Large Deformation Modelling of Thin, Flexible and Tunable Plates

Literature Review

AE 493 B.Tech. Project Stage-I

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Nomenclature

Δ	2 Dimensional Laplacian Operator
∇	2 Dimensional Gradient Operator
$\mathbf{C}$	Right Cauchy-Green Tensor
$\mathbf{E}$	Membrane Green Strain tensor
$\mathbf{e}_i$	Standard orthonormal basis in Euclidean Space. ($i \in \{1, 2, 3\}$)
$\mathbf{F}$	Deformation gradient tensor
$\mathbf{I}$	2 Dimensional Identity Tensor
$\mathbf{u}$	Vector of in-plane displacements
Ω_0	Domain of points in Undeformed Configuration
Ω_t	Domain of points in the Deformed Configuration at time t
ϕ	3 Dimensional Total Strain Energy Density
Ψ	2 Dimensional Total Strain Energy Density at mid-plane
Ψ_b	2 Dimensional Bending Strain Energy Density at mid-plane
Ψ_m	2 Dimensional Membrane Strain Energy Density at mid-plane
φ	Deformation map
w	Transverse displacement

Abstract

Thin plates exhibit a rich spectrum of mechanical responses when subjected to in-plane or transverse loading, particularly when the deformation is large enough for geometric nonlinearities to become significant. Classical theories such as the Kirchhoff–Love and Föppl–von Kármán (FvK) models accurately describe small to moderately large deflections, but they fail to predict several important behaviours observed in experiments, including wrinkling, post-buckling paths, and non-classical bifurcations. Motivated by these limitations, this report develops a large-deformation plate model based on the Saint Venant–Kirchhoff (SVK) constitutive law. The formulation incorporates full Green–Lagrange strains, a linearised curvature tensor, and a decoupled membrane–bending energy structure derived using the Kirchhoff–Love assumptions. The resulting Euler–Lagrange equations naturally couple in-plane and transverse responses and remain valid for moderate to large displacements.

To study the model numerically, the governing fourth-order system is recast into a second-order form and solved using the finite element method in COMSOL Multiphysics. Two loading scenarios are analysed in detail: uniaxial stretching and uniform transverse loading. The simulations reveal key nonlinear features of the model, including membrane stiffening, the sensitivity of wrinkling to imperfections, and the strong influence of boundary constraints on deformation patterns. The results demonstrate that the large-deformation framework captures behaviours beyond the reach of classical plate theories, making it a promising basis for future studies on stability and post-buckling of thin elastic structures.

Chapter 1

Classical Models

Thin plates are structural components whose thickness is very small compared to their length and width. Because of this slender geometry, their response to loads differs from that of bulk three-dimensional bodies. When a thin plate is subjected to forces normal to its surface, it prefers to bend rather than compress through its thickness, and most of the important stresses develop in the mid-surface. These features allow the behaviour of plates to be described using simplified two-dimensional models instead of the full three-dimensional elasticity framework.

In this chapter, I present an overview of thin plate behaviour under small deformations. I begin by outlining the key assumptions that form the foundation of plate models. This is followed by a discussion of the Classical Plate Theory, where its governing equations and limitations are introduced. To address these shortcomings, the Föppl–von Kármán model is then presented as an improved formulation that incorporates geometric nonlinearities. After describing its equations and limitations, the chapter concludes by highlighting why more advanced models are required for studying complex plate behaviour beyond the scope of these classical theories.

1.1 Primary Assumptions

1.1.1 Dimension Reduction

Because of the small thickness, the out-of-plane normal and shear stresses are typically much smaller than the in-plane stresses and strains. Moreover, when subjected to transverse loads (i.e., loads normal to the plate surface), thin plates primarily deform through bending rather than through direct through-thickness compression. This bending deformation leads to the development of in-plane stress components that are sufficient to balance the applied loads and maintain equilibrium.

These observations justify the use of the plane stress assumption for thin plates, wherein the transverse normal and shear stresses are considered negligible compared to the in-plane components. Adopting this assumption enables a more concise and tractable formulation of the governing equations for thin plate behaviour. This forms the basis for the modelling approach followed in this work.

1.1.2 Kirchhoff – Love Assumptions

The Kirchhoff–Love assumptions describe how the material fibers through the thickness of a thin plate behave during deformation. These fibers are initially straight and perpendicular to the mid-surface, and because the plate is very thin, their deformation is highly restricted.

1. Normals remain straight after deformation. A line that is normal to the mid-surface in the undeformed plate does not bend or warp as the plate deforms. This eliminates any through-thickness distortion and allows the full displacement field of the plate to be expressed in terms of the mid-surface variables.

2. Normals keep their length and remain perpendicular to the deformed surface. This means no stretching occurs along the plate thickness, and the transverse shear strains vanish ($\gamma_{xz} = \gamma_{yz} = 0$). As a result, the plate undergoes pure bending without shear deformation, which is a reasonable approximation for thin plates.

1.1.3 Small Deformations

Classical plate models rely on the assumption that the deformations and rotations experienced by the plate are sufficiently small. Under this condition, the displacement gradients remain small, and the nonlinear terms that appear in the exact strain–displacement relations—such as products of displacement derivatives—become negligible compared to the linear terms. As a result, the strain tensor can be approximated using only its linear part, greatly simplifying the mathematical formulation.

This linearisation has two important consequences. First, the governing equations become linear differential equations, which are far easier to analyse analytically and numerically. Second, the differences among the various stress measures used in continuum mechanics (Cauchy, first Piola–Kirchhoff, and second Piola–Kirchhoff stresses) are insignificant when strains are small. In this regime, the deformation gradient is close to the identity, and the push-forward and pull-back operations relating these stress tensors produce only minor differences. Thus, any of these stress measures can be used within a small-deformation theory without affecting the model's accuracy.

1.1.4 Isotropic Material

Although not a strict constraint, plate models are often discussed in the context of Isotropic bodies in much of the literature. So, I will also assume the material to be linearly isotropic for the entire discussion. This leads to the following 2D constitutive relation between the stress and strain.

$$\boldsymbol{\sigma} = \frac{\mathrm{E}}{1-\nu^2}\left[\nu(tr\mathbf{E})\mathbf{I} + (1-\nu)\mathbf{E}\right]$$

1.2 Classical Plate Theory

Classical Plate Theory (CPT), also known as the Kirchhoff–Love plate theory, is one of the earliest and most widely used models for analysing thin plates. It provides a fully linear formulation, allowing closed-form or semi-analytical solutions for many practical loading and boundary conditions. The theory is built directly on the assumptions introduced earlier: plane stress, small strains, and the Kirchhoff–Love kinematic hypotheses.

1.2.1 Kinematic Relations

Under the Kirchhoff–Love assumptions, the normal fibers to the mid-surface remain straight, inextensible, and perpendicular to the deformed mid-surface. This restricts the displacement field across the thickness to the form

$$w = w(x,y), \qquad u(x,y,z) = -z\, w_{,x}(x,y), \qquad v(x,y,z) = -z\, w_{,y}(x,y),$$

where w is the transverse displacement, and (u, v) are the in-plane displacements.
Using the small-strain assumption, the strain tensor reduces to

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right),$$

which gives the in-plane strain components

$$\varepsilon_{xx} = u_{,x}, \qquad \varepsilon_{yy} = v_{,y}, \qquad \gamma_{xy} = u_{,y} + v_{,x}.$$

Substituting the expressions for u and v, we obtain

$$\varepsilon_{xx} = -z\, w_{,xx}, \qquad \varepsilon_{yy} = -z\, w_{,yy}, \qquad \gamma_{xy} = -2z\, w_{,xy}.$$

These relations clearly show that bending induces linear variations of strain through the thickness.

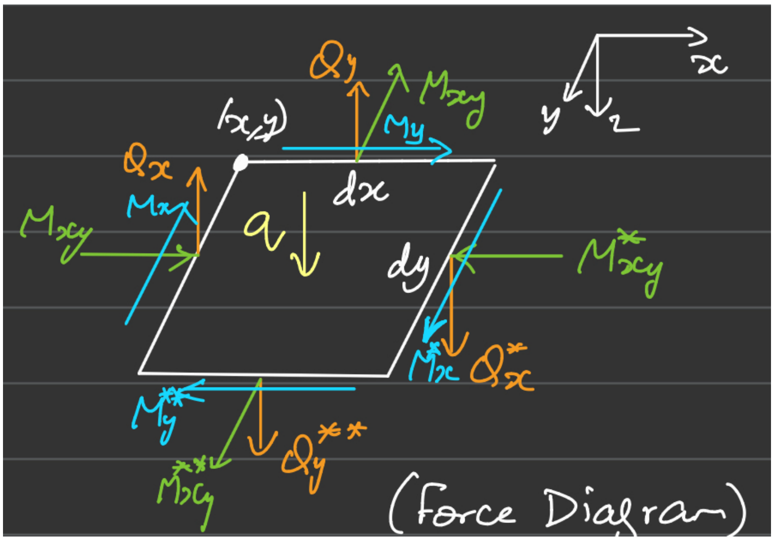

Figure 1.1: Plate Under External Loads

1.2.2 Governing Equation

From the strain expressions and the linear elastic constitutive law, the bending moments per unit length are

$$M_{xx} = -D(w_{,xx} + \nu w_{,yy}), \qquad M_{yy} = -D(w_{,yy} + \nu w_{,xx}), \qquad M_{xy} = -D(1-\nu)w_{xy},$$

where the flexural rigidity of the plate is

$$D = \frac{Eh^3}{12(1-\nu^2)}.$$

Force equilibrium in the transverse direction yields the classical plate equation:

$$D\nabla^4 w = q(x,y),$$

where $q(x,y)$ denotes the distributed transverse load. This biharmonic equation governs the deflection of an isotropic thin plate under small deformations.
The transverse shear forces per unit length follow as

$$Q_x = -D\,(\nabla^2 w)_{,x}, \qquad Q_y = -D\,(\nabla^2 w)_{,y}.$$

Although CPT assumes shear strains vanish, these shear forces appear as reaction quantities obtained from moment equilibrium.

1.2.3 Drawbacks

While Classical Plate Theory is extremely useful for small-deflection problems, its accuracy deteriorates when the plate experiences moderate to large transverse displacements. In such cases:

- in-plane stretching of the mid-surface cannot be ignored,
- nonlinear strain terms such as $(w_{,x})^2$ and $(w_{,y})^2$ become comparable to the linear terms,
- bending and stretching become strongly coupled, altering the plate's stiffness,
- the model fails to predict essential behaviours such as post-buckling, wrinkling, and membrane stiffening.

These limitations indicate that CPT is suitable only for small deflections. To analyze plates undergoing moderate rotations while still maintaining small strains, a geometrically nonlinear model is needed. This leads naturally to the Föppl–von Kármán (FvK) equations, discussed in the next section.

