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Large Deformation Analysis

of Magneto-Active Polymer Membranes under varying Mechanical Loads

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Thin Membrane Characteristics

Bulk Body

- Compression and Tension dominate
- Transverse stresses balance transverse loads
- Needs complete 3D formulation
- Tougher to buckle

Thin Membrane

- Bending effects also dominate
- In-plane stresses balance transverse loads
- Dimension reduction to 2D using Plane Stress
- Buckles easily by wrinkling

Energy Relations

- The thickness is much smaller than the other dimensions.
- As a result,

Energy with through-thickness deformations	<<	Bending Energy	<<	Energy with in-plane deformations

- In terms of the ratio $z = h/L$,

Energy associated with in-plane deformations	$\sim O(z)$
Energy associated with bending deformations	$\sim O(z^3)$
Energy associated with through-thickness def.	$\sim O(z^5)$

Assumptions

List of Assumptions

1. Membrane is flat in material frame
2. Kirchhoff - Love Assumptions
    a. Normals to the flat surface remain straight after deformation.
    b. Normals remain of the same length and stay normal to the deformed surface after deformation.
3. Total Strain Energy can be decoupled into Membrane and Bending parts
4. Hyperelastic polymer matrix obeying Mooney–Rivlin model
5. Small local curvatures
6. Small Magnetisation of Membrane

Constitutive Model

Equilibrium Equations

$$\nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] + \mathbf{f}_m \quad = \mathbf{0}$$

$$\nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N} \nabla w] + f_b = 0$$

Natural Boundary Conditions

$$\oint_{\Gamma_0} \left([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{n} - \mathbf{t}_m \right) \cdot \eta \; d\Gamma = 0$$

$$\oint_{\Gamma_0} \Big(([\mathbf{N} \nabla w + \nabla \cdot \mathbf{M}] \cdot \mathbf{n} - t_b)\zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla \zeta \Big) d\Gamma = 0$$

Strain Tensors

$$\mathbf{E}_m = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T (\nabla \mathbf{u}) + \nabla w \otimes \nabla w \right]$$

$$\mathbf{K} = -\nabla^2 w$$

Strain Energy Density per unit Area

$$\Psi^{3D}_{Total} = \Psi^{3D}_{Isochoric} + \Psi^{3D}_{Volumetric} + \Psi^{3D}_{Magneto-elastic} + \Psi^{3D}_{Magneto-elastic}$$

$$\Psi^{3D}_{Isochoric} = h[c_1(\overline{I}_1^{3D} - 3) + c_2(\overline{I}_2^{3D} - 3)]$$

$$\Psi^{3D}_{Volumetric} = h\left[\frac{1}{2}\kappa(J^{3D} - 1)^2\right]$$

$$\Psi^{3D}_{Magneto-elastic} = h[\hat{c}_2(\overline{I}_2^{3D} - 3)]$$

$$\Psi^{2D}_{Bending} = \frac{Yh^3}{24(1-\nu^2)}[\nu(\mathrm{tr}\mathbf{K}^{2D})^2 + (1-\nu)\mathbf{K}^{2D} : \mathbf{K}^{2D}]$$

Magneto-Elastic Coupling Coefficient

$$\hat{c}_2 = \zeta_2 \tan^{-1}(\eta_2 \sqrt{I_4})$$

$$\zeta_2 = b_1 \ln(b_2 \sqrt{I_4} + 1)$$

$$\eta_2 = a_1 (e^{a_2 \sqrt{I_4}} - 1)$$

Invariants

$$I_1 = \mathrm{tr}(\mathbf{C})$$

$$I_2 = \frac{1}{2}[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)]$$

$$I_3 = \det(\mathbf{C})$$

$$I_4 = \mathbf{H} \cdot \mathbf{H}$$

Compressibility Relations

$$J = \det\mathbf{F}$$

$$J^2 = \det\mathbf{C}$$

$$\overline{\mathbf{F}} = J^{-1/3}\mathbf{F}$$

$$\overline{\mathbf{C}} = J^{-2/3}\mathbf{C}$$

$$\overline{I_1} = J^{-2/3} I_1$$

$$\overline{I_2} = J^{-4/3} I_2$$

$$\overline{I_3} = 1$$

Advantages

- Applicable under large deformations
- Able to capture Magneto-stiffening effects
- Can be used to predict post-buckling and bifurcation behaviours
- Applicable for any flat geometry in material configuration

Disadvantages

- The plate equations are nonlinear PDEs and have no known analytical or semi-analytical solutions. The solution can only be found via numerical methods
- Works only if the geometry is flat in material frame
- Assumes decoupling of membrane and bending energies in Cartesian coordinate system, which introduces error

