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OF TECHNOLOGY
KANPUR

Brownian Escape from Equilateral Triangle

Anusua Paul (241080056)

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Abstract

In this SURGE project, we read the research paper [1] focusing on a closed form expression of the expected exit time of a Brownian motion from an equilateral triangle. For that we first construct a random walk on a triangular lattice in the 2-d plane which is analogous to a suitable ruin problem of a special variant of three person zero sum game. Finally a proper scaling of this random walk helps us to connect between the explicit exit time of a Brownian motion and the time to ruin the game. This gives us the required unique solution of the Poisson equation given by $\frac{1}{2}\Delta u = -1$ which vanishes on the boundary of the triangle.

Introduction

Firstly let us introduce about the Peter, Paul, Mary game which is a particular variant of three person zero sum game. We can select a pair among this three person in three possible ways. After selecting the pair we toss a coin to choose who among the pair wins. The person who wins get 1 rupee from the person who looses in the pair. The third one who was not selected does not gain or loose anything. Now we have to follow the steps below to get into our final closed form [1, p. 44].

- First we show this Peter, Paul, Mary ruin problem with initial fortunes a, b, c such that $a + b + c = S$, where S is a positive integer is analogous to the exit problem of a random walk from an equilateral triangle of side length S which is placed on the triangular lattice of unit side length in such a way that every vertex of this triangle is any of the vertices of the lattice.
- Then we show that after doing proper scaling of this random walk the exit time of it from an equilateral triangle converges in distribution to the exit time of a Brownian motion in 2-d plane for the same.
- Comparing this random walk to the ruin problem we finally reach into our closed form solution.

The lattice and the Laplace equation

Let S be a positive integer. Consider an equilateral triangle of side length S which is placed on the triangular lattice of unit length such that the vertices of Δ_S are the vertices of the lattice. Suppose (α, β) is the cartesian co ordinate of a vertex of a lattice inside Δ_S. $(0,0), (S,0), (S/2, \sqrt{3}S/2)$ are the vertices of Δ_S then we introduce three new co ordinates a, b, c corresponds to $(2/\sqrt{3})\beta, \alpha - (1/\sqrt{3})\beta, S - \alpha - (1/\sqrt{3})\beta$. WLG assume that a, b, c are positive integers. Suppose $h(a, b, c)$ is expected time to ruin of Peter, Paul, Marry problem with beginning fortunes as rupees a, b, c respectively. The difference equation which is required to solve is given by [1, p. 46]

$$\begin{aligned} h(a,b,c) = 1 + \frac{1}{6}\big(&h(a-1,b+1,c) + h(a+1,b-1,c) + \\ &h(a,b-1,c+1) + h(a,b+1,c-1) + \\ &h(a-1,b,c+1) + h(a+1,b,c-1)\big) \end{aligned}$$

with boundary condition

$$h(a,b,c) = 0 \quad \text{whenever} \quad \min\{a,b,c\} = 0$$

Now this difference equation along with this boundary condition has the unique solution.

$$h(a,b,c) = \frac{3abc}{a+b+c}$$

Convergence of Expected Exit Time

Suppose S is a positive real number. Now we want to get an explicit expression for the exit time of a 2-D Brownian motion from an equilateral triangle of length S. To do this first we construct a sequence of approximating random walk converging to a standard Brownian motion. Then we show that the expected exit time of the the random walk will also converge to the expected exit time of the limiting distribution.

Define $T = inf\{t \geq 0 : X_t \notin \triangle_S\}$ the exit time of the process X from an equilateral triangle of side length S. Let ξ_i be a sequence of iid random variables taking values $(\cos(k\pi/3), \sin(k\pi/3))$ for $k = 1, 2, ..5, 6$ taking equal probability for each k. It may be thought as a single step on the triangular lattice. Fix $(\alpha, \beta) \in \Delta_S$. Define [1, p. 47]

$$X_t^n = Y_{nt}^n = (\alpha, \beta) + \frac{\sqrt{2}}{\sqrt{n}}\left(\sum_{i=1}^{[nt]} \xi_i + (nt - [nt])\xi_{[nt]+1}\right) \tag{1}$$

where $[\cdot]$ denotes the integer part.

