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JSS ACADEMY OF TECHNICAL EDUCATION, NOIDA
DEPARTMENT OF MATHEMATICS

UNIT-1

MATRICES

C103.1 Understand the concept of complex matrices, Eigen values, Eigen vectors and apply the concept of rank to evaluate linear simultaneous equations.

TOPICS COVERED

1.1 INVERSE OF A MATRIX USING ELEMENTARY TRANSFORMATION

(Gauss Jordan Method)

The following transformations are called elementary transformation of a matrix:

a) Interchanging of rows (columns).

b) Multiplication of a row (column) by a non-zero scalar.

c) Adding or subtracting k multiple of a row (column) to another row (column).

1.1.1 Definition

A matrix B is said to be row (column) equivalent to a matrix A if it is obtained from A by applying a finite number of elementary row (column) transformations. In such case, we write B~A. In Gauss Jordan method, we perform the sequence of elementary row transformations on A and I simultaneously, keeping them side by side.

1. Employing elementary row transformations, find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.
2. Use elementary row transformation, find the inverse of the matrix $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.
   [SOLVED]
3. Use elementary row transformation, find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$.
   [SOLVED]
4. Applying elementary row transformation, find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 2 \end{bmatrix}$.
5. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ by using elementary row transformation.

1.2 Rank of a Matrix

1.2.1 Definition

A matrix is said to be of rank r if it has at least one non-singular submatrix of order r but has no non-singular submatrix of order more than r.

Rank of a matrix A is denoted by $\rho(A)$.

A matrix is said to be of rank zero if and only if all its elements are zero.

1.2.2 METHOD OF FINDING RANK

To determine the rank of a matrix A, we adopt the following different methods:

1.2.2.1 Normal form method

If A is an $m \times n$ matrix and by a series of elementary (row or column or both) operations, it can be put into one of the following forms (called normal or canonical forms):

$\begin{bmatrix} \boldsymbol{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\begin{bmatrix} \boldsymbol{I_r} \\ \mathbf{0} \end{bmatrix}$, $[\boldsymbol{I_r} \quad \mathbf{0}]$, $[\boldsymbol{I_r}]$where I_r is the unit matrix of order r.

This method is also called sweep out method or pivotal method.

1.2.2.2 Echelon Form Method

A matrix is said to be in row reduced echelon form if

(i) The first non-zero entry in each non-zero row is 1.

(ii) The rows containing only zeroes occur below all the non-zeroes rows.

(iii) The number of zeroes before the first non-zeroes element in a row is less than the number of such zeroes in the next row.

The rank of a matrix in row reduced echelon form is equal to the number of non-zero rows of the matrix. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in the row reduced echelon form and its rank is 2 (the number of non-zero rows).

Corollary: If A is an $m \times n$ matrix of rank r, there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

6. Use elementary transformation to reduce the following matrix A to triangular form and hence find the rank of A where $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$. [SOLVED]

7. Find the rank of the matrix A by reducing it to echelon form. $A = \begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$.

8. Find the rank of the matrix A by reducing it to echelon form. $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$.

9. Find the rank of the matrix A by reducing it to echelon form. $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}$.

10. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ by reducing it to normal form.

11. Reduce the matrix A to its normal form and hence find its rank, where $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$. [SOLVED]

12. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$, find non-singular matrices P and Q such that PAQ is in the normal form.

13. If $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$, determine two non-singular matrices P and Q such that PAQ=I. Hence find rank of the matrix. [SOLVED]

14. If $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$, determine two non-singular matrices P and Q such that PAQ=I. Hence find rank of the matrix.

1.3 CONSISTENCY OF A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

Consider the system of equation $AX = B$.

If $B = 0$,i.e $AX = 0$, the system of equation is called homogeneous linear equations.

If $B \neq 0$,i.e $AX = B$, the system of equation is called non-homogeneous linear equations.

1.3.1 NON HOMOGENEOUS LINEAR EQUATIONS

Conditions for consistency for **non-homogeneous linear equations**$AX = B$

1. If $\rho(A{:}B) \neq \rho(A)$, the system is inconsistent.
2. If $\rho(A{:}B) = \rho(A) = no.of\ unknowns$, the system has a unique solution.
3. If $\rho(A{:}B) = \rho(A) < no.of\ unknowns$, the system has a infinite number of solutions.

