Module II

A set can be described as a well defined collection of objects or a set is an unodered collection of objects.

Ex: $V = \{a, e, i, o, u\}$

The set V of all vowels in the English alphabet

The set ~~of~~ O of all ~~odd~~ +ve integers can be expressed by $O = \{1, 3, 5, 7, 9\}$

The objects in a set are called the elements or members of the set. A set is said to contain its elements. There are several ways to describe a set. One way is to list all the members of a set, when this is possible. It is called list method

Ex: $A = \{a, b, c, d\}$ another way to describe a set is to use set builder notation. we characterize all those elements in the set by stating the property or properties they must have to be members.

$O = \{x / x$ is an odd +ve Integer less than $10\}$

$$O = \{x \in Z^+ / x \text{ is odd and } x < 10\}$$

Remark ~~Note that~~

Note that the concept of a datatype of type in c.s is buildup on the concept of a set. In particular a datatype/type is the name of a set together with a set operations that can be performed on objects from that set.

Ex: Boolean is the name of the set $\{0,1\}$ together with operators on one or more elements of the set such as AND, OR & NOT.

- Two sets are equal if & only if they have the same elements. i.e If A & B are sets then A & B are equal if & only if $\forall x (x \in A \leftrightarrow x \in B)$

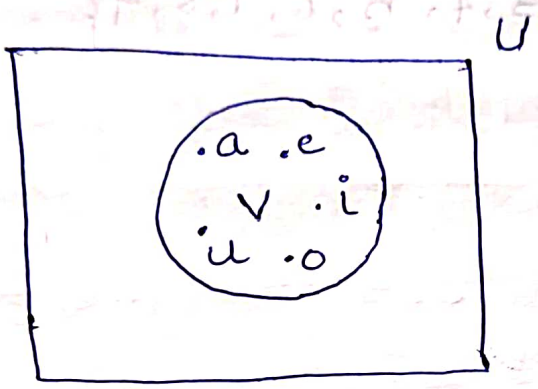
We write $A = B$ if A & B are equal sets

The sets $A = \{0,1,2,3\}$
$B = \{0,1,2,3\}$ are equal.

Sets can be represented graphically using Venn diagrams. In Venn diagrams the universal set U, which contains all the objects under

Consideration, is represented by a rectangle. Inside this rectangle circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular element of the set.

Venn diagrams for the set of vowels.



- The set that has no element is called null set / empty set. It is denoted by ϕ or $\{\}$

A set with 1 element is called single ~~term~~ ton set.

Ex: $A = \{1\}$, $B = \{\phi\}$

The set A is said to be a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. i.e $A \subseteq B$ if & only if

$$\forall x \, (x \in A \rightarrow x \in B)$$

Ex: The set of all odd +ve integers less than 10 is the subset of set of all +ve integers less than 10.

$A = \{1, 3, 5, 7, 9\}$

$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A \subseteq B$

<u>Remarks</u>

1) $\phi \subseteq s$

2) $s \subseteq s$

<u>Theorem 1</u>

For every set s

1) $\phi \subseteq s$

2) $s \subseteq s$

$\boxed{A \subseteq B \text{ means } \forall x (x \in A \rightarrow x \in B)}$

To prove that $\phi \subseteq s$

i.e to prove that $\forall x (x \in \phi \rightarrow x \in s)$ is true

We have the empty set contains no elements

$\therefore$ $x \in \phi$ is always false

But $x \in s$ is either true or false. Hence the conditional statement $x \in \phi \rightarrow x \in s$

is always true i.e $\forall x\,(x \in \phi \rightarrow x \in S)$ is true
i.e $\phi \subseteq S$

2. To prove that $S \subseteq S$
i.e to prove $\forall x\,(x \in S \rightarrow x \in S)$
it is clear that $x \in S \rightarrow x \in S$ is always true. Hence the conditional statement is always true. $\therefore$ $\forall x\,(x \in S \rightarrow x \in S)$ is true.

Proper Subset

A set A is said to be proper subset of B if $A \subseteq B$ & $A \neq B$. It is denoted by $A \subseteq B$
i.e A is a proper subset of B if ~~$\forall x$~~
$\forall x\,(x \in A \rightarrow x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$

$A = \{1, 2\}$
$B = \{1, 2, 3\}$
A is a proper subset of B

Note: $A = B$ iff $A \subseteq B$ & $B \subseteq A$
$A = B$ iff $\forall x\,(x \in A \rightarrow x \in B)$

- Let S be a set if there are exactly n distinct element in X where n is a non-negative Integer, we Say that S is a finite Set and that n is the cardinality of S.

The Cardinality of S is denoted by $|S|$.

Ex: let A be the set of odd +ve Integer less than 10.

$|A| = 5$

let S be the Set of letters in the English alphabet

$|S| = 26$

$|\phi| = 0$

- A Set is Said to be infinite if it is not finite.

Ex: Set of natural numbers.

- Given a Set S the power Set of S is the Set of all subsets of the set S. The power Set of S is denoted by P(S)

→ Find the powerset of the set $= \{0, 1, 2\}$

let $S = \{0, 1, 2\}$

$P(S) = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

3^2

2^3

$\{\phi\ 1, 2\}\ \{0, 1, 2\}\}$

→ what is the powerset of the empty set. what is the powerset of the set $\{\phi\}$

$P(\phi) = \{\phi\}$

$P(S) = \{\phi, \{\phi\}\}$

Definition

The ordered n-tuple ~~$A_1 A_2 \ldots A$~~ $(a_1, a_2 \ldots a_n)$ is the odered collection that has a_1 as its 1st element, a_2 as its second element, a_n as its nth element. we say that two ordered ~~entu~~ n-tuples are equal if & only if each corresponding pairs of their elements is equal. i.e $(a_1, a_2, \ldots a_n) = (b_1, b_2 \ldots b_n)$ iff $a_1 = b_1, a_2 = b_2, \ldots a_n = b_n$.

Cartesian products

Let A & B be sets the cartesian product of A & B denoted by $A \times B$, is the set of all odered pairs (a, b) where $a \in A$, $b \in B$ Hence

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Ex: what is the cartesian product of $A = \{1,2\}$ $B = \{a,b,c\}$

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$

→ S.T the cartesian product $B \times A \neq$ the cartesian Product $A \times B$. where $A = \{1,2\}$ $B = \{a,b,c\}$

$$B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

clearly $A \times B \neq B \times A$

Relation

A subset R of the cartesian product $A \times B$ is called a Relation from the set A to the set B. The elements of R are odered pairs where the first element belongs to A & the second to B.

Ex: $R = \{(a,0), (a,1), (a,3), (b,1), (b,2), (c,0), (c,3)\}$ is a relation from the set $\{a,b,c\}$ to $\{0,1,2,3\}$

The cartesian products of the sets $A_1, A_2 \cdots A_n$ denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of odered n-tuples $(a_1, a_2 \cdots a_n)$ where a_i belongs to A_i for $i = 1, 2, \cdots n$. In other words $A_1 \times A_2 \times \cdots A_n = \{(a_1, a_2 \cdots a_n) / a_i \in A_i$ for $i = 1, 2, \cdots n\}$

- what is the cartesian product $A \times B \times C$

$A = \{0, 1\}$

$B = \{1, 2\}$

$C = \{0, 1, 2\}$

$(0,1), (0,2) (1,1) (1,2)$

$(0,1,0) (0,1,1) (0,1,2), (0,2,0)$

$(0,2,1), (0,2,2), \cancel{(0,1,0}(1,1,0), (1,1,1)$

$(1,1,2)$

$A \times B \times C = \{(0,1,0) (0,1,1) (0,1,1,2) (0,2,0)$

$(1,2,0) (1,2,1) (1,2,2)$

$(0,2,1) (0,2,2) (1,1,0) (1,1,1)$

$(1,1,2) (1,2,0) (1,2,1) (1,2,2)\}$

- what do the statements $\forall x \in R (x^2 \geq 0)$ and $\exists x \in Z (x^2 = 1)$ mean?

This statement $\forall x \in R (x^2 \geq 0)$ states that $\forall$ ~~R x~~ for every real no x ~~$x \in R$~~ $(x^2 \geq 0)$ This statement can be expressed as The Square of every

$A \times B \times C$ $\{1, 2, 3, 4 \cdots n\}$

real no. is non-negative. The statement $\exists x \in Z\ (x^2=1)$ states that There exist an integer x such that $x^2=1$. This statement can be expressed as "There is an Integer whose Square is 1". The above two statements are true.

