EE2703 Assignment 8: The Digital Fourier Transform

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1 Abstract

The main objectives of the Assignment are:

- Learning about DFT (Discrete Fourier Transform) and how it can be implemented in Python using the fft and fftshift function in numpys fft module. (*numpy.fft.fft()*)
- Using the fft function to find the DFTs of variuos periodic functions
- Plotting and analysing the relevent graphs

2 Introduction to DFT

2.1 Theory

Suppose f[n] is a periodic sequence of samples, with a period N. i.e:

$$f[n] = f[n+N] \tag{1}$$

Then the DTFT of the sequence is also a periodic sequence F[k] with the same period N. So we have:

$$F[k] = \sum_{n=0}^{N-1} f[n]e^{-2\pi\frac{nk}{N}j} = \sum_{n=0}^{N-1} f[n]W^{nk} \tag{2}$$

$$f[n] = \frac{1}{N}\sum_{n=0}^{N-1} F[k]W^{-nk} \tag{3}$$

Where,

$$W = e^{-2\pi\frac{j}{N}} \tag{4}$$

The values F[k] are what remains of the Digital Spectrum $F(e^{j})$. We can consider them as the values of $F(e^{j})$ for $= 2\pi\frac{k}{N}$, since the first N terms in the expression for $F(e^{j2\pi\frac{k}{N}})$ yield:

$$F(e^{j2\pi\frac{k}{N}}) = \sum_{n=0}^{N-1} f[n]e^{-\frac{2\pi kn}{N}j} + ... \tag{5}$$

Which is the same as the DFT expression. The remaining terms are just repetitions of the above sum and help build up the delta function that is needed to take us from a continuous transform to a discrete impulse.

From the above, we realize that the DFT is nothing but a sampled version of the DTFT, which is the digital version of the analog Fourier Transform.

In this assignment, we explore how to obtain the DFT, and how to recover the analog Fourier Transform for some known functions by the proper sampling of the function.

2.2 Implementing DFT in Python

There are two commands in Python, one to compute the forward fourier transform and the other to compute the inverse transform. They are:

- *numpy.fft.fft()*
- *numpy.fft.ifft()*

When we import pylab, both functions are imported into the local namespace.

3 Spectrum of sin(5t)

given the signal is sin(5t) which is periodic, we find the spectrum.

$$y = sin(5t) \tag{6}$$

$$y = \frac{e^{j(5t)} - e^{j(-5t)}}{2j} \tag{7}$$

So, the spectrum is:

$$Y(w) = \frac{1}{2j}[\delta(w-5) - \delta(w+5)] \tag{8}$$

We observe peaks at w= 5 and w= -5 with peak magnitudes of 0.5 and the respective phases are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

3.1 The Plots

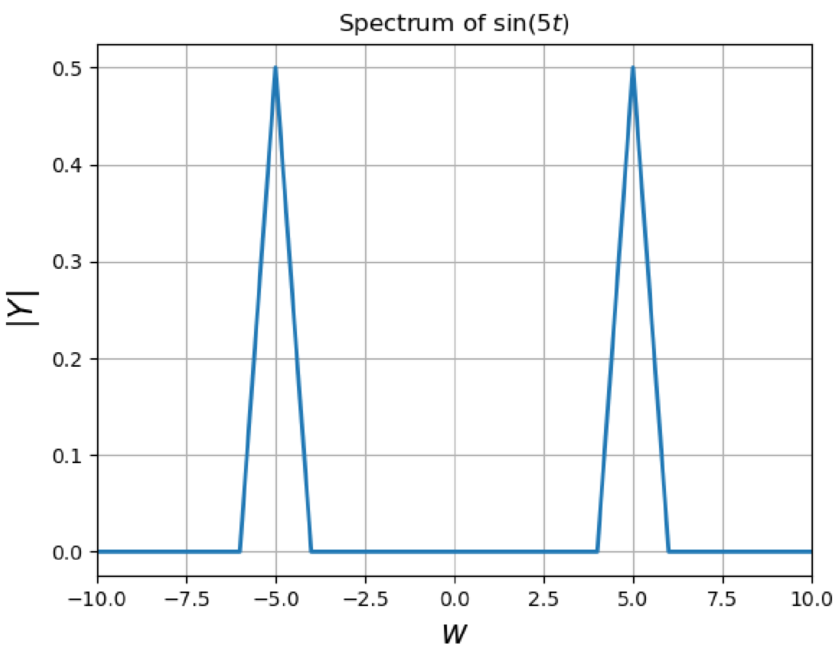

Figure 1: Spectrum of sin(5t)

