Linear Algebra
Practice Problems for Quiz III

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Introduction

Herein we will discuss the solutions for problems relating to the third assignment in linear algebra. The goal of this is to become proficient in the skills required to succeed in both the class and the practical applications of it.

**The book from which the problems are taken is Steve Leon's Linear Algebra with Applications, 9e.*

Problem Set

§	**EXERCISES**
3.6	1, 4, 5, 12, 22, 25
4.2	1, 2, 5, 6, 14, 17
4.3	1, 2, 5, 7, 9, 11
5.2	1, 2, 3, 4, 13

3.6 Row Space and Column Space

Problem 1. For each of the following matrices, find a basis for the row space, a basis for the column space, and a basis for the null space.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

Solution. The solutions will be broken down into the basis, B, of the **row space** ($RS(A)$), **column space** ($CS(A)$), and **null space** ($N(A)$), in that order.

(a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$

I) $B(RS(A))$, the basis of the row space of A, can be found by first reducing the matrix to its row echelon form U to eliminate linearly dependent rows.

The row echelon form of A, $U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

This leaves two linearly independent rows, $(1, 3, 2)$, $(0, 1, 0)$. These two rows form the basis of the row space of A, as all other rows of A can be obtained through addition or scalar multiplication of these rows.

$\therefore B(RS(A)) = \{(1, 3, 2), (0, 1, 0)\}$.

II) $B(CS(A))$, the basis of the column space of A, can be found by reducing A to a row echelon form U and obtaining from U the columns with leading ones. These columns of A form $B(CS(A))$.

The row echelon form of A, $U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The columns with leading ones are $\boldsymbol{c_1}$, $\boldsymbol{c_2}$, meaning that these are the columns from A that form $B(CS(A))$.

$\therefore B(CS(A)) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}$.

III) $B(N(A))$, the basis of the null space of A, is the set of linearly independent vectors that span the entirety of $N(A)$, the set of vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$.

First, we solve for $N(A)$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 4 & 7 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And so it follows that

$$\begin{aligned} x_1 + \quad + 2x_3 &= 0, \\ + x_2 &= 0 \end{aligned}$$

Let $x_3 = \alpha$, where $\alpha \in \mathbb{R}$, then $N(A) = \begin{bmatrix} -2\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore B(N(A)) = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ as linear combinations of these linearly independent vectors span the entirety of $N(A)$.

(b) Let $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

I) Through row operations, we obtain the reduced row echelon form of A,

$$U^* = \begin{bmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Where we find that all three rows are linearly independent.

$\therefore B(RS(A)) = \left\{(1,0,0,-\frac{10}{7}),(0,1,0,-\frac{2}{7}),(0,0,1,0)\right\}$

II) Through the same reduced row echelon form, we can see that the columns containing leading ones are $\boldsymbol{c_1},\ \boldsymbol{c_2},\ \boldsymbol{c_3}$.

$\therefore B(CS(A)) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}$

III) Solve for the $N(A)$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{10}{7} & 0 \\ 0 & 1 & 0 & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Let $x_4 = \alpha,\ \alpha \in \mathbb{R}$, then

$$\begin{aligned} x_1 + \quad & -\frac{10}{7}\alpha = 0, \\ x_2 + \quad & -\frac{2}{7}\alpha = 0, \\ x_3 \quad & = 0 \end{aligned}$$

And $N(A) = \begin{bmatrix} \frac{10}{7}\alpha \\ \frac{2}{7}\alpha \\ 0 \\ \alpha \end{bmatrix}$.

$$\therefore B(N(A)) = \left\{ \begin{bmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 2. In each of the following, determine the dimension of the subspace of $\mathbb{R}^3$ spanned by the given vectors.

(a) $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

4.2 Matrix Representations of Linear Transformations

4.3 Similarity

5.2 Orthogonal Subspaces