

Solving Predator-Prey System using Exponential Time Differencing Method

MM6103 Term Project

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Predator-Prey System

Lotka-Volterra Model:

$$\frac{dU}{dT} = \text{growth of prey} - \text{loss due to predation}$$

$$\frac{dV}{dT} = \text{gain from predation} - \text{natural death}$$

The ***predator-prey cycle*** arises due to delayed feedback:

- When prey density increases → predators thrive.
- High predation reduces prey → predator population then declines.
- Prey recovers → cycle repeats.

Reaction-Diffusion Model

U : Prey Density

V : Predator Density

The model is a **reaction–diffusion system**:

- **Reaction terms:** describe local biological interactions (growth, death, predation).
- **Diffusion terms:** capture spatial spread (migration or movement).

This coupling creates **spatiotemporal patterns** (e.g., waves, oscillations, spots).

Such dynamics are similar to **nonlinear chemical systems** like the

Belousov–Zhabotinsky reaction — both exhibit *self-organization* and *pattern formation* through diffusion-driven instability.

Observable Phenomena

- **Oscillatory behavior:** energy exchange between predator and prey populations.
- **Spatial structures:** diffusion causes Turing-like patterns (spots, stripes).
- **Stability transitions:** sensitive to parameter perturbations: similar to nonlinear instabilities in physical systems.

Governing Equation

$$\begin{aligned}\frac{\partial U}{\partial T} &= D_1 \frac{\partial^2 U}{\partial X^2} + U \left[\alpha \left(1 - \frac{U}{K} \right) - \frac{\gamma V}{U + \delta} \right], \\ \frac{\partial V}{\partial T} &= D_2 \frac{\partial^2 V}{\partial X^2} + V \left[\beta \left(1 - \frac{\gamma V}{U} \right) \right],\end{aligned}$$

$\alpha U(1 - U/K)$: Logistic Growth

K : Carrying Capacity

α : Intrinsic Growth Rate

β : Predation Intensity

γV : per-Capita prey reduction

D_i : Diffusion Coefficient

Non-Dimensional Form

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u(1-u) - \frac{\mu u}{u+\phi} v = f(u, v), \\ \frac{\partial v}{\partial t} &= D \frac{\partial^2 v}{\partial x^2} + \psi v - \frac{\psi v^2}{u} = g(u, v). \end{aligned} \right\}$$

3 Parameter System !

$$u(t) = \frac{U(T)}{K}, \quad v(t) = \frac{hV(T)}{K}, \quad t = \alpha T, \quad \mu = \frac{\gamma}{h\alpha}, \quad \psi = \frac{\beta}{\alpha}, \quad \phi = \frac{\delta}{K}, \quad D = \frac{D_2}{D_1}$$

Numerical Strategy

1. Use 4th Order Centered finite difference for representing second order spatial terms.
2. Integrate the Linear term using exponential Integrator to remove stiffness from the system.
3. Approximate the Integral using 4th Order Runge-Kutta method for Exponential Time Difference methods.
4. Use CF Approximations so that we can use ETDRK4 method for non-diagonal matrices.

4th Order Centered Finite Difference Approximation

$$\frac{\partial^2 w}{\partial x^2} = \frac{-w_{i-2,j} + 16w_{i-1,j} - 30w_{i,j} + 16w_{i+1,j} - w_{i+2,j}}{12h_x^2}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{-w_{i,j-2} + 16w_{i,j-1} - 30w_{i,j} + 16w_{i,j+1} - w_{i,j+2}}{12h_y^2}$$

Exponential Time Differencing Method

$$\frac{dw}{dt} = D(L_1 + L_2)w + N(w, t)$$

$$w(t_{n+1}) = w(t_n) e^{L\Delta t} + e^{L\Delta t} \int_0^{\Delta t} e^{-L\tau} N(w(t_n + \tau), t_n + \tau) d\tau$$

4th Order ETD Runge Kutta Method

$$a_n = w_n e^{L\Delta t/2} + (e^{L\Delta t/2} - I) N_n / L,$$

$$b_n = w_n e^{L\Delta t/2} + (e^{L\Delta t/2} - I) N(a_n, t_n + \Delta t/2) / L,$$

$$c_n = w_n e^{L\Delta t/2} + (e^{L\Delta t/2} - I) (2N(b_n, t_n + \Delta t/2) - N_n) / L.$$

$$\begin{aligned} w_{n+1} = & w_n e^{L\Delta t} + N_n [-4 - L\Delta t + e^{L\Delta t} (4 - 3L\Delta t + L^2\Delta t^2)] \\ & + 2(N(a_n, t_n + \Delta t/2) + N(b_n, t_n + \Delta t/2)) [2 + L\Delta t + e^{L\Delta t} (-2 + L\Delta t)] \\ & + N(c_n, t_n + \Delta t) [-4 - 3L\Delta t - L^2\Delta t^2 + e^{L\Delta t} (4 - L\Delta t)] / L^3\Delta t^2, \end{aligned}$$