Derivation

Procedure

1. Define the mathematical framework (model description)
2. Obtain Kinematical Relations (Strain Tensors, etc.)
3. Decoupling the Total Strain Energy Density
4. Evaluate the expression for 2D Bending Strain Energy Density
5. Evaluate the expression for 3D Membrane Strain Energy Densities
6. Evaluating Stresses
7. Enforcing Plane Stress condition ($\mathbf{P}_{33} = 0$)
8. Reduce the 3D membrane stress tensor into 2D
9. Formulate Action Integral & find Equilibrium equations + Natural BCs

Model Description

- Let there be a thin, flat membrane Ω_0, with thickness h, which is deformed to Ω_t under the deformation map φ: $\Omega_0 \rightarrow \Omega_t$
- Let **X** and **x** be the position vectors in undeformed and deformed configurations respectively.

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 \in \Omega_0 \qquad \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \Omega_t$$

- Let **u**(**X**) and w(**X**) be the in-plane and out-of-plane displacements, resp.

$$\mathbf{u}(\mathbf{X}) = u_1(\mathbf{X})\mathbf{e}_1 + u_2(\mathbf{X})\mathbf{e}_2$$

- Under the deformation map φ,

$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}) + w(\mathbf{X})\,\mathbf{e}_3$$

- Deformation Gradient,

$$\mathbf{F} = \nabla\varphi = \mathbf{I} + \nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w.$$

Membrane Strain Tensor

$$\mathbf{E} = \frac{1}{2}(\ \mathbf{F}^T\mathbf{F} - \mathbf{I}\ )$$

$$\begin{aligned}\mathbf{E} = \frac{1}{2}[\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + (\nabla w \otimes \mathbf{e}_3)(\mathbf{e}_3 \otimes \nabla w)\ ] \\ + \frac{1}{2}[\ \nabla w \otimes \mathbf{e}_3 + (\nabla w \otimes \mathbf{e}_3)\nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w + (\nabla\mathbf{u})^T(\mathbf{e}_3 \otimes \nabla w)\ ]\end{aligned}$$

$$\mathbf{E}_m = \frac{1}{2}[\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + \nabla w \otimes \nabla w]$$

Bending Strain Tensor (Curvature Tensor)

$$\mathbf{K} = -\nabla^2 w$$

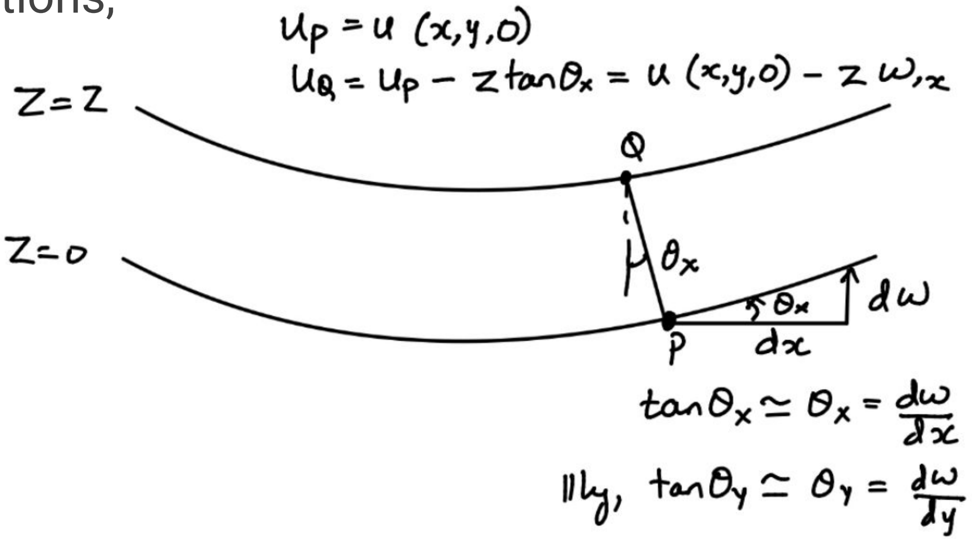
Kinematic Relations

- Based on Kirchhoff - Love Assumptions,

$$u_1(x,y,z) = u_1(x,y,0) - z\frac{\partial w}{\partial x}\Big|_{z=0}$$

$$u_2(x,y,z) = u_2(x,y,0) - z\frac{\partial w}{\partial y}\Big|_{z=0}$$

$$w(x,y,z) = w(x,y,0)$$



$$\mathbf{F}^{3D} = \begin{pmatrix} \mathbf{F}^{2D} & 0 \\ 0 & \lambda_3 \end{pmatrix} \qquad \mathbf{C}^{3D} = \begin{pmatrix} \mathbf{C}^{2D} & 0 \\ 0 & \lambda_3^2 \end{pmatrix}$$