1.3 The Föppl–von Kármán Model

The Föppl–von Kármán (FvK) model extends the Classical Plate Theory by incorporating the leading-order geometric nonlinearities that arise when a plate undergoes moderate rotations while still experiencing small strains. In this regime, the nonlinear term $\nabla w \otimes \nabla w$, which contains products of transverse displacement gradients, becomes comparable in magnitude to the in-plane displacement gradient $\nabla \mathbf{u}$. Consequently, this term must be retained in the strain–displacement relation. Higher-order nonlinearities such as $(\nabla \mathbf{u})^T(\nabla \mathbf{u})$ remain much smaller and are neglected. This balance produces a minimal nonlinear model that captures bending–stretching coupling without the full complexity of general nonlinear elasticity.

1.3.1 Kinematic Relations

When the plate deflects significantly in the transverse direction, the slopes $w_{,x}$ and $w_{,y}$ can no longer be considered infinitesimal. The Green–Lagrange strain tensor, truncated to include only the dominant nonlinear terms, yields the membrane strains

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2}(w_{,x})^2, \qquad \varepsilon_{yy} = v_{,y} + \frac{1}{2}(w_{,y})^2,$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y}.$$

The terms involving $w_{,x}$ and $w_{,y}$ represent in-plane stretching induced by bending, a feature absent in the classical theory. These contributions are responsible for phenomena such as membrane stiffening and post-buckling behaviour.

1.3.2 Governing Equations

To satisfy in-plane equilibrium automatically, the in-plane stress resultants are expressed through the Airy stress potential χ:

$$\sigma_{xx} = \chi_{,yy}, \qquad \sigma_{yy} = \chi_{,xx}, \qquad \sigma_{xy} = -\chi_{,xy}.$$

This ensures the equilibrium equations $\sigma_{xx,x} + \sigma_{xy,y} = 0$ and $\sigma_{xy,x} + \sigma_{yy,y} = 0$ hold identically in the absence of body forces.
For compact notation, the FvK model uses the von Kármán bracket

$$[f, g] = f_{,xx}g_{,yy} + f_{,yy}g_{,xx} - 2f_{,xy}g_{,xy},$$

which naturally appears when expressing compatibility between in-plane strains and bending curvatures.
With these definitions, the coupled Föppl–von Kármán equations are written as

$$(1) \quad D\nabla^4 w \; - \; [\chi, w] = q(x, y),$$

$$(2) \quad \nabla^4 \chi \; + \; \frac{Eh}{2}\,[w, w] = 0.$$

The first equation represents transverse force equilibrium and includes the additional nonlinear term $[\chi, w]$ that couples the bending deformation to the in-plane stresses. The second equation enforces compatibility between the in-plane strains generated by (u, v) and the membrane strain induced by $w(x, y)$.

1.3.3 Drawbacks

Although the FvK model captures moderate rotations and the leading-order geometric nonlinearities, it is not suitable for large deformations or for capturing highly nonlinear structural responses. Its major limitations include:

- Inaccuracy under large deflections, where higher-order nonlinear strain terms become significant,
- Failure to accurately describe post-buckling behaviour for many loading and boundary conditions,
- Incorrect predictions of bifurcation types and stability characteristics in certain plate configurations.

A notable example is provided by Healey *et al.*, who studied a uniaxially stretched plate fixed at one end and subjected to an imposed strain at the opposite boundary. The FvK model predicts a pitchfork bifurcation in the transverse displacement, whereas more complete nonlinear models correctly yield an isola-centre bifurcation. Experimental observations support the latter behaviour, indicating that the FvK theory may qualitatively misrepresent the stability landscape in some regimes. The adjoining figure illustrates the difference in the bifurcation diagrams obtained from the FvK formulation and from more advanced nonlinear plate models. These limitations highlight the need for more sophisticated plate theories that can accurately

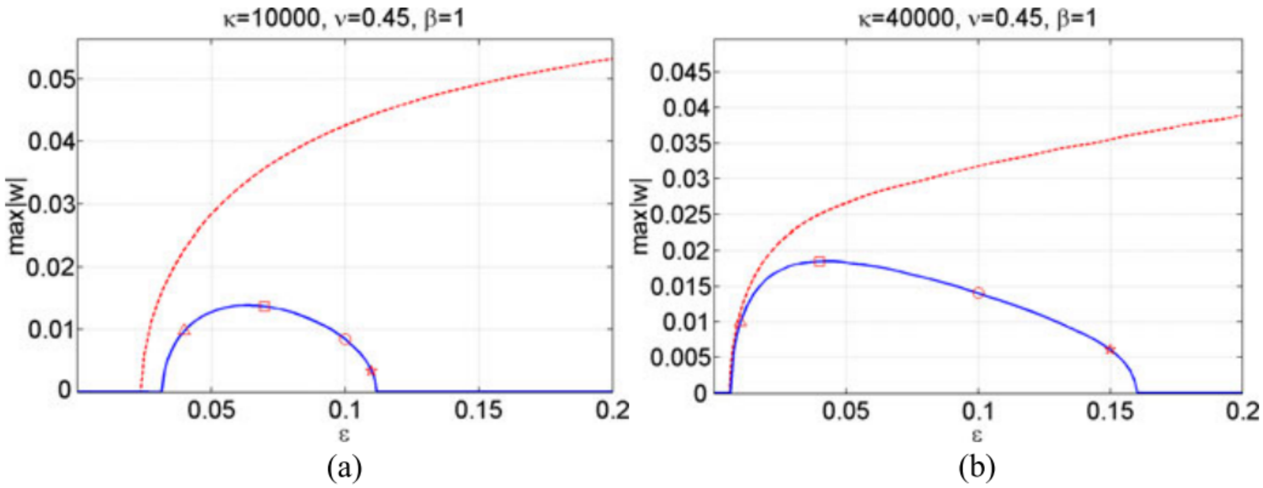

Figure 1.2: Bifurcation diagrams, maximum amplitude, max $|w|$, vs. macroscopic strain ϵ; film thicknesses (a) h = 0.1 mm; (b) h = 0.05 mm; red: Fv-K, blue: Healey (Advanced)

describe large deformations, complex stability phenomena, and post-buckling behaviour. The next chapter will examine these advanced models in greater detail.

$$\begin{aligned}\mathbf{E} = &\frac{1}{2}[\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + (\nabla w\otimes\mathbf{e}_3)(\mathbf{e}_3\otimes\nabla w)\ ]\\ &+\frac{1}{2}[\ \nabla w\otimes\mathbf{e}_3 + (\nabla w\otimes\mathbf{e}_3)\nabla\mathbf{u} + \mathbf{e}_3\otimes\nabla w + (\nabla\mathbf{u})^T(\mathbf{e}_3\otimes\nabla w)\ ]\end{aligned}$$

Chapter 2

Large Deformation Model

In this chapter, we develop the mathematical model used to describe the large-deformation behaviour of flat membranes. Healey *et al.* demonstrated that the Saint Venant–Kirchhoff (SVK) model provides an effective framework for analysing large deformations in flat plates subjected to uniaxial loading. Their work shows that wrinkling arises as a bifurcation phenomenon and that the SVK model captures this behaviour more accurately than the classical Föppl–von Kármán model, whose bifurcation predictions do not qualitatively agree with experimental observations.

We begin by listing the assumptions underlying the model; these assumptions are referenced throughout the chapter and become clearer as the development progresses. Section 2.2 establishes the geometric and kinematic framework required for the subsequent derivations. Sections 2.3 through 2.7 present the general form of the governing equations for the membrane.

2.1 Assumptions

1. The plate is very thin ($\frac{h}{L} \ll 1$)
2. Kirchhoff-Love Assumptions:
    1) Normals to the flat surface remain straight after deformation.
    2) Normals remain of the same length and stay normal to the deformed surface after deformation.
3. Linearised strain energy is assumed for decoupling the membrane and bending strain energies.
4. Linear Isotropic Material is assumed for both the membrane constitutive relation and the bending (curvature) constitutive relation.
5. Linearised Curvature tensor (Bending Strain, $\mathbf{K}$) is used instead of the exact non-linear expression.

2.2 Model Description

Consider a thin rectangular membrane of in-plane dimensions $L \times W$ and uniform thickness h. We introduce a Cartesian coordinate system such that the origin lies at the bottom-left corner of the reference (undeformed) mid-plane, with the $\mathbf{e}_1$- and $\mathbf{e}_2$-axes aligned along the length L and width W of the membrane, respectively. The $\mathbf{e}_3$-axis is taken to be the unit normal to the mid-plane, pointing upward.

Let Ω_0 and Ω_t denote the domains occupied by the membrane in the undeformed and deformed configurations, respectively. Since the membrane is thin, we represent its mid-surface in the reference configuration by

$$\Omega_0 = (0, L) \times (0, W) \subset \mathbb{R}^2.$$

A material point on the mid-surface is identified by its reference position vector

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 \in \Omega_0.$$

After deformation, the same material point occupies the position

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \Omega_t.$$

We decompose the displacement into in-plane and transverse (out-of-plane) components. Let $\mathbf{u}(\mathbf{X})$ be the in-plane displacement field and $w(\mathbf{X})$ the transverse displacement:

$$\mathbf{u}(\mathbf{X}) = u_1(\mathbf{X})\mathbf{e}_1 + u_2(\mathbf{X})\mathbf{e}_2.$$

The 2-dimensional gradient operator associated with the reference mid-surface is

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial X_1} + \mathbf{e}_2 \frac{\partial}{\partial X_2}.$$

The deformation map $\varphi : \Omega_0 \to \Omega_t$ relates the reference and deformed positions as

$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}) + w(\mathbf{X})\,\mathbf{e}_3.$$

This kinematic description assumes that the membrane undergoes moderate to large transverse displacement but remains thin enough for a mid-surface representation to be appropriate (Assumption 1).