Truth Sets of Quantifiers

Given a predicate P and a domain D be defined the truth set of P to be the set of elements x in D ~~be the s~~ for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D / P(x)\}$

→ What are the truth sets of the predicates $P(x)$, $Q(x)$, $R(x)$ where the domain is the set of Integers and $P(x)$ is $|x|=1$, $Q(x)$ is $x^2=2$ and $R(x)$ is $|x|=x$

The truth set of $P(x)$ is $\{x \in Z / P(x)\}$

i.e $\{x \in Z / |x|=1\}$

$\therefore$ The truth set is $\{-1, 1\}$ $\quad x = |-2| = 2$

The truth set of $Q(x)$ is $\{x \in Z / Q(x)\}$

i.e $\{x \in Z / x^2 = 2\}$ i.e ϕ

$\therefore$ The truth set of $R(x)$ is $\{x \in Z / R(x)\}$

i.e $\{x \in Z / |x| = x\}$ i.e {~~positive~~ set of all non-negative Integers}

Russel's paradox

Russell proposed the following paradox:

Let the domain valey set of all sets and define S to be the following set

$S = \{x \mid x \notin x\}$. Thus S is the set of all sets which are not members of themselves

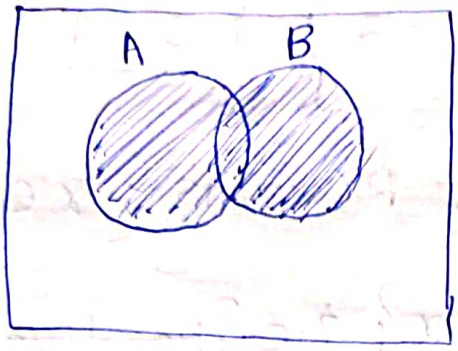
→ Is S a member of itself?

Set Operations

Let A & B be Sets the Union of the sets A & B denoted by $A \cup B$ is the set that contains those elements that are either in A or B ~~or in both~~.

$A \cup B = \{x / x \in A \vee x \in B\}$

Venn diagram representing $A \cup B$



$A = \{1, 3, 5\}$

$B = \{1, 2, 3\}$

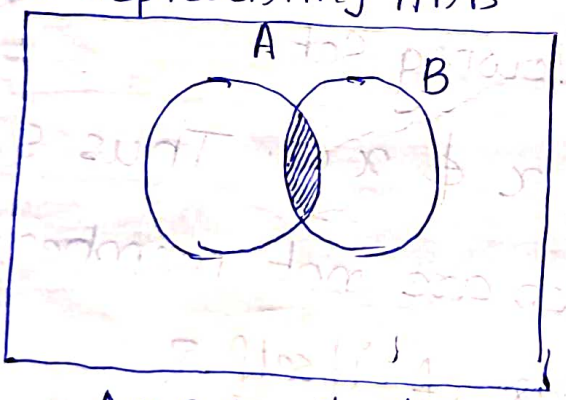
$A \cup B = \{1, 2, 3, 5\}$

$A \cup B$ is shaded

Let A & B be sets the intersection of the sets A & B denoted by $A \cap B$ is the set containing those elements in both A & B

$A \cap B = \{x / x \in A \wedge x \in B\}$

Venn diagram representing $A \cap B$



$A = \{1, 3, 5\}$

$B = \{1, 2, 3\}$

$A \cap B = \{1, 3\}$

$A \cap B$ is shaded

- Two Sets are called disjoint if their intersection is the empty set.

Let $A = \{1, 3, 5, 7, 9\}$

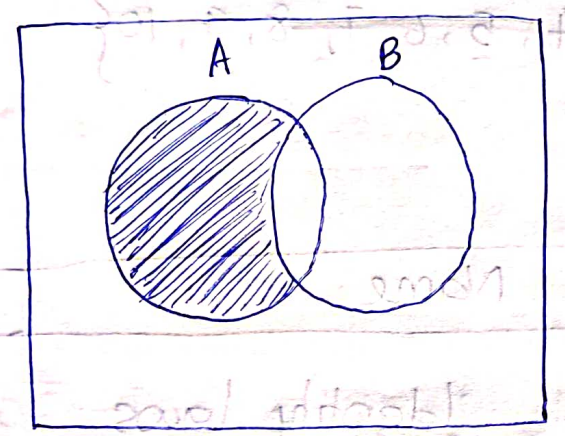
$B = \{2, 4, 6, 8, 10\}$ $\quad A \cap B = \phi$

$\therefore$ A & B are disjoint

Let A & B be Sets the diffrence of A & B denoted by A−B is the set containing those elements that are in A but not in B. The diffrence of A & B is also called the compliment of B with respect to A.

$$A-B = \{x / x \in A \wedge x \notin B\}$$

Venn Diagram for A−B



$A = \{1, 3, 5\}$

$B = \{1, 2, 3\}$

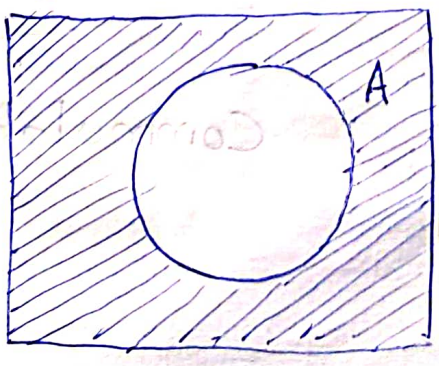
$A - B = \{5\}$

$B - A = \{2\}$

Let U be the Universal set. The compliment of the set A denoted by $\bar{A}$ is the compliment of A with respect to U. In other words the compliment of the set A is U−A

An element belongs to $\bar{A}$ if & only if $x \notin A$

Venn diagram for the compliment of the set A.



Let $A = \{a, e, i, o, u\}$ where the Universal Set is the set of letters of the English alphabet

$\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

→ Let A be the set of +ve Integers greater than 10 where the Universal set is the set of all +ve Integers

$\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Set Identities

Identity	Name
$A \cup \phi = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\bar{A})} = A$	complimentation laws.
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorptn laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Compliment laws

* P.T $\overline{A \cap B} = \overline{A} \cup \overline{B}$

To prove that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

i.e To P.T $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ & $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A \cap B} \Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in \overline{A}$ or $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cup \overline{B}$

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

To P.T

* $\bar{A} \cup \bar{B} = \overline{A \cap B}$

$x \in \bar{A} \cup \bar{B} \Rightarrow x \notin A \cup B$

$\Rightarrow$ ~~$x \notin A$~~ $x \notin A$ or $x \notin B$

$\Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

$\Rightarrow$ ~~$x \in \overline{A \cup B}$~~ $x \notin A \cap B$

$\Rightarrow x \in \overline{A \cap B}$

$\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$

- Use membership table to show that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

A	B	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$	$A \cap B$	$\overline{A \cap B}$
1	1	0	0	0	1	0
1	0	0	1	1	0	1
0	1	1	0	1	0	1
0	0	1	1	1	0	1

$\bar{A} \cup \bar{B} = \overline{A \cap B}$

Use membership table to show that ~~A∩B~~

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

* Use set builder notation & logical equallences to establish the Second Demorgan law:-

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \{x \mid x \in A \cap B\}$ By definition of complement

$= \{x \mid \neg (x \in A \cap B)\}$

$= \{x \mid \neg (x \in A \wedge x \in B)\}$

$= \{x \mid \neg (x \in A) \vee \neg (x \in B)\}$

$= \{x \mid x \notin A \vee x \notin B\}$

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$

$= \bar{A} \cup \bar{B}$

Prove the 1st Distributive law which states that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Ans) To P.T $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Take any $x \in A \cap (B \cup C)$

$x \in A \cap (B \cup C) \Rightarrow x \in A$ & $x \in (B \cup C)$

$\Rightarrow x \in A$ & $x \in B$ or $x \in C$

$\Rightarrow x \in A$ & $x \in B$ or $x \in A$ & $x \in C$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Take any $x \in (A \cap B) \cup (A \cap C)$

$x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow (x \in A$ & $x \in B)$ or $(x \in A$ & $x \in C)$

$\Rightarrow x \in A$ & $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ & $x \in (B \cup C)$

$\Rightarrow x \in A \cap (B \cup C)$

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

→ Let A, B & C be sets show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$\overline{A \cup (B \cap C)}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{C} \cup \overline{B}$
1	1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	0	1	1
1	0	1	0	1	0	0	1	0	1
1	0	0	0	1	0	0	1	1	1
0	1	1	1	1	0	1	0	0	0
0	1	0	0	0	1	1	0	1	1
0	0	1	0	0	1	1	1	0	1
0	0	0	0	0	1	1	1	1	1

