LA Unit -2
Assignment Solutions

1a) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 3 \\ 1 & -1 & 7 \\ -2 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 2R_1}]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -3 & 6 \\ 0 & 6 & 0 \end{bmatrix} \xrightarrow[R_4 \to R_4 + 6R_2]{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

rank $= n = 3$ $\Rightarrow$ Independent

1b) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & +1 & -2 \\ 0 & 3 & -6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

rank $= 2 < n$ $\Rightarrow$ Dependent

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$c_1 + c_2 + 5c_3 = 0$ and $c_2 - (2c_3) = 0$

$\boxed{c_2 = 2c_3}$

$\boxed{c_1 = -7c_3}$

1c) $p_1(t)(t^2 - t + 5) + p_2(t)(2t^2 - 3t) + p_3(t)(-t^2 + 2t + 5) = 0$

$t^2(p_1(t) + 2p_2(t) - p_3(t)) + t(-p_1(t) - 3p_2(t) + 2p_3(t)) + 1(5p_1(t) + 5p_3(t)) = 0$

$p_1(t) + 2p_2(t) - p_3(t) = 0$

$-p_1(t) - 3p_2(t) + 2p_3(t) = 0$

$5p_1(t) + 0p_2(t) + 5p_3(t) = 0$

$Ax = b$ form:

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ 5 & 0 & 5 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ 5 & 0 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -10 & 10 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 10R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, dependent (rank $= 2 < n$)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-p_2(t) + p_3(t) = 0 \implies p_2(t) = p_3(t)$

$p_1(t) + 2p_2(t) - p_3(t) = 0 \implies -p_3(t) = p_1(t)$

1d) $c_1\begin{bmatrix}1 & -5\\ -4 & 2\end{bmatrix} + c_2\begin{bmatrix}1 & 1\\ -1 & 5\end{bmatrix} + c_3\begin{bmatrix}-2 & 3\\ 5 & 2\end{bmatrix} + c_4\begin{bmatrix}1 & -2\\ -5 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$\begin{bmatrix}c_1 & -5c_1\\ -4c_1 & 2c_1\end{bmatrix} + \begin{bmatrix}c_2 & c_2\\ -c_2 & 5c_2\end{bmatrix} + \begin{bmatrix}-2c_3 & 3c_3\\ 5c_3 & 2c_3\end{bmatrix} + \begin{bmatrix}c_4 & -2c_4\\ -5c_4 & 3c_4\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}c_1 + c_2 - 2c_3 + c_4 & -5c_1 + c_2 + 3c_3 - 2c_4\\ -4c_1 - c_2 + 5c_3 - 5c_4 & 2c_1 + 5c_2 + 2c_3 + 3c_4\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, the equations are :-

$$c_1 + c_2 + c_4 - 2c_3 = 0$$

$$-5c_1 + c_2 + 3c_3 - 2c_4 = 0$$

$$-4c_1 - c_2 + 5c_3 - 5c_4 = 0$$

$$2c_1 + 5c_2 + 2c_3 + 3c_4 = 0$$

$$A = \begin{bmatrix}1 & 1 & -2 & 1\\ -5 & 1 & 3 & -2\\ -4 & -1 & 5 & -5\\ 2 & 5 & 2 & 3\end{bmatrix} \qquad b = \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to R_2 + 5R_1$
$R_3 \to R_3 + 4R_1$
$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix}1 & 1 & -2 & 1\\ 0 & 6 & -7 & 3\\ 0 & 3 & -3 & -1\\ 0 & 3 & 6 & 1\end{bmatrix} \xrightarrow[R_4 \to R_4 - \frac{1}{2}R_2]{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix}1 & 1 & -2 & 1\\ 0 & 6 & -7 & 3\\ 0 & 0 & +1/2 & -5/2\\ 0 & 0 & 19/2 & -1/2\end{bmatrix}$$

$R_4 \to R_4 - 19R_3$

$$\begin{bmatrix}1 & 1 & -2 & 1\\ 0 & 6 & -7 & 3\\ 0 & 0 & 1/2 & -5/2\\ 0 & 0 & 0 & 47\end{bmatrix}$$

rank = n = 4

Hence, independent.