The deformation gradient is

$$\mathbf{F} = \nabla\varphi.$$

We evaluate each term:

$\nabla\mathbf{X} = \dfrac{\partial}{\partial X_\alpha}(X_\beta\mathbf{e}_\beta) \otimes \mathbf{e}_\alpha = \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \mathbf{I}$

$\nabla(w\mathbf{e}_3) = \dfrac{\partial}{\partial X_\alpha}(w\mathbf{e}_3) \otimes \mathbf{e}_\alpha = \mathbf{e}_3 \otimes (\dfrac{\partial w}{\partial X_\alpha}\mathbf{e}_\alpha) = \mathbf{e}_3 \otimes \nabla w$

Hence, the deformation gradient is

$$\boxed{\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w.}$$

2.3 Membrane Strain Tensor

By definition, the 3-dimensional strain is

$$\mathbf{E} = \frac{1}{2}(\ \mathbf{F}^T\mathbf{F} - \mathbf{I}\ )$$

Upon expansion, we get

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2}(\ \mathbf{F}^T\mathbf{F} - \mathbf{I}\ ) \\
&= \frac{1}{2}[\ (\mathbf{I} + \nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w)^T(\mathbf{I} + \nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w) - \mathbf{I}\ ] \\
&= \frac{1}{2}[\ (\mathbf{I}^T + (\nabla\mathbf{u})^T + (\mathbf{e}_3 \otimes \nabla w)^T)(\mathbf{I} + \nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w) - \mathbf{I}\ ] \\
&= \frac{1}{2}[\ (\mathbf{I} + (\nabla\mathbf{u})^T + \nabla w \otimes \mathbf{e}_3)(\mathbf{I} + \nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w) - \mathbf{I}\ ] \\
&= \frac{1}{2}[\ \mathbf{I} + (\nabla\mathbf{u})^T + \nabla w \otimes \mathbf{e}_3 + \nabla\mathbf{u} + (\nabla\mathbf{u})^T\nabla\mathbf{u} + (\nabla w \otimes \mathbf{e}_3)\nabla\mathbf{u} + \mathbf{e}_3 \otimes \nabla w + (\nabla\mathbf{u})^T(\mathbf{e}_3 \otimes \nabla w) \\
&\quad + (\nabla w \otimes \mathbf{e}_3)(\mathbf{e}_3 \otimes \nabla w) - \mathbf{I}\ ] \\
&= \frac{1}{2}[\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + (\nabla w \otimes \mathbf{e}_3)(\mathbf{e}_3 \otimes \nabla w)\ ] + \frac{1}{2}[\ \nabla w \otimes \mathbf{e}_3 + (\nabla w \otimes \mathbf{e}_3)\nabla\mathbf{u} \\
&\quad + \mathbf{e}_3 \otimes \nabla w + (\nabla\mathbf{u})^T(\mathbf{e}_3 \otimes \nabla w)\ ]
\end{aligned}$$

The three-dimensional strain tensor is reduced to a two-dimensional form by separating the membrane strains from the curvature-induced strains. This is justified by the small thickness of the membrane, due to which the transverse strains are significantly smaller than the in-plane strains. I've provided a detailed justification of this assumption in section 2.4.
Consistent with this assumption, we eliminate out-of-plane contributions from the membrane strain tensor by evaluating the relevant tensor products on arbitrary in-plane vectors $\mathbf{v} \in \Omega_0$.
First, the purely out-of-plane quadratic term reduces to an in-plane dyadic product

$$(\nabla w \otimes \mathbf{e}_3)(\mathbf{e}_3 \otimes \nabla w) = (\mathbf{e}_3 \cdot \mathbf{e}_3)(\nabla w \otimes \nabla w) = \nabla w \otimes \nabla w$$

Next, all mixed terms vanish when acting on in-plane vectors due to orthogonality with $\mathbf{e}_3$.

$$(\nabla w \otimes \mathbf{e}_3)\mathbf{v} = (\mathbf{e}_3 \cdot \mathbf{v})\nabla w = 0$$
$$[(\nabla w \otimes \mathbf{e}_3)(\nabla\mathbf{u})]\mathbf{v} = (\nabla w \otimes \mathbf{e}_3)\mathbf{v}' = (\mathbf{e}_3 \cdot \mathbf{v}')\nabla w = 0 \quad (\mathbf{v}' = (\nabla\mathbf{u})\mathbf{v} \in \mathbb{R}^2)$$
$$[(\nabla\mathbf{u})^T(\mathbf{e}_3 \otimes \nabla w)]\mathbf{v} = (\mathbf{v} \cdot \nabla w)((\nabla\mathbf{u})^T\mathbf{e}_3) = (\mathbf{v} \cdot \nabla w)(u_{\alpha\beta}(\mathbf{e}_\beta \otimes \mathbf{e}_\alpha) \cdot \mathbf{e}_3) = 0$$

Finally, the remaining term is purely out of plane: $(\mathbf{e}_3 \otimes \nabla w)\mathbf{v} = (\mathbf{v} \cdot \nabla w)\mathbf{e}_3$, and therefore does not contribute to in-plane stretching.
Hence, the Membrane Strain Tensor $\mathbf{E}_m$ is given as

$$\boxed{\mathbf{E}_m = \frac{1}{2}\left[\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + \nabla w \otimes \nabla w\right]}$$

In all subsequent discussions, the symbol $\mathbf{E}$ is used to denote $\mathbf{E}_m$ for simplicity.
Note that the membrane strain tensor is symmetric.

$$\begin{aligned}
\mathbf{E}_m^T &= \frac{1}{2}\left[\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + \nabla w \otimes \nabla w\right]^T \\
&= \frac{1}{2}\left[\ (\nabla\mathbf{u})^T + (\nabla\mathbf{u}) + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + \nabla w \otimes \nabla w\right] \\
&= \mathbf{E_m}
\end{aligned}$$

(We used the identities $(\mathbf{A}^T)^T = \mathbf{A}$, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, and $(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$.)

2.4 Decoupling Total Strain Energy

For sufficiently thin plates, the total strain energy can be decomposed into two independent contributions: an in-plane (membrane) part and an out-of-plane (bending) part. This separation is possible because the strain components vary linearly through the thickness, and the plate thickness h is small compared to in-plane dimensions.

$$\Psi(\mathbf{E}, \mathbf{K}) = \Psi_m(\mathbf{E}) + \Psi_b(\mathbf{K}),$$

where Ψ is the total strain energy density per unit mid-surface area, Ψ_m is the membrane (in-plane) energy, and Ψ_b is the bending energy associated with curvature changes.
A key observation is that

$$\Psi_m \sim O(h), \qquad \Psi_b \sim O(h^3),$$

reflecting the relative dominance of membrane and bending contributions.

2.4.1 Kinematic Preliminaries

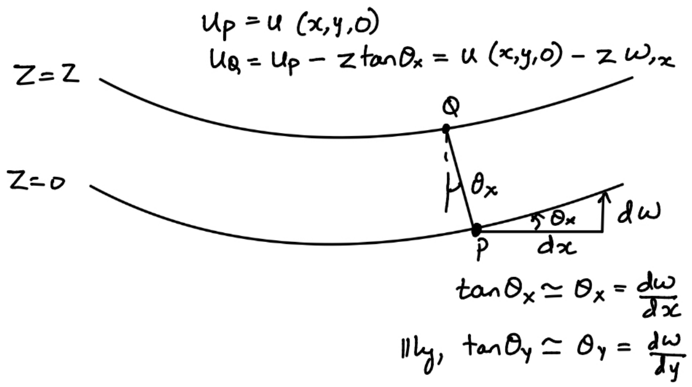

Figure 2.1: Side view of deformed membrane

Under the assumptions of small curvature and the Kirchhoff–Love hypotheses (normals to the mid-surface remain straight and inextensible), the displacement field takes the form:

$$u_1(X_1, X_2, X_3) = u_1(X_1, X_2, 0) - X_3\, w_{,x},$$

$$u_2(X_1, X_2, X_3) = u_2(X_1, X_2, 0) - X_3\, w_{,y},$$
$$w(X_1, X_2, X_3) = w_0(X_1, X_2),$$

where $X_3 = z$ is the thickness coordinate. Let

$$u_{10} = u_1(X_1, X_2, 0), \qquad u_{20} = u_2(X_1, X_2, 0), \qquad w_0 = w(X_1, X_2, 0).$$

Then the displacement field becomes

$$u_1 = u_{10} - z w_{0,x}, \qquad u_2 = u_{20} - z w_{0,y}, \qquad w = w_0.$$

2.4.2 Strain Expansion Through Thickness

Using the Green–Lagrange strain tensor truncated to second order,

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\gamma,\alpha}u_{\gamma,\beta} + w_{,\alpha}w_{,\beta}\right),$$

and substituting the displacement field, we obtain

$$\begin{aligned} E_{\alpha\beta} = &\frac{1}{2}[(u_{\alpha 0,\beta} + u_{\beta 0,\alpha} + u_{\gamma 0,\alpha}u_{\gamma 0,\beta} + w_{0,\alpha}w_{0,\beta})] \\ &- \frac{1}{2}[2w_{0,\alpha\beta} + u_{\gamma 0,\beta}w_{0,\gamma\alpha} + u_{\gamma 0,\alpha}w_{0,\gamma\beta}]\, z \\ &+ \frac{1}{2}w_{0,\gamma\alpha}w_{0,\gamma\beta}\, z^2. \end{aligned}$$

Since $z \ll L$ (plate slenderness), the $O(z^2)$ terms are negligible, giving the linear expansion:

$$E_{\alpha\beta} \approx E_{\alpha\beta,0} + E_{\alpha\beta,z}\, z,$$

where

$$E_{\alpha\beta,0} = \frac{1}{2}(u_{\alpha 0,\beta} + u_{\beta 0,\alpha} + w_{0,\alpha}w_{0,\beta}),$$
$$E_{\alpha\beta,z} = -w_{0,\alpha\beta}.$$

2.4.3 Strain Energy Density

Let ϕ denote the 3D strain energy density:

$$\phi = \frac{1}{2}\,\mathbf{E} : \mathbb{C} : \mathbf{E}.$$

Using $E_{\alpha\beta} = E_0 + E_{,z}z$:

$$\phi \approx \frac{1}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_0) + \frac{1}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_{,z} + \mathbf{E}_{,z} : \mathbb{C} : \mathbf{E}_0)z.$$

Define Ψ as the strain energy per unit mid-surface area:

$$\Psi = \int_{-h/2}^{h/2} \phi \, dz.$$

Integrating,

$$\Psi = \frac{h}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_0) + \frac{h^3}{24}(\mathbf{E}_{,z} : \mathbb{C} : \mathbf{E}_{,z}).$$

Thus, the decoupled forms are:

$$\Psi_m = \frac{h}{2}(\mathbf{E}_0 : \mathbb{C} : \mathbf{E}_0), \qquad \Psi_b = \frac{h^3}{24}(\mathbf{E}_{,z} : \mathbb{C} : \mathbf{E}_{,z}).$$

2.4.4 Isotropic Case

For a linearly elastic isotropic material,

$$\boldsymbol{\sigma}_0 = \mathbb{C} : \mathbf{E}_0 = \frac{E}{1-\nu^2}\big[\nu\,(\mathrm{tr}\mathbf{E}_0)\mathbf{I} + (1-\nu)\mathbf{E}_0\big],$$

so the membrane energy becomes

$$\Psi_m = \frac{Eh}{2(1-\nu^2)}\left[\nu(\mathrm{tr}\mathbf{E}_0)^2 + (1-\nu)\mathbf{E}_0 : \mathbf{E}_0\right].$$

The linearised curvature tensor is

$$K_{\alpha\beta} = -w_{0,\alpha\beta},$$

so

$$\Psi_b = \frac{Eh^3}{24(1-\nu^2)}\left[\nu(\mathrm{tr}\mathbf{K})^2 + (1-\nu)\mathbf{K} : \mathbf{K}\right].$$