$(\overline{C} \cup \overline{B}) \cap \overline{A}$

0
0
0
0
0
1
1
1

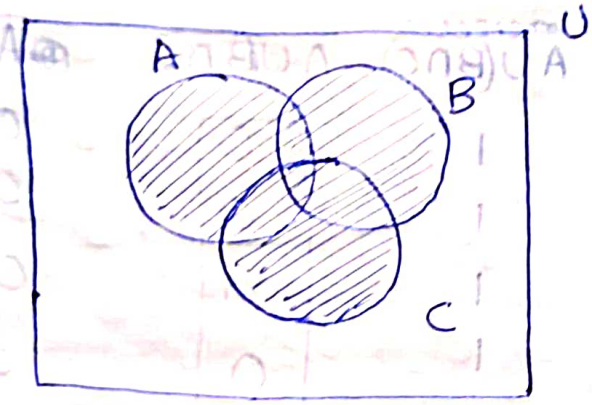
$\overline{A \cup (B \cap C)} = \bar{A} \cap (\overline{B \cap C})$ By Demorgan

$= \bar{A} \cap (\bar{B} \cup \bar{C})$

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ commutative law for intersection

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ By commutative law for union

Generalized Unions & Intersections

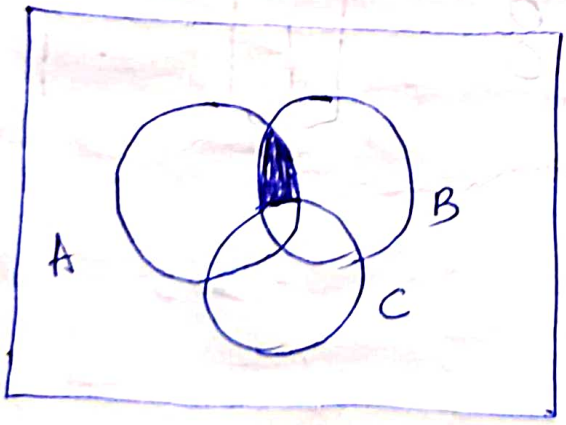


$A \cup B \cup C$ is shaded

$A = \{0, 2, 4, 6, 8\}$
$B = \{0, 1, 2, 3, 4\}$
$C = \{0, 3, 6, 9\}$

$A \cup B \cup C$ is shaded

what are $A \cup B \cup C$
$A \cap B \cap C$



$A \cup B \cup C$
$= \{0, 1, 2, 3, 4, 6, 8, 9\}$

$A \cap B \cap C$
$= \{0\}$

The Union of a Collection of set is the set that contains those elements that are members atleast one set in the collection.

We use the notation $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ to denote the union of the set $A_1, A_2, \cdots A_n$

The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection. we use the notation

$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

To denote the intersection of the set $A_1, A_2 \cdots A_n$

Ex: Let $A_i = \{i, i+1, i+2 \cdots\}$ then

Let $A_1 = \{1, 2, 3 \cdots\}$

$A_2 = \{2, 3, 4 \cdots\}$

$A_3 = \{3, 4, 5, \cdots\}$

$A_4 = \{4, 5, 6 \cdots\}$

$\vdots$

$A_n = \{n, n+1, n+2 \cdots\}$

$A_1 \cup A_2 \cup A_3 \cup \cdots A_n = \bigcup_{i=1}^{n} A_i$

$= \{1, 2, 3, 4, 5 \cdots n, n+1\}$

$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i = \{n, n+1, n+2 \cdots\}$

* Suppose that $A_i = \{1, 2, 3 \cdots i\}$ for $i = 1, 2, 3 \ldots$

Find $\bigcup_{i=1}^{\infty} A_i$ & $\bigcap_{i=1}^{\infty} A_i$

$$\cancel{A_1 = \{1, 2, 3, \ldots \infty\}}$$
$$\cancel{A_2 = \{1, 2, 3 \ldots 2\}}$$
$$\cancel{A_3 = \{1, 2, 3 \ldots 3\}}$$

$$A_1 = \{1\}$$
$$A_2 = \{1, 2\}$$
$$A_3 = \{1, 2, 3\}$$
$$A_4 = \{1, 2, 3, 4\}$$
$$\vdots$$
$$A_n = \{1, 2, 3, 4, \cdots n\}$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$$
$$= \{1\} \cup \{1, 2\} \cup \{1, 2 3\} \cup \cdots$$
$$= \{1, 2, 3 \cdots n\}$$
$$= \mathbb{Z}^+ = \mathbb{N}$$

$$\bigcap_{i=1}^{\infty} A_i = \{1\} \cap \{1, 2\}, \{1, 2, 3\}$$
$$= \{1\}$$

Computer representation of sets

There are various ways to represent sets using a computer. One method is to store the elements

of the sets in an unodered fashion. Here we consider a method for storing elements using an arbitrary odering of the elements of the universal set.

Assume that the universal set U is finite. 1st specify an arbitrary odering of the elements of U For instance $a_1, a_2 \ldots\ldots a_n$ represent a subset A(U) with the bit string of length n where the ith bit in the string is 1 if $a_i \in A$ & is zero if $a_i \notin A$.

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the odering of elements of U has the elements in increasing oder i.e $a_i = i$. what bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U.

Consider the subset of all odd integers in A

$A = \{1, 3, 5, 7, 9\}$. Then

1010101010 is the bit string that represent subset A (U).

Let B be the subset of all even integers in B

$B = \{2, 4, 6, 8, 10\}$ then,

$= \{0101010101$

is the bit string represent B.

$C = \{1, 2, 3, 4, 5\}$

$= 1111100000$

→ What is the bit string representation of complement of the set $\{1, 3, 5, 7, 9\}$ where the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The bit strings for the sets $= \{1, 2, 3, 4, 5\}$ & $\{1, 3, 5, 7, 9\}$ are 1111100000 & 1010101010 respectively use bit strings to find union & intersection of these sets.

~~union~~ $U = \{$

Functions

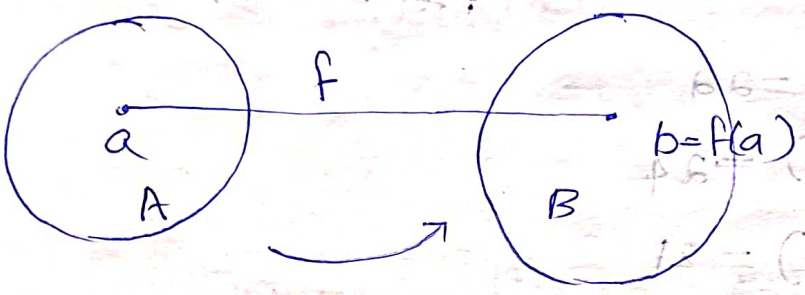
Let A & B be non-empty sets. A function from A to B is an assignment of exactly one element of B to each element of A. We write

~~f(A)~~ $f(a) = b$ if b is the unique element of B

Assign by the function f to the element of A

a(A) if f is a function from A to B we write $f: A \to B$. Functions are sometimes

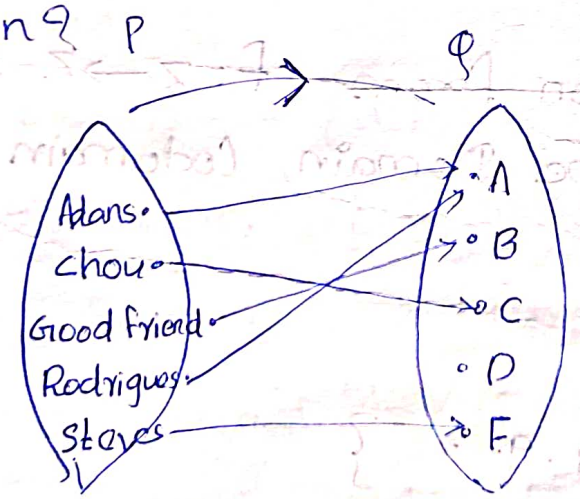
called mappings or transformations. If f is a function from A to B we say that A is the Domain of f and B is the co-domain of f. If $f(a) = b$ we say that b is the image of a & a is the pre-image of b. The range of f is the set of all images of elements of 'A'. If f is a function from A to B we say that f maps A to B



The function f maps A to B

→ What are the domain, codomain & range of the function? P

Ans):



Domain$_P$ = { Adams, Chou, Good friend, Rodriquez, Stevez }

Codomain$_Q$ = { A, B, C, D, F }

Range = {A, B, C, F}

Let R consisting of (Abdul 22) (Brenda, 24) (Carla, 21) (Desire, 22) (Eddie 24), and (Edicia 22) where each pair consists of a graduate student & the age this student. what is the function that this relation determines. let f be the function then the relation becomes ~~f(abdul)=22~~

f(abdul) = 22
f(Brenda) = 24
f(carla) = 21
f(Desire) = 22
f(Eddie) = 24
f(Edicia) = 22.