Decoupling Total Strain Energy

- Total Strain Energy Density can be decomposed into Membrane and Bending components under small thickness assumption

$$\Psi(\mathbf{E}, \mathbf{K}) = \Psi_m(\mathbf{E}) + \Psi_b(\mathbf{K})$$

Derivation

- Substituting the kinematic relations obtained from Kirchhoff – Love assumptions into the membrane strain expression,

$$E_{\alpha\beta} \approx E_{\alpha\beta,0} + E_{\alpha\beta,z}\, z$$

$$E_{\alpha\beta,0} = \frac{1}{2}(u_{\alpha 0,\beta} + u_{\beta 0,\alpha} + w_{0,\alpha} w_{0,\beta})$$

$$E_{\alpha\beta,z} = -w_{0,\alpha\beta}.$$

- Substituting the Strain expression into the linear strain energy density expression,

$$\phi = \frac{1}{2}\mathbf{E} : \mathbb{C} : \mathbf{E}$$

$$\phi \approx \frac{1}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_0) + \frac{1}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_{,z} + \mathbf{E}_{,z} : \mathbb{C} : \mathbf{E}_0)z$$

- Integrating along thickness to get thickness-averages strain energy,

$$\Psi = \frac{h}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_0) + \frac{h^3}{24}(\mathbf{E}_{,z} : \mathbb{C} : \mathbf{E}_{,z})$$

Membrane Strain Energy Density

$$\Psi^{3D}_{Membrane} = \Psi^{3D}_{Isochoric} + \Psi^{3D}_{Volumetric} + \Psi^{3D}_{Magneto-elastic}$$

$$\Psi^{3D}_{Isochoric} = h[c_1(\overline{I}_1 - 3) + c_2(\overline{I}_2 - 3)]$$

$$\Psi^{3D}_{Volumetric} = h\left[\frac{1}{2}\kappa(J-1)^2\right]$$

$$\Psi^{3D}_{Magneto-elastic} = h[\hat{c}_2(\overline{I}_2 - 3)]$$

Bending Strain Energy Density

$$\Psi^{2D}_{Bending} = \frac{Yh^3}{24(1-\nu^2)}\left[\nu(\mathrm{tr}\mathbf{K})^2 + (1-\nu)\mathbf{K}:\mathbf{K}\right]$$

Membrane PK2 Stress Tensor

$$\left.\frac{\partial \Psi^{3D}_{Isochoric}}{\partial \mathbf{C}^{3D}}\right|_{2D} = hc_1(J\lambda_3)^{-2/3}\left[\mathbf{I} - \frac{I_1+\lambda_3^2}{3}\mathbf{C}^{-1}\right] + hc_2(J\lambda_3)^{-4/3}\left[(I_1+\lambda_3^2)\mathbf{I} - \mathbf{C} - \frac{2}{3}(I_2+\lambda_3^2 I_1)\mathbf{C}^{-1}\right]$$

$$\left.\frac{\partial \Psi^{3D}_{Volumetric}}{\partial \mathbf{C}^{3D}}\right|_{2D} = h\kappa(J\lambda_3)(J\lambda_3-1)\mathbf{C}^{-1}$$

$$\left.\frac{\partial \Psi^{3D}_{Magneto-elastic}}{\partial \mathbf{C}^{3D}}\right|_{2D} = h\hat{c}_2(J\lambda_3)^{-4/3}\left[(I_1+\lambda_3^2)\mathbf{I} - \mathbf{C} - \frac{2}{3}(I_2+\lambda_3^2 I_1)\mathbf{C}^{-1}\right]$$

$$\mathbf{N} = 2\left(\left.\frac{\partial \Psi^{3D}_{Isochoric}}{\partial \mathbf{C}^{3D}}\right|_{2D} + \left.\frac{\partial \Psi^{3D}_{Volumetric}}{\partial \mathbf{C}^{3D}}\right|_{2D} + \left.\frac{\partial \Psi^{3D}_{Magneto-elastic}}{\partial \mathbf{C}^{3D}}\right|_{2D}\right)$$

Couple Stress Tensor

$$\mathbf{M} = \frac{\partial \Psi^{2D}_{Bending}}{\partial \mathbf{K}} = \frac{Yh^3}{12(1-\nu^2)}\left[\nu(\mathrm{tr}\mathbf{K})\mathbf{I} + (1-\nu)\mathbf{K}\right]$$