1.3.2 HOMOGENEOUS LINEAR EQUATIONS

Conditions for consistency for **homogeneous linear equations**$AX = 0$

$X = 0$ is always a solution in which each unknown has the value zero called null solution or trivial solution.

It has either the trivial solution or an infinite number of solutions.

1. If $\rho(A) = no. of\ unknowns$, the system has only the trivial solution.
2. If $\rho(A) < no. of\ unknowns$, the system has infinite number of non-trivial solution.

15. Show that the system of equations $x + y + z = -3$, $3x + y - 2z = -2$, $x + 4y + 7z = 7$ is not consistent?
16. Apply Matrix method to solve system of equations $x + 2y - z = 3, 3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$.
17. Investigate for what values of λ and μ do the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) unique solution, (iii) infinite solution?
18. Investigate for what values of λ and μ the equations $x + 2y + z = 8$, $2x + 2y + 2z = 13$, $3x + 4y + \lambda z = \mu$ have (i) no solution, (ii) unique solution and (iii) many solutions. [SOLVED]
19. Find the values of k for which the system of equations
    $(3k - 8)x + 3y + 3z = 0$, $3x + (3k - 8)y + 3z = 0, 3x + 3y + (3k - 8)z = 0$
    has non-trivial solution.
20. Discuss for all values of k, the system of equations $2x + 3ky + (3k + 4)z = 0$, $x + (k + 4)y + (4k + 2)z = 0$, $x + 2(k + 1)y + (3k + 4)z = 0$. [SOLVED]
21. Solve the following system of equations:
    $$3x - y - z = 0, \qquad x + y + 2z = 0, \qquad 5x + y + 3z = 0$$
22. Solve the following system of equations:
    $x + y - z + w = 0$, $x - y + 2z - w = 0$, $3x + y + w = 0$

1.4 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

1.4.1 Linear Dependence

A set of r n-vectors $X_1, X_2, X_3 \dots \dots X_r$ is said to be linearly dependent if there exists r scalars $k_1, k_2, k_3 \dots \dots k_r$ not all zero such that $k_1X_1 + k_2X_2 + k_3X_3 + \cdots \dots . + k_rX_r = 0$ (linear combination)

1.4.2 Linear Independence

A set of r n-vectors $X_1, X_2, X_3 \dots \dots X_r$ is said to be linearly independent if every relation of the type $k_1X_1 + k_2X_2 + k_3X_3 + \cdots \dots . + k_rX_r = 0$implies $k_1 = k_2 = k_3 = \cdots \dots = k_r = 0$.

Note:

1. If rank of a matrix =no. of vectors, then given set of vectors is L.I.

2. If rank of a matrix<no. of vectors, then given set of vectors is L.D.
3. If a set of vectors is L.D. then at least one vector of the set can be expressed as a linear combination of the remaining vectors.
4. If a set of vectors is L.I. then no vector of the set can be expressed as a linear combination of the remaining vectors.

23. Examine whether following vectors are linearly independent or dependent:

$$X_1 = (2,2,1)^T, \quad X_2 = (1,3,1)^T, \qquad X_3 = (1,2,2)^T$$

24. Show that the rows of the following matrix are linearly dependent and find the relationship between them. [SOLVED]

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 4 & 6 & 2 & -4 \\ -6 & 0 & -3 & -4 \end{bmatrix}.$$

25. Find the value of λ for which the vectors (1,-2, λ), (2,-1,5) and (3,-5,7λ) are linearly dependent.
26. Show that the vectors (3,1,-4), (2,2,-3) and (0,-4,1) are linearly dependent.
27. Show that the vectors [2,-1,3,2], [3,-5,2,2], [1,3,4,2] are linearly dependent. Express one of the vector as a linear combination of the other. [SOLVED]

1.5 EIGEN VALUES AND EIGEN VECTORS

1.5.1 DEFINITION

Eigen values and Eigen vectors are important concepts in linear algebra. They are derived from German word 'Eigen' which means proper or characteristic. Eigen vectors are non-zero vectors that get mapped into scalar multiples of themselves under a linear operator. Any non-zero vector X is said to be a characteristic vector or Eigen vector of a matrix A if there exists a number λ such that$AX = \lambda X$ where $A = [a_{ij}]_{n\times n}$is an n- rowed square matrix and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.

The matrix $A - \lambda I$ is called characteristic matrix of A where I is the unit matrix of order n.