→ Let ~~f is a function from~~ $f: Z \to Z$ such that $f(x) = x^2$. find the Domain, Codomain & range.

Domain – $\mathbb{Z}$
codomain – $\mathbb{Z}$
Range = {0, 1, 4, 9. . . }

<u>Definition</u>

Let f_1 & f_2 be functions from A to R f_1+f_2 and f_1f_2 are also functions from

(27)

A to R defined by $(f_1+f_2)(x) = f_1(x)+f_2(x)$

$(f_1f_2)(x) = f_1(x) f_2(x)$

Let f_1 & f_2 be functions from R to R such that $f_1(x)=x^2$ and $f_2(x)=x-x^2$. what are the functions f_1+f_2 and f_1f_2

$$(f_1+f_2)(x) = f_1(x)+f_2(x)$$
$$= x^2+x-x^2$$
$$(f_1+f_2)(x) = x$$
$$(f_1f_2)(x) = f_1(x)f_2(x)$$
$$= x^2(x-x^2)$$
$$= x^3-x^4$$

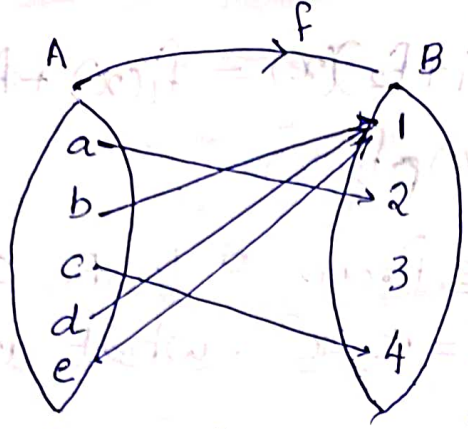
- Let f be a function from the set A to the set B & then S be a Subset of N. The image of S under the function f is the subset of B that consists of the images of the elements of S. we denote the image of S by f(S) i.e $f(S)=\{t \mid \exists s\in S,\ f(s)=t\}$ or

$$f(S)=\{f(s) \mid s\in S\}$$

- Let $A=\{a,b,c,d,e\}$ and $B=\{1,2,3,4\}$ with

$f(a)=2$
$f(b)=1$
$f(c)=4$
$f(d)=1$
$f(e)=1$

Find the image of the subset $S=\{b,c,d\}$

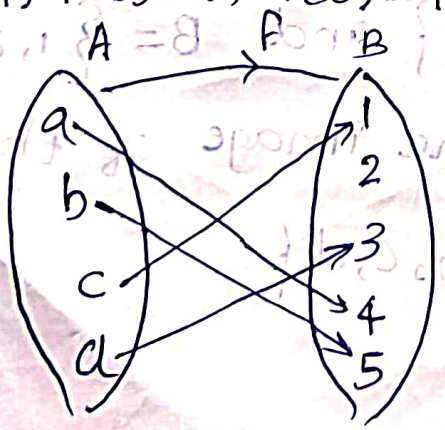


$$f(S) = \{f(s) \mid s \in S\}$$
$$= \{f(b), f(c), f(d)\}$$
$$= \{1, 4\}$$

Definition:- A function f is said to be one to one or injective if & only if $f(a) = f(b) \Rightarrow a = b$ for all a & b in the Domain of f. A function is said to be an injection if it is one to one.

Note that a function f is one to one if & only if $f(a) \neq f(b)$ whenever $a \neq b$. We can express that f is one to one using quantifiers as $\forall a \forall b (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$ where the universe of dispose is the domain of the function.

→ Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, & $f(d) = 3$ is one to one



from the fig it is clear that it is one-one

→ Determine whether the function $f(x)=x^2$ from the set of Integers to the set of Integers is one to one

$f: Z \to Z \quad f(x)=x^2$

To P.T f is one to One.

i.e to prove $f(a)=f(b) \Rightarrow a=b$

$f(a)=f(b) \Rightarrow a^2=b^2$

Let $a=-1$ & $b=1$ then $f(a)=f(-1)=1$

$f(b)=f(1)=1$

$f(-1)=f(1)$ but $-1 \neq 1$

∴ The given function is not one to One.

→ Determine whether the function $f(x)=x+1$ from the set of real no's to itself is One to One. Given that f is a function from R to R ∴ $f(x)=x+1$

Ex: ~~f(x)~~ To prove or disprove. $f(x)=x+1$ is one to one

~~Let f(a)~~

~~i.e to prove~~

Let $f(a)=f(b) \Rightarrow a+1=b+1$

$\Rightarrow a=b$

$f(a)=f(b) \Rightarrow a=b$

∴ f is One to One function

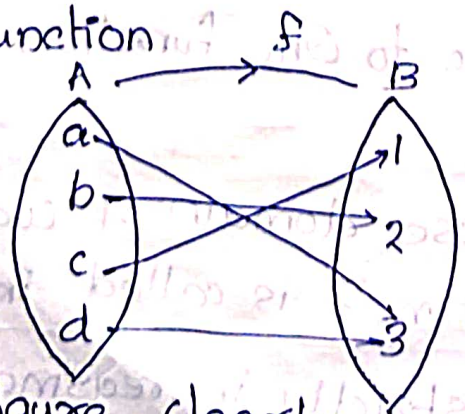
<u>Definition</u>

A function f whose domain & codomain are subsets of the set of real no's is called increasing if $f(x) <= f(y)$ & strictly increasing if $f(x) < f(y)$

whenever $x<y$ & x & y are in the domain of f.
Similarly, f is called decreasing if $f(x) \geq f(y)$ & strictly decreasing if $f(x) > f(y)$ whenever $x<y$ & x & y are in the domain of f. A function f is increasing if for every x for every y $x<y \Rightarrow f(x) \leq f(y)$ strictly increasing if $\forall x \forall y\ (x<y \rightarrow f(x) < f(y))$ decreasing if $\forall x \forall y\ (x<y \rightarrow f(x) \geq f(y))$ and strictly decreasing if $\forall x \forall y\ (x<y \rightarrow f(x) > f(y))$

→ A function $f: A$ to B is called Onto or Surjective if & only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a Surjection if it is onto. A function f is onto if $\forall y \exists x\ (f(x) = y)$ where the Domain for x is the Domain of the function & the Domain for y is the Codomain of the function. ~~Let~~

Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a)=3$, $f(b)=2$, $f(c)=1$, & $f(d)=3$ is a and Onto function.



from the figure, clearly it is an onto function.

→ Is the function $f(x) = x^2$ from the set of Integers to the set of Integers onto. Here f is a function from $f: \mathbb{Z} \to \mathbb{Z}$ s.t. $f(x) = x^2$

The function f is not onto bcz there is no integer x with $x^2 = -1$.

- Is the function $f(x) = x+1$ from the set of Integers to the set of Integers onto.

Here f is a function from $\mathbb{Z} \to \mathbb{Z}$ s.t. $f(x) = x+1$

To prove that f is onto. Take any y belongs to Z

Let $x = y-1$ $\quad y = x+1$

$f(x) = x+1$ $\quad f(x) = x+1 = y-1+1$

$= y-1+1$

$f(x) = \underline{y}$

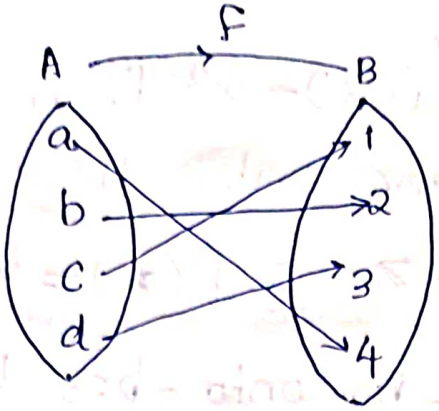
$\therefore \exists\, x \in \mathbb{Z}$ s.t. $f(x) = y$

$\therefore$ f is Onto

<u>Definition</u>

The function f is a One to One Correspondence or a bijection if it is both ~~onto~~ one to one & onto.

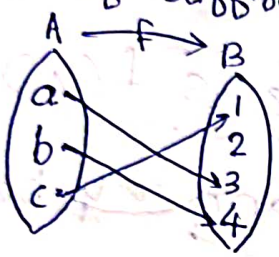
- Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ & $f(d) = 3$. Is f a bijection.



Clearly, f is one-one & onto
∴ f is a bijection.

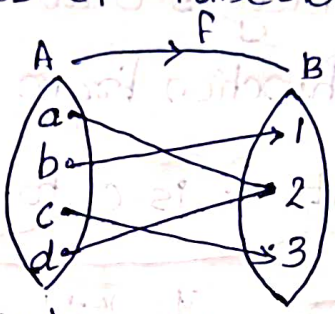
Examples of different types of function:-

a.