Plane Stress Condition

$$\mathbf{P}_{33} = (\mathbf{FN})_{33} = 0$$

$$2h\lambda_3 \left[\frac{1}{3}c_1(J\lambda_3)^{-2/3}\left(2 - \frac{I_1}{\lambda_3^2}\right) + (c_2 + \hat{c}_2)(J\lambda_3)^{-4/3}\left(-\frac{2I_2}{3\lambda_3^2}\right) + \kappa(J\lambda_3)(J\lambda_3 - 1)\frac{1}{\lambda_3^2}\right] = 0$$

Nonlinear Algebraic Equation for finding suitable value of λ_3

Variational Analysis

- Let $\mathbf{f}(\mathbf{X}) = \mathbf{f}_m(\mathbf{X}) + f_b(\mathbf{X})\mathbf{e}_3$ be the body force and $\mathbf{t}$ be the traction acting over the boundary Γ_0.
- Then for quasi-static loading, the Lagrangian can be written as

$$\Pi[\mathbf{u}, w] = V - W_{\text{ext}} = \int_{\Omega_0} \left[\Psi_{\text{Total}}(\mathbf{E}, \mathbf{K}) - \mathbf{f}_m \cdot \mathbf{u} - f_b\, w\right] \mathrm{d}\Omega - \oint_{\Gamma_0} \mathbf{t} \cdot (\mathbf{u} + w\mathbf{e}_3)\, \mathrm{d}\Gamma.$$

- Assume $\boldsymbol{\eta}$ and ζ be some arbitrary admissible functions. Then for some small parameter ε, the perturbed displacement vectors about the equilibrium position can be written as

$$\tilde{\mathbf{u}} = \mathbf{u} + \varepsilon\boldsymbol{\eta}, \qquad \tilde{w} = w + \varepsilon\zeta;$$

- Evaluating First Variation of Lagrangian and putting it to zero at equilibrium condition yields –

$$\int_{\Omega_0} \Big[-\left[\nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] + \mathbf{f}_m\right] \cdot \eta - \left[\nabla \cdot (\mathbf{N}\nabla w) + \nabla \cdot (\nabla \cdot \mathbf{M}) + f_b\right]\zeta \Big] d\Omega$$

$$+ \oint_{\Gamma_0} \Big[\Big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\,\mathbf{n} - \mathbf{t}_m \Big) \cdot \eta + \Big((\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}) \cdot \mathbf{n} - t_b \Big)\zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla \zeta \Big] d\Gamma = 0$$

- Euler–Lagrange Equations:

$$\nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] + \mathbf{f}_m \quad = \mathbf{0}$$

$$\nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N}\nabla w] + f_b = 0$$

- Natural Boundary Conditions

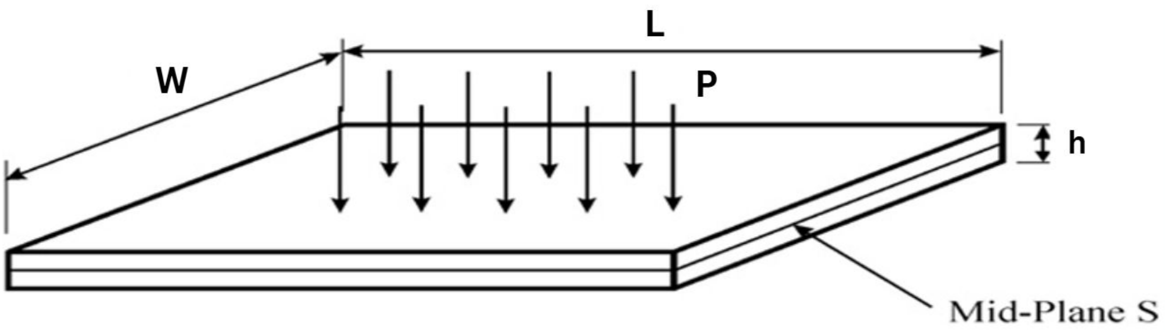
$$\oint_{\Gamma_0} \Big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\,\mathbf{n} - \mathbf{t}_m \Big) \cdot \eta \; d\Gamma = 0$$

$$\oint_{\Gamma_0} \Big(([\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}] \cdot \mathbf{n} - t_b)\zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla \zeta \Big) d\Gamma = 0$$

Membrane under Transverse Loading

Problem Description

A thin cuboidal hyperelastic membrane made up of a magneto-active polymer matrix and having dimensions L x W x h, is kept such that all boundary edges are kept fixed. A uniform load P acts over the surface of the membrane in the downward direction.