2.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 6 & 8 \\ 1 & 2 & 4 & 5 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + \frac{1}{2}R_3} \begin{bmatrix} \boxed{1} & 1 & 2 & 2 \\ 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{-2} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As $\rho(A) = 3 \neq 4$, These 4 vectors do not form the basis of R_4

The dimension they span is 3

To extend the basis :

a) Make an augmented matrix $[A : I]$ and perform gaussian elemination Take the ~~first column~~ necessary columns [1 in this case] that have non zero pivots, from the ~~I~~ transformed I matrix to get the remaining vectors for basis

b) Find the orthogonal compliment of the current basis (this is there in the 3rd unit)

c) As ~~dim (C(A)) + dim (N(A^T)) = 4~~ $[C(A)]^{\perp} = N(A^T)$, [column space is ⊥r to Left null space]

find the $N(A^T)$ of the matrix and $C(A)$ and $N(A^T)$ form the basis.

Using (c),

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & b_1 \\ 1 & 2 & 5 & 6 & b_2 \\ 1 & 3 & 6 & 8 & b_3 \\ 1 & 2 & 4 & 5 & b_4 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & b_1 \\ 0 & 1 & 3 & 4 & b_2 - b_1 \\ 0 & 2 & 4 & 6 & b_3 - b_1 \\ 0 & 1 & 2 & 3 & b_4 - b_1 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & b_1 \\ 0 & 1 & 3 & 4 & b_2 - b_1 \\ 0 & 0 & -2 & -2 & b_3 - 2b_2 + b_1 \\ 0 & 0 & -1 & -1 & b_4 - b_2 \end{array}\right] \xrightarrow{R_4 \to R_4 + \frac{R_3}{2}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & b_1 \\ 0 & 1 & 3 & 4 & b_2 - b_1 \\ 0 & 0 & -2 & -2 & b_3 - 2b_2 + b_1 \\ 0 & 0 & 0 & 0 & b_4 - b_2 + \frac{1}{2}(b_3 - 2b_2 + b_1) \end{array}\right]$$

$$b_4 - b_2 + \frac{1}{2}\left[b_3 - 2b_2 + b_1\right] = 0.$$

$$b_4 - b_2 + \frac{b_3}{2} - b_2 + \frac{b_1}{2} = 0.$$

$$b_1 - 4b_2 + b_3 + 2b_4 = 0.$$

$$N(A^T) = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 2 \end{bmatrix} \qquad C(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix} \right\}.$$

$$\text{Basis of } \mathbb{R}^4 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

3 a)

$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 4 & -9 & 7 & -3 \\ 1 & -4 & 8 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & 6 & 1 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{8} \end{bmatrix} \xleftarrow[\substack{R_2 \to -R_2 \\ R_3 \to R_3/8}]{R_1 \to R_1/2} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 8 & -1 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_2 \longrightarrow R_2 - R_3 \\ R_1 \longrightarrow R_1 - 2R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3/4 \\ 0 & 1 & 0 & -7/8 \\ 0 & 0 & 1 & -\frac{1}{8} \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -5/2 \\ 0 & 1 & 0 & -7/8 \\ 0 & 0 & 1 & -1/8 \end{bmatrix} \\ \begin{matrix} x & \quad y & \quad z & \quad t \end{matrix} \end{array}$$

pivot variables $\rightarrow x, y, z$

free variable $\rightarrow t$

$x - \frac{5}{2}t = 0 \Rightarrow x = \frac{5}{2}t$

$y - \frac{7}{8}t = 0 \Rightarrow y = \frac{7}{8}t$

$z - \frac{1}{8}t = 0 \Rightarrow z = \frac{1}{8}t$

special sol^n: $t=1$

$$\left(\frac{5}{2}, \frac{7}{8}, \frac{1}{8}, 1\right)$$

3 b)