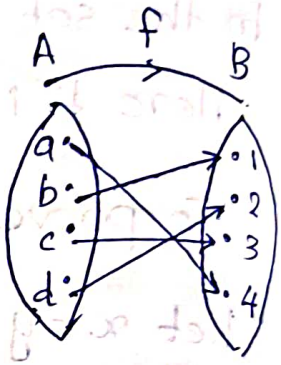
One one not onto

b



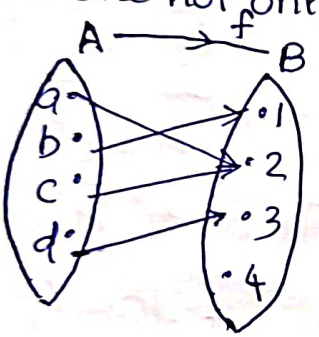
onto, not one-one

c.



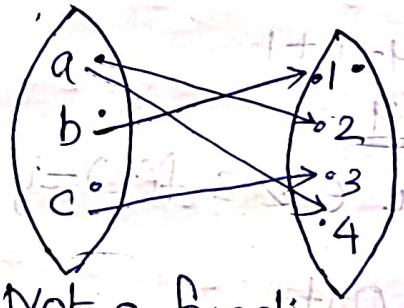
one one & onto.

d.



Neither one-one nor onto

e)



Not a function.

Identity function

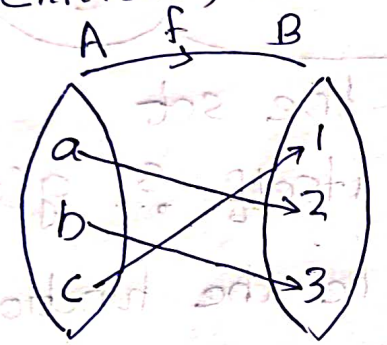
Let A be a set the identity function on A is the function $I_A : A \to A$ where $I_A(x) = x \quad \forall x \in A$

The identity function is one-one & onto ∴ It is a bijection.

Inverse functions & composition of functions

Let f be a one one correspondence from the set A to the set B.

The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a)=b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b)=a$ where $f(a)=b$. A one-one correspondence is called invertible. A function is not invertible if it is not a one to one correspondence. Because the inverse of such a function does not exists. Let f be a function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a)=2$, $f(b)=3$, $f(c)=1$ is f invertible, and if it is, what is its inverse.



Clearly, f is one one & onto $\therefore$ f is a One one correspondence from a set A to B

$\therefore$ f is invertible

$f^{-1}(1)=c$, $f^{-1}(2)=a$, $f^{-1}(3)=b$.

→ Let $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x)=x+1$. Is f invertible & if it is what is its inverse.

→ We already proved that f is one one & onto

$\therefore$ f is a bijection

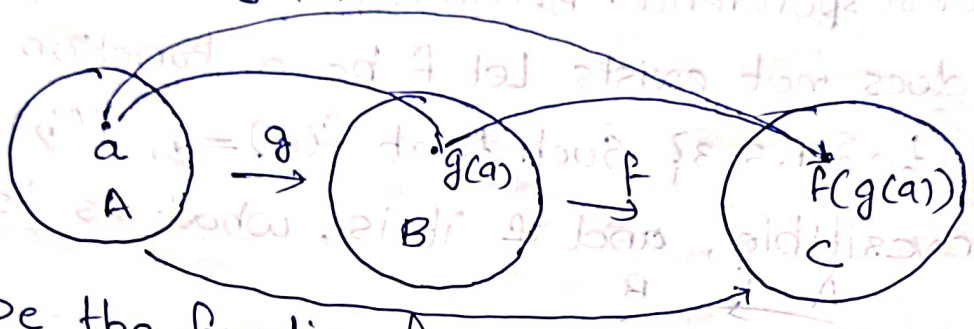
Hence f is invertible.

Let $y=x+1$, $x=y-1$

$f^{-1}(y)=y-1$

→ let f be the function from R to R with $f(x) = x^2$ is ~~of~~ invertible.

→ Let g be a function A to set B and let f be a function from the set B to the set C. The composition of the function f and g denoted by $f \circ g$ is defined by $(f \circ g)a = f(g(a))$



Let g be the function from the set Set $\{a, b, c\}$ to itself s.t. $g(a) = b$ $g(b) = c$ $g(c) = a$. let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ s.t. $f(a) = 3$ $f(b) = 2$, $f(c) = 1$. What is the composition of f and g and what is the composition of ~~f and~~ g and f.

The composition of $f \circ g$ is defined by $f \circ g(a) = f(g$

$$\cancel{f \circ g(a) = f \circ g(b) \quad f(b)}$$
$$= \cancel{2}$$

$$f(g(a) = f(b)$$
$$= 2$$

$$\boxed{\begin{array}{l} g(a) = b. \\ f(b) = 2 \end{array}}$$

~~g(f(a) = g(2)~~

~~g(2) =~~

$$(f \circ g)(b) = f(g(b))$$
$$= f(c)$$
$$= 1$$

$$(f \circ g)(c) = f(g(c))$$
$$= f(a)$$
$$= 3$$

Note that $g \circ f$ is not defined because the range of f is not a subset of the domain of g.

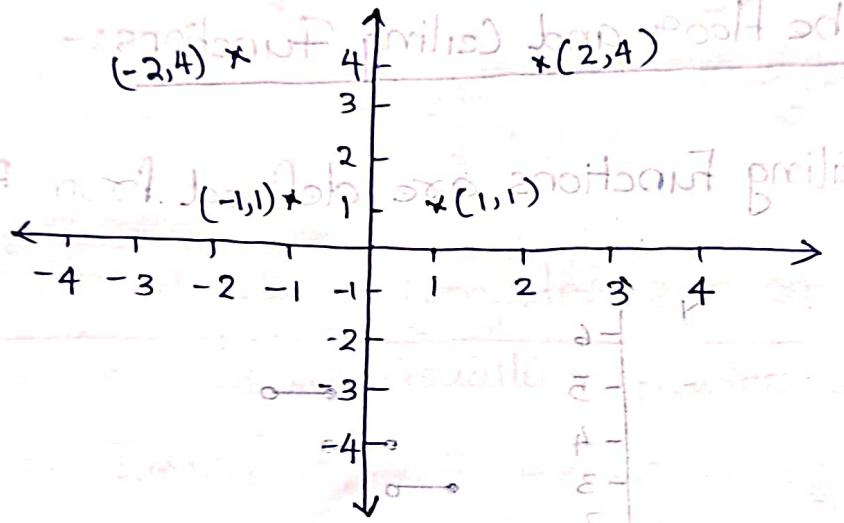
Ans Let f & g be the functions from the set of Integers to the set of Integers defined by $f(x) = 2x+3$, $g(x) = 3x+2$. what is the composit^n of f and g. what is the composition of g and f.

$$f \circ g(x) = f(g(x))$$
$$= f(3x+2)$$
$$= f(2(3x+3)+2)$$
$$= f(6x+7)$$

$$g \circ f(x) = g(f(x))$$
$$= g(2x+3)$$
$$= g(3(2x+3)+2$$
$$= g(6x+9+2$$
$$= \underline{6x+11}$$

→ Display the graph of the function $f(x) = x^2$ from the set of Integers to the set of Integers.

x	-2	-1	0	1	2
x^2	4	1	0	1	4



<u>Some Important Functions:-</u> (Floor function & ceiling funct^n)

The floor function assigns to the real no. x, the largest integer i.e less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$

The ceiling function assigns to the real no. x the smallest Integer i.e greater than or equal to x.

The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Note: The floor function is always called the greatest Integer Function. It is always denoted by $[x]$.