Works Only when \mathbf{L} is scalar or Diagonal

Carathéodory-Fejér approximation

Approximates a function with a rational function on the negative real line with 1e-12 order of error.

$$r_n(z) = \frac{p_n(z)}{q_n(z)} = r_\infty + \sum_{j=1}^n \frac{c_j}{z - z_j},$$

Here, $R_n(z) = f_n(z)$ approximately. When z is a matrix \mathbf{M} , we get this form. Additionally, we operate $f(\mathbf{M})$ over a vector \mathbf{v} to get this:

$$f(M)v \approx r_\infty v + \sum_{j=1}^n c_j (M - z_j I)^{-1} v.$$

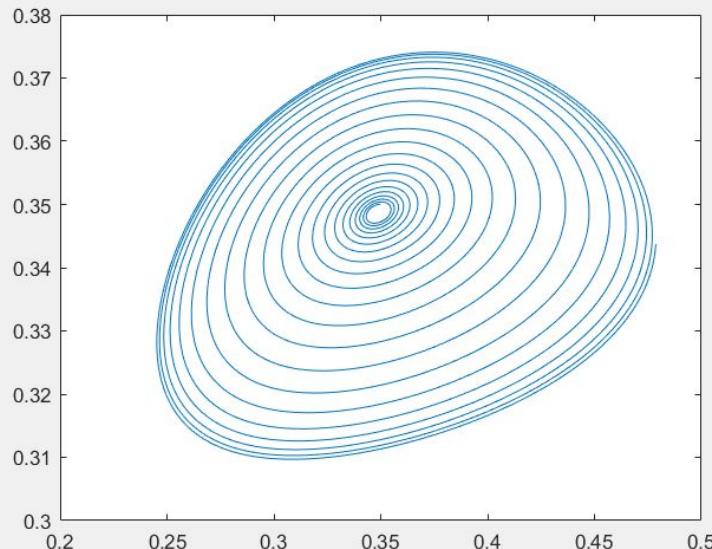
Now we solve the system of Linear Equations $(\mathbf{M} - z_j \mathbf{I})^{-1} \mathbf{v}$ to evaluate $f(\mathbf{M})\mathbf{v}$.

This approximation allows us to use ETDRK4 method to solve our system!