$$\begin{bmatrix} 0 & 3 & 1 & 4 \\ 1 & 1 & 2 & 1 \\ 3 & 4 & 5 & 2 \\ 4 & 8 & 8 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 3 & 4 & 5 & 2 \\ 4 & 8 & 8 & 7 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 4R_1]{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 4 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - \frac{4}{3}R_2]{R_3 \to R_3 - \frac{1}{3}R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & -4/3 & -7/3 \\ 0 & 0 & -4/3 & -7/3 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & -4/3 & -7/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

$$\xrightarrow[R_3 \to -\frac{3}{4} R_3]{R_2 \to R_2 \times 1/3} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 1 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - \frac{1}{3}R_3]{R_1 \to R_1 - 2R_3} \begin{bmatrix} 1 & 1 & 0 & -5/2 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{array}{c} \begin{bmatrix} \underline{1} & 0 & 0 & -13/4 \\ 0 & \underline{1} & 0 & 3/4 \\ 0 & 0 & \underline{1} & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x & \; y & \;\; z & \quad t \end{matrix} \end{array}$$

Pivots : x, y, z

Free : t

Special solution = $\left\{ t \begin{bmatrix} 13/4 \\ -3/4 \\ -7/4 \\ 1 \end{bmatrix}, t \in R \right\}$

4. (a) $A = \begin{bmatrix} 2 & 4 & -2 & 2 \\ -2 & 5 & 7 & 3 \\ -3 & 6 & -8 & 6 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 + \frac{3}{2}R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 9 & 5 & 5 \\ 0 & 12 & -11 & 9 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{4}{3}R_2} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 9 & 5 & 5 \\ 0 & 0 & -53/3 & 7/3 \end{bmatrix}$$

Pivots : $2, 9, -53/3$.

$C(A)$ = columns that have pivots

$$= \left\{ c_1 \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in R \right\}$$

Basis $= \left\{ \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix} \right\}$ $\dim(C(A)) = 3$

$N(A) = Ax = 0 = Ux = 0$ [U = Row Echlon form]

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 9 & 5 & 5 \\ 0 & 0 & -53/3 & 7/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

Solving,

$2x + 4y - 2z + 2t = 0 \implies x + 2y - z + t = 0$.

$9y + 5z + 5t = 0$.

$-\frac{53z}{3} + \frac{7t}{3} = 0 \implies -53z + 7t = 0$.

Let $t = k$.

$z = 7k/53$.

$y = -\frac{5}{9}(z + t)$
$= -\frac{5}{9}\left(\frac{7k}{53} + k\right)$
$= -\frac{5}{9}\left(\frac{60k}{53}\right) = \frac{-100k}{159}$

$x = z - t - 2y$
$= \frac{7k}{53} - k - \frac{200k}{159}$
$= \frac{21k - 159k - 200k}{159}$ = -3[illegible] / 159

$$N(A) = \left\{ k \begin{bmatrix} -338/159 \\ -100/159 \\ 7/53 \\ 1 \end{bmatrix} \;\middle|\; k \in R \right\} = \left\{ k_1 \begin{bmatrix} -338 \\ -100 \\ 21 \\ 159 \end{bmatrix} \;\middle|\; k_1 \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -338 \\ -100 \\ 21 \\ 159 \end{bmatrix} \right\} \qquad \dim(N(A)) = 1$$

(b) $$A = \begin{bmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & -1 & -2 \\ 2 & -3 & 0 & 4 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + \frac{1}{2}R_1]{R_3 \to R_3 + R_1} \begin{bmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & -1 & -2 \\ 0 & -2 & 2 & 5 \\ 0 & -1/2 & 3 & -3/2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + \frac{1}{6}R_2]{R_3 \to R_3 + \frac{2}{3}R_2} \begin{bmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 4/3 & 11/3 \\ 0 & 0 & 17/6 & -11/6 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - \frac{17}{8}R_3} \begin{bmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 4/3 & 11/3 \\ 0 & 0 & 0 & -77/8 \end{bmatrix}$$

Pivots : $-2, 3, 4/3, -77/8$. $k = 4$.