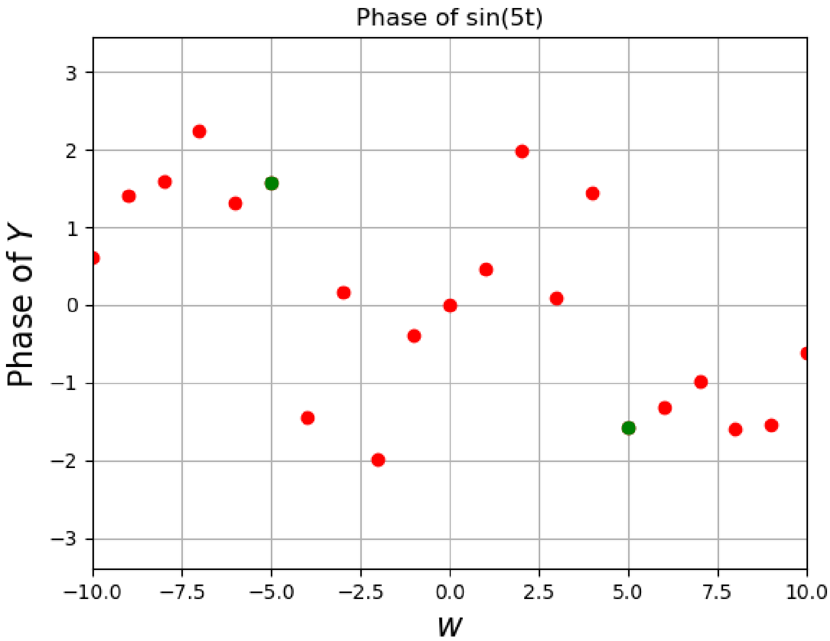

Figure 2: Phase of sin(5t)

3.2 The Code

```
#Q1: Working through the examples in the Assignment

#DFT of sin(5t)
x=linspace(0,2*pi,129);x=x[:-1]
y=sin(5*x)
Y=fftshift(fft(y))/128.0
w=linspace(-64,63,128)

#Magnitude plot of the DFT of sin(5t)
plot(w,abs(Y),lw=2)
xlim([-10,10])
ylabel(r"$|Y|$",size=16)
xlabel(r"$w$",size=16)
title(r"Spectrum of $\sin(5t)$")
grid(True)
show()

#Phase plot of the DFT of sin(5t)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)        #plots the points in green only where magnitude is
                                 greater than 0.001
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-10,10])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$w$",size=16)
title('Phase of sin(5t)')
grid(True)
show()
```

4 Spectrum of (1 + 0.1cos(t))cos(10t)

$$(1 + 0.1cos(t))cos(10t) = cos(10t) + 0.1(cos(t) * cos(10t)) \tag{9}$$

$$cos(10t) + 0.1(cos(t) * cos(10t)) = cos(10t) + 0.05 * cos(11t) + 0.05 * cos(9t) \tag{10}$$

$$\begin{gathered} cos(10t) + 0.05cos(11t) + 0.05cos(9t) \\ = 0.5e^{j(10t)} + 0.5e^{j(-10t)} + 0.025e^{j(11t)} + 0.025e^{j(-11t)} + 0.025e^{j(9t)} + 0.025e^{j(-9t)} \end{gathered} \tag{11}$$

Hence,

$$Y(w) = 0.5\delta(w-10)+0.5\delta(w+10)+0.025\delta(w-11)+0.025\delta(w+11)+0.025\delta(w-9)+0.025\delta(w+9) \tag{12}$$

There fore, we have to have **6 peaks** in the spectrum

4.1 With only 128 Samples

Plots:

