CIA - II [A Group]

Solution Set.

Section-A.

(1) (a)

$$L^{-1}\left\{\frac{1}{(p+2)^5}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{1}{p^5}\right\}$$ ½ mark

$$= e^{-2t}\frac{t^4}{\Gamma 5} \text{ or } e^{-2t}\frac{t^4}{\lfloor 4}$$ ½ mark **Ans.**

(b)

$$L^{-1}\left\{\frac{1}{p(p^2+1)}\right\}.$$

$$= L^{-1}\left\{\frac{1}{p} - \frac{p}{p^2+1}\right\}$$ ½ mark

$$= 1 - \cos t$$ ½ mark **Ans.**

(c)

$$\langle a_n \rangle = \frac{n+1}{n}$$

Here,

$$a_n = \frac{n+1}{n}$$

$$a_{n+1} = \frac{n+2}{n+1}$$

$$a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} < 0$$ ½ mark

$$\Rightarrow \langle a_n \rangle \text{ is decreasing.}$$

Now.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$ ½ mark

$$= 1.$$

$\langle a_n \rangle$ is decreasing and bounded below by '1'.

$\Rightarrow \langle a_n \rangle$ is $\boxed{\text{Convergent}}$

(2)

(d) $f(x) = x\sin x \;; \quad 0 < x < 2\pi.$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[(x)(-\cos x) - (1)(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-x\cos x + \sin x\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\big[-2\pi\big] = -2 \quad \text{Ans}$$

1 mark

(e) **N-C** The N-C for a function $f(z)$ to be analytic at all the points in a region R are (i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Provided u_x, u_y, v_x, v_y Exist.

S-C The S-C for a function $f(z)$ to be analytic at all the points in a region R are (i) $u_x = v_y$ (ii) $u_y = -v_x$ (iii) u_x, u_y, v_x, v_y are cts. function of x and y in region R.

1 m

Section B.

(2)(a) $L^{-1}\left\{\frac{p}{(p^2+1)(p^2+4)}\right\}$.

Let $f(p) = \frac{1}{p^2+4}$, $g(p) = \frac{p}{p^2+4}$

$\Rightarrow F(t) = \frac{1}{2}\sin 2t$

& $G(t) = \cos t$

1 mark

$$L^{-1}\left\{\frac{p}{(p^2+1)(p^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u \cdot \cos(t-u)\,du.$$

$$= \frac{1}{4}\int_0^t \left\{\sin(u+t) + \sin(3u-t)\right\} du$$

$$= -\frac{1}{4}\left[\cos(u+t) + \frac{\cos(3u-t)}{3}\right]_0^t$$

$$= \frac{1}{3}\left[\cos t - \cos 2t\right]$$

2 marks

Ans.

(2)(b) $\frac{dx}{dt} - y = e^t$; —(1) $\frac{dy}{dt} + x = \sin t$ —(2) | $x(0)=1$, $y(0)=1$

Taking Laplace Transformation of (1) & (2)

$[p\bar{x} - x(0)] - \bar{y} = \frac{1}{p-2} \Rightarrow [p\bar{x} - 1] - \bar{y} = \frac{1}{p-2}$

$[p\bar{y} - y(0)] + \bar{x} = \frac{1}{p^2+1} \Rightarrow [p\bar{y} - 1] + \bar{x} = \frac{1}{p^2+1}$

1.5 mark

On solving. $\bar{x} = \frac{1}{2(p-1)} + \frac{1}{2}\frac{(p+1)}{(p^2+1)} + \frac{1}{(p^2+1)^2}$

Apply Laplace Inv. $\Rightarrow$ $x(t) = \frac{1}{2}\left[e^t + \cos t + 2\sin t - t\cos t\right]$

Similarly, $y(t) = \frac{1}{2}\left[-e^t - \sin t + \cos t + t\sin t\right]$

1.5 mark

Ans.