$C(A)$ = columns containing pivots

$$= \left\{ c_1 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -2 \\ 4 \\ -2 \end{bmatrix} \right\}.$$

$$\text{Basis} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -2 \end{bmatrix} \right\} \qquad \dim(C(A)) = 4$$

$N(A) = Ax = 0 = Ux = 0.$

Solving :

$-\frac{77t}{8} = 0 \Rightarrow t = 0.$

$\frac{4}{3}z + \frac{11}{3}t = 0 \Rightarrow z = 0.$

$3y - z - 2t = 0 \Rightarrow y = 0.$

$-2x + y + 2z + t = 0 = x = 0.$

$$N(A) = \left\{ c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid c_1 \in R \right\}.$$

basis $= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\dim(N(A)) = 4$

[origin].

5) $Ax = b$

$$\begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 5 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$A:b = \left[\begin{array}{ccc:c} 2 & 2 & 3 & a \\ 3 & -1 & 5 & b \\ 1 & -3 & 2 & c \end{array}\right] \xrightarrow[R_3 \to R_3 - \frac{1}{2} R_1]{R_2 \to R_2 - \frac{3}{2} R_1} \left[\begin{array}{ccc:c} 2 & 2 & 3 & a \\ 0 & -4 & 1/2 & b - \frac{3}{2} a \\ 0 & -4 & 1/2 & c - \frac{1}{2} a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc:c} 2 & 2 & 3 & a \\ 0 & -4 & 1/2 & b - \frac{3}{2} a \\ 0 & 0 & 0 & c - b + a \end{array}\right]$$

$\text{rank}(A) = 2 < n(3)$

For a solution to exist; $\text{rank}(A) = \text{rank}(A:b) = 2$

$$\Rightarrow \boxed{c - b + a = 0}$$

6. $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{bmatrix}$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & 1 & -4 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$C(A)$ = columns containing pivots

$$= \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For a vector (a, b, c) to belong to $C(A)$, forming an augmented matrix, $[C(A) : b]$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & -3 & b-a \\ 0 & 1 & 1 & c \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & -3 & b-a \\ 0 & 0 & 4 & c-b+a \end{array}\right]$$

As $\rho[C(A) : b] = \rho[C(A)]$, there exists a solution for $C(A)x = b$ or $C(A)c = b$, where $c = (c_1, c_2, c_3)$

$\Rightarrow$ Any vector b can be expressed as a linear combinati of $C(A)$

$\therefore$ There is no constraint on the vector (a, b, c).

[As $C(A)$ spans a 3D space in $\mathbb{R}^3$, so they form the basis of $\mathbb{R}^3$ and can represent any vector]

7) We have only 1 equation

$$2x - 3y + 4z = 0$$

<u>$Ax = b$ form</u>

$$\begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

A is already in echelon form having pivot variable 'x'
y & z are free variables.

Hence a general soln:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (3y - 4z)/2 \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3y}{2} \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -2z \\ 0 \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

2 basis : (3, 2, 0) and (-2, 0, 1)

8. $[A : b]$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 3R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 1/3 R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - \frac{b_4 - 3b_1}{3} \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

$\rho(A:b) = \rho(A)$ for there to be a solution to this.

$\therefore$ We have, $b_2 - 2b_1 = 0 \quad \Rightarrow \quad \underline{\underline{b_2 = 2b_1}}$

$$b_3 - 2b_1 - \frac{b_4}{3} + b_1 = 0 \quad \Rightarrow \quad b_3 - b_1 - \frac{b_4}{3} = 0$$

$$3b_3 - 3b_1 - b_4 = 0$$

$$\underline{\underline{3b_1 - 3b_3 + b_4 = 0}}$$

9) u, v, w are linearly independent

$\rightarrow \quad \alpha u + \beta v + \gamma w = 0$ only when $\alpha = \beta = \gamma = 0$

$x_1 = u + v$

$x_2 = u - v$

$x_3 = u - 2v + w$

$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$

$c_1(u+v) + c_2(u-v) + c_3(u - 2v + w) = 0$

$u\underbrace{(c_1 + c_2 + c_3)}_{\alpha} + v\underbrace{(c_1 - c_2 - 2c_3)}_{\beta} + w\underbrace{(0c_1 + 0c_2 + c_3)}_{\gamma} = 0$