2.4.5 Final Result

Replacing $\mathbf{E}_0$ by $\mathbf{E}$ for notational simplicity, we obtain the standard decoupled membrane and bending strain energy densities:

$$\boxed{\begin{aligned} \Psi_m &= \frac{Eh}{2(1-\nu^2)}\left[\nu(\mathrm{tr}\mathbf{E})^2 + (1-\nu)\mathbf{E} : \mathbf{E}\right], \\ \Psi_b &= \frac{Eh^3}{24(1-\nu^2)}\left[\nu(\mathrm{tr}\mathbf{K})^2 + (1-\nu)\mathbf{K} : \mathbf{K}\right]. \end{aligned}}$$

2.5 Stress Tensors

Each strain energy density term has an associated stress measure. Let $\mathbf{N}$ denote the second Piola–Kirchhoff *membrane stress tensor* corresponding to the membrane strain energy $\Psi_m(\mathbf{E})$, and let $\mathbf{M}$ denote the second Piola–Kirchhoff *couple stress tensor* associated with the bending strain energy $\Psi_b(\mathbf{K})$. They are defined as

$$\mathbf{N} = \frac{\partial \Psi_m(\mathbf{E})}{\partial \mathbf{E}}, \qquad \mathbf{M} = \frac{\partial \Psi_b(\mathbf{K})}{\partial \mathbf{K}}.$$

2.5.1 Derivation

Let $\mathbf{U}$ be some second order tensor and $\mathbf{V}$ be some arbitrary second-order tensor having a small Frobenius Norm. The Fréchet derivative is denoted by

$$D_{\mathbf{U}} f(\mathbf{U})[\mathbf{V}] = \frac{\partial f(\mathbf{U})}{\partial \mathbf{U}} : \mathbf{V}.$$

Now,

Derivative of $(\operatorname{tr}\mathbf{U})^2$

$$\begin{aligned} D_{\mathbf{U}}[(\operatorname{tr}\mathbf{U})^2][\mathbf{V}] &= (\operatorname{tr}(\mathbf{U}+\mathbf{V}))^2 - (\operatorname{tr}\mathbf{U})^2 + O(\|\mathbf{V}\|^2) \\ &= 2(\operatorname{tr}\mathbf{U})(\operatorname{tr}\mathbf{V}) + O(\|\mathbf{V}\|^2) \\ &= 2(\operatorname{tr}\mathbf{U})\,\mathbf{I} : \mathbf{V}. \end{aligned}$$

Thus,

$$\frac{\partial (\operatorname{tr}\mathbf{U})^2}{\partial \mathbf{U}} = 2(\operatorname{tr}\mathbf{U})\,\mathbf{I}.$$

Derivative of $\mathbf{U} : \mathbf{U}$

$$\begin{aligned} D_{\mathbf{U}}(\mathbf{U}:\mathbf{U})[\mathbf{V}] &= (\mathbf{U}+\mathbf{V}) : (\mathbf{U}+\mathbf{V}) - \mathbf{U}:\mathbf{U} + O(\|\mathbf{V}\|^2) \\ &= 2\mathbf{U} : \mathbf{V} + O(\|\mathbf{V}\|^2). \end{aligned}$$

Hence,

$$\frac{\partial (\mathbf{U}:\mathbf{U})}{\partial \mathbf{U}} = 2\mathbf{U}.$$

Upon substituting $\mathbf{U}$ for $\mathbf{E}$ and $\mathbf{K}$ and using the relations in the expression of $\mathbf{N}$ and $\mathbf{M}$ respectively, we get

$$\boxed{\begin{aligned} \mathbf{N} &= \frac{\mathrm{E}h}{(1-\nu^2)}[\nu(tr\mathbf{E})\mathbf{I} + (1-\nu)\mathbf{E}] \\ \mathbf{M} &= \frac{\mathrm{E}h^3}{12(1-\nu^2)}[\nu(tr\mathbf{K})\mathbf{I} + (1-\nu)\mathbf{K}] \end{aligned}}$$

2.5.2 Symmetry of N and M

Since **N** and **M** are second Piola–Kirchhoff stresses, they must be symmetric. Using the fact that **E** and **K** are symmetric tensors:

$$\mathbf{N}^T = \frac{Eh}{1-\nu^2}\left[\nu(\operatorname{tr}\mathbf{E})\mathbf{I} + (1-\nu)\mathbf{E}^T\right] = \mathbf{N},$$

$$\mathbf{M}^T = \frac{Eh^3}{12(1-\nu^2)}\left[\nu(\operatorname{tr}\mathbf{K})\mathbf{I} + (1-\nu)\mathbf{K}^T\right] = \mathbf{M}.$$

2.6 Non-dimensionalisation

Let L be the characteristic in-plane length of the membrane. Using this scale, all geometric and kinematic quantities are non-dimensionalised as

$$\mathbf{X} = L\mathbf{X}^*, \qquad \mathbf{u} = L\mathbf{u}^*, \qquad w = Lw^*, \qquad h = Lh^*,$$

where $(\cdot)^*$ indicates non-dimensional variables.

Gradient Operator

$$\nabla = \mathbf{e}_1\frac{\partial}{\partial X_1} + \mathbf{e}_2\frac{\partial}{\partial X_2} = \frac{1}{L}\left(\mathbf{e}_1\frac{\partial}{\partial X_1^*} + \mathbf{e}_2\frac{\partial}{\partial X_2^*}\right) = \frac{1}{L}\nabla^*.$$

Consequently,

$$\nabla\mathbf{u} = \left(\frac{1}{L}\nabla^*\right)(L\mathbf{u}^*) = \nabla^*\mathbf{u}^*, \qquad \nabla w = \left(\frac{1}{L}\nabla^*\right)(Lw^*) = \nabla^* w^*.$$

Strain Measures The Green–Lagrange in-plane strain tensor is

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + (\nabla w\otimes\mathbf{e}_3)(\mathbf{e}_3\otimes\nabla w)\right).$$

Using the relations above,

$$\mathbf{E} = \frac{1}{2}\left(\nabla^*\mathbf{u}^* + (\nabla^*\mathbf{u}^*)^T + (\nabla^*\mathbf{u}^*)^T(\nabla^*\mathbf{u}^*) + (\nabla^* w^*\otimes\mathbf{e}_3)(\mathbf{e}_3\otimes\nabla^* w^*)\right) = \mathbf{E}^*.$$

The curvature tensor is

$$\mathbf{K} = -\nabla^2 w = -\left(\frac{1}{L}\nabla^*\right)^2(Lw^*) = -\frac{1}{L}\nabla^{*,2}w^* = \frac{1}{L}\mathbf{K}^*.$$

Define the dimensional parameter

$$\lambda = \frac{Eh}{12(1-\nu^2)}.$$

Membrane energy

$$\Psi_m = \frac{Eh}{2(1-\nu^2)}\left[\nu(\operatorname{tr}\mathbf{E})^2 + (1-\nu)\,\mathbf{E}:\mathbf{E}\right].$$

Using $\mathbf{E} = \mathbf{E}^*$:

$$\Psi_m = 6\lambda\left[\nu(\operatorname{tr}\mathbf{E}^*)^2 + (1-\nu)\,\mathbf{E}^*:\mathbf{E}^*\right] = \lambda\Psi_m^*,$$

where

$$\Psi_m^* = 6\left[\nu(\operatorname{tr}\mathbf{E}^*)^2 + (1-\nu)\,\mathbf{E}^*:\mathbf{E}^*\right].$$

Bending energy

$$\Psi_b = \frac{Eh^3}{24(1-\nu^2)}\left[\nu(\operatorname{tr}\mathbf{K})^2 + (1-\nu)\mathbf{K}:\mathbf{K}\right].$$

Substituting h and $\mathbf{K}$:

$$\Psi_b = \frac{\lambda(Lh^*)^2}{2}\left[\nu\left(\operatorname{tr}\left(\frac{1}{L}\mathbf{K}^*\right)\right)^2 + (1-\nu)\left(\frac{1}{L}\mathbf{K}^*\right):\left(\frac{1}{L}\mathbf{K}^*\right)\right] = \lambda\Psi_b^*,$$

where

$$\Psi_b^* = \frac{h^{*2}}{2}\left[\nu(\operatorname{tr}\mathbf{K}^*)^2 + (1-\nu)\mathbf{K}^*:\mathbf{K}^*\right].$$

Thus the total energy becomes

$$\Psi = \Psi_m + \Psi_b = \lambda\left(\Psi_m^* + \Psi_b^*\right) = \lambda\Psi^*(\mathbf{E}^*, \mathbf{K}^*).$$

Stress Measures Membrane stress

$$\mathbf{N} = \frac{Eh}{1-\nu^2}\left[\nu(\operatorname{tr}\mathbf{E})\mathbf{I} + (1-\nu)\mathbf{E}\right] = 12\lambda\left[\nu(\operatorname{tr}\mathbf{E}^*)\mathbf{I} + (1-\nu)\mathbf{E}^*\right] = \lambda\mathbf{N}^*,$$

$$\mathbf{N}^* = 12\left[\nu(\operatorname{tr}\mathbf{E}^*)\mathbf{I} + (1-\nu)\mathbf{E}^*\right].$$

Bending stress

$$\mathbf{M} = \frac{Eh^3}{12(1-\nu^2)}\left[\nu(\operatorname{tr}\mathbf{K})\mathbf{I} + (1-\nu)\mathbf{K}\right] = \lambda h^{*2}\left[\nu(\operatorname{tr}\mathbf{K}^*)\mathbf{I} + (1-\nu)\mathbf{K}^*\right] = \lambda\mathbf{M}^*,$$

$$\mathbf{M}^* = h^{*2}\left[\nu(\operatorname{tr}\mathbf{K}^*)\mathbf{I} + (1-\nu)\mathbf{K}^*\right].$$

Throughout the rest of this report, all quantities will be assumed non-dimensional, and the * notation will be omitted for clarity.