Proposition 1. *[1] The sequence of processes $\{X_t^n,\ t \geq 0\}$ converges in law to a standard Brownian motion starting at (α, β).*

Using Donsker's invariance principle [2] and multivariate CLT we can prove this.

Proposition 2. *[1] The following convergence in law holds:*

$$T(X^n, \triangle_S) \Rightarrow T(B, \triangle_S).$$

Lemma 3. *[1] The sequence $\{T(X^n, \triangle_S)\}_{n\in\mathbb{N}}$ is uniformly integrable.*

Convergence in probability and Uniform convergence together implies the proposition below.

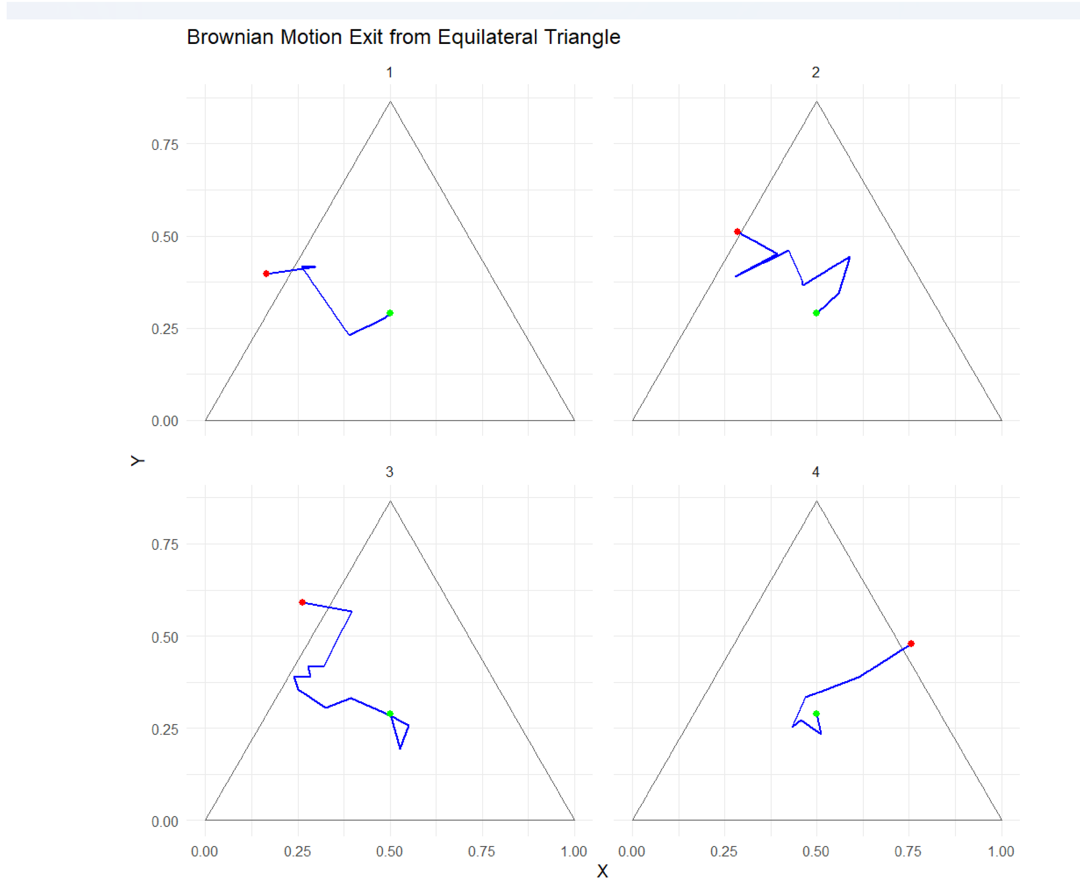
Proposition 4. *[1]*

$$\lim_{n\to\infty} \mathbb{E}\left[T(X_n, \triangle_S)\right] = \mathbb{E}\left[T(B, \triangle_S)\right]$$

Define $Z^n := \frac{\sqrt{n}}{\sqrt{2}} Y^n$ which is a scaled version of Y^n of unit size. Using this we can use properties related to the random walk described earlier. The below lemma is followed by some proper geometric arguments.