3 (a) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots \frac{x^n}{n^2+1}$

Here. $u_n = \frac{x^n}{n^2+1}$

$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}}{\left(1+\frac{1}{n^2}\right)} \cdot \frac{1}{x}$

$= \frac{1}{x} = \frac{1}{x}$ >1 $\boxed{\text{Conv.}}$; <1 $\boxed{\text{DIV.}}$; $=1$ $\boxed{\text{Test fail.}}$

1½ mar

if $x = 1$ $\quad \Sigma u_n = \frac{1}{n^2+1}$

By Applying Comparison Test..

Let $\Sigma v_n = \frac{1}{n^2}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \times \frac{n^2}{1} = 1$

(Non-zero finite)

$\Rightarrow \Sigma u_n$ & Σv_n both conv & diverge together.

But Σv_n is convergent by P-series.

$\Rightarrow \Sigma u_n$ is also convergent.

1½ mar

Finally. Series. Σu_n is convergent $\boxed{x \le 1}$

Divergent $\boxed{x > 1}$

A

3 (b) $\frac{2}{1} + \frac{2 \cdot 5}{1 \cdot 5} + \frac{2 \cdot 5 \cdot 8}{1 \cdot 5 \cdot 9} + \cdots$

Here,

$$u_n = \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{1 \cdot 5 \cdot 9 \cdots (4n-3)}$$

$$u_{n+1} = \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)(3n+2)}{1 \cdot 5 \cdot 9 \cdots (4n-3)(4n+1)}$$

2 mark

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{4n+1}{(3n+2)} = \frac{4}{3} > 1$$

1 mark

By D'Alembert's Ratio Test

Σu_n is $\boxed{\text{convergent.}}$ Ans.

4 (a) $f(z) = \sin z.$

$f(z) = \sin(x+iy)$

$= \sin x \cdot \cos iy + \cos x \cdot \sin iy$

$= \sin x \cdot \cosh y + i \cos x \sinh y$

$\Rightarrow u = \sin x \cdot \cosh y, \ v = \cos x \cdot \sinh y.$

$1\frac{1}{2}$ mark

Now,
$u_x = \cos x \cosh y$
$u_y = \sin x \sinh y$
$v_x = -\sin x \sinh y$
$v_y = \cos x \cosh y.$

clearly,
$u_x = v_y$
& $u_y = -v_x$

$1\frac{1}{2}$ mark

$\Rightarrow$ C-R Eqn are satisfied and u_x, u_y, v_x, v_y are continuous $\Rightarrow f(z) = \sin z$ is analytic.

Ans.

4(b) Let $f(z) = u + iv = r^2\cos 2\theta + i r^2 \sin P\theta$

$\Rightarrow u = r^2\cos 2\theta, \quad v = r^2 \sin P\theta$

By C-R Eqn

$u_r = 2r\cos 2\theta$

$v_r = 2r\sin P\theta$

$u_\theta = -2r^2\sin 2\theta$

$v_\theta = Pr^2\cos P\theta$

2 mark

$\because f(z)$ is analytic

$\Rightarrow u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$ 1 mark

$\Rightarrow \boxed{P=2}$ Ans.

Section-C

5(a) $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & 0 \le x \le \pi \end{cases}$

Here $f(-x) = f(x) \Rightarrow$ Even f^n.

$\Rightarrow \boxed{b_n = 0.}$ 1 mark

$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ —— (1).

Now, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$ 1 mar

$\boxed{a_0 = 0}$

& $a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$

$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$ 3 r

$a_n = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$

$$\Rightarrow f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\cos nx}{n^2}$$ 1 mar Ans. 7

5 (b) HRC

$$f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l-x) & \frac{l}{2} \le x \le l. \end{cases}$$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$\boxed{a_0 = \frac{kl}{2}}$$

$1\frac{1}{2}$ mark

Now, $a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$

$$a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right].$$

If n is odd $\boxed{a_n = 0}$

If n is even $a_n \Rightarrow a_2 = -\frac{8kl}{2^2\pi^2}$

$$a_4 = 0,\quad a_6 = -\frac{8kl}{6^2\pi^2}$$

4.5 mar

$$\Rightarrow f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \cdots\right]$$

Put $x = l$.

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}.}$$

Prove

Ans