2.7 Plate Equations

Let the body force be decomposed as

$$\mathbf{f}(x,y) = \mathbf{f}_m(x,y) + f_b(x,y)\,\mathbf{e}_3,$$

and let $\mathbf{t}$ denote the force per unit length acting on the boundary Γ_0. For quasi-static loading the Lagrangian reduces to the potential functional

$$\Pi[\mathbf{u}, w] = V - W_{\text{ext}} = \int_{\Omega_0} \left[\Psi_{\text{Total}}(\mathbf{E},\mathbf{K}) - \mathbf{f}_m \cdot \mathbf{u} - f_b\, w\right] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\mathbf{u} + w\mathbf{e}_3)\, d\Gamma.$$

Let $\boldsymbol{\eta} : \Omega_0 \to \mathbb{R}^2$ and $\zeta : \Omega_0 \to \mathbb{R}$ be arbitrary admissible variations (satisfying essential boundary conditions). For a perturbed field under the small perturbation parameter ε,

$$\tilde{\mathbf{u}} = \mathbf{u} + \varepsilon\boldsymbol{\eta}, \qquad \tilde{w} = w + \varepsilon\zeta,$$

the first variation of Π (keeping only $O(\varepsilon)$ terms) reads

$$\begin{aligned}
\delta\Pi &= \tilde{\Pi} - \Pi \\
&= \left(\int_{\Omega_0} [\tilde{\Psi}_{Total} - \mathbf{f}_m \cdot \tilde{\mathbf{u}} - f_b\tilde{w}] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\tilde{\mathbf{u}} + \tilde{w}\mathbf{e}_3) d\Gamma\right) \\
&\quad - \left(\int_{\Omega_0} [\Psi_{Total} - \mathbf{f}_m \cdot \mathbf{u} - f_b w] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\mathbf{u} + w\mathbf{e}_3) d\Gamma\right) \\
&= \int_{\Omega_0} [\delta\Psi_{Total} - \mathbf{f}_m \cdot \delta\mathbf{u} - f_b\delta w] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\delta\mathbf{u} + \delta w\mathbf{e}_3) d\Gamma \\
&= \int_{\Omega_0} [\frac{\partial \Psi_{Total}}{\partial \mathbf{E}} : \delta\mathbf{E} + \frac{\partial \Psi_{Total}}{\partial \mathbf{K}} : \delta\mathbf{K} - \mathbf{f}_m \cdot \delta\mathbf{u} - f_b\delta w] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\delta\mathbf{u} + \delta w\mathbf{e}_3) d\Gamma \\
&= \int_{\Omega_0} [\frac{\partial \Psi_{m}}{\partial \mathbf{E}} : \delta\mathbf{E} + \frac{\partial \Psi_{b}}{\partial \mathbf{K}} : \delta\mathbf{K} - \mathbf{f}_m \cdot \delta\mathbf{u} - f_b\delta w] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\delta\mathbf{u} + \delta w\mathbf{e}_3) d\Gamma \\
&= \int_{\Omega_0} [\mathbf{N} : \delta\mathbf{E} + \mathbf{M} : \delta\mathbf{K} - \mathbf{f}_m \cdot \delta\mathbf{u} - f_b\delta w] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\delta\mathbf{u} + \delta w\mathbf{e}_3) d\Gamma
\end{aligned}$$

Thus,

$$\delta\Pi = \int_{\Omega_0} \left[\mathbf{N} : \delta\mathbf{E} + \mathbf{M} : \delta\mathbf{K} - \mathbf{f}_m \cdot \boldsymbol{\eta} - f_b\,\zeta\right] d\Omega - \oint_{\Gamma_0} \mathbf{t}\cdot(\boldsymbol{\eta} + \zeta\mathbf{e}_3)\, d\Gamma,$$

$$\begin{aligned}
\delta\mathbf{E} &= \tilde{\mathbf{E}} - \mathbf{E} \\
&= \frac{1}{2}\left[\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T + (\nabla\tilde{\mathbf{u}})^T(\nabla\tilde{\mathbf{u}}) + \nabla\tilde{w}\otimes\nabla\tilde{w}\right] - \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) + \nabla w\otimes\nabla w\right] \\
&= \frac{1}{2}\left[\nabla\delta\mathbf{u} + (\nabla\delta\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\delta\mathbf{u}) + (\nabla\delta\mathbf{u})^T(\nabla\mathbf{u}) + (\nabla\delta\mathbf{u})^T(\nabla\delta\mathbf{u}) + \nabla w\otimes\nabla\delta w\right. \\
&\quad \left.+ \nabla\delta w\otimes\nabla w + \nabla\delta w\otimes\nabla\delta w\right] \\
&= \frac{1}{2}\left[\nabla\eta + (\nabla\eta)^T + (\nabla\mathbf{u})^T(\nabla\eta) + (\nabla\eta)^T(\nabla\mathbf{u}) + \nabla w\otimes\nabla\zeta + \nabla\zeta\otimes\nabla w\right]\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

$\delta\mathbf{K} = \tilde{\mathbf{K}} - \mathbf{K} = -\nabla^2\tilde{w} - (-\nabla^2 w) = (-\nabla^2\zeta)\varepsilon$

For a Symmetric tensor $\mathbf{S}$ and some tensor $\mathbf{A}$ of same order, we have the following relation:

$$\mathbf{S} : \left(\frac{\mathbf{A} + \mathbf{A}^T}{2}\right) = \mathbf{S} : \mathbf{A}$$

Since $\mathbf{N}$ is a symmetric tensor,

$$\begin{aligned}
\mathbf{N} : \delta\mathbf{E} &= \mathbf{N} : \frac{1}{2}\left[\nabla\eta + (\nabla\eta)^T + (\nabla\mathbf{u})^T(\nabla\eta) + (\nabla\eta)^T(\nabla\mathbf{u}) + \nabla w \otimes \nabla\zeta + \nabla\zeta \otimes \nabla w\right]\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= \mathbf{N} : \left(\frac{\nabla\eta + (\nabla\eta)^T}{2}\right)\varepsilon + \mathbf{N} : \left(\frac{(\nabla\mathbf{u})^T(\nabla\eta) + ((\nabla\mathbf{u})^T(\nabla\eta))^T}{2}\right)\varepsilon \\
&\quad + \mathbf{N} : \left(\frac{\nabla w \otimes \nabla\zeta}{2} + \frac{(\nabla w \otimes \nabla\zeta)^T}{2}\right)\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= \mathbf{N} : \left[\nabla\eta + (\nabla\mathbf{u})^T(\nabla\eta) + \nabla w \otimes \nabla\zeta\right]\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

$\mathbf{N} : ((\nabla\mathbf{u})^T(\nabla\eta)) = tr(\mathbf{N}^T(\nabla\mathbf{u})^T(\nabla\eta)) = tr(((\nabla\mathbf{u})\mathbf{N})^T(\nabla\eta)) = ((\nabla\mathbf{u})\mathbf{N}) : \nabla\eta$

$\mathbf{N} : (\nabla w \otimes \nabla\zeta) = N_{\alpha\beta}w_{,\alpha}\zeta_{,\beta} = N^T_{\beta\alpha}w_{,\alpha}\zeta_{,\beta} = (\mathbf{N}^T\nabla w)_\beta\zeta_{,\beta} = (\mathbf{N}\nabla w) \cdot \nabla\zeta$

Thus,

$$\mathbf{N} : \delta\mathbf{E} = \left[\left[(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}\right] : \nabla\boldsymbol{\eta} + (\mathbf{N}\nabla w) \cdot \nabla\zeta\right]\varepsilon + \mathcal{O}(\varepsilon^2), \qquad \mathbf{M} : \delta\mathbf{K} = (-\mathbf{M} : \nabla^2\zeta)\varepsilon$$

Substituting back these relations, we get

$$\delta\Pi = \varepsilon\int_{\Omega_0}\left[\left[(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}\right] : \nabla\eta + (\mathbf{N}\nabla w) \cdot \nabla\zeta - \mathbf{M} : \nabla^2\zeta - \mathbf{f}_m \cdot \eta - f_b\zeta\right]d\Omega - \varepsilon\oint_{\Gamma_0}\mathbf{t} \cdot (\eta + \zeta\mathbf{e}_3)d\Gamma$$

At Equilibrium conditions, $\delta\Pi = 0$. Therefore,

$$\int_{\Omega_0}\left[\left[(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}\right] : \nabla\eta + (\mathbf{N}\nabla w) \cdot \nabla\zeta - \mathbf{M} : \nabla^2\zeta - \mathbf{f}_m \cdot \eta - f_b\zeta\right]d\Omega - \oint_{\Gamma_0}\mathbf{t} \cdot (\eta + \zeta\mathbf{e}_3)d\Gamma = 0$$

Note:

For some second-order tensor $\mathbf{A}$ and some vector $\mathbf{v}$, we can write

$$\nabla \cdot (\mathbf{A}^T\mathbf{v}) = ((\mathbf{A}^T)_{ij}v_j)_{,i} = A_{ji,i}v_j + A_{ji}v_{j,i} = (\nabla \cdot \mathbf{A}) \cdot \mathbf{v} + \mathbf{A} : \nabla\mathbf{v}$$

$$\therefore \mathbf{A} : \nabla\mathbf{v} = \nabla \cdot (\mathbf{A}^T\mathbf{v}) - (\nabla \cdot \mathbf{A}) \cdot \mathbf{v}$$

$$\int_\Omega \mathbf{A} : \nabla\mathbf{v}d\Omega = -\int_\Omega(\nabla \cdot \mathbf{A}) \cdot \mathbf{v}d\Omega + \int_\Omega\nabla \cdot (\mathbf{A}^T\mathbf{v})d\Omega$$

Using Gauss-Divergence Theorem,

$$\int_\Omega \mathbf{A} : \nabla\mathbf{v}d\Omega = -\int_\Omega(\nabla \cdot \mathbf{A}) \cdot \mathbf{v}d\Omega + \oint_\Gamma(\mathbf{A}^T\mathbf{v}) \cdot \mathbf{n}d\Gamma$$

Similarly, for some vector $\mathbf{v}$ and some scalar function ϕ, we have

$$\nabla \cdot (\phi\mathbf{v}) = ((\phi v_i)_{,i} = \phi, iv_i + \phi v_{i,i} = \mathbf{v} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{v}$$

$$\therefore \mathbf{v} \cdot \nabla\phi = \nabla \cdot (\phi\mathbf{v}) - \phi(\nabla \cdot \mathbf{v})$$

Using Gauss-Divergence Theorem, we get

$$\int_\Omega \mathbf{v} \cdot \nabla\phi d\Omega = -\int_\Omega \phi\nabla \cdot \mathbf{v}d\Omega + \oint_\Gamma \phi\mathbf{v} \cdot \mathbf{n}d\Gamma$$

Using these relations, as well as putting $\mathbf{t} = \mathbf{t}_m + t_b\mathbf{e}_3$, we can simplify our equilibrium equation as follows

$$\begin{aligned}
&\int_{\Omega_0} \Big[-\big[\nabla \cdot [(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}]\big] \cdot \eta - \nabla \cdot (\mathbf{N}\nabla w)\zeta + (\nabla \cdot \mathbf{M}) \cdot \nabla\zeta - \mathbf{f}_m \cdot \eta - f_b\zeta \Big] d\Omega \\
&\quad + \oint_{\Gamma_0} \Big[[(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}]^T \eta + \mathbf{N}\nabla w\zeta - \mathbf{M} \cdot \nabla\zeta \Big] \cdot \mathbf{n} - (\mathbf{t}_m \cdot \eta + t_b\zeta)\ d\Gamma = 0 \\
\Rightarrow &\int_{\Omega_0} \Big[-\big[\nabla \cdot [(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}] + \mathbf{f}_m\big] \cdot \eta - \big[\nabla \cdot (\mathbf{N}\nabla w) + \nabla \cdot (\nabla \cdot \mathbf{M}) + f_b\big]\zeta \Big] d\Omega \\
&\quad + \oint_{\Gamma_0} \Big[\big([(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}]\,\mathbf{n} - \mathbf{t}_m \big) \cdot \eta + \big((\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}) \cdot \mathbf{n} - t_b \big)\zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla\zeta \Big] d\Gamma = 0
\end{aligned}$$

