x = -1$,

Therefore, the eigen vector is

$$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 9$.

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-4x+4y=0, $4x - 4y = 0 \Rightarrow x = y$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = 1$ be the free variable, then $x = 1$,

Therefore, the eigen vector is

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore $S = [X_1, X_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$,

$$\wedge = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \qquad S^{-1} = \frac{-1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

We have $A = S \wedge S^{-1}$ and

$$A^{\frac{1}{2}} = S \wedge^{\frac{1}{2}} S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{\frac{1}{2}} & 0 \\ 0 & 9^{\frac{1}{2}} \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

We have 4 square roots in $\wedge^{\frac{1}{2}}$, i.e., $\sqrt{1} = \pm 1, \sqrt{9} = \pm 9$. We get different eigen values for different values.

One square root is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. There are 4 of them.

16) Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write two different diagonalizing matrices S.

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda^3 - (sum\ of\ element\ of\ diagonal\ of\ A)\lambda^2 + (sum\ of\ the\ mionors\ of\ A)\lambda - Determinant\ of\ A$$

Solving we get $\lambda = 0, 0, 3$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Case 1: When $\lambda = 0$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y + z = 0$$

$$x + y + z = 0$$

$$x + y + z = 0$$

All the equations are same, 3(unknowns)-1(equation)=2 free variable.

Let $y\ and\ z\ be\ the\ free\ varaible\ with\ y = 1$ and $z = 0$, then $x = -1$,

If we take $y = 0\ and\ z = 1, then\ x = -1$.

Therefore, the eigen vector are $X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Similarly, if we take $y = 1$ and $z = 1, then\ x = -2$, then eigen vector we can take $X_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\ and\ y = 0\ and\ z = 1\ we\ have\ X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case 2: $\lambda = 3$.

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + z = 0$$

$$x - 2y + z = 0$$

$$x + y - 2z = 0$$

Taking any two different equations say

$$-2x + y + z = 0$$

$$x - 2y + z = 0$$

($ax + by + cz = 0$ and $dx + ey + fz = 0$, then $\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$)

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{-y}{-3} = \frac{z}{3}$$

$$X_3(Eigen\ vector) = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Therefore $S = \begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$ and $S = \begin{bmatrix}-2 & -1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}$ can be diagonalizing matrices

17) Find the matrices $\wedge$ and S to diagonalize $A = \begin{bmatrix}0.6 & 0\cdot 4\\ 0.4 & 0.6\end{bmatrix}$. What are limits of $\wedge^k$ and $S \wedge^k S^{-1}$ as $k \to \infty$.

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

i.e., $|A - \lambda I| = \begin{vmatrix}0.6-\lambda & 0.4\\ 0.4 & 0.6-\lambda\end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - 1.2\lambda + 0.2 = 0$$

$$\lambda = 1, 0.2$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix}0.6-\lambda & 0.4\\ 0.4 & 0.6-\lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{bmatrix}-0.4 & 0.4\\ 0.4 & -0.4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$-0.4x + 0.4y = 0,\ 0.4x - 0.4y = 0 \Rightarrow x = y$$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = 1$ be the free variable, then $x1$,

Therefore, the eigen vector is

$$X_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

Case 2: When $\lambda = 0.2$.

$$\begin{bmatrix}0.4 & 0.4\\ 0.4 & 0.4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$0.4x + 0.4y = 0,\ 0.4x + 0.4y = 0 \Rightarrow x = -y$$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = 1$ be the free variable, then $x = -1$,

Therefore, the eigen vector is

$$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Therefore $S = [X_1, X_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

$$\wedge = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We have $A = S \wedge S^{-1}$ and

$$A^k = S \wedge^k S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 0.2^k \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + 0.2^k & 1 - 0.2^k \\ 1 - 0.2^k & 1 + 0.2^k \end{bmatrix},$$

$\wedge^k \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $and\ A^k = S \wedge^k S^{-1} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ as $k \rightarrow \infty$