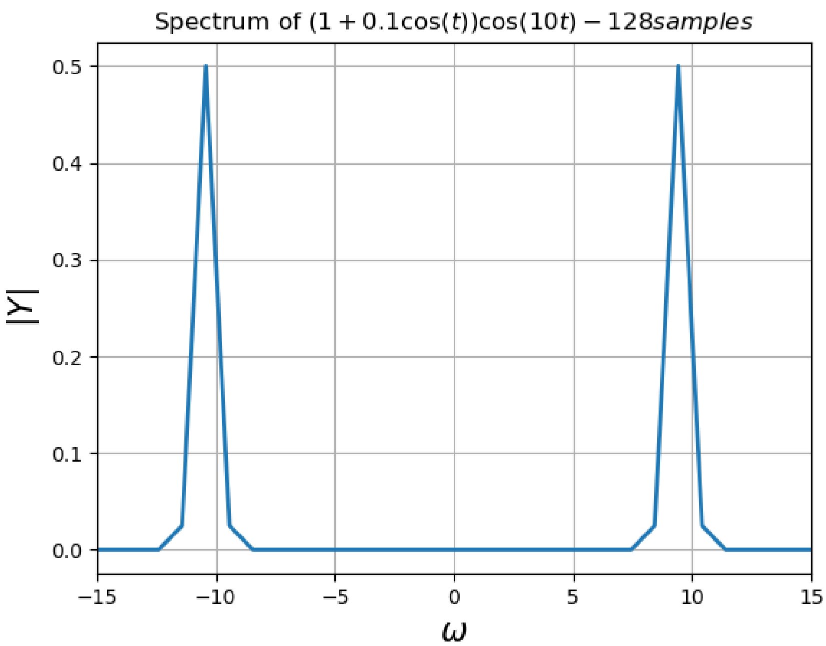

Figure 3: Spectrum of (1 + 0.1cos(t))cos(10t) - 128 samples

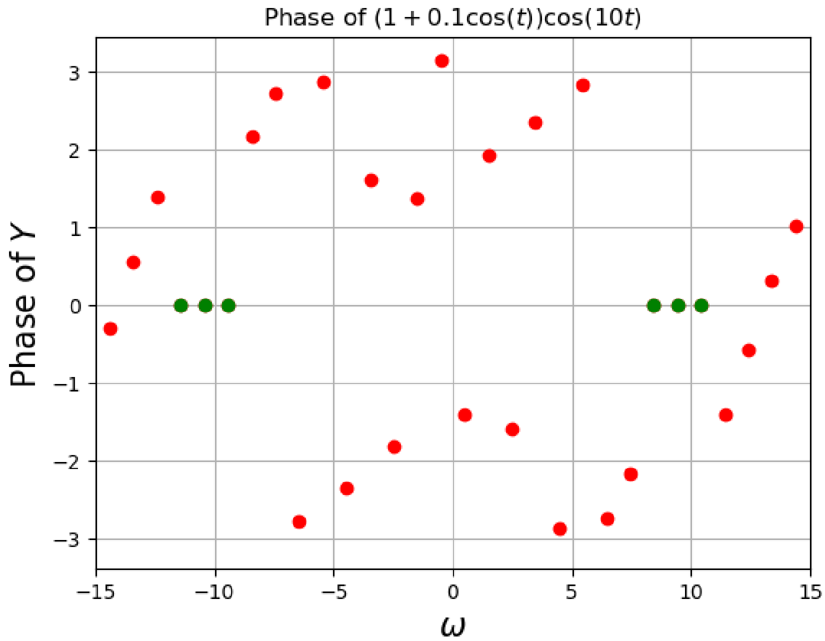

Figure 4: Phase of (1 + 0.1cos(t))cos(10t) - 128 samples

4.2 With only 512 Samples

Plots:

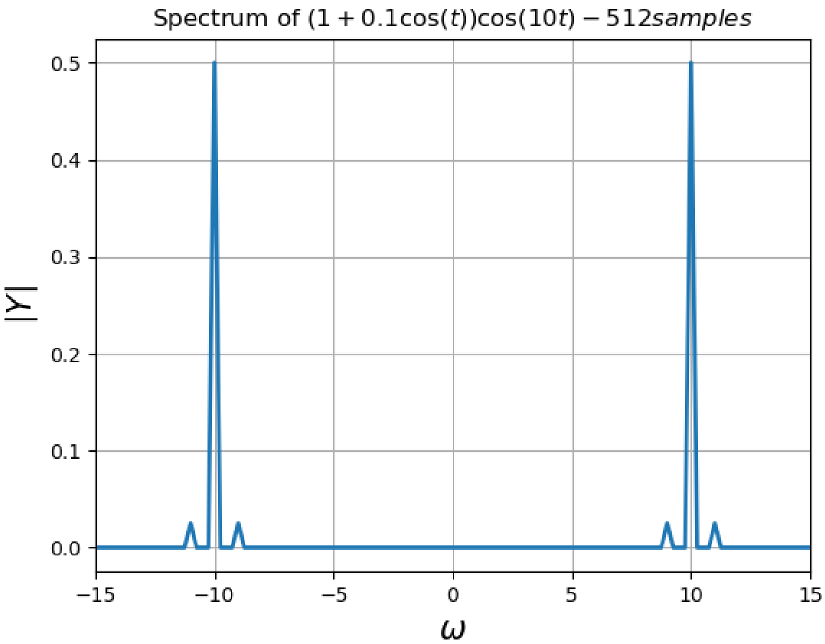

Figure 5: Spectrum of (1 + 0.1cos(t))cos(10t) - 512 samples

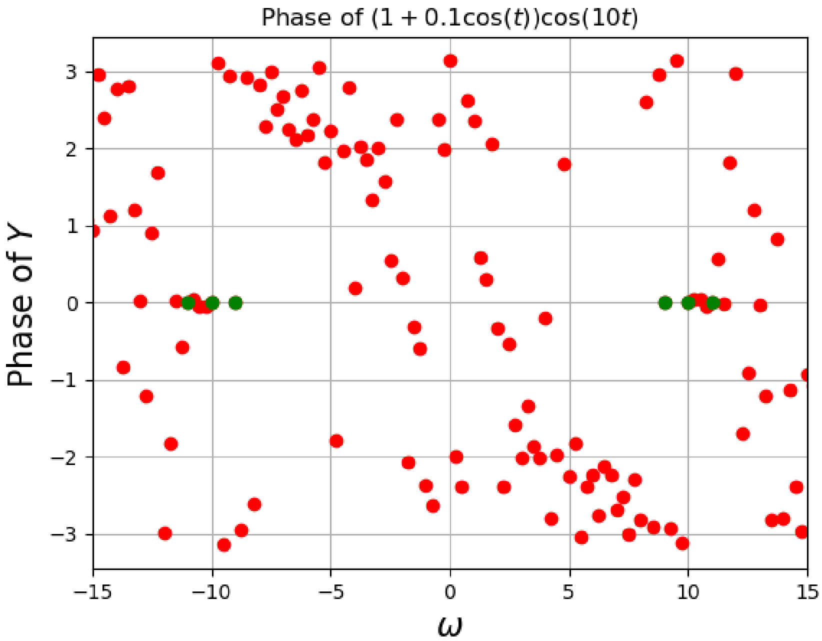

Figure 6: Phase of (1 + 0.1cos(t))cos(10t) - 512 samples

We can clearly see that 6 peaks are not visible when we take a lower number of samples (i.e 128) but we the 6 peaks are clearly visible when we take a higher number of samples (i.e 512).