Since η and ζ are arbitrary, their coefficients must be zero at all points to satisfy the equation. Therefore, we arrive at our Euler-Lagrange Equations

$$\boxed{\begin{aligned} \nabla \cdot [(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}] + \mathbf{f}_m &= \mathbf{0} \\ \nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N}\nabla w] + f_b &= 0 \end{aligned}}$$

The Natural Boundary Conditions are

$$\boxed{\begin{gathered} \oint_{\Gamma_0} \big([(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}]\,\mathbf{n} - \mathbf{t}_m \big) \cdot \eta\ d\Gamma = 0 \\ \oint_{\Gamma_0} \Big(\big([\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}] \cdot \mathbf{n} - t_b \big)\zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla\zeta \Big) d\Gamma = 0 \end{gathered}}$$

Biharmonic Form

$$\begin{aligned}
&\nabla \cdot (\nabla \cdot \mathbf{M}) = M_{\alpha\beta,\alpha\beta} = h^2[\nu K_{\gamma\gamma}\delta_{\alpha\beta} + (1-\nu)K_{\alpha\beta}]_{,\alpha\beta} = h^2[\nu(-w_{,\gamma\gamma})\delta_{\alpha\beta} + (1-\nu)(-w_{,\alpha\beta})]_{,\alpha\beta} \\
&= -h^2[\nu w_{,\gamma\gamma\alpha\beta}\delta_{\alpha\beta} + (1-\nu)w_{\alpha\alpha\beta\beta}] = -h^2[\nu w_{,\gamma\gamma\alpha\alpha} + (1-\nu)w_{\alpha\alpha\beta\beta}] = -h^2\Delta^2 w
\end{aligned}$$

Replacing this relation in the second plate equation, we get

$$\begin{aligned} \nabla \cdot [(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}] + \mathbf{f}_m &= \mathbf{0} \\ h^2\Delta^2 w - \nabla \cdot (\mathbf{N}\nabla w) - f_b &= 0 \end{aligned}$$

Note: $\Delta^2 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$ is the biharmonic operator.

Remarks

- The formulation above is general and contains both geometric nonlinearity (through $\nabla \mathbf{u}$ and the nonlinear terms in $\mathbf{E}$) and the coupling between membrane and bending responses (through the $\mathbf{N}\nabla w$ term).

- To recover the linear Kirchhoff–Love plate equation one neglects $\nabla \mathbf{u}$ and the nonlinear in-plane strain contributions; then $\mathbf{N}$ reduces to a pre-stress or zero and the transverse equation reduces to $\nabla\cdot(\nabla\cdot\mathbf{M}) + q = 0$ which is equivalent to $D\nabla^4 w = q$.

- $\mathbf{f}_m$, f_b and $\mathbf{t}$ are inherently assumed to be non-dimensional. To make them dimensional for practical purposes, use

$$\mathbf{f}_m = \frac{\lambda}{L}\mathbf{f}_m^*, \qquad f_b = \frac{\lambda}{L} f_b^* \qquad \mathbf{t} = \frac{\lambda}{L}\mathbf{t}^*$$

,

Chapter 3

Simulation Results

COMSOL Multiphysics was used to numerically evaluate the equilibrium configurations of the nonlinear membrane model developed in the previous chapter. Because the governing equations form a fully coupled system of nonlinear partial differential equations, analytical solutions are generally not attainable. COMSOL's finite element environment enables direct implementation of the weak form of the equations and computation of equilibrium solutions for prescribed load parameters.

In this chapter, we first describe the COMSOL setup used for the simulations, including the choice of interface, the introduction of auxiliary variables, and the implementation of boundary conditions. We then present the two problems studied in detail. For each problem, we outline the complete formulation—from the PDE system to the natural boundary conditions—and subsequently use COMSOL to determine the corresponding equilibrium configurations and comment on the qualitative behaviour of the solutions.

3.1 COMSOL Setup

Component A two-dimensional component was created because the membrane is represented by its mid-surface, and all quantities of interest (in-plane and transverse displacements, strains, etc.) depend only on the in-plane coordinates.

Physics The physical behaviour of the membrane was implemented through the `Coefficient Form PDE` interface, which offers sufficient flexibility to specify custom nonlinear PDEs involving vector fields, gradients, and mixed derivatives. Four dependent variables were introduced:

$$u, \quad v, \quad w, \quad q$$

corresponding to the in-plane displacements, the out-of-plane displacement, and the auxiliary field introduced during the reduction of the governing equations from fourth order to second order.

Study A stationary study was used since the formulation is quasi-static and seeks the equilibrium configuration corresponding to a prescribed load level. No inertial or time-dependent effects are considered, making the stationary solver appropriate.

Geometry The computational domain was created as a 2D rectangle of dimensions $L \times W$, representing the undeformed mid-surface of the membrane. These dimensions were defined as global parameters to allow easy modification during parametric sweeps and to keep the model consistent with the analytical formulation.

Parameter List All material properties, geometric parameters, and loading magnitudes were defined as global parameters to maintain consistency between sections of the model and to ensure reproducibility. These parameters are listed in Figure 3.1. Defining them globally also allows for automated continuation in loading and facilitates parameter studies such as varying P, L, W, or material constants.

Parameters

Name	Expression	Value	Description
L	0.2[m]	0.2 m	Length of membrane
W	0.1[m]	0.1 m	Width of membrane
H	1e-3[m]	0.001 m	Thickness of membrane
kappa	1e9 [m^-2]	1E9 1/m²	parameter: 1/H^2
nu	0.45	0.45	Poisson Ratio
epsilon	0.03	0.03	Applied strain

Figure 3.1: List of global parameters used in the COMSOL model.

Variables Several auxiliary variables were defined to simplify the implementation and make the model easier to interpret. These include the membrane strain components, bending strain components, membrane stresses, and couple stresses, all written either directly in terms of the displacement fields or other variables using the relations derived in Chapter 2. Each component was defined as an individual variable, allowing COMSOL to evaluate these quantities automatically and use them consistently throughout the formulation. The complete list of variables is shown in Figure 3.2. Note that all the expressions are non-dimensionalised with the use of [m], [m^2], etc. since Comsol treats the independent variable x and y as parameters having the dimension of length.

Solver We solved the nonlinear PDE system using COMSOL's default solver, which employs a fully-coupled Newton–Raphson method with an internal direct linear solver. In future work, we will implement an arc-length continuation scheme to track post-buckling paths and capture the bifurcation behaviour more accurately.

Variables

Name	Expression
E11	ux [m] + 0.5*(ux^2 + vx^2 + wx^2) [m^2]
E12	0.5*(uy + vx) [m] + 0.5*(ux*uy + vx*vy + wx*wy) [m^2]
E22	vy [m] + 0.5*(uy^2 + vy^2 + wy^2) [m^2]
N11	12*(E11 + nu*E22)
N12	12*(1-nu)*E12
N22	12*(nu*E11 + E22)
K11	-wxx [m^2]
K12	-wxy [m^2]
K22	-wyy [m^2]
M11	(1/kappa) * (K11 + nu*K22) [1/m^2]
M12	(1/kappa) * (1 - nu) * K12 [1/m^2]
M22	(1/kappa) * (nu*K11 + K22) [1/m^2]
fm1	0
fm2	0
fb	0

Figure 3.2: List of variables used in the COMSOL model.

Boundary Conditions We used `Dirichlet Boundary Condition` node to implement geometric boundary conditions and `Flux/Source` Node to implement the natural boundary conditions.

3.1.1 Equation Setup

The `Coefficient Form PDE` interface in COMSOL accepts systems of PDEs in the following format

$$\nabla \cdot (-c\nabla \mathbf{u} - \alpha \mathbf{u} + \gamma) + \beta \cdot \nabla \mathbf{u} + \mathrm{a}\mathbf{u} = \mathrm{f}$$

As we can see, the interface allows equations of at most second order. In our case, however, the governing equations form a fourth-order PDE system. To make the equations compatible with COMSOL's requirements, we reduce them to a second-order system by introducing an additional dependent variable

$$q = \Delta w.$$

With this definition, the original fourth-order system is converted into a system of four coupled second-order PDEs, as shown

$$\begin{gathered}\nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] + \mathbf{f}_m = \mathbf{0}, \\ \Delta q - \kappa \nabla \cdot (\mathbf{N}\nabla w) - \kappa f_b = 0, \\ q = \Delta w\end{gathered}$$

where, $\kappa = 1/h^2$ is a useful parameter.

Below, we list all non-zero inputs required by the interface. Any input not included in the list should be taken as zero. Note that each entry of the c matrix is itself a 2×2 matrix.

$$c = \begin{bmatrix} -\mathbf{N} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \kappa\mathbf{N} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad f = \begin{bmatrix} -f_{m1} \\ -f_{m2} \\ \kappa f_b \\ 0 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3.2 Plate under uniaxial stretching

3.2.1 Problem Description

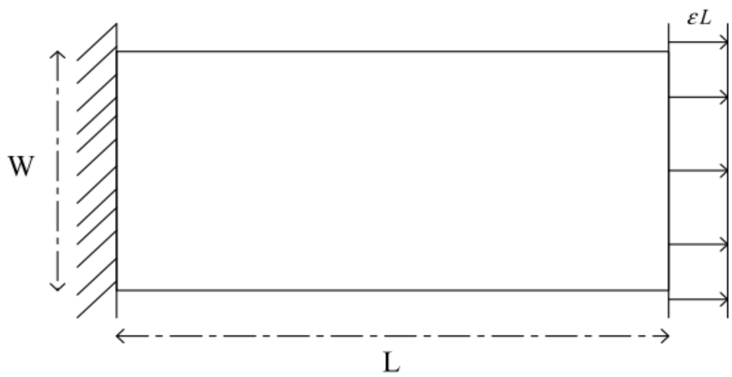

Figure 3.3: Plate under uniform uniaxial stretch

The plate is rectangular with in-plane dimensions $L \times W$ and uniform thickness h. It is fixed along the edge $x = 0$, and a uniform load is applied along the edge $x = L$ such that a prescribed macroscopic axial strain ε is imposed at that boundary. The corresponding displacement boundary conditions are

$$\mathbf{u}(0, y) = \mathbf{0}, \qquad u_1(L, y) = \varepsilon L, \qquad u_2(L, y) = 0, \qquad w(0, y) = w(L, y) = 0.$$

No body forces act on the plate; therefore,

$$\mathbf{f}_m = \mathbf{0}, \qquad f_b = 0.$$

The only tractions arise from the fixed and loaded edges. Hence,

$$t_b = 0, \qquad \mathbf{t}_m(L, y) = P\,\mathbf{e}_1, \qquad \mathbf{t}_m(0, y) = -P\,\mathbf{e}_1.$$

Under these conditions, our equilibrium equations become

$$\begin{aligned} \nabla \cdot [(\mathbf{I} + \nabla\mathbf{u})\mathbf{N}] &= \mathbf{0} \\ \nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N}\nabla w] &= 0 \end{aligned}$$