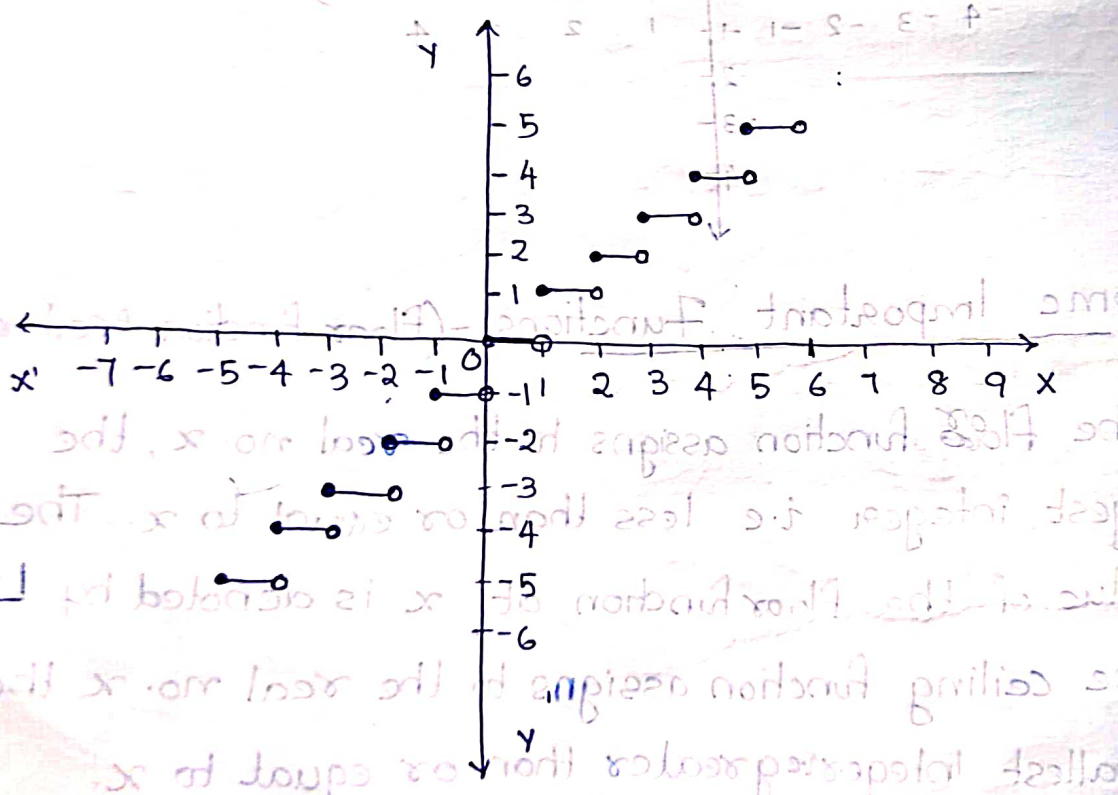
Examples

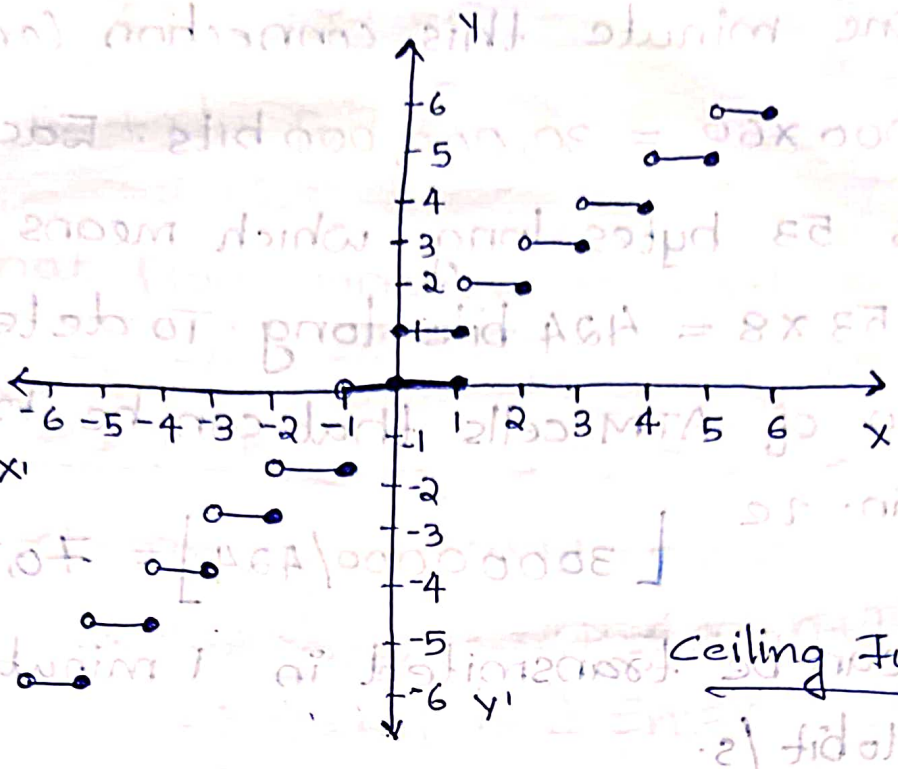
$\lfloor 1/2 \rfloor = 0$ $\quad \lfloor -1/2 \rfloor = -1$ $\quad \lfloor 3.1 \rfloor = 3$ $\quad \lceil 7 \rceil = 7$

$\lceil 1/2 \rceil = 1$, $\quad \lceil -1/2 \rceil = 0$, $\quad \lceil 3.1 \rceil = 4$, $\quad \lfloor 7 \rfloor = 7$

Graph of the Floor and Ceiling Functions:-

Floor and Ceiling Functions are defined from R to R.





Ceiling Function

- Data stored on a Computer disc or transmitter over a data network as usually represented as a string of Bytes. Each byte is made up of 8 bits.

How many bytes are req to encode 100 bits of data.

Ans): $\lceil 12.57 \rceil = 13$

$\therefore$ 13 bytes are required.

$$\begin{array}{r} 12.5 \\ 8\overline{)100} \\ 8 \\ \hline 20 \\ 16 \\ \hline 40 \end{array}$$

Asynchronous transfer mode (ATM)

A communications protocol used on backbone networks & data are organized into cells of 53 bytes. How many ATM cells can be transmited in 1 minute over a connection that transmits data at the rate of 500 kilobytes per second.

- In one minute this connection can transmit $500{,}000 \times 60 = 30{,}000{,}000$ bits. Each ATM cell is 53 bytes long which means that it is $53 \times 8 = 424$ bits long. To determine the no. of ATM cells that can be transmited in 1 min. i.e

$$\lfloor 30000000/424 \rfloor = 70{,}754 \text{ ATM}$$

Cells can be transmited in 1 minute over a 500 kilobit/s.

Useful properties of floor & ceiling functions:-

(A) 1) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
[where n is an integer]

2] $\lceil x \rceil = n$ if and only if $n-1 < x \le n$

3) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$

4) $\lceil x \rceil = n$ if and only if $x \le n < n+1$

(B) 1. $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil$

(C) 1. $\lfloor -x \rfloor = -\lceil x \rceil$

2. $\lceil -x \rceil = -\lfloor x \rfloor$

(D) 1. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

(5) 2. $\lceil x+n \rceil = \lceil x \rceil + n$

- Show that floor function of $x+n$

Let floor function of $x = m$ where m is a +ve Integer.

$$\lfloor x \rfloor = m \Rightarrow m \le x < m+1$$

$$\Rightarrow m+n \le x+n \le m+n+1$$

$$\Rightarrow \lfloor x+n \rfloor = m+n$$

$$\Rightarrow \lfloor x+n \rfloor = \lfloor x \rfloor + n.$$

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n$$

- Show that $\lceil x+n \rceil = \lceil x \rceil + n$

Let ceiling function of $x = m$ we have ~~ceiling function of~~ $\lceil x \rceil = m$ if and only if

$$m-1 \le x \le m$$

$$\lceil x \rceil = m \Rightarrow m-1 \le x \le m$$

$$\Rightarrow m+n-1 \le x+n \le m+n$$

$$\Rightarrow \lceil x+n \rceil = m+n$$

$$= \lceil x+n \rceil = \lceil x \rceil + n$$

imp

P.T If x is a real no. then floor function

$$\lfloor 2x \rfloor = \lfloor x \rfloor = \lfloor x + 1/2 \rfloor$$

Let $x = n+\varepsilon$ where n is a +ve Integer and $0 \le \varepsilon < 1$.