Therefore, as we take more number of samples, a more accurate spectrum is achieved.

4.3 Code

```
#DFT of (1 + 0.1cos(t))cos(10t) with 128 samples
t=linspace(0,2*pi,129);t=t[:-1]
y=(1+0.1*cos(t))*cos(10*t)
Y=fftshift(fft(y))/128.0
w=linspace(-64,63,129);w=w[:-1]

#Magnitude plot of the DFT of (1 + 0.1cos(t))cos(10t) with 128 samples
plot(w,abs(Y),lw=2)
xlim([-15,15])
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Spectrum of $\left(1+0.1\cos\left(t\right)\right)\cos\left(10t\right) -
                                        128 samples$")
grid(True)
show()

#Phase plot of the DFT of (1 + 0.1cos(t))cos(10t) with 128 samples
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-15,15])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Phase of $\left(1+0.1\cos\left(t\right)\right)\cos\left(10t\right)$")
grid(True)
show()
```

```
#DFT of (1 + 0.1cos(t))cos(10t) with 512 samples
t=linspace(-4*pi,4*pi,513);t=t[:-1]
y=(1+0.1*cos(t))*cos(10*t)
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513);w=w[:-1]

#Magnitude plot of the DFT of (1 + 0.1cos(t))cos(10t) with 512 samples
plot(w,abs(Y),lw=2)
xlim([-15,15])
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Spectrum of $\left(1+0.1\cos\left(t\right)\right)\cos\left(10t\right)-
                                        512 samples$")
grid(True)
show()

#Phase plot of the DFT of (1 + 0.1cos(t))cos(10t) with 512 samples
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-15,15])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Phase of $\left(1+0.1\cos\left(t\right)\right)\cos\left(10t\right)$")
grid(True)
show()
```

5 Spectrum of sin^3t and cos^3t

To compare the DFT of sin^3t and cos^3t, we use the following:

$$sin^3t = \frac{3sin(t)}{4} - \frac{sin(3t)}{4} \tag{13}$$

$$cos^3t = \frac{3cos(t)}{4} - \frac{cos(3t)}{4} \tag{14}$$

Hence we observe the following:

for sin^3t:

- peaks at : w= 1 and -1, have a magnitude of 0.75
- peaks at : w= 3 and -3, have a magnitude of 0.25
- phase at w = -1,3 is $\frac{\pi}{2}$
- phase at w = 1,-3 is $-\frac{\pi}{2}$

Similarly for for cos^3t:

- peaks at : w= 1 and -1, have a magnitude of 0.75
- peaks at : w= 3 and -3, have a magnitude of 0.25 of all the peaks will be 0.