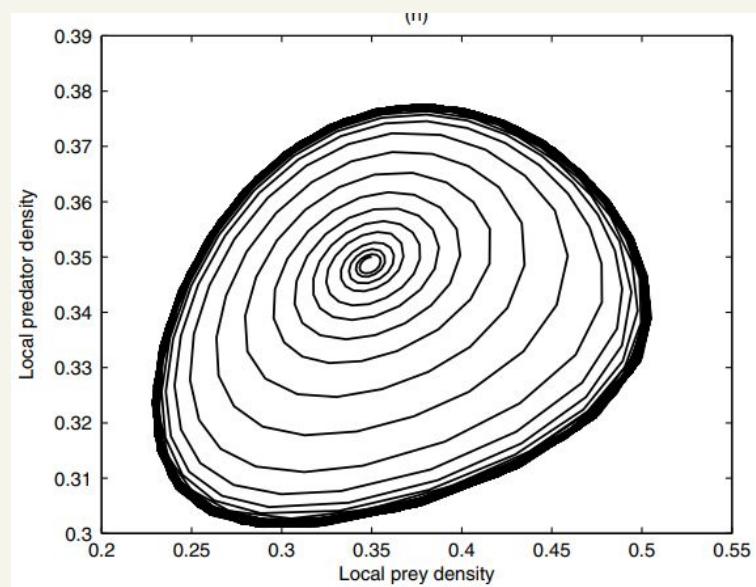
1 Dimensional Behavior

Phase Planes

Calculated



Paper



Oscillatory Nature

Parameter Values

$$D = 0.1$$

$$\mu = 1$$

$$\phi = 0.2$$

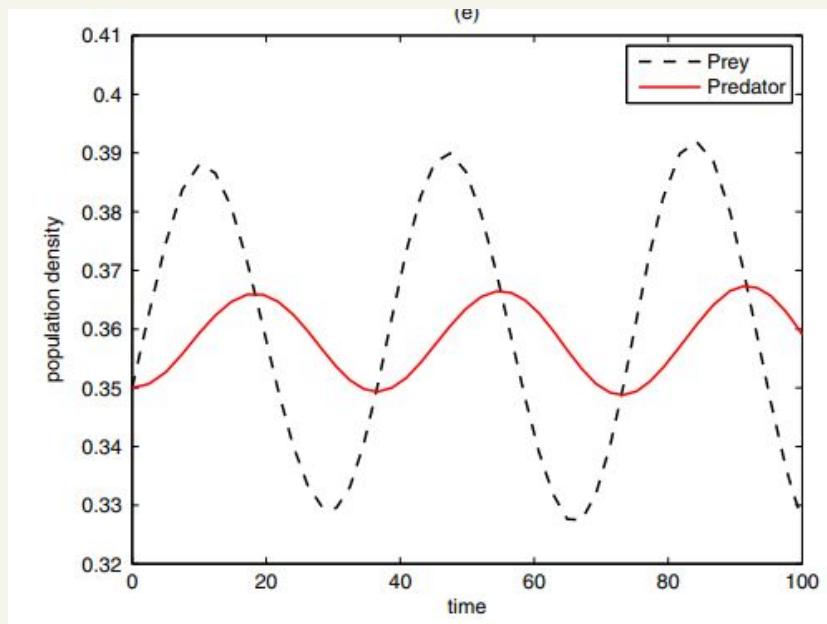
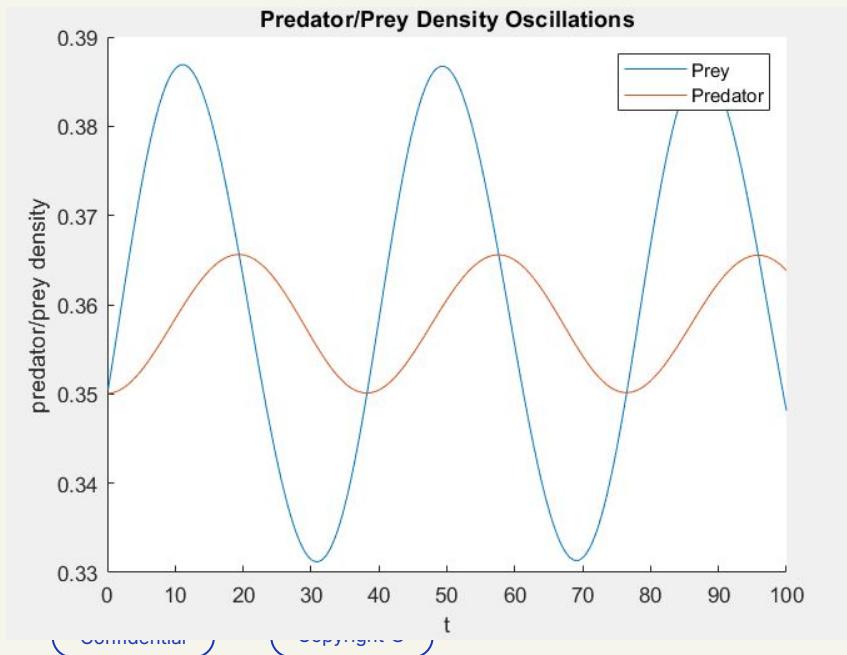
$$\psi = 0.05$$