3.2.2 Natural Boundary Conditions

Adapting the general equations for natural boundary conditions to our problem, we get

$$(A) \quad \oint_{\Gamma_0} \big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{n} - \mathbf{t}_m \big) \cdot \eta \; d\Gamma = 0$$

$$(B) \quad \oint_{\Gamma_0} \Big(\big([\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}] \cdot \mathbf{n} \big) \zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla \zeta \Big) d\Gamma = 0$$

Let us first simplify equation (A). η is chosen such that it has special behaviour at points where dirichlet boundary conditions are applied. Essentially, $\eta = \mathbf{0}$ at $x = 0, 0 < y < W$ and $x = L, 0 < y < W$.

$$\oint_{\Gamma_0} \big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{n} - \mathbf{t}_m \big) \cdot \eta \; d\Gamma = 0$$

$$\Rightarrow \int_0^L \big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, (-\mathbf{e}_2) \big) \cdot \eta \; d\Gamma + \int_L^0 \big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{e}_2 \big) \cdot \eta \; d\Gamma = 0$$

$$\Rightarrow \int_0^L \big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{e}_2 \big) \cdot \eta \; d\Gamma = 0$$

Therefore, for the integral to be zero, the coefficient of η must be zero along edges $y = 0$ and $y = W$. This leads to

$$[(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{e}_2 = 0 \qquad 0 < x < L, \quad y = 0, W$$

Now, we proceed to the discussion for Equation (B). First, we evaluate the integral along the edge $y = 0$.

$$\int_0^L \left[\big([\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}] \cdot (-\mathbf{e}_2) \big) \zeta - (\mathbf{M}(-\mathbf{e}_2)) \cdot \left(\frac{\partial \zeta}{\partial x}\mathbf{e}_1 + \frac{\partial \zeta}{\partial y}\mathbf{e}_2 \right) \right] dx$$

$$\Rightarrow -\int_0^L \left[\big([\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}] \cdot \mathbf{e}_2 \big) \zeta - (\mathbf{e}_1 \cdot \mathbf{M}\mathbf{e}_2) \frac{\partial \zeta}{\partial x} - (\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_2) \frac{\partial \zeta}{\partial y} \right] dx$$

$$\Rightarrow -\int_0^L \left[\left(\left[\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_1}{\partial x} \right] \cdot \mathbf{e}_2 \right) \zeta - \frac{\partial (\zeta \mathbf{e}_1 \cdot \mathbf{M}\mathbf{e}_2)}{\partial x} - (\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_2) \frac{\partial \zeta}{\partial y} \right] dx$$

$$\Rightarrow -\int_0^L \left[\left(\left[\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_1}{\partial x} \right] \cdot \mathbf{e}_2 \right) \zeta - (\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_2) \frac{\partial \zeta}{\partial y} \right] dx - [\zeta \mathbf{e}_1 \cdot \mathbf{M}\mathbf{e}_2]_{x=0,y=0}^{x=L,y=0}$$

$\zeta = 0$ at $x = 0$ and $x = L$ for all y. Thus, the last term in brackets will be reduced to zero.

$$-\int_0^L \left[\left(\left[\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_1}{\partial x} \right] \cdot \mathbf{e}_2 \right) \zeta - (\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_2) \frac{\partial \zeta}{\partial y} \right] dx$$

Since, ζ and $\dfrac{\partial \zeta}{\partial y}$ are arbitrary, hence their coefficients must be zero. The same logic applies to the integral along the edge $y = W$. This leads us to the following boundary conditions

$$\left(\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_1}{\partial x} \right) \cdot \mathbf{e}_2 = 0 \qquad 0 < x < L, \quad y = 0, W$$

$$\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_2 = 0 \qquad 0 < x < L, \quad y = 0, W$$

Similarly, we evaluate the integral along the edges $x = 0$ and $x = L$ and get the following reduced integral

$$-\int_0^W \left[\left(\left[\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_2}{\partial y} \right] \cdot \mathbf{e}_1 \right) \zeta - (\mathbf{e}_1 \cdot \mathbf{M}\mathbf{e}_1) \frac{\partial \zeta}{\partial x} \right] dy$$

Note that $\zeta = 0$ for $0 < y < W$ at $x = 0$ and $x = L$. Thus, the only boundary condition we get from these integrals is

$$\mathbf{e}_1 \cdot \mathbf{M}\mathbf{e}_1 = 0 \qquad x = 0, L, \quad 0 < y < L$$

3.2.3 Summary

PDE System

$$
\begin{aligned}
&(1) \quad \nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] \quad = \mathbf{0} \\
&(2) \quad \nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N}\nabla w] = 0 \\
&(3) \quad \mathbf{N} = 12 \left[\nu (tr\mathbf{E})\mathbf{I} + (1-\nu)\mathbf{E} \right] \\
&(4) \quad \mathbf{M} = h^2 \left[\nu (tr\mathbf{K})\mathbf{I} + (1-\nu)\mathbf{K} \right] \\
&(5) \quad \mathbf{E} = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T (\nabla \mathbf{u}) + \nabla w \otimes \nabla w \right] \\
&(6) \quad \mathbf{K} = -\nabla^2 w
\end{aligned}
$$

Dirichlet Boundary Conditions

$$
\begin{aligned}
u_1(0,y) &= 0 \qquad & u_1(L,y) &= \varepsilon L \\
u_2(0,y) &= 0 \qquad & u_2(L,y) &= 0 \\
w(0,y) &= 0 \qquad & w(L,y) &= 0
\end{aligned}
$$

Natural Boundary Conditions

$$
\begin{aligned}
&\mathbf{n} \cdot \mathbf{M}\mathbf{n} = 0 \qquad \text{along all 4 edges} \\
&[(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{e}_2 \,|_{y=0} = 0 \qquad 0 < x < L \\
&[(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{e}_2 \,|_{y=W} = 0 \qquad 0 < x < L \\
&\left(\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_1}{\partial x} \right) \cdot \mathbf{e}_2 \Bigg|_{y-0} = 0 \qquad 0 < x < L \\
&\left(\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_1}{\partial x} \right) \cdot \mathbf{e}_2 \Bigg|_{y=W} = 0 \qquad 0 < x < L
\end{aligned}
$$

3.2.4 Results

(a) In-plane displacement $u_1(x,y)$ The field u_1 increases monotonically from the fixed left edge to the displaced right edge. This nearly linear variation is expected because a uniform axial strain is imposed along the x-direction and the plate is thin, so membrane stiffness dominates

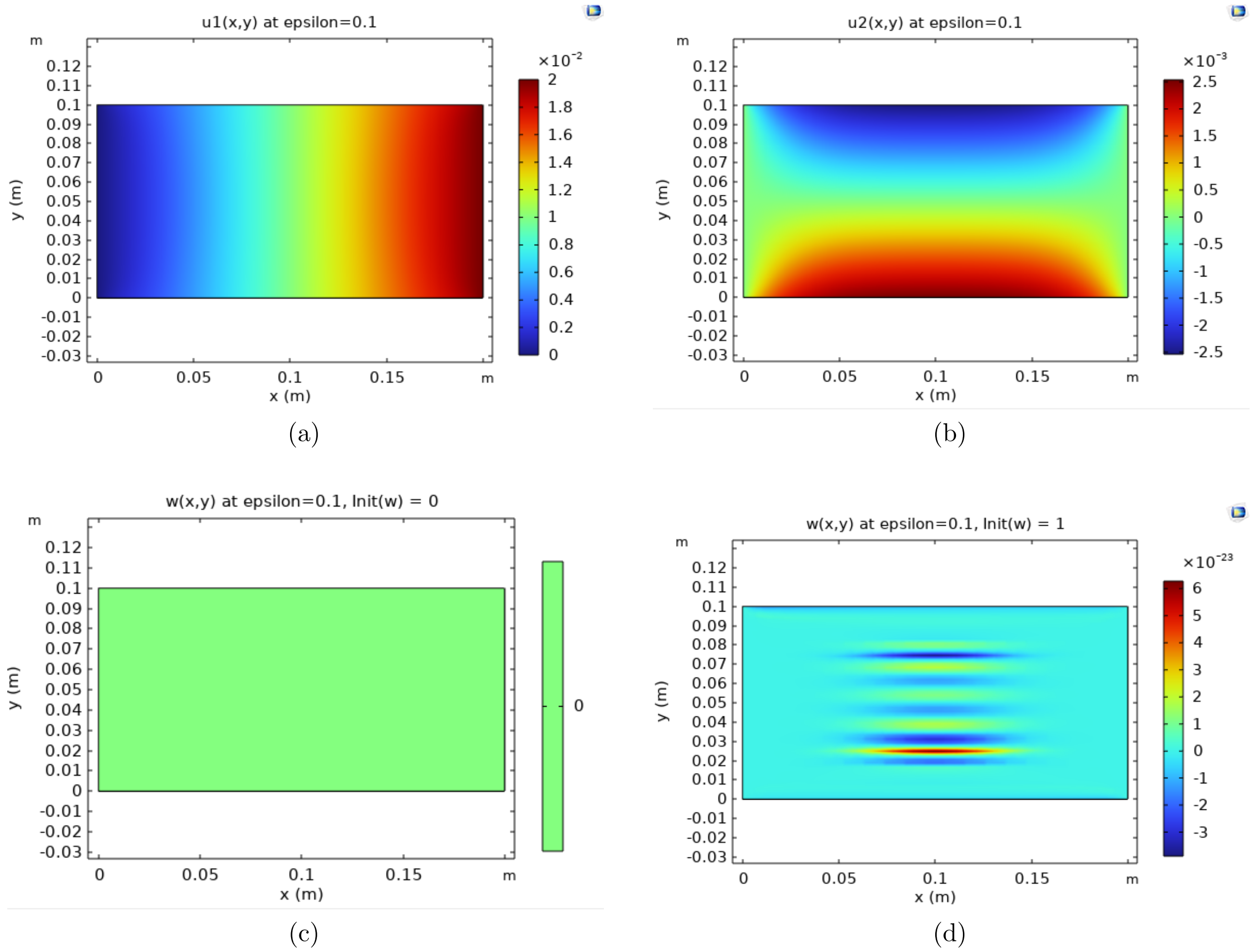

Figure 3.4: Plots of (a) u_1, (b) u_2 and (c) - (d) w at $\varepsilon = 0.1$. (c) Init(w) = 0, (d) Init(w) = 1

bending stiffness for this loading. Thus, the plate elongates uniformly, producing the smooth gradient.

(b) In-plane displacement $u_2(x, y)$ Although no vertical displacement is prescribed, the plate exhibits contraction in the y-direction. This behaviour arises from the Poisson effect: stretching along x induces a transverse contraction proportional to $\nu\varepsilon$. Since the boundaries restrict the transverse motion non-uniformly, the resulting contraction field is curved rather than uniform. This behaviour matches classical linear elasticity predictions for an axially stretched membrane.