5.1 Plots of sin^3t

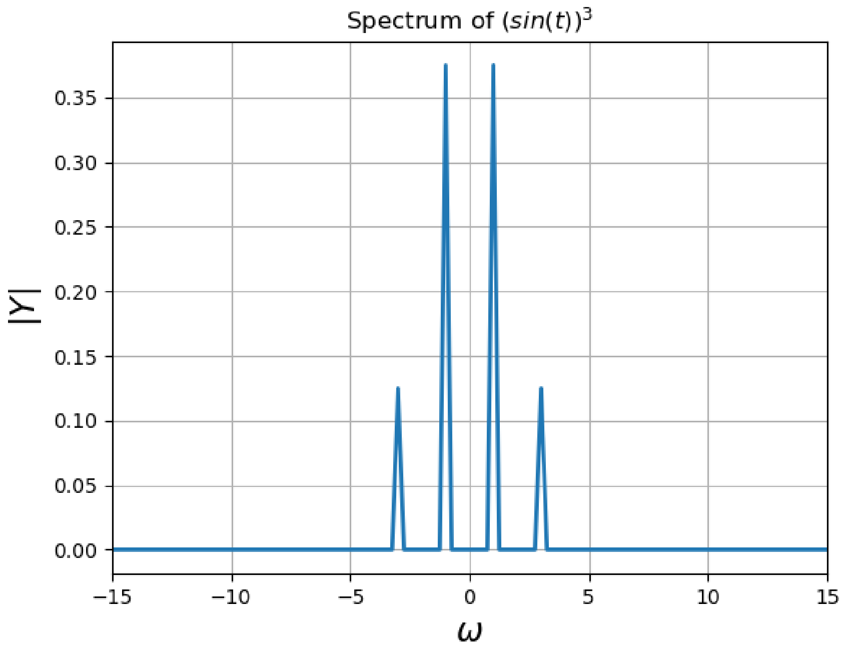

Figure 7: Spectrum of $\sin^3 t$

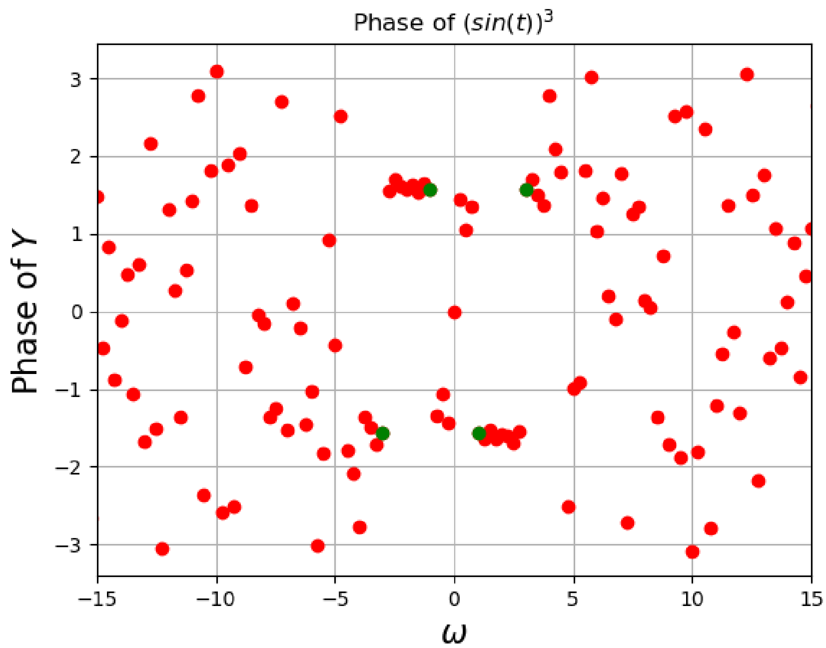

Figure 8: Phase of $\sin^3 t$

5.2 Plots of cos^3t

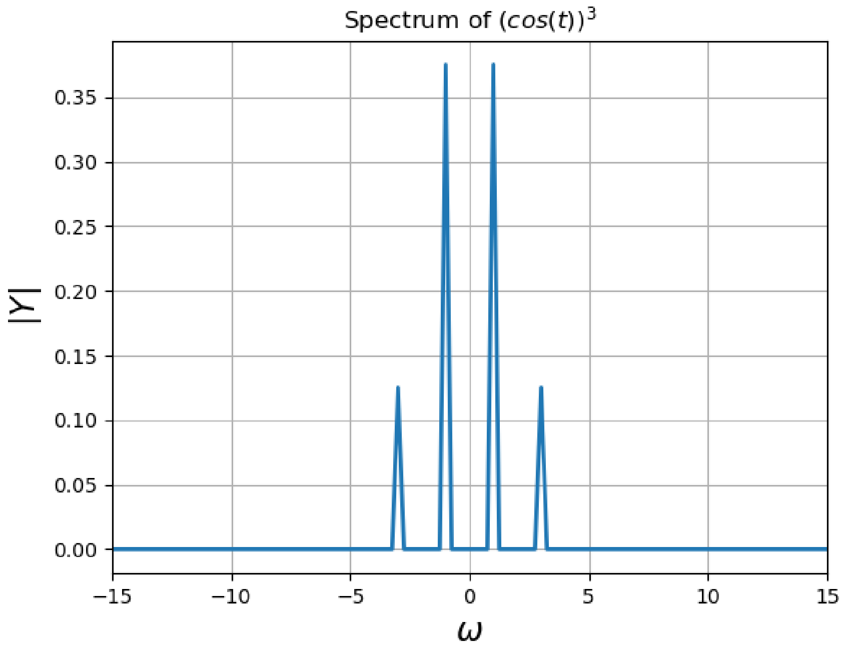

Figure 9: Spectrum of $\cos^3 t$

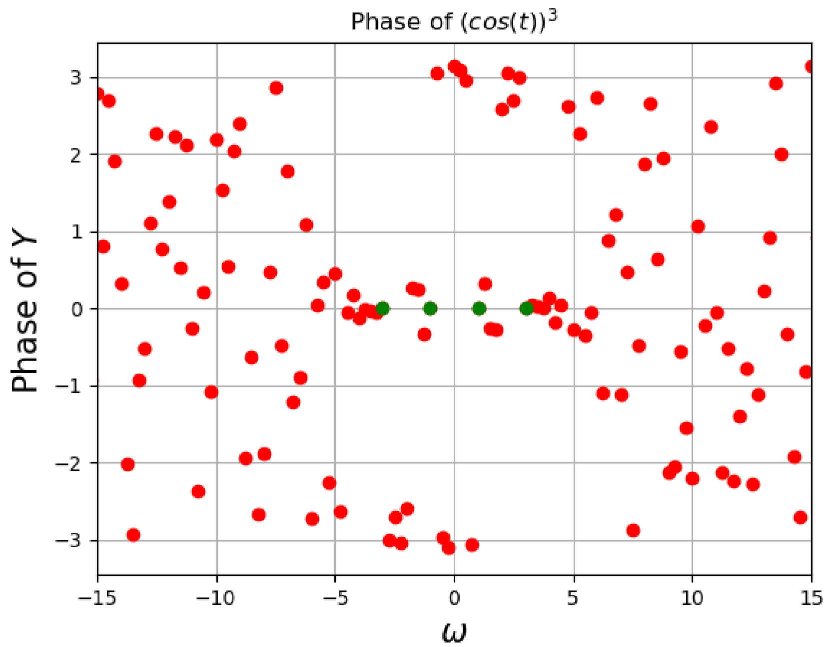

Figure 10: Phase of $\cos^3 t$

5.3 Code

```
#Q2: spectrums of (sin(t))^3 and (cos(t))^3

#DFT of (sin(t))^3 with 512 samples
```

```
t=linspace(-4*pi,4*pi,513);t=t[:-1]
y=(sin(t))**3
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513);w=w[:-1]

#Magnitude plot of the DFT of (sin(t))^3
plot(w,abs(Y),lw=2)
xlim([-15,15])
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Spectrum of $(sin(t))^3$")
grid(True)
show()

#Phase plot of the DFT of (sin(t))^3
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-15,15])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Phase of $(sin(t))^3$")
grid(True)
show()



#DFT of (cos(t))^3 with 512 samples
t=linspace(-4*pi,4*pi,513);t=t[:-1]
y=(cos(t))**3
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513);w=w[:-1]