$$u_0 = 0.35$$

$$v_0 = 0.35$$

Calculated

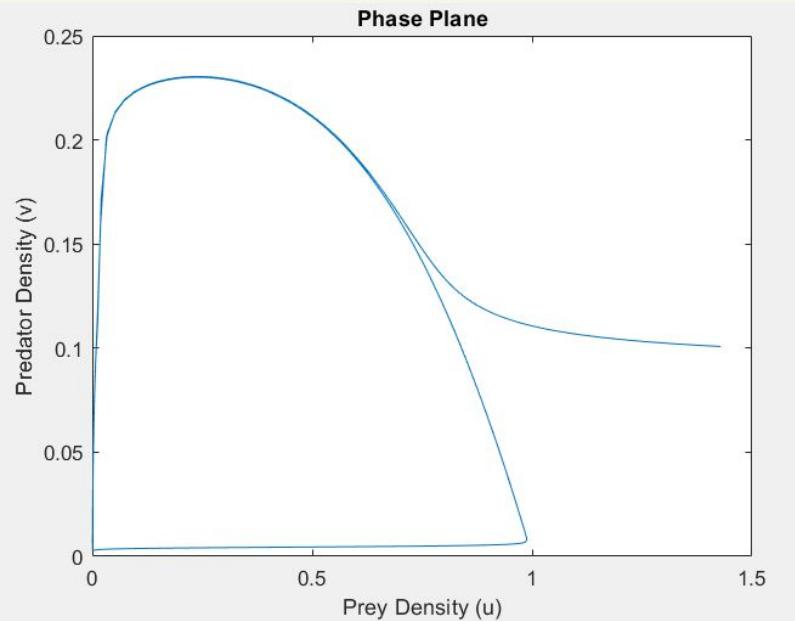
Paper



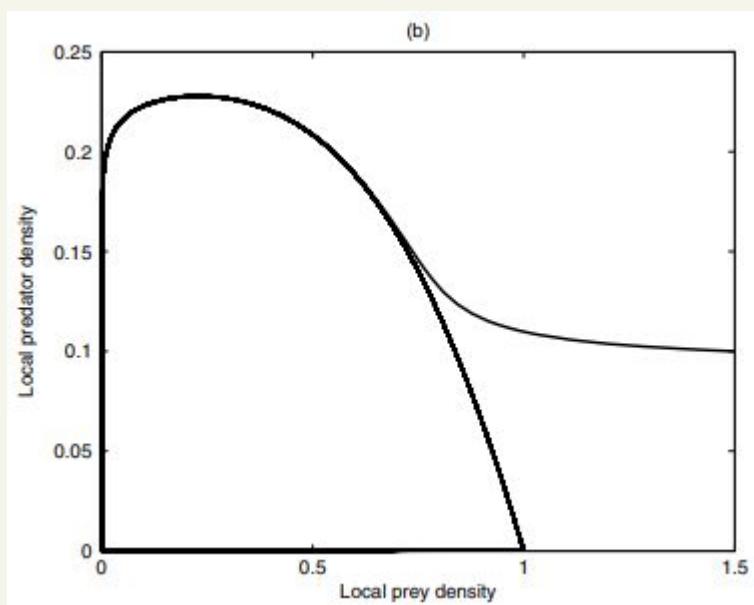
Alternate Case

Phase Planes

Calculated



Paper



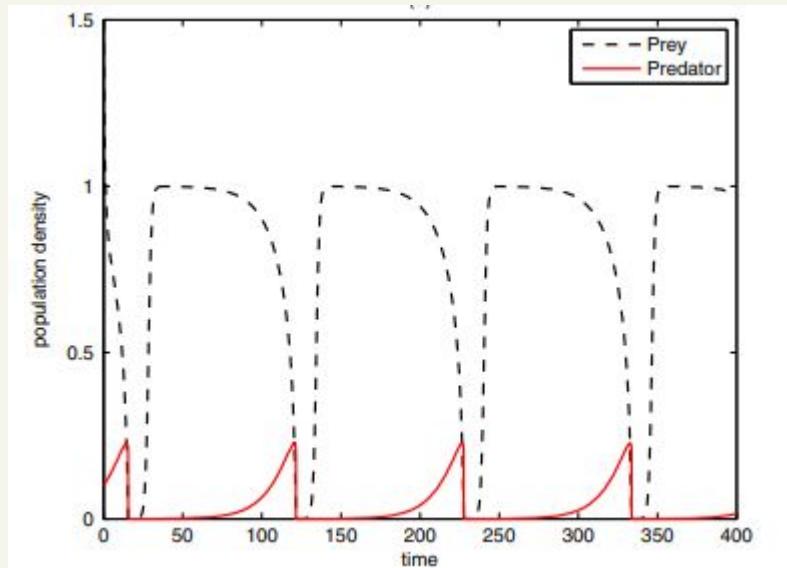
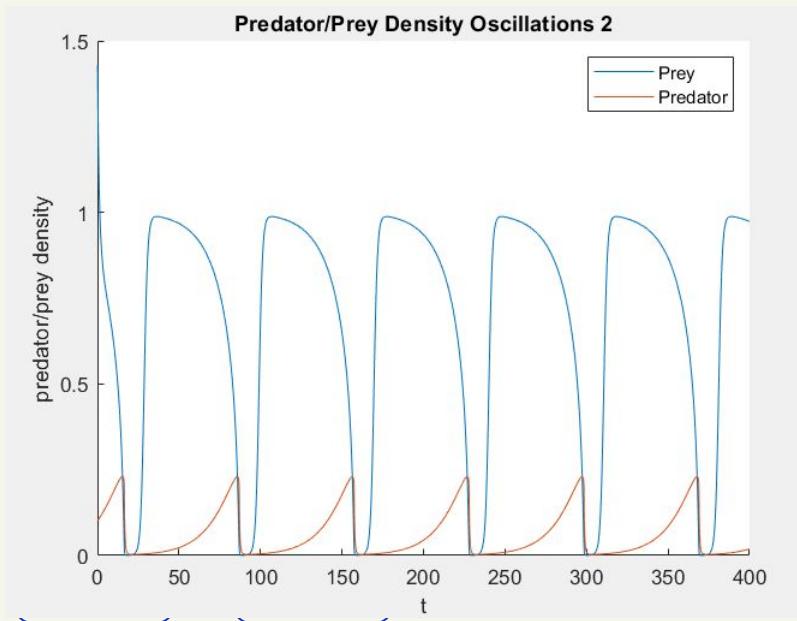
Oscillatory Nature