Lemma 5. *[1]*

$$T(X^n, \triangle_S) = \frac{1}{n}T(Y^n, \triangle_S) = \frac{1}{n}T\left(Z^n, \triangle_{\frac{\sqrt{n}}{\sqrt{2}}S}\right)$$



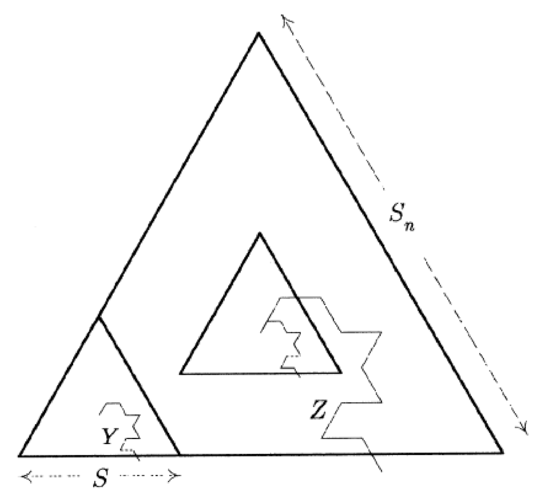


Figure 2. The random walks Z and Y. Here $S_n = \sqrt{n/2}S$.

Conclusion

Proposition 6. *[1] The limit of the expected exit times from $\triangle_S$ of the approximating random walks X^n is*

$$\frac{\sqrt{3}\beta\left(\alpha - \frac{1}{\sqrt{3}}\beta\right)\left(S - \alpha - \frac{1}{\sqrt{3}}\beta\right)}{S}, \tag{14}$$

where (α, β) is the starting point.

Proof. The set

$$\left\{(\alpha, \beta) \in \Delta_s : \ \sqrt{n/2}(\alpha, \beta) \ \text{ is a vertex of the triangular lattice for infinitely many } n\right\}$$

is dense in Δ_s. Define

$$m = [\sqrt{n/2}S] + 1$$

For $k = m - 1, m$,

$$\begin{aligned} a_k &= \frac{\sqrt{n/2} \cdot 2}{\sqrt{3}}\beta \\ b_k &= \sqrt{n/2}\left(\alpha - \frac{1}{\sqrt{3}}\beta\right) \\ c_k &= k - \sqrt{n/2}\left(\alpha + \frac{1}{\sqrt{3}}\beta\right) \end{aligned}$$

which follows that

$$E\left(T(Z^n, \Delta_k)\right) = n\sqrt{3}\beta\left(\alpha - \frac{1}{\sqrt{3}}\beta\right)\frac{k - \sqrt{n/2}\alpha - \sqrt{n/2}\left(\frac{1}{\sqrt{3}}\beta\right)}{k}$$

From the previous lemma,

$$\frac{1}{n}T\left(Z^n, \Delta_k\right) = T\left(X^n, \Delta_{k/\sqrt{n/2}}\right)$$

Thus we have

$$\Delta_{(m-1)/\sqrt{n/2}} \subseteq \Delta_S \subseteq \Delta_{m/\sqrt{n/2}}$$

Taking $n \to \infty$

$$\lim_{n\to\infty} \mathbb{E}[T(X_n, \Delta_S)] = \frac{\sqrt{3}\beta\left(\alpha - \frac{1}{\sqrt{3}}\beta\right)\left(S - \alpha - \frac{1}{\sqrt{3}}\beta\right)}{S}$$

$\square$

References

[1] Aureli Alabert, Mercè Farré, and Rahul Roy. Exit times from equilateral triangles. *Appl. Math. Optim.*, 49(1):43–53, 2004.

[2] Peter Mörters and Yuval Peres. *Brownian motion*, volume 30 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.