#Magnitude plot of the DFT of (cos(t))^3
plot(w,abs(Y),lw=2)
xlim([-15,15])
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Spectrum of $(cos(t))^3$")
grid(True)
show()

#Phase plot of the DFT of (cos(t))^3
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-15,15])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Phase of $(cos(t))^3$")
grid(True)
show()
```

6 Spectrum of Phase Modulated signal cos(20t +5cos(t))

6.1 Plots

6.2 Plots of cos(20t +5cos(t))

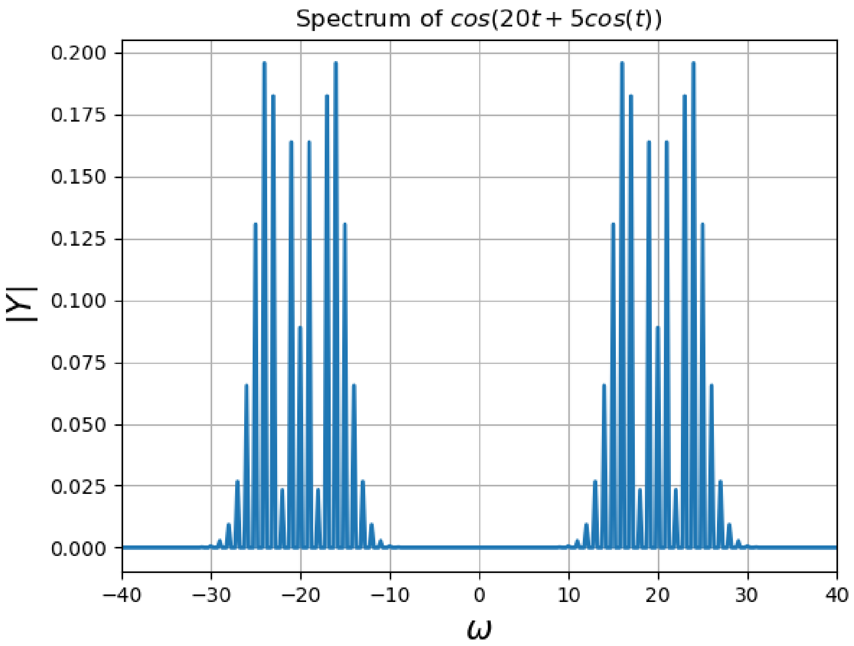

Figure 11: Spectrum of cos(20t +5cos(t))

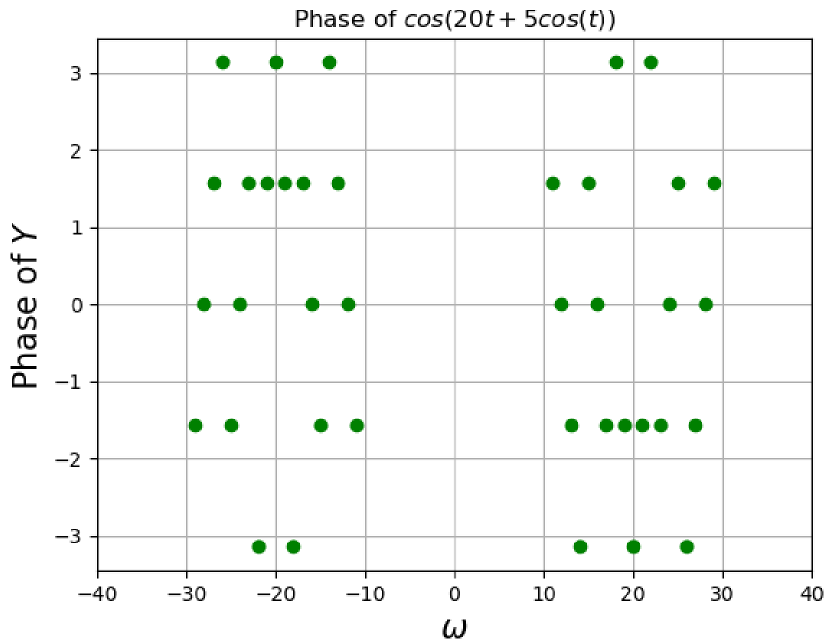

Figure 12: Phase of cos(20t +5cos(t))

6.3 Analysis

- We have plotted the spectrum of cos(20t +5cos(t)) for a time interval of $(-4\pi, 4\pi)$ and for 512 number of samples.
- We have only plotted the phase for points with a magnitude greater than $1e^{-3}$.
- We can clearly see that in the phase plot of the signal, all the points are centered around the carrier frequency of w=20
- Hence, its clear that the signal is a Phase Modulated signal.