(c) Out-of-plane displacement $w(x, y)$ with initial condition $w = 0$ When the plate begins perfectly flat and is subjected to pure in-plane stretching, the out-of-plane displacement remains zero. This occurs because there are no imperfections in the model. That is, the load at the end is being applied uniformly as well as the initial transverse displacement is identically zero. So, the system is in an ideal state and thus, won't show any buckling, which in this case would be wrinkling. Hence, the plate remains flat throughout the loading, resulting in plot (c).

(d) Out-of-plane displacement $w(x, y)$ with initial condition $w = 1$ When a small initial imperfection is introduced, the plate no longer remains flat under the same in-plane loading. In this case, the plate converges to a non-trivial solution giving rise to a wrinkling pattern. This suggests the presence of multiple bifurcation branches and instability in the problem. This behaviour reflects the imperfection sensitivity characteristic of nonlinear plate models.
Note that the order of magnitude of w in plot (d) is extremely low. A possible explanation for this is that the Newton-Raphson solver is not strong enough to capture the bifurcation behaviour exactly. A better solver well-suited for post-buckling analysis (say, involving arc-length method) will be needed for effective analysis.

3.3 Plate under Transverse loading

3.3.1 Problem Description

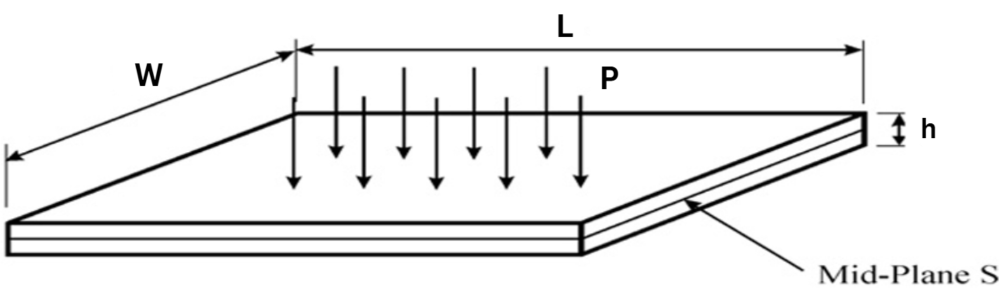

Figure 3.5: Plate under uniform transverse loading

The plate is rectangular with in-plane dimensions $L \times W$ and uniform thickness h. All four edges are fixed, i.e. at $x = 0$, $x = L$, $y = 0$, and $y = W$. A uniform transverse load of magnitude P acts downward over the entire plate. The corresponding displacement boundary conditions are

$$\mathbf{u}(0, y) = \mathbf{u}(x, 0) = \mathbf{u}(L, y) = \mathbf{u}(x, W) = \mathbf{0},$$

$$w(0, y) = w(x, 0) = w(L, y) = w(x, W) = 0.$$

A transverse body force of magnitude P acts on the plate; therefore,

$$\mathbf{f}_m = \mathbf{0}, \qquad f_b = -\frac{PL}{\lambda}.$$

Since all edges are fixed, reaction tractions arise along the boundaries. These are unknown a priori:

$$t_b = t_b, \qquad \mathbf{t}_m = \mathbf{t}_m.$$

Under these conditions, the equilibrium equations become

$$\nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] = \mathbf{0}$$
$$\nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N}\nabla w] = \frac{PL}{\lambda}$$

Adapting the general equations for natural boundary conditions to our problem, we get

$$(A) \quad \oint_{\Gamma_0} \big([(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}]\, \mathbf{n} - \mathbf{t}_m \big) \cdot \eta \; d\Gamma = 0$$
$$(B) \quad \oint_{\Gamma_0} \Big(\big([\mathbf{N}\nabla w + \nabla \cdot \mathbf{M}] \cdot \mathbf{n} - t_b \big)\zeta - (\mathbf{M}\mathbf{n}) \cdot \nabla\zeta \Big) d\Gamma = 0$$

In this problem, the equation (A) is trivially satisfied since $\eta = \mathbf{0} \quad \forall (x, y) \in \Gamma_0$. So we only have to deal with equation (B). Again, we first evaluate the integral along the edge $y = 0$, apply by parts on the suitable term to get the following expression

$$\int_0^L \left[\left(\left[\mathbf{N}\nabla w + \nabla \cdot \mathbf{M} + \frac{\partial \mathbf{M}\mathbf{e}_2}{\partial x} \right] \cdot \mathbf{e}_1 - t_b \right) \zeta - (\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_1) \frac{\partial \zeta}{\partial y} \right] dx - [\zeta \mathbf{e}_1 \cdot \mathbf{M}\mathbf{e}_1]_{x=0,y=0}^{x=L,y=0}$$

$\zeta = 0$ along all edges. Thus, only 1 term remains in the expression.

$$-\int_0^L \left[-(\mathbf{e}_2 \cdot \mathbf{M}\mathbf{e}_2) \frac{\partial \zeta}{\partial y} \right] dx$$

Since $\dfrac{\partial \zeta}{\partial y}$ is arbitrary, its coefficients must be zero. The same logic applies to the integral along the other edges. This leads us to the following boundary conditions

$$\mathbf{n} \cdot \mathbf{M}\mathbf{n} = 0 \qquad \text{along all 4 edges}$$

3.3.2 Summary

PDE System

(1) $\nabla \cdot [(\mathbf{I} + \nabla \mathbf{u})\mathbf{N}] = \mathbf{0}$

(2) $\nabla \cdot [\nabla \cdot \mathbf{M} + \mathbf{N}\nabla w] = \dfrac{PL}{\lambda}$

(3) $\mathbf{N} = 12\,[\nu(tr\mathbf{E})\mathbf{I} + (1 - \nu)\mathbf{E}]$

(4) $\mathbf{M} = h^2\,[\nu(tr\mathbf{K})\mathbf{I} + (1 - \nu)\mathbf{K}]$

(5) $\mathbf{E} = \dfrac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T(\nabla \mathbf{u}) + \nabla w \otimes \nabla w\right]$

(6) $\mathbf{K} = -\nabla^2 w$

Dirichlet Boundary Conditions

$\mathbf{u}(0, y) = \mathbf{u}(x, 0) = \mathbf{u}(L, y) = \mathbf{u}(x, W) = \mathbf{0}$,
$w(0, y) = w(x, 0) = w(L, y) = w(x, W) = 0$.

Natural Boundary Conditions

$\mathbf{n} \cdot \mathbf{M}\mathbf{n} = 0 \qquad$ along all 4 edges

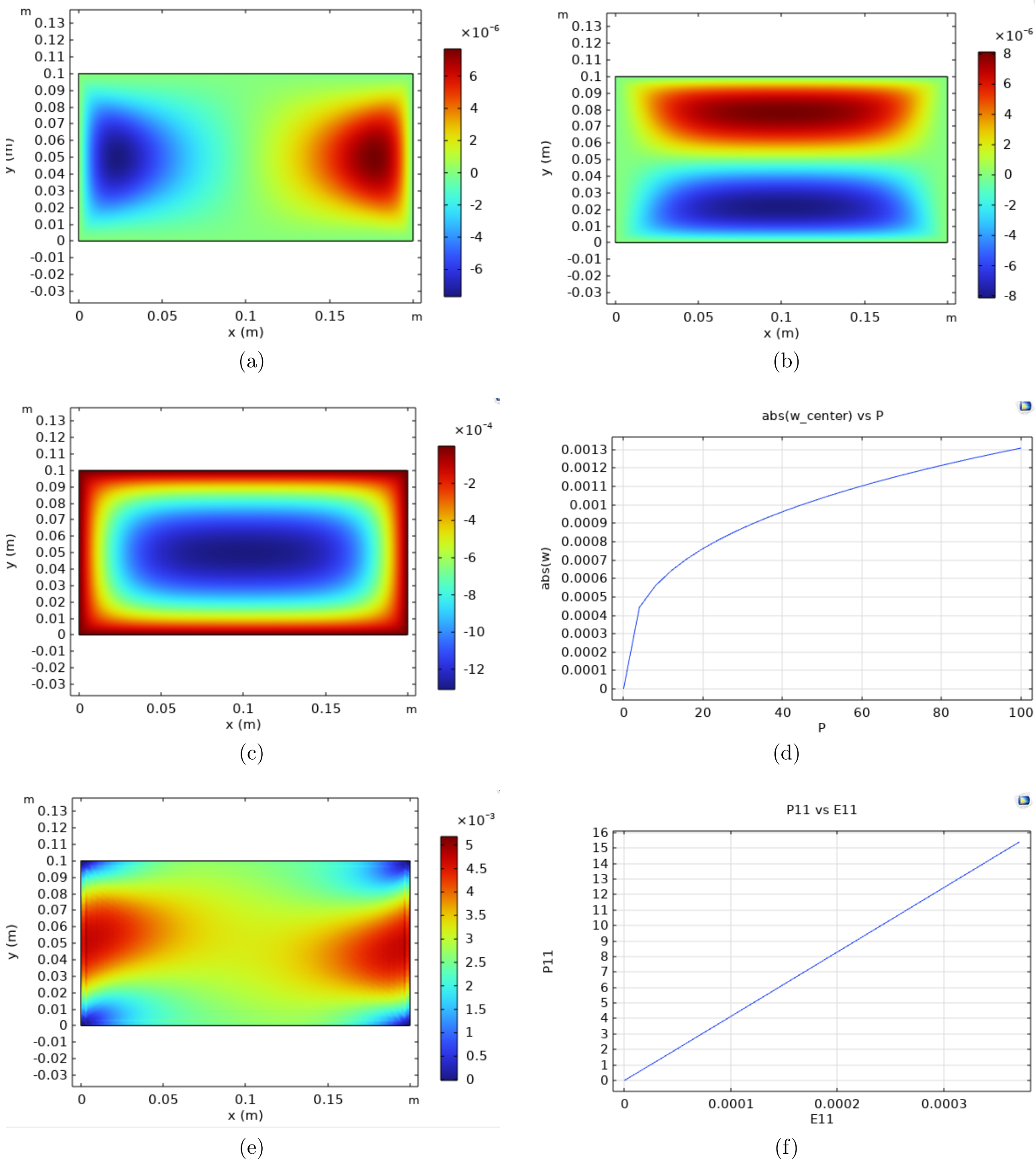

Figure 3.6: Plots of (a) u_1, (b) u_2, (c) w, (e) $P_{11} = (\mathbf{FN})_{11}$ at P = 100. (d) shows the plot of abs(w)_center vs P. (f) shows the plot of P_{11} vs E_{11} at (L/20, W/2).

3.3.3 Results

The final plot (f) P_{11} v.s. E_{11} is coming as a straight line due to our assumption of linear isotropy of the material. Such inaccuracy can be corrected by using a more appropriate constitutive relations like Mooney-Rivlin or Neo-Hookean model as in the case of Hyperelastic materials.

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