Body Forces

$\mathbf{f}_m = \mathbf{0}, \qquad f_b = -P$

Natural Boundary Conditions

$\mathbf{n} \cdot \mathbf{M}\mathbf{n} = 0 \qquad \text{along all edges}$

Dirichlet Boundary Conditions

$$\mathbf{u}(0,y) = \mathbf{u}(L,y) = \mathbf{u}(x,0) = \mathbf{u}(x,W) = \mathbf{0} \qquad \forall\ x,y$$

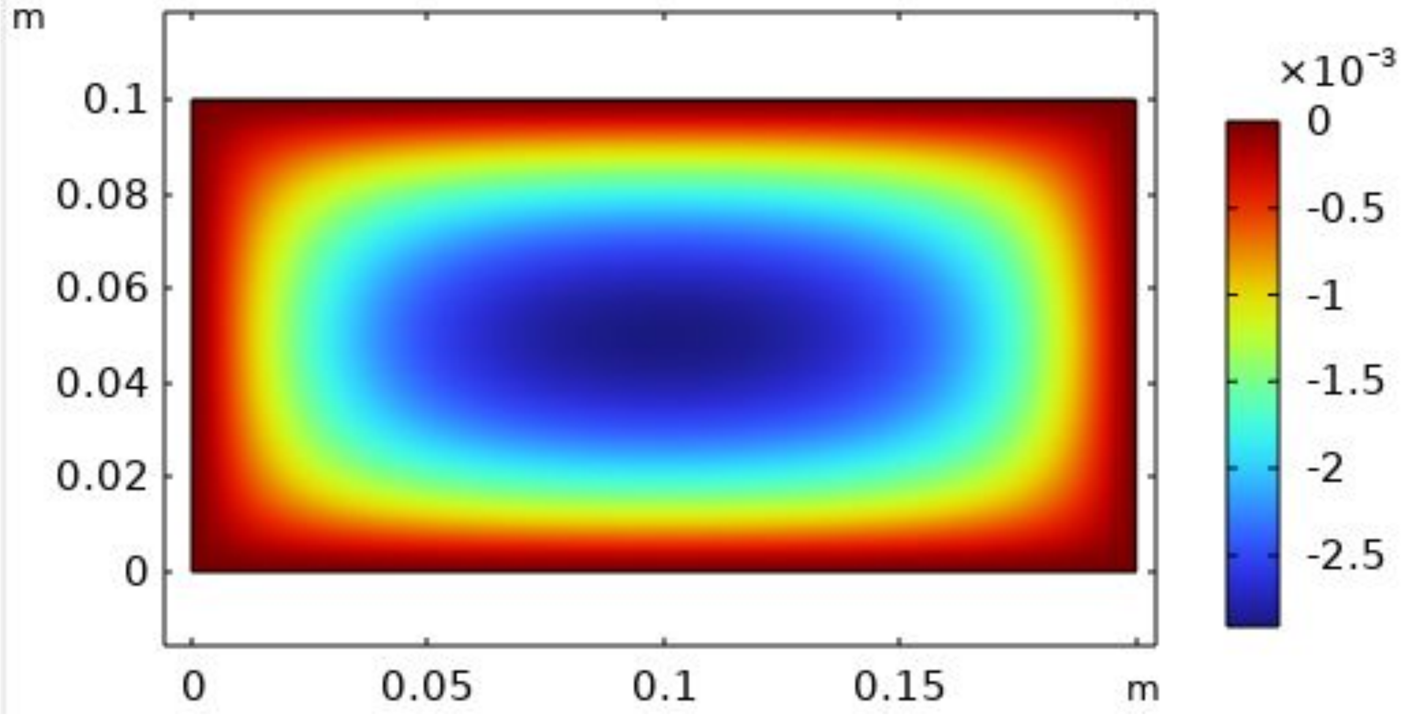
$$w(0,y) = w(L,y) = w(x,0) = w(x,W) = 0 \qquad \forall\ x,y$$