The $|A - \lambda I|$ is called characteristic polynomial of A.

The equation $|A - \lambda I| = 0$ is called characteristic equation of A and the roots of this equation are called eigenvalues of the matrix A. The set of all eigenvectors is called the eigenspace of A corresponding to λ. The set of all eigenvalues of A is called the spectrum of A.

1.5.2 Properties of Eigenvalues

1. If λ is an eigenvalue of the matrix A then λ is also an eigenvector of A^T.
2. If λ is an eigenvalue of the matrix A then $\frac{1}{\lambda}$is an eigenvector of A^{-1}.
3. If λ is an eigenvalue of the matrix A then λ^k is an eigenvector of A^k.
4. If λ is an eigenvalue of the matrix A then $\lambda \pm k$ is an eigenvector of $A \pm kI$.
5. If λ is an eigenvalue of the matrix A then $k\lambda$ is an eigenvector of kA.
6. The eigenvalues of a triangular matrix are the diagonal elements of the matrix.
7. The eigenvalues of a real symmetric matrix are real.

1.5.3 Working rule for finding Eigenvalues and Eigenvectors

1. Write the characteristic equation $|A - \lambda I| = 0$ for the given square matrix.
2. Find the eigenvalues of the matrix by solving characteristic equation.
3. Find eigenvectors corresponding to each eigenvalues from the equation $(A - \lambda I)X = 0$.

28. If 2 is an eigenvalue of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, find the other two eigen values.

29. If 2, 3 are the eigenvalues of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, find the value of a. [SOLVED]

30. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

31. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. [SOLVED]

32. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

33. Find a and b such that $A = \begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and -2 as eigenvalues.

34. Two of the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1} and A^3.

35. Form the matrix whose eigenvalues are $\alpha - 5, \beta - 5, \gamma - 5$ where α, β, γ are the eigen values of $\begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$.

1.6 CAYLEY HAMILTON THEOREM

1.6.1 Definition

Every square matrix satisfies its own characteristic equation.

36. Apply Cayley Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625I$.

[SOLVED]

37. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

38. Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley Hamilton theorem for this matrix. Find A^{-1} and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. [SOLVED]

39. Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley Hamilton theoremand hence find A^{-1}, if it exists.

40. Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and hence use it to find A^{-1} and A^4.

1.7 Complex Matrices, Hermitian and skew Hermitian Matrices, unitary Matrices

1.7.1 Complex Matrix

If at least one element of a matrix is a complex number $a + ib$,where a,b are real and $i = \sqrt{-1}$, then the matrix is called a complex matrix.

1.7.2 Conjugate of a matrix

The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called conjugate of the matrix A and is denoted by $\bar{A}$. e.g.

$$A = \begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} 1-3i & 2-5i & 8 \\ i & 6 & 9+i \end{bmatrix}$$

1.7.3 Transposed conjugate of the matrix

The conjugate of the transpose of a matrix A is called the conjugate transpose or transposed conjugate of A and is denoted by A^θ, e.g $A^\theta = (\bar{A})^T = \overline{(A^T)}$

e.g. If $A = \begin{bmatrix} 1-2i & 2+3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$, $A^T = \begin{bmatrix} 1-2i & 4-5i & 8 \\ 2+3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$

Then $A^\theta = \begin{bmatrix} 1+2i & 4+5i & 8 \\ 2-3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

1.7.4 Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called Hermitian if $a_{ij} = \overline{a_{ji}}$ for all i and j, i.e $A = A^\theta$, eg.

$$\begin{bmatrix} 1 & 2+3i & 3-4i \\ 2-3i & 0 & 2-7i \\ 3+4i & 2+7i & 2 \end{bmatrix}$$

1.7.5 Skew Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called skew Hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i and j, i.e $A = -A^\theta$.

Hence diagonal elements of a skew Hermitian matrix must be either purely imaginary or zero, eg.

$$\begin{bmatrix} i & 2+3i \\ 2-3i & 0 \end{bmatrix}$$

1.7.6 Unitary Matrix

A square matrix A is called unitary if $AA^\theta = I$.