Ans) Consider $0 \le \varepsilon < 1/2$ then $2x = 2(n+\varepsilon) = 2n+2\varepsilon$

$\lfloor 2x \rfloor = 2n$ because $0 \le 2\varepsilon < 1$

$\therefore \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+\frac{1}{2} \rfloor$

$\varepsilon < \frac{1}{2}$ $\quad x = n+\varepsilon$

$2\varepsilon < 1$ $\quad 0 \le \varepsilon < 1/2$

$x = n+\varepsilon$ $\quad 1/2 \le \varepsilon < 1$

$2x = 2n+2\varepsilon$ $\quad$ also $x+\frac{1}{2} = n+\varepsilon+\frac{1}{2}$

$\lfloor 2x \rfloor = 2n$ $\quad = n+\left(\frac{1}{2}+\varepsilon\right)$

~~$x+1/2 = n$~~

we have $0 \le \varepsilon < 1/2$

$0 \le 1/2 \, \varepsilon < 1$

Hence $\lfloor x+\frac{1}{2} \rfloor = n$

also we have $\lfloor x \rfloor = \lfloor n+\varepsilon \rfloor = n$

$\therefore \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+\frac{1}{2} \rfloor$

Consider $\frac{1}{2} \le \varepsilon < 1$

To P.T $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+\frac{1}{2} \rfloor$

We have $x = n+\varepsilon$

$2x = 2n+2\varepsilon$

(7) $= 2n+1+2\varepsilon-1$

$\lfloor 2x \rfloor = 2n+1 \quad (\because 0 \le 2\varepsilon-1 < 1)$

$\lfloor x \rfloor = n$

$\cancel{x+\frac{1}{2}}$

$x+\frac{1}{2} = n+\varepsilon+1/2$

$= n+1+(\varepsilon-1+1/2)$

$= n+1+(\varepsilon-1/2)$

$\lfloor x+\frac{1}{2} \rfloor = n+1 \qquad \because 0 \le \varepsilon-1/2 < 1$

$\therefore \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+\frac{1}{2} \rfloor$

From the two cases we can conclude that

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+1/2 \rfloor$

Factorial function

The factorial function $f: N \to Z^+$ denoted by

$f(n) = n!$

$f(1) = 1! = 1$

$f(2) = 1\times 2 = 2$

$f(3) = 1\times 2\times 3 = 6$

$f(4) = 1\times 2\times 3\times 4 = 24$

Sequences and Summations

A sequence is a function from a subset of the set of Integers (usually either the set $\{0, 1, 2 \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. we use the notation a_n to denote the image of Integer n, we call a_n a term of the sequence. we use the notation n to describe the sequence

Ex: Consider the sequence $\{a_n\}$ where $a_n = 1/n$ $n \in N$

$a_1 = 1$

$a_2 = 1/2$

$a_3 = 1/3$

$a_4 = 1/4$

$\therefore$ the given seqence becomes $\{1, 1/2, 1/3, 1/4 \ldots\}$

- A geometric progression is a sequence of the form $a, ar, ar^2, ar^3 \ldots a^n$, etc. where the initial term and the common ratio r are real no's.

(9) Ex: 1) $\{b_n\}$ where $b_n = (-1)^n$

$\{-1, 1, -1, 1 \ldots\}$

2) $\{c_n\}$ where $c_n = 2 \cdot 5^n$

- Find the list of terms begins with $\{c_0, c_1, c_2 \ldots\}$

$c_0 = 2 \times 5^0 = 2.$

$c_1 = 2 \times 5^1 =$

$= \{2, 10, 50, 250\}$

$\{d_n\}$ where $d_n = 6 \cdot \left(\frac{1}{3}\right)^n$

Find the list of terms $\{d_0, d_1, d_2 \ldots\}$

- An arithmetic progression is a sequence of the form $a, a+d, a+2d, a+3d \ldots\ldots a+nd$;

where the initial term a & the common difference d are real nos.

Ex: $\{S_n\}$ where $S_n = -1 + 4n$

$-1 + 4 \times 0$
$-1 \times 4 \times 1$
$-1 + 4 \times 2$
$-1 + 8$
$-1 + 4 \times 4$

Find the list of terms $\{S_0, S_1, S_2, S_3\}$

$S_0 = -1$
$S_1 = 3$
$S_2 = 7$
$S_3 = 15$

$\{-1, 3, 7, 15\}$

$\Rightarrow \{t_n\}$ where $t_n = 7 - 3n$

find the seq starting with $n = 0$

$t_0 = 7 - 3 \times 0 = 7$
$t_1 = 7 - 3 \times 1$
$= 7 - 3 = 4$
$t_2 = 7 - 3 \times 2 = 1$
$t_3 = 7 -$

$\{7, 4, 1 \ldots\}$

- Seq of the form $\{a_1, a_2 \cdots a_n$ are often used in CS. These finite sequences are called strings. This string is also denoted by $a_1 a_2 a_3 \cdots a_n$
- The length of the string S is the no. of terms in this string.
- The Empty string, denoted by λ, is the string that has no terms
- The Empty string has length 0.

  Ex: The string abcd is a string of length 4.
- Find formulae for the sequences with the following 1st 5 terms.

a) $1, 1/2, 1/4, 1/8$

$$a_n = \left(\frac{1}{2}\right)^n \quad n = 0, 1, 2, 3 \cdots$$

This is a geometric progression with $a = 1/2$ & $r = 1/2$.

b) $1, 3, 5, 7, 9$

$$a_n = 2n+1, \quad n = 0, 1, 2, \cdots$$

It is an arithmetic progression with $a = 1$, $d = 2$.

H.W

c) $1, -1, 1, -1, 1$

11 How can we produce the terms of a sequence if the 1st 10 terms are 1, 2, 2, 3, 3, 3, 4, 4 4, 4

Note that the Integer 1 appear once, 2 appears twice, 3 appears thrice & 4 appears 4 times.

∴ The next 5 terms of sequence will be 5.

How can we produce the terms of a sequence if the 1st 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59.

Prove / disprove that ceiling function $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$

when $x = 1/2$, $y = 1/2$

$$x+y = \frac{1}{2}+\frac{1}{2} = 1$$

$$\lceil x+y \rceil = \lceil 1 \rceil = 1$$

$$\lceil x \rceil = \lceil 1/2 \rceil = 1$$

$$\lceil y \rceil = \lceil 1/2 \rceil = 1$$

$$\lceil x \rceil + \lceil y \rceil = 2$$

∴ $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ need not be true

→ The nth term of sequence is $a+(n-1)d$

$$= 5+(n-1)6$$

$$= 5+6n-6$$

$$= 6n-1$$

bcz the given Sequence is a arithmetic progression

· Conjecture a Simple formulae for a_n If the 1st 10 terms of the Sequence $\{a_n\}$ are

1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047

$1 = 3-2$

$7 = 3^2-2$

$25 = 3^3-2$

$79 = 3^4-2$ . . . .

$a_n = 3^n - 2$

Summations

We begin by describing the notation used to express the sum of the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$ from the Sequence $\{a_n\}$ we use the notation $\sum_{j=M}^{N} a_j$, $\sum_{j=M}^{n} a_j$ or $\sum_{m \le j \le n} a_j$

(13) To represent $a_m + a_{m+1} + \cdots + a_n$

Here the variable j is called the Index of Summations.

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$$

- Express the sum of the 1st 100 terms of the Sequence $\{a_n\}$ where $a_n = 1/n$ for $n = 1, 2, \ldots$

$$\sum_{n=1}^{100} a_n = \sum_{n=1}^{100} 1/n$$

$$= \sum_{j=1}^{100} 1/j$$

- what is the value of $\sum_{j=1}^{5} j^2$

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= \underline{\underline{55}}$$

- what is the value of $\sum_{k=4}^{8} (-1)^k$

$$= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$

$$1 - 1 + 1 - 1 + 1$$

$$= 1$$

Note: $\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$

(14) $j=1$

$j-1=0$

put $k=j-1$

$j=1 \quad k=0$

$j=5 \quad , k=4$

$j=k+1$

<u>Theorem</u>

If a & r are real nos and $r \neq 0$ ~~then~~ $\cancel{\sum_{j=0}^{n} a}$

then $\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1}, & \text{if } r \neq 1 \\ (n+1)a, & \text{if } r = 1 \end{cases}$

Let $S = \sum_{j=0}^{n} ar^j$

Multiply both sides of equality by r.

$$rS = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

$$= \sum_{k=1}^{n+1} ar^k$$

$$= \sum_{k=0}^{n+1} ar^k - a$$

15. $\sum_{k=0}^{n} m^k + ar^{n+1} - a$

$$rS = S + ar^{n+1} - a$$

$$rS - S = ar^{n+1} - a.$$

$$(r-1)S = ar^{n+1} - a$$

$$S = \frac{ar^{n+1} - a}{r-1}$$

$$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r-1}, \text{ if } r \neq 1$$

If $r = 1$

$$\sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a \times 1^j$$

$$= \sum_{j=0}^{n} a \cdot 1^j$$

$$= a + a + \cdots + a$$

$$= (n+1)a$$

$$\therefore \sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r-1}, & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

• What is the value of $\sum_{i=1}^{4} \sum_{j=1}^{3} i,j$

$$= \sum_{i=1}^{4} \left[i\times1 + i\times2 + i\times3 \right]$$

$$= \sum_{i=1}^{4} \left[i + 2i + 3i \right]$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6\times1 + 6\times2 + 6\times3 + 6\times4$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

What is the value of $\sum_{s \in \{0,2,4\}} s$

$$\sum_{s \in \{0,2,4\}} s = 0+2+4 = 6$$

Some Useful Sumation formulae

Sum	closed form
$\sum_{k=0}^{n} a r^k \ (r \neq 0)$	$\frac{a r^{n+1} - a}{r-1} \quad r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^3$	$\frac{n(n+1)(2n+1)}{6}$