$c_1 + c_2 + c_3 = 0$ and $c_1 - c_2 - 2c_3 = 0$ and $c_3 = 0$

$\Rightarrow c_1 + c_2 = 0 \qquad c_1 - c_2 = 0$

$c_1 = -c_2 \qquad c_1 = c_2$

Holds true iff $c_1 = c_2 = 0$

$\therefore c_1 = c_2 = c_3 = 0$

The given set of vectors are linearly independent.

10. Considering each matrix as a vector : We get

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ -5 & 1 & -4 & -7 \\ -4 & -1 & -5 & -5 \\ 2 & 5 & 7 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 + 4R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 6 & 6 & -2 \\ 0 & 3 & 3 & -1 \\ 0 & 3 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_2/2]{R_3 \to R_3 - R_2/2} \begin{bmatrix} \boxed{1} & 1 & 2 & 1 \\ 0 & \boxed{6} & 6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As there are only 2 pivots, there exists only 2 linearly independent columns, thus forming the basis. The other two can be expressed using the previous 2 matrices.

$\therefore$ Basis of W $= \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \right\}$.

$\dim(W) = 2$.

11) $V = \{ (a,b,c,d) \mid a-2b-4c=0 ,\ 2a-c-3d=0 \}$

Ax=b form :-

$$\begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{array}{c} \begin{bmatrix} 1 & -2 & -4 & 0 \\ 0 & 4 & 7 & -3 \end{bmatrix} \\ \begin{array}{cccc} a & b & c & d \end{array} \end{array}$$

pivots: a, b | free : c, d Dim = rank = 2

$a - 2b - 4c = 0$
$4b + 7c - 3d = 0$
$\Rightarrow b = (3d - 7c)/4$

$a - \frac{(3d-7c)}{2} - 4c = 0$
$\Rightarrow a = \frac{c + 3d}{2}$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} (c+3d)/2 \\ (3d-7c)/4 \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{c}{2} \\ -7c/4 \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} 3d/2 \\ 3d/4 \\ 0 \\ d \end{pmatrix}$$

Basis of V = $(1, \frac{1}{2}, 0, \frac{2}{3})$ and $(0, -2, 1, -\frac{1}{3})$

$(d = \frac{2}{3})$ $(c=1, d=-\frac{1}{3})$

12. If v should belong to span (u_1, u_2, v_3), v should be written as a linear combination of u_1, u_2 and u_3.

i.e $v = c_1u_1 + c_2v_2 + c_3u_3$.

i.e $AC = v$, where $A = [u_1\ v_2\ v_3]$ and $C = [c_1, c_2, c_3]^T$

Solving, (by $Ax = b$, using $[A:b]$)

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 1 & 0 & b \\ 0 & 2 & -4 & c \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 2 & -4 & c \end{array}\right] \xrightarrow{R_3 \to R_3 - \frac{2}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 0 & 0 & c - \frac{2}{3}(b-2a) \end{array}\right]$$

$\rho(A) = 2 \neq 3$.

(i) $\therefore$ No, u_1, v_2, v_3 do not span R_3 as they have only 2 pivots.

If $v = (a, b, c)$ must belong to span (u_1, v_2, v_3),

$\rho(A:B) = \rho(A) = 2$. [For infinite solutions].

For this to happen, $c - \frac{2}{3}(b-2a) = 0$.

$$\Rightarrow \frac{3c - 2b + 4a}{3} = 0.$$

$$\Rightarrow 3c - 2b + 4a = 0$$

(ii) Yes W is a subspace of R^3

(iii) Basis for span = Basis for column space $C(A)$

$$C(A) = \left\{ c_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}, c_1, c_2 \in R.$$

$\therefore$ Basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

$= \{ u_1, u_2 \}$.