41. Express the matrix $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrix. [SOLVED]
42. Express the matrix $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrix.
43. Prove that $U = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a unitary matrix. Hence find A^{-1}. [SOLVED]
44. Show that the matrix $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.
45. Every square matrix can be uniquely expressed as $P + iQ$, where P and Q are Hermitian.
46. If A and $B$are unitary matrices, show that AB is a unitary matrix. [SOLVED]

47. Show that every Hermitian matrix can be written as $P + iQ$ where P is a real symmetric matrix and Q is a real skew symmetric matrix.
48. Define a unitary matrix. If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ is a matrix, then show that $(I - N)(I + N)^{-1}$ is a unitary matrix where I is an identity matrix.

1.8 APPLICATION OF MATRICES TO ENGINEERING PROBLEM

Matrix inverse can provide a simple and effective procedure for encoding and decoding messages. Assign the numbers 1-26 to the letters in the alphabet. Assign 27 to blank (space). (A more sophisticated code could include both capital and lower case letters and punctuation symbols)

A	B	C	D	E …………	………..	X	Y	Z	Blank
1	2	3	4	5 …………..	………..	24	25	26	27

Any matrix 'A' whose elements are positive integers and whose inverse exists can be used as an 'encoding matrix'.

A message can be decoded by multiplying with A^{-1}, the 'decoding matrix'.

49. The message SECRET CODE correspondence to the sequence

    19 5 3 18 5 20 27 3 15 4 5

    Encoding matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$.

50. Encode the message 'THE SUN ALSO RISES' using $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. [SOLVED]
51. The message 46 84 85 55 101 100 31 59 64 57 102 99 29 57 38 65 111 122 was encoded with the matrix A. Decode this message. Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$.

JSS ACADEMY OF TECHNICAL EDUCATION, NOIDA

Department of Mathematics

Solution of Question Bank

Engineering Mathematics - I (BAS-103)

Q2

$$A \sim I$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$I \sim A^{-1}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} = A$$

$$A \sim I$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$ ~~&~~ then $R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\therefore \quad \cancel{A} \sim A^{-1}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$, $R_4 \to R_4 - 9R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \therefore \ \rho(A) = 3$$

10 $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$C_2 \to C_2 - C_1, \; C_3 \to C_3 - C_1, \; C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix} \longrightarrow R_2 \to (-1)R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & -6 & -2 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 28 & 54 \end{bmatrix} \longrightarrow \begin{matrix} C_3 \to C_3 - 5C_2 \\ C_4 \to C_4 - 10C_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 28 & 54 \end{bmatrix}$$

$C_3 \to \frac{1}{28} C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 54 \end{bmatrix} \longrightarrow C_4 \to C_4 - 54C_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim [I_3 \,|\, 0] \Rightarrow \rho(A) = 3.$$

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$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 8 & 0 \\ 0 & 1 & -2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 5 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 18 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + 2C_2$, $C_4 \to C_4 + 8C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 18 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to \frac{1}{18} C_3 \quad , \quad C_4 \to \frac{1}{40} C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4/18 & -8/40 \\ 0 & 0 & 1/18 & 0 \\ 0 & 1 & 2/18 & 8/40 \\ 0 & 0 & 0 & 1/40 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4/18 & 1/45 \\ 0 & 0 & 1/18 & -1/18 \\ 0 & 1 & 2/18 & 4/45 \\ 0 & 0 & 0 & 1/40 \end{bmatrix}$$

$$[I_3 \vdots 0] = P A Q$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} \;\&\; Q = \begin{bmatrix} 1 & -1 & -4/18 & 1/45 \\ 0 & 0 & 1/18 & -1/18 \\ 0 & 1 & 2/18 & 4/45 \\ 0 & 0 & 0 & 1/40 \end{bmatrix}$$

$$\rho(A) = 3.$$

18 The matrix form of the system is

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

The augmented matrix of the system is

$$[A \vdots B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 2 & 2 & 13 \\ 3 & 4 & \lambda & \mu \end{array}\right]$$

Reducing ~~Echelon~~ augmented matrix to Echelon form

$R_2 \to R_2 - 2R_1 , \quad R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & -2 & \lambda-3 & \mu-24 \end{array}\right]$$

$R_2 \to \left(-\frac{1}{2}\right) R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 3/2 \\ 0 & -2 & \lambda-3 & \mu-24 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & \lambda-3 & \mu-21 \end{array}\right]$$

(i) If $\lambda = 3$ & $\mu \neq 21$, system is inconsistent & has no solⁿ

(ii) If $\lambda \neq 3$ & μ has any value, system is consistent & has unique solⁿ.