17 $\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert<1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} k x^{k-1}, \lvert x\rvert<1$	$\frac{1}{(1-x)^2}$

- Find $\sum_{k=50}^{100} k^2$

Ans

we have $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$

$$\sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2 = \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\frac{100(100+1)(2\times100+1)}{6} - \frac{49(49+1)(2\times49+1)}{6}$$

$$338350 - 40425 = \underline{\underline{297{,}925}}$$

- Let x be a real no. with $|x| < 1$. Find
- $\sum_{n=0}^{\infty} x^n$

We have $\sum_{n=0}^{k} x^n = \frac{1(x^{k+1}-1)}{x-1}$

$= \frac{x^{k+1}-1}{x-1}$

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \sum_{n=0}^{k} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1}$$

$$= \frac{0-1}{x-1}$$

$$= \frac{1}{1-x}$$

- A set S is finite with cardinality $n \in N$ if there is a bijection from the set ~~$n \in N$~~ $\{0, 1, 2, \ldots, n-1\}$ to S. A set is infinite if it is not finite.
- A set S is infinite if there exist an injection $f: S \to S$ ~~from S to S~~ such that $f(S)$ is a proper subset of S.

Ex:- The set N of Natural nos is an infinite set

Consider the injection (one-one) $f: N \to N$ defined by $f(n) = 2n+1$.

The range of f is a subset of domain f.

$\therefore$ N is an infinite set.

- A set that is either finite or has the same cardinality as the set of +ve Integers is called countable.

A set that is not countable is called uncountable

- when an infinite set S is countable we denote the cardinality of S by $\aleph_0$ (aleph-null) where ($\aleph$ aleph is the 1st letter of the hebrew alphabet). we write $|S| = \aleph$ and say that S has Cardinality Aleph-null

S.T the set of odd +ve Integers is a countable set.

TO S.T set of odd +ve Integers is a countable set.

i.e to P.T there exist a one to one correspondance b/w the set of odd +ve Integers and set of +ve Integers

Consider the function $Z^+ \to O$ such that

$f(n) = 2n-1$ $\quad$ $O \to$ set of all odd +ve Integers.

To P.T f is one one & onto

Let $f(n) = f(m)$

$$2n-1 = 2m-1$$
$$2n = 2m$$
$$n = m.$$

hence, f is one one function

· Next we will prove f is onto. Take any $t \in O$ then $t = 2k-1$ for Some Integer k.

i.e $t = 2k-1 = f(k)$

i.e $f(k) = t$.

Hence f is onto.

∴ The Set of odd +ve Integers is a Countable Set.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 3 & 5 & 7 & 9 & 11 \end{array} \cdots$$

S.T set of all Integers is countable.

We can list all Integers in a Sequence by Starting with 0 and alternating between +ve and -ve Integers.

21 i.e $0, 1, -1, 2, -2, 3, -3, \ldots$

we can find a one-one correspondance between the set of all +ve Integers and set of all Integers

we define a function $f(n) = \begin{cases} \frac{n}{2}, & \text{when } n \text{ is even} \\ -\frac{(n-1)}{2}, & \text{when } n \text{ is odd} \end{cases}$

clearly f is a One-One, Onto function

- S.T the set of +ve rational nos is Countable.

we list +ve rational nos in a sequence such that p/q with $p+q = 2$ followed by $p+q=3$ followed by $p+q=4$ and so on.

when we encounter a no p/q i.e already listed we do not list it again

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \ldots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} \ldots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} \ldots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} \ldots
\end{array}$$

The initial terms in the list of +ve ~~natural~~ rational numbers we have constructed are

$1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5$ etc.

These nos are shown circle, the uncircled nos in the list are those we leave out bcz they are already listed. Hence the set of +ve rational nos is countable.

→ S.T the set of real nos is a set of uncountable set

Suppose that set of real nos is countable. Then the subset of all real nos that fall b/w 0 & 1 also be countable. The real nos b/w 0 & 1 can be listed in some order say $r_1, r_2, r_3 \ldots\ldots$ Let the decimal representation of these real nos be

$$r_1 = 0.d_{11}\, d_{12}\, d_{13}\, d_{14} \ldots$$
$$r_2 = 0.d_{21}\, d_{22}\, d_{23}\, d_{24} \ldots$$
$$r_3 = 0.d_{31}\, d_{32}\, d_{33}\, d_{34} \ldots$$
$$r_4 = 0.d_{41}\, d_{42}\, d_{43}\, d_{44} \ldots$$
$$\vdots$$

Then form a new real no with decimal expansion $r = 0.d_1 d_2 d_3 d_4 \ldots$

where the decimal digits are determine by the following rule.

(23)

$$d_i = \begin{cases} 4 & \text{if } d_{ij} \neq 4 \\ 5 & \text{if } d_{ij} = 4 \end{cases}$$

Ex: $r_1 = 0.23794102 \ldots$ $r_2 = 0.44590138 \ldots$

$r_3 = 0.09118704 \ldots$ $r_4 = 0.80553900 \ldots$

and soon $r = 0.d_1 d_2 d_3 d_4$

$= 0.4544 \ldots$

where $d_1 = 4$ because $d_{11} \neq 4$. $d_2 = 5$ $d_{22} = 4$ $d_3 = 4$ because ~~$d_2 = 5$ because d_{22}~~ bc3 $d_{33} \neq 4$, $d_4 = 4$ bc3 $d_{44} \neq 4$ and soon.

Then the real no r is not equal to any of $\langle r_1, r_2, \ldots \rangle$ because the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point for each i. Hence we get a real no r. between 0 & 1 that is not in the list.

This contradicts our assumption that all the real nos b/w 0 & 1 must be listed.

$\therefore$ all the real nos b/w 0 & 1 cannot be listed

Hence the set of real nos b/w 0 & 1 is uncountable.

We know that any set with an uncountable subset is uncountable. Hence the set of real nos is uncountable.

<u>Definition</u>

Let S be a set. Enumeration of S be a set & enumeration of S is subjective function f from an initial segment of $\mathbb{N}$ to S. If f is a injective also (& Hence bijective, then f is an enumeration without repeatation). If f is not injective then f is an enumeration with repeatations.

Ex: If $S = \{\alpha, \beta, \gamma, \delta\}$ then $\langle \alpha, \beta, \gamma, \delta \rangle$ is an enumeration.

$\langle \alpha, \gamma, \beta, \beta, \delta \rangle$ is an enumeration with repeatation $\langle \gamma, \alpha, \beta, \delta \rangle$ is an enumeration without repeatation.

- Let S be the set of Natural nos of the form $3N$, $n \in \mathbb{N}$ $\langle 0, 3, 6, 9 \ldots \rangle$ is an

enumeration defined by $f(n) = 3n$, $n \in \cancel{N}\ n \in Z^+$

$\langle 6, 3, 0, 15, 12, 9 \rangle$ is also an enumeration

defined by $f(n) = 3n+6$, if $n = 3k$, for some k

Note: If a set S is countable if & only if there exists an enumeration of S.

- The union of a countable collection of countable sets is countable.

Let $S_0, S_1, S_2 \ldots\ldots$ be a countable collection of countable sets and $S_i = \langle a_{i0}, a_{i1}, a_{i2} \ldots \rangle$ $i = 1, 2, 3 \ldots$

$\cancel{S_0 = \langle a_{00}, a_{01}, a_{02}}$

$$S_0 = \langle a_{00}, a_{01}, a_{02}, a_{03} \ldots \rangle$$

$$S_1 = \langle a_{10}, a_{11}, a_{12}, a_{13} \ldots \rangle$$

$$S_2 = \langle a_{20}, a_{21}, a_{22}, a_{23} \ldots \rangle$$

$$S_3 = \langle a_{30}, a_{31}, a_{32}, a_{33} \ldots \rangle$$

enumerate the elements as shown in the above diagram.

$b_0 = a_{00}$
$b_1 = a_{01}$
$b_2 = a_{10}$
$b_3 = a_{02}$
$b_4 = a_{11}$
$b_5 = a_{20}$ etc.

- This is an enumeratⁿ of the elements of $\bigcup_{i=0}^{\infty} S_i$

  hence $\bigcup_{i=0}^{\infty} S_i$ is countable.

- A set S is of cardinality c if there is a bijection from $[0,1]$ to S. The label c is given to these bcs the set $[0,1]$ is often called continuum.

- Let S & T be sets then S & T are equipotent, or have the same cardinality denoted by $|S| = |T|$ if there is a bijection from S to

~~Relations~~