Parameter Values

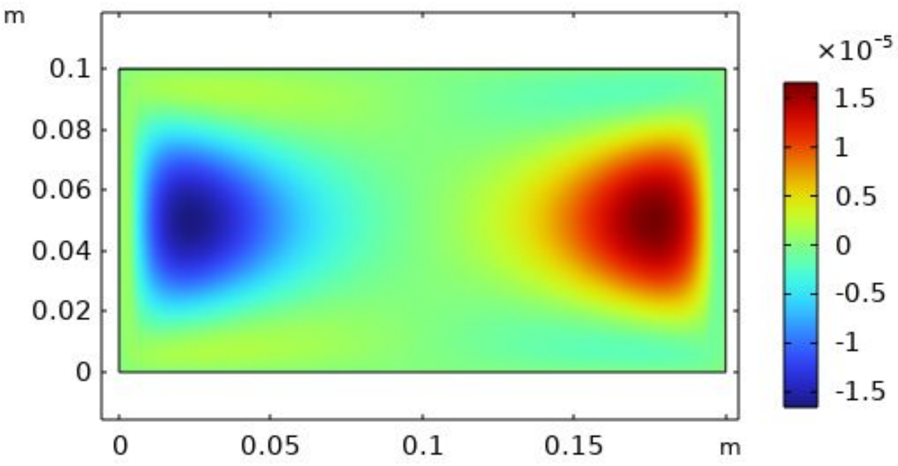
Name	Value	Description
L	0.2 m	Length
W	0.1 m	Width
h	0.002 m	Thickness
μ_0	$4\pi \times 10^{-7}$ H/m	Magnetic Permeability
c_1	108 kPa	Mooney-Rivlin Constant 1
c_2	137 kPa	Mooney-Rivlin Constant 2
a_1	4.97 m/A	Magneto-elastic coupling constant 1
a_2	$9.147\mu_0$ m/A	Magneto-elastic coupling constant 2
b_1	174.685 kPa	Magneto-elastic coupling constant 3
b_2	$13.024\mu_0$ m/A	Magneto-elastic coupling constant 4
ν	0.49	Poisson Ratio
μ	$2(c_1 + c_2)$	Shear Modulus
Y	$2\mu(1 + \nu)$	Young's Modulus
κ	$Y/(3(1 - 2\nu))$	Bulk Modulus