(iii) If $\lambda = 3$ & $\mu = 21$, the system is consistent & has infinite solⁿ.

20 The matrix form of the system is $AX = 0$

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix} = (k-8)(-k+2) + 5k(k-2)$$

$$= (k-2)(-k+8+5k)$$

$$= 4(k-2)(k+2)$$

(i) When $k \neq \pm 2$, $|A| \neq 0$, system has trivial solⁿ i.e, $x = y = z = 0$.

(ii) When $k = \pm 2$ $|A| = 0$, system has non-trivial solⁿ

Case 1 : When $k = 2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & -6 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to \left(-\frac{1}{6}\right) R_2$

$$\left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & 1 & 10/6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 < 3$ (no. of unknowns)

$\therefore$ system has non-trivial solⁿ

No. of parameters $= 3 - 2 = 1$

The corresponding system of eqⁿ is

$x = 0$, $y + \frac{10}{6} z = 0$

Let $z = t \Rightarrow y = -\frac{10}{6} t \Rightarrow -\frac{5}{3} t$

$\therefore$ $x = 0,\ y = -\frac{5}{3} t,\ z = t$ is the solⁿ of system.

Case 2 : When $k = -2$, augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right]$$

$R_2 \to -\frac{1}{10} R_2$, $R_3 \to \left(-\frac{1}{4}\right) R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 < 3$ (no. of unknowns)

$\therefore$ system has non-trivial solⁿ

$x - 4z = 0$ Let $z = t$ $x = 4t,\ y = t$

$y - z = 0.$

2.4 Let the row vectors of the matrix be given by x_1, x_2, x_3, x_4

Let $k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$.

$$k_1(1,0,2,1) + k_2(3,1,2,1) + k_3(4,6,2,-4) + k_4(-6,0,-3,-4) = (0,0,0,0)$$

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$
$$k_2 + 6k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 - 3k_4 = 0$$
$$k_1 + k_2 - 4k_3 - 4k_4 = 0$$

$AX = 0$.

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing matrix A to echelon form

$R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & -4 & -6 & 9 \\ 0 & -2 & -8 & 2 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$, $R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$R_3 \to \frac{1}{18}R_3$, $R_4 \to \frac{1}{4}R_4$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3 < 4$ (No of unknowns)

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0$$
$$k_2 + 6k_3 = 0$$
$$k_3 + \frac{1}{2}k_4 = 0$$

Let $k_4 = t$

$k_3 = -\frac{1}{2}t$, $k_2 = -6k_3 = 3t$, $k_1 = -3k_2 - 4k_3 + 6k_4$
$= -9t + 2t + 6t$
$= -t$

$\because$ k_1, k_2, k_3, k_4 are not all zero, the vectors are L.D.

$$-tX_1 + 3tX_2 - \frac{1}{2}tX_3 + tX_4 = 0$$
$$2X_1 - 6X_2 + X_3 - 2X_4 = 0$$

27 $X_1 = (2,-1,3,2)$, $X_2 = (3,-5,2,2)$ $X_3 = (1,3,4,2)$

Let $k_1X_1 + k_2X_2 + k_3X_3 = 0$.

$$k_1(2,-1,3,2) + k_2(3,-5,2,2) + k_3(1,3,4,2) = (0,0,0,0)$$

Equating corresponding components,

$$2k_1 + 3k_2 + k_3 = 0$$
$$-k_1 - 5k_2 + 3k_3 = 0$$
$$3k_1 + 2k_2 + 4k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 = 0$$

The matrix form of the system is

$$AX = 0$$

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix is

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ -1 & -5 & 3 & 0 \\ 3 & 2 & 4 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & -2 & 4 \\ -1 & -5 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & 8 & -8 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$R_2 \to \left(-\frac{1}{7}\right) R_2$, $R_3 \to \left(\frac{1}{8}\right) R_3$, $R_4 \to \frac{1}{6} R_4$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 < 3$ (no. of unknowns)

$\therefore$ the system has non-trivial sol^n

corresponding system of eq^{n}s is

$$k_1 - 2k_2 + 4k_3 = 0$$
$$k_2 - k_3 = 0$$

Let $k_3 = t \Rightarrow k_2 = t$, $k_1 = 2k_2 - 4k_3 = 2t - 4t$
$= -2t$

$\because$ k_1, k_2, k_3 are not all zero, the vectors are L.D.