6.4 Code

```
#Q3: spectrum of phase modulated signal cos(20t +5cos(t))

#DFT of cos(20t +5cos(t)) with 512 samples
t=linspace(-4*pi,4*pi,513);t=t[:-1]
y=cos(20*t+5*cos(t))
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513);w=w[:-1]

#Magnitude plot of the DFT of cos(20t +5cos(t))
plot(w,abs(Y),lw=2)
xlim([-40,40])
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Spectrum of $cos(20t +5cos(t))$")
grid(True)
show()

#Phase plot of the DFT of cos(20t +5cos(t))
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-40,40])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
title(r"Phase of $cos(20t +5cos(t))$")
grid(True)
show()
```

7 Spectrum and Analysis of the Gaussian function

A Gaussian Function is a function of the form:

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}} \tag{15}$$

In our case, the function is:

$$f(t) = e^{-\frac{t^2}{2}} \tag{16}$$

This function is very useful as its fourier transform is also a gaussian function.

The fourier transform of the given signal is:

$$Y(w) = \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}} \tag{17}$$

- Its important to remember that the given signal is **aperiodic**, hence it is not bandlimited in the frequency domain.
- Hence, the spectrum depends on the time interval and the number of samples we take.
- since the signal is real, the phase plot will be completely 0. Hence we dont have to plot it.

To plot the spectrum, we consider 2 different number of samples and time intervals,

- N=1024, $-16\pi, 16\pi$
- N=512, $-8\pi, 8\pi$

7.1 For $-16\pi, 16\pi$ and 1024 samples

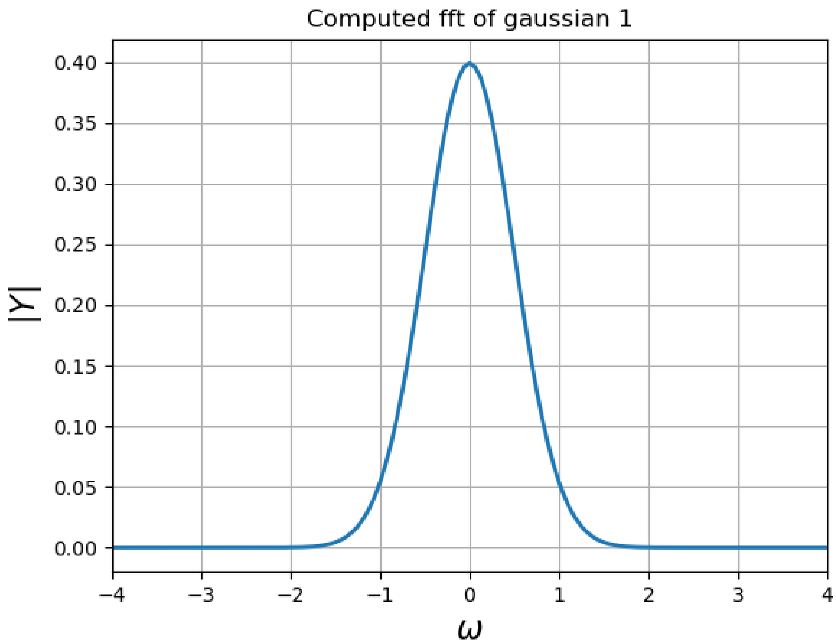

Figure 13: Spectrum of gaussian function

7.2 For $-8\pi, 8\pi$ and 512 samples

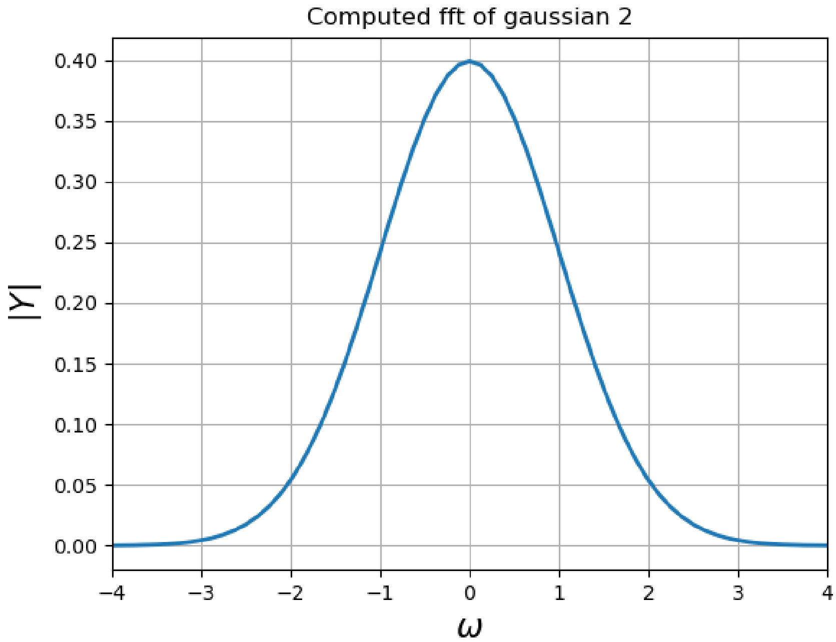

Figure 14: Spectrum of gaussian function

Hence, we can see that the spectrum of the gaussian sharpens as we increase the number of samples taken and reduce the time interval (i.e, we increase the sampling rate).