13)

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \begin{array}{c} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x & y & z & t \end{matrix} \end{array}$$

Pivot variables : x, y

Free variables : z, t

$x + 2y + 4z + 3t = 0$

$y + 2z + t = 0$

$\Rightarrow y = -2z - t$

$x + 2(-2z - t) + 4z + 3t = 0$

$x = -t$

N(A) comprises of : $\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$b = (-2, -2, 0, 2) = 2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow b \in N(A)$

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 4 & 2 & 6 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Dimension of column space of $A^T = 2$

Dimension of $N(A^T) = 1$

Bases of $C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

Basis of $N(A^T) = \left\{ \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} \right\}$

14 . Let A be a matrix with m rows and n columns. Then there are four fundamental subspaces for A:

1. **C(A): the column space of A, it contains all linear combinations of the columns of A**
2. **C(A^T): the row space of A, it contains all linear combinations of the rows of A (or, columns of A^T)**
3. **N(A): the nullspace of A, it contains all solutions to the system Ax=0**
4. **N(A^T): the left nullspace of A, it contains all solutions to the system $A^T y = 0$**

14. $$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ 4 & -2 & 1 & -5 & -7 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + \frac{4}{3}R_1]{R_2 \to R_2 + \frac{1}{3}R_1,\quad R_3 \to R_3 + \frac{2}{3}R_1} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 \\ 0 & -6 & -1/3 & -11/3 & -49/3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & -6 & -1/3 & -11/3 & -49/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - \frac{5}{13}R_3} \begin{bmatrix} \boxed{-3} & 6 & -1 & 1 & -7 \\ 0 & \boxed{-6} & -1/3 & -11/3 & -49/3 \\ 0 & 0 & \boxed{13/3} & 26/3 & -26/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots : $-3, -6, 13/3$

$$C(A) = \left\{ c_1 \begin{bmatrix} -3 \\ 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -2 \\ -4 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 5 \\ 1 \end{bmatrix} \;\middle|\; c_1, c_2, c_2 \in R \right\}.$$

basis = $\{ [-3\ 1\ 2\ 4]^T, [-6\ -2\ -4\ -2]^T, [-1\ 2\ 5\ 1]^T \}$.

$$C(A^T) = \left\{ c_1 \begin{bmatrix} -3 \\ 6 \\ -1 \\ 1 \\ -7 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -4 \\ 5 \\ 8 \\ -4 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in R \right\}.$$

basis = $\{ [-3\ 6\ -1\ 1\ -7]^T, [-1\ -2\ 2\ 3\ -1]^T, [2, -4\ 5\ 8\ -4]^T \}$

for N(A), Solving $Ax = 0$, we get

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & -6 & -1/3 & -11/3 & -49/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ w \end{bmatrix} = 0$$

x, y, z - pivot variables

t, w - free variables

Let $t = k$, $w = l$.

$$\frac{13z}{3} + \frac{26t}{3} - \frac{26w}{3} = 0 \Rightarrow z + 2t - 2w = 0$$

$$z = 2w - 2t$$
$$= 2l - 2k$$

$$-6y - \frac{z}{3} - \frac{11t}{3} - \frac{49w}{3} = 0 \Rightarrow 18y + z + 11t + 49w = 0$$

$$y = \frac{-49l - 11k - 2l + 2k}{18}$$

$$= \frac{-51l - 9k}{18}$$

$$-3x + 6y - z + t - 7w = 0 \Rightarrow x = \frac{6y - z + t - 7w}{3}$$

$$= \frac{6\left(\frac{-51l - 9k}{18}\right) - (2l - 2k) + k - 7l}{3}$$

$$= -17l - 3k - 2l + 2k + k - 7l \,/\, 3$$

$$= \frac{-26l}{3}$$

$$N(A) = \begin{bmatrix} x \\ y \\ z \\ t \\ w \end{bmatrix} = \begin{bmatrix} \frac{-26l}{3} \\ -\frac{17l}{6} - \frac{k}{2} \\ 2l - 2k \\ k \\ l \end{bmatrix} = k \begin{bmatrix} 0 \\ -1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + l \begin{bmatrix} -26/3 \\ -17/6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \left\{ c_1 \begin{bmatrix} 0 \\ -1 \\ -4 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -52 \\ -17 \\ 12 \\ 0 \\ 6 \end{bmatrix} \;\middle|\; c_1, c_2 \in R \right\}$$