$$-2tX_1 + tX_2 + tX_3 = 0$$
$$\Rightarrow \quad 2X_1 - X_2 - X_3 = 0$$
$$\Rightarrow \quad 2X_1 = X_2 + X_3 .$$

29. Let λ_1, λ_2 & λ_3 be the eigen values of the matrix A.

$\lambda_1 = 2$, $\lambda_2 = 3$.

sum of eigen values of A = sum of principal diagonal element

$$2 + 3 + \lambda_3 = 2 + 2 + 2$$
$$\lambda_3 = 1$$

Product of eigenvalues of $A = |A|$

$$2 \times 3 \times 1 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$$

$$6 = 2(4-0) - 0(0-0) + 1(0-2a)$$

$$= 8 - 2a$$

$$2a = 2 \Rightarrow \boxed{a=1}$$

31 $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic eq^n is

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = sum of principal diagonal element of A

$$= -2+1+0 = -1$$

S_2 = sum of minors of principal diagondl element

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (0-12) + (0-3) + (-2-4)$$

$$= -12 - 3 - 6$$

$$= -21$$

$$S_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) - 3(-4+1)$$

$$= 24 + 12 + 9$$

$$= 45$$

$\therefore$ characteristic eq^n is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow \quad \lambda = 5, -3, -3.$$

(a) $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x+2y-3z=0$$
$$2x-4y-6z=0$$
$$-x-2y-5z=0.$$

By Cramer's rule

$$\frac{x}{\begin{vmatrix}2 & -3\\ -4 & -6\end{vmatrix}} = \frac{-y}{\begin{vmatrix}-7 & -3\\ 2 & -6\end{vmatrix}} = \frac{z}{\begin{vmatrix}-7 & 2\\ 2 & -4\end{vmatrix}} = t$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = t$$

$$X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}t\\ 2t\\ -t\end{bmatrix} = t\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

$X_1 = \begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$ is eigen vector corresponding to $\lambda = 5$.

b) $\lambda = -3$

$$[A-\lambda I]X = 0.$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x+2y-3z=0$$

Let $y = t_1$ & $z = t_2$

$\Rightarrow x = -2t_1 + 3t_2$

$$X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2t_1+3t_2\\ t_1\\ t_2\end{bmatrix} = \begin{bmatrix}-2t_1\\ t_1\\ 0\end{bmatrix} + \begin{bmatrix}3t_2\\ 0\\ t_2\end{bmatrix}$$

$$= t_1\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix} + t_2\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$$

$$X_2 = \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}, \quad X_3 = \begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - S_1\lambda + S_2 = 0.$$

S_1 = sum of principal diagonal elements

$= 1 - 1 = 0$

$S_2 = |A| = -1 - 4 = -5$

$\therefore$ characteristic eq^n is $\lambda^2 - 5 = 0$

By Cayley Hamilton theorem,

$$A^2 - 5I = 0$$
$$A^2 = 5I$$
$$A^4 = 25I$$
$$A^8 = 625I$$

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$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - S_1\lambda + S_2 = 0.$$

$S_1 = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

$$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda = -1, 5 \quad \text{——} (1)$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

$\Rightarrow$ Matrix A satisfies its own characteristic eq^n

Pre-multiplying eq.(1) by A^{-1}

$$A^{-1}(A^2 - 4A - 5I) = 0$$
$$A - 4I - 5A^{-1} = 0$$

$$4A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$$= A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I) + 3(A^2 - 4A - 5I) + A + 5I$$

$$= 0 + (A + 5I)$$

$$= A + 5I$$

which is a linear polynomial in A.

4b

$$A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} + \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}$$

$Q = \frac{1}{2}(A - A^\theta) = \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} - \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

We know, P is a Hermitian matrix & Q is a skew-Hermitian matrix

$$A = P + Q = \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

43 $A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$A^T = \frac{1}{2}\begin{bmatrix} \sqrt{2} & i\sqrt{2} & 0 \\ -i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $A^\theta = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$AA^\theta = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow \quad AA^\theta = I$

$\therefore$ A is a unitary matrix

& $A^{-1} = A^\theta = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

45 $AA^\theta = A^\theta A = I$. Since A is unitary matrix

$BB^\theta = B^\theta B = I$

$(AB)(AB)^\theta = (AB)(B^\theta A^\theta)$

$= A(BB^\theta)A^\theta = AIA^\theta = AA^\theta = I$

Again $(AB)^\theta(AB) = (B^\theta A^\theta)(AB) = B^\theta(A^\theta A)B$

$= B^\theta I B = B^\theta B = I$

Hence AB is a unitary matrix.