Parameter Values

$$\begin{array}{ll} D = 0.1 & \psi = 0.08 \\ \mu = 1.5 & u_0 = 1.5 \\ \phi = 0.01 & v_0 = 0.1 \end{array}$$

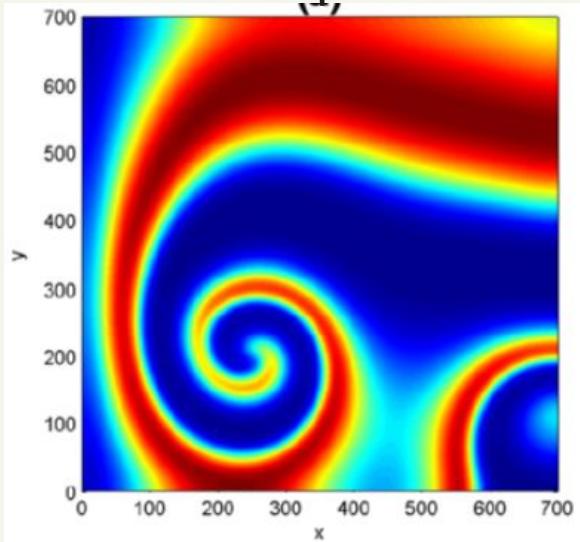
Calculated

Paper

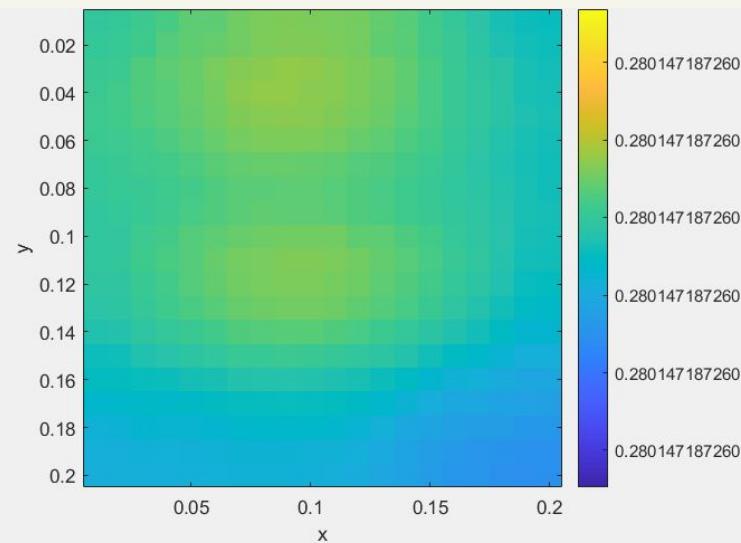


2 Dimensional Behavior (Prey Distribution)

Paper



Calculated



Possible Reasons for Deviation:
1. Smaller Stencil (40×40)
2. Smaller #iterations ($N_t = 50$)



Takes too much time (> 2 hrs)
and space ($> 5-10$ GBs)

→ My laptop cannot handle it :(

References

1. K. M. Owolabi and K. C. Patidar, "Numerical simulations of multicomponent ecological models with adaptive methods", *Theor. Biol. Med. Model.*, 13:1, 2016, doi. 10.1186/s12976-016-0027-4
2. S. M. Cox and P. C. Matthews, "Exponential Time Differencing for Stiff Systems", *J. Comput. Phys.*, vol. 176, pp. 430–455, 2002, doi. 10.1006/jcph.2002.6995
3. T. Schmelzer and L. Trefethen, "Evaluating matrix functions for exponential integrators via Caratheodory-Fejer approximation and contour integrals", *Elect Trans Numer Anal.*, vol. 29, pp. 1–18, 2007, link:
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