[2] Garai & Haldar

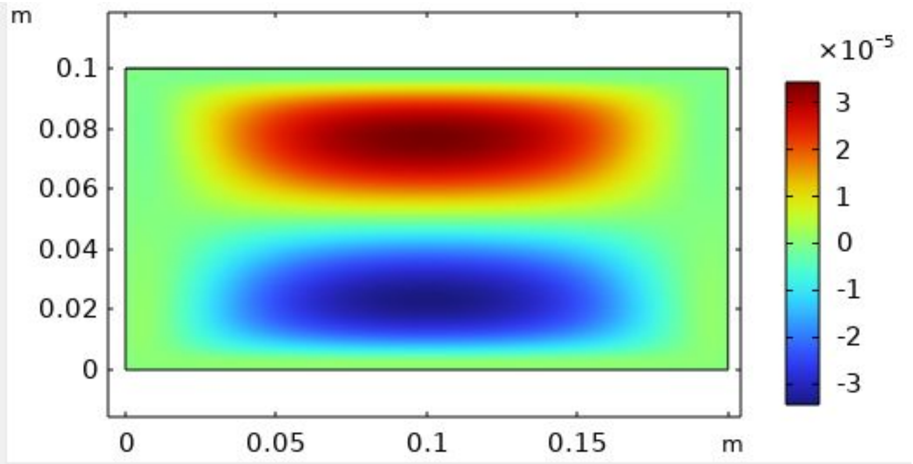
Plot of w(x,y) at P = 1 kPa



Plots of $u_1(x,y)$ and $u_2(x,y)$ at P = 1 kPa

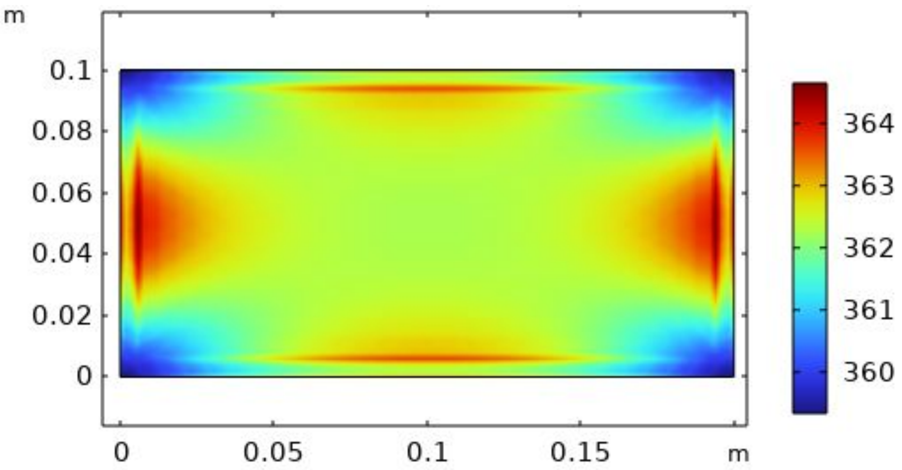


Plot of $u_1(x,y)$

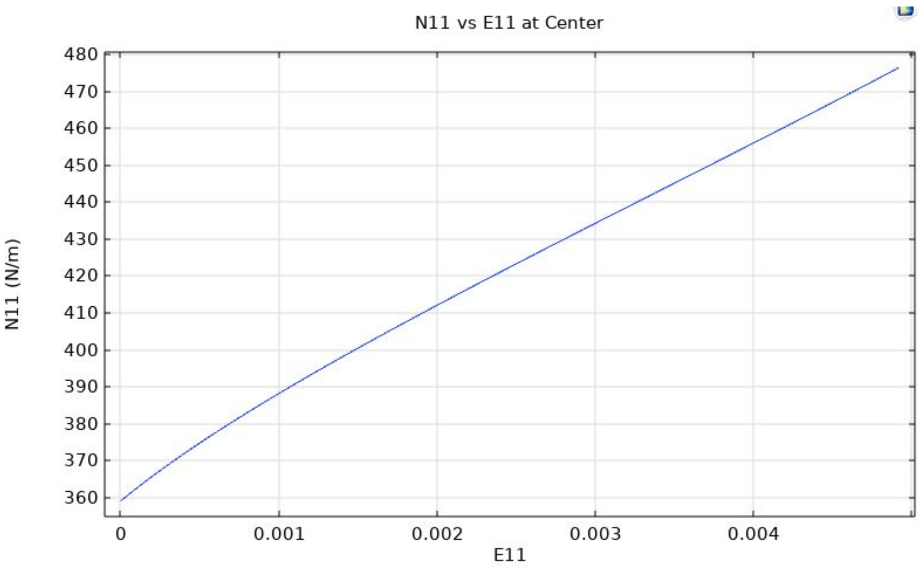


Plot of $u_2(x,y)$

Stress Plot and Stress - Strain Curve

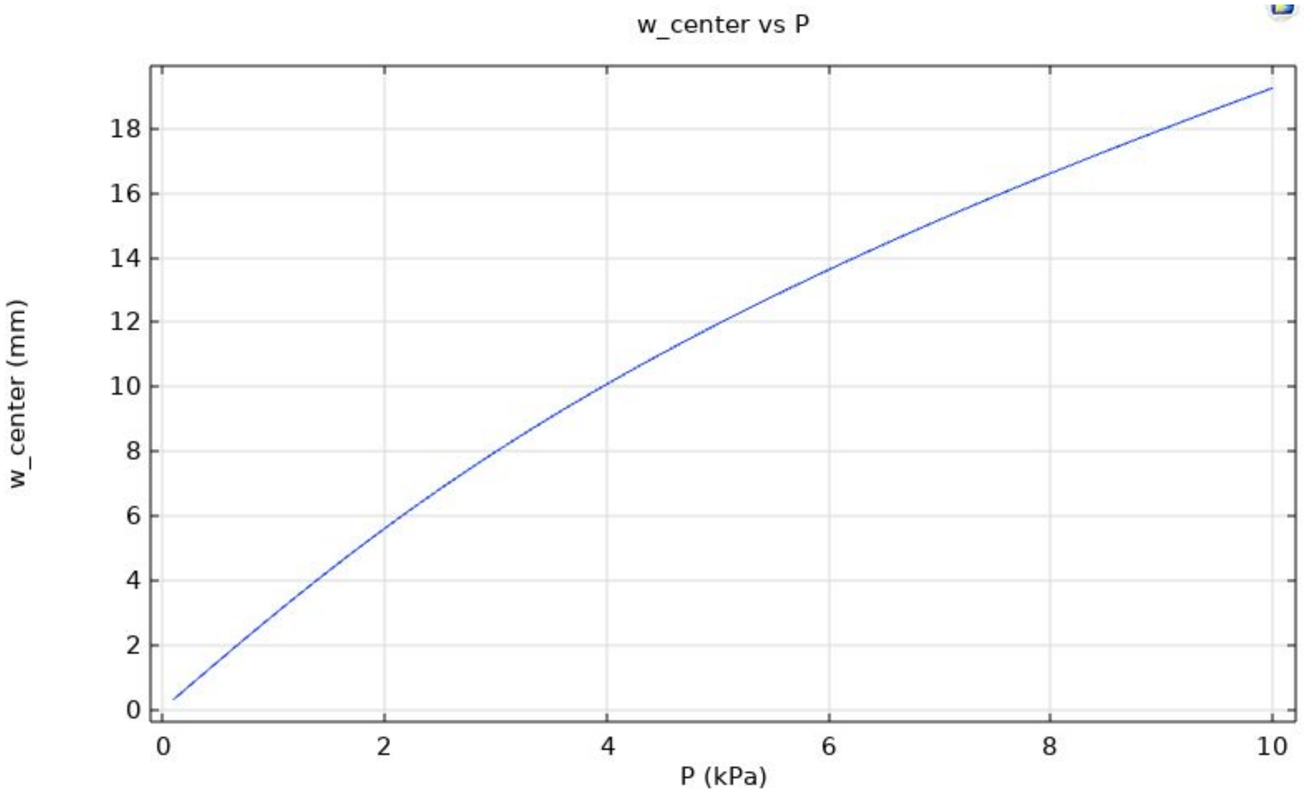


Plot of N_{11}

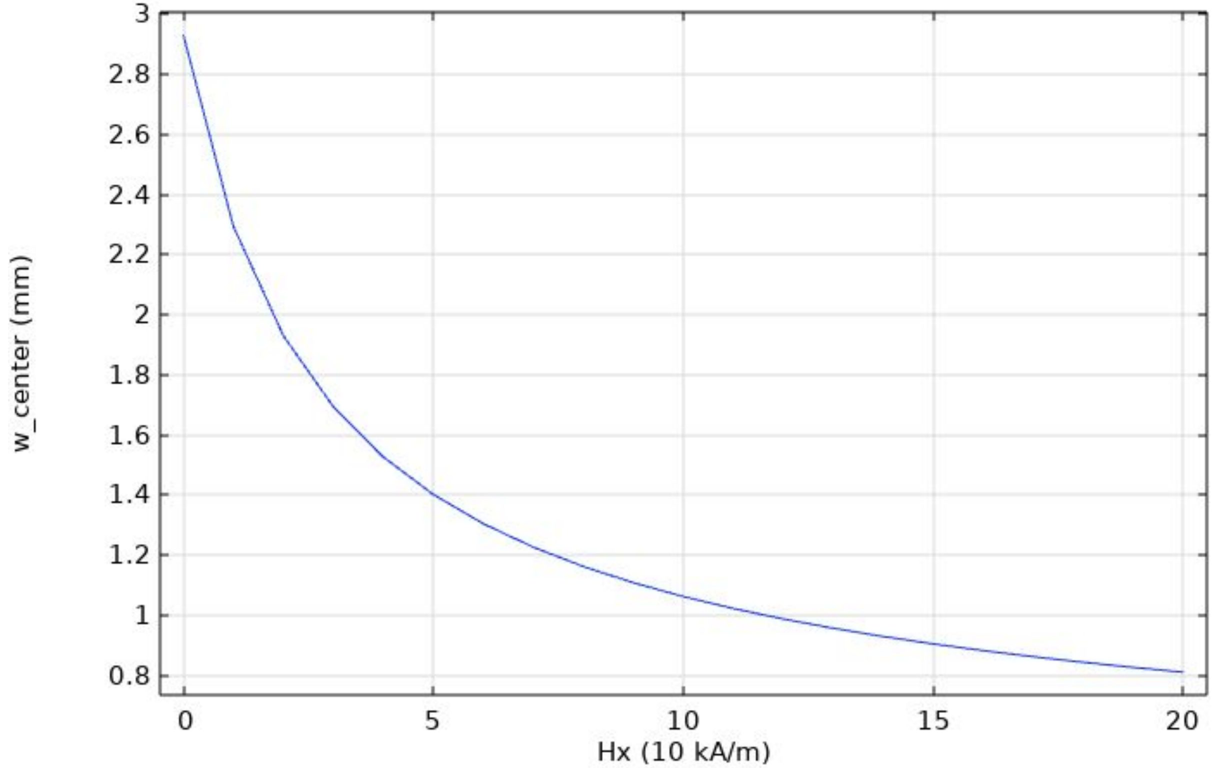


N_{11} vs E_{11}

w_{Center} vs Load P



w_{Center} vs H_X at P = 1 kPa



Future Direction

Future Directions

1. Improving the solving algorithm to analyse post buckling behaviour and capture wrinkling behaviour

2. Formulate a model for thin membranes having some general curvilinear geometry in material frame

3. Understand and implement a suitable energy decoupling method that induce lesser error at arbitrary surface geometries

4. Device method to control wrinkling using magnetic field actuation

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