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THE SUN ALSO RISES

20 8 5 27 19 21 14 27 1 12 19 15 27 18 9 19 5 19 27

$$B = \begin{bmatrix} 20 & 5 & 19 & 14 & 1 & 19 & 27 & 9 & 5 & 27 \\ 8 & 27 & 21 & 27 & 12 & 15 & 18 & 19 & 19 & 27 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 & 5 & 19 & 14 & 1 & 19 & 27 & 9 & 5 & 27 \\ 8 & 27 & 21 & 27 & 12 & 15 & 18 & 19 & 19 & 27 \end{bmatrix}$$

Coded Message

36 44 59 86 61 82 68 95 25 37 49 64 63

81 47 66 43 62

Answer Key

1. $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

2. $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & 3 \end{bmatrix}$

3. $A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$

4. $A^{-1} = \frac{1}{40}\begin{bmatrix} -12 & 14 & 2 \\ 14 & -23 & 11 \\ 2 & 11 & -7 \end{bmatrix}$

5. $A^{-1} = \begin{bmatrix} 1 & 4 & 8 \\ 1 & 3 & 7 \\ 0 & -1 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $\rho(A) = 3$.

7. $\rho(A) = 4$

8. $\rho(A) = 3$

9. $\rho(A) = 2$

10. $\rho(A) = 3$

11. $[\, I_3 \,\vdots\, 0 \,]$; $\rho(A) = 3$

12. $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

13 $P = \begin{bmatrix} +1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & -1 & -4/18 & 1/45 \\ 0 & 0 & 1/18 & -1/18 \\ 0 & 1 & 2/18 & 4/45 \\ 0 & 0 & 0 & 1/40 \end{bmatrix}$

$\rho(A) = 3$.

14 $P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 9/28 & 1/28 & -6/28 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\rho(A) = 3$.

16 $x = -1, \quad y = 4, \quad z = 4$.

17 (i) $\lambda = 3, \ \mu \neq 10$; (ii) $\lambda \neq 3$, μ may have any value

(iii) $\lambda = 3$ & $\mu = 10$

18 (i) $\lambda = 3, \ \mu \neq 21$ (ii) $\lambda \neq 3$, μ may have any value

(iii) $\lambda = 3, \ \mu = 21$

19 $k = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$

20 $K \neq \pm 2, \quad x = 0, y = 0, z = 0$

$K = 2, \quad x = 0, \quad y = -\frac{5}{3}t, \quad z = t$

$K = -2, \quad x = 4t, \quad y = t, \quad z = t$.

21 $x = -\frac{1}{4}t, \quad y = -\frac{7}{4}t, \quad z = t$.

22 $x = -\frac{1}{2}t_1, \quad y = \frac{3}{2}t_1 - t_2, \quad z = t_1, \omega = t_2$

23 L.I

24 L.D ; $2X_1 - 6X_2 + X_3 - 2X_4 = 0$

25 $\lambda = \frac{5}{14}$

26 L.D

27 $2x_1 = x_2 + x_3$

28 $2, -2$

29 $a = 1$

30 $\lambda = 0, 3, 15, ; \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

31 $\lambda = 5, -3, -3 ; \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

32 $\lambda = 1, 1, 1 ; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

33 $b = 2$ or $b = -1$ $\therefore a = -1$ or $a = 2$

34 $\frac{1}{3}, \frac{1}{6}, \frac{1}{2} ; 3^3, 6^3, 2^3$.

35 $\begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}$

37 $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 ; \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

38 $\lambda = -1, 5 ; A^{-1} = \frac{1}{20}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} ; A + 5I$

39 A^{-1} does not exist.

40 $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}, A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$

42 $A = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} + \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$

43 $A^{-1} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

49 $\begin{bmatrix} 19 & 3 & 5 & 27 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 27 \end{bmatrix}$

50 36 44 59 86 61 82 68 95 25 37 49 64 63 81
47 66 43 62

51 WHO IS CARL GAUSS