We plot the actual gaussian function and see that it strongly coincides with the Figure 14. (i.e $-8\pi, 8\pi$ range and 512 samples) with minimal deviation.

We cross check this by finding the maximum deviation, which is equal to $1.608160 * 10^{-16}$ which is extremely small.

7.3 Actual Gaussian function plot

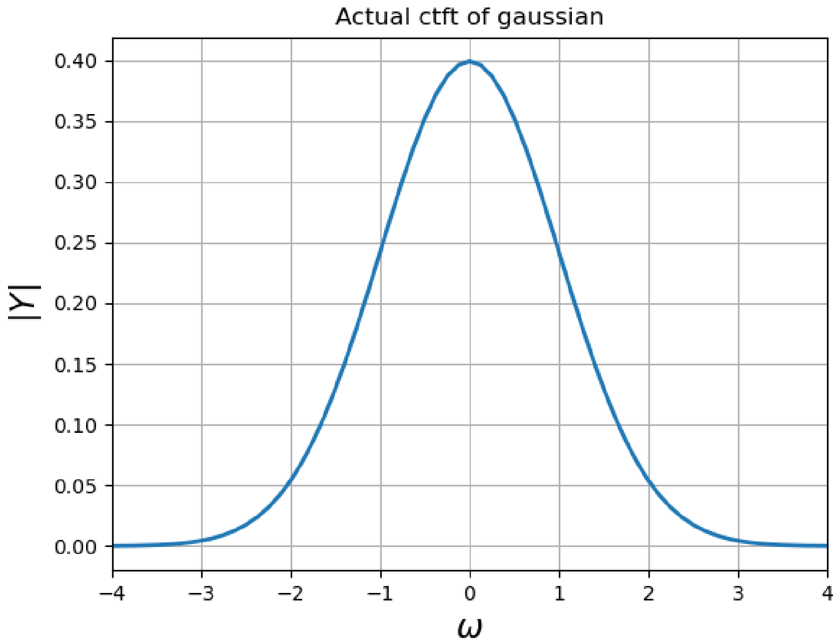

Figure 15: Spectrum of actual gaussian function

7.4 Code

```
#Q4: spectrum of Gaussian Function with different sampling rates

#case1 : sampling rate is higher
T=16*pi
N=1024
N2=512

#spectrum of the gaussian function
t = linspace(-T/2,T/2,N+1);t=t[:-1]
y = exp(-(t**2)/2)
Y = fftshift(abs(fft(y)))/N # Finding DFT
Y = Y/(max(Y)*sqrt(2*pi))
w = linspace(-N2*pi/T,N2*pi/T,N+1);w=w[:-1]
fft_gf = (1/sqrt(2*pi))*exp(-(w**2)/2)

#plotting the spectrum
plot(w,abs(Y),lw=2)
xlim(-4,4)
title('Computed fft of gaussian 1')
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()


#case2 : sampling rate is lower
T=16*pi
N=512
N2=512
```

```
#spectrum of the gaussian function along with the ctft of the ACTUAL gaussian
    function
t = linspace(-T/2,T/2,N+1);t=t[:-1]
y = exp(-(t**2)/2)
Y = fftshift(abs(fft(y)))/N # Finding DFT
Y = Y/(max(Y)*sqrt(2*pi))
w = linspace(-N2*pi/T,N2*pi/T,N+1);w=w[:-1]
fft_gf = (1/sqrt(2*pi))*exp(-(w**2)/2)
error = max(abs(abs(Y)-fft_gf)) #finds the maximum error between the calculated
    and actual functions

#plotting the spectrum
plot(w,abs(Y),lw=2)
xlim(-4,4)
title('Computed fft of gaussian 2')
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()

#plotting the spectrum of the ACTUAL gaussian function
plot(w,abs(fft_gf),lw=2)
xlim(-4,4)
title('Actual ctft of gaussian')
ylabel(r"$|Y|$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()

print('The Maximum error between the computed and actual fft of the gaussian
    function is %e:' %error)
```

8 Conclusion

- In this assignment, we have learnt how to find the DFT of various signals with the help of the fft numpy module (fft and fftshift).
- We analysed the data and plotted the respective spectrum and phase plot.
- We analysed the DFT for:
  1. Sinusoids
  2. Amplitude modulated signals
  3. Phase modulated signals and
  4. The Gaussian function
- We have seen how for the phase modulated signal, the phase plot is centered around w=20 and is zero for most values far away from it.
- We have seen how the gaussian function has multiple spectrums based on the time interval and the samples taken,
  1. The Plot sharpens as we decrease the time interval and
  2. The plot sharpens as we decrease the number of samples, i.e
  3. the plot sharpens for a higher sampling rate.