for $N(A^T)$, either solve $A^T x = 0$ OR use $[A \vdots b]$

Using the latter,

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & b_1 \\ 1 & -2 & 2 & 3 & -1 & b_2 \\ 2 & -4 & 5 & 8 & -4 & b_3 \\ 4 & -2 & 1 & -5 & -7 & b_4 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 + 2/3 R_1 \\ R_4 \to R_4 + \frac{4}{3} R_1}]{R_2 \to R_2 + \frac{1}{3} R_1} \left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & b_1 \\ 0 & 0 & 5/3 & 10/3 & -10/3 & b_2 + b_1/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 & b_3 + 2b_1/3 \\ 0 & -6 & -1/3 & -11/3 & -49/3 & b_4 + 4b_1/3 \end{array}\right]$$

$$\xrightarrow{R_2 \longleftrightarrow R_4} \left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & b_1 \\ 0 & -6 & -1/3 & -11/3 & -49/3 & b_4 + 4b_1/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 & b_3 + 2b_1/3 \\ 0 & 0 & 5/3 & 10/3 & -10/3 & b_2 + b_1/3 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - \frac{5R_3}{13}} \left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & b_1 \\ 0 & -6 & -1/3 & -11/3 & -49/3 & b_4 + 4b_1/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 & b_3 + 2b_1/3 \\ 0 & 0 & 0 & 0 & 0 & b_2 + b_1/3 - 5/13\,[b_3 + 2b_1/3] \end{array}\right]$$

$$b_2 + \frac{b_1}{3} - \frac{5}{13}\left[b_3 + \frac{2b_1}{3}\right] = 0$$

$$b_2 + \frac{b_1}{3} - \frac{5}{13 \cdot 3}[3b_3 + 2b_1] = 0 \Rightarrow 39b_2 + 13b_1 - 15b_3 - 10b_1 = 0$$

$$3b_1 + 39b_2 - 15b_3 = 0$$

$$N(A^T) = \left\{ a \begin{bmatrix} 3 \\ 39 \\ -15 \\ 0 \end{bmatrix} \;\middle|\; a \in R \right\}$$

basis $= \{ [3 \;\; 39 \;\; -15 \;\; 0] \}$.

15

(i) $\begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$$\xrightarrow{R_2 \to R_2 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & -7/3 \end{bmatrix}$$

$\rho(A) = 2 = m$

$\therefore$ Right inverse exists

$$RI = A^T (AA^T)^{-1}.$$

$$AA^T = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix}$$

$$(AA^T)^{-1} = \frac{1}{49} \begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix}$$

$$RI = A^T (AA^T)^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix} \frac{1}{49}$$

$$= \frac{1}{49} \begin{bmatrix} 14 & 7 \\ 0 & 0 \\ 7 & -21 \end{bmatrix}$$

//

15) (ii)

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}_{3\times 2} \xrightarrow[R_3 \to R_3 - \frac{1}{3}R_1]{R_2 \to R_2 + \frac{1}{3}R_1} \begin{bmatrix} 3 & 1 \\ 0 & 4/3 \\ 0 & -4/3 \end{bmatrix}$$

$$\downarrow R_4 \to R_4 + R_3$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 4/3 \\ 0 & 0 \end{bmatrix}$$

rank = n = 2
(left inverse exists)

$$B = (A^T A)^{-1} A^T$$

$$A^T A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & +1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 3 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 11 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{32}\begin{bmatrix} 3 & -1 \\ -1 & 11 \end{bmatrix}$$

$$B = \frac{1}{32}\begin{bmatrix} 3 & -1 \\ -1 & 11 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$B = \frac{1}{32}\begin{bmatrix} 8 & -4 & 4 \\ 8 & 12 & -12 \end{bmatrix}$$