MIT Lectures: Single Variable Calculus

Lecture No. 1

A) What is a derivative?

A derivative measures how one quantity changes when another quantity changes.

- **Geometrical interpretation:** slope of the tangent line to a curve
- **Physical interpretation:** rate of change (speed, velocity, current, etc.)

In simple words, a derivative tells us how fast something is changing at an instant.

Why derivatives are important

Any real measurement involves change:

- Physics: velocity, acceleration
- Engineering: stress, optimization
- Economics: marginal cost, profit
- Political science: growth rates, trends

B) How to differentiate any function you know

Examples of functions we eventually differentiate:

$$\frac{d}{dx}(e^x), \quad \frac{d}{dx}(\tan^{-1} x)$$

In this lecture we focus on the *meaning* of derivatives, not techniques. Rules will come later.

Geometric Viewpoint on Derivatives

Let

$$y = f(x)$$

Choose a fixed point on the curve:

$$P = (x_0, f(x_0))$$

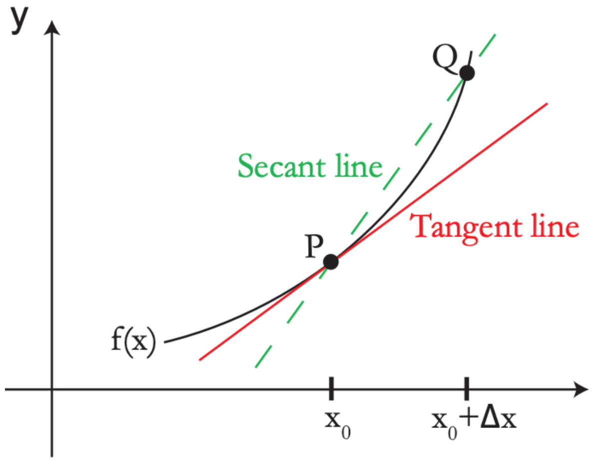
Choose another nearby point:

$$Q = (x_0 + \Delta x, f(x_0 + \Delta x))$$

The straight line joining P and Q is called a **secant line**.

As Q moves closer to P, the secant line rotates and approaches a limiting position. This limiting line is called the **tangent line** at P.

Figure 1: Secant and Tangent on a Curve



A tangent line is NOT just a line touching the curve once. It is defined using a limit of secant lines.

Geometric Definition of the Derivative

The derivative of $f(x)$ at x_0 is defined as:

The limit of the slopes of secant lines PQ as $Q \to P$, keeping P fixed.

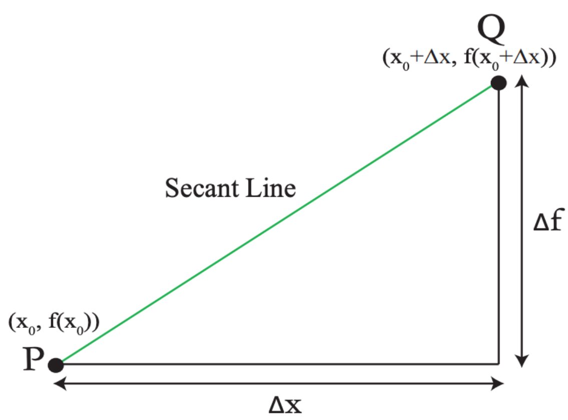
Slope of the secant line:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta f}{\Delta x}$$

where

$$\Delta x = (x_0 + \Delta x) - x_0, \quad \Delta f = f(x_0 + \Delta x) - f(x_0)$$

Figure 2: Pure Geometric Definition



We never substitute $\Delta x = 0$ directly. Doing so gives $0/0$, which is undefined. We must simplify first.

Limit Definition of the Derivative

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

This expression is called the **difference quotient**.

Example 1: $f(x) = \dfrac{1}{x}$

Start from the definition:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Substitute $f(x) = \frac{1}{x}$:

$$= \frac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x}$$

Take a common denominator in the numerator:

$$= \frac{\frac{x_0-(x_0+\Delta x)}{(x_0+\Delta x)x_0}}{\Delta x}$$

Simplify the numerator:

$$x_0 - (x_0 + \Delta x) = -\Delta x$$

So:

$$= \frac{-\Delta x}{(x_0 + \Delta x)x_0 \, \Delta x}$$

Cancel Δx:

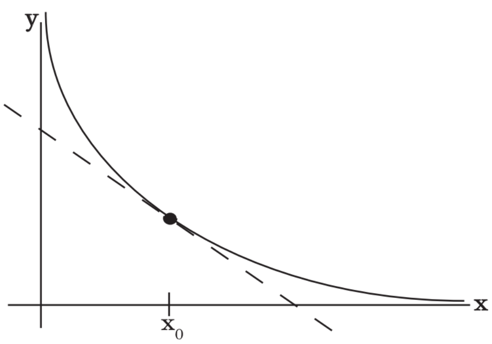
$$= -\frac{1}{(x_0 + \Delta x)x_0}$$

Now take the limit:

$$f'(x_0) = -\frac{1}{x_0^2}$$

The slope is negative everywhere. Hence the function $y = \frac{1}{x}$ is decreasing.

Figure 3: Graph of $y = \frac{1}{x}$ with Tangent



Triangle Area Formed by the Tangent

Set $y = 0$ to find x-intercept:

$$0 - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$$

Multiply both sides by $-x_0^2$:

$$x_0 = x - x_0 \Rightarrow x = 2x_0$$

By symmetry of $y = \frac{1}{x}$:

$$y = 2y_0$$

$$\text{Area} = \frac{1}{2}(2x_0)(2y_0) = 2$$

Surprisingly, the area is always constant, no matter where the tangent is drawn.

Example 2: $f(x) = x^n$, $n \in \mathbb{N}$

Start from the definition:

$$\frac{(x + \Delta x)^n - x^n}{\Delta x}$$

Using binomial theorem:

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}(\Delta x)^2 + \cdots$$

Subtract x^n:

$$nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}(\Delta x)^2 + \cdots$$

Divide by Δx:

$$nx^{n-1} + \binom{n}{2}x^{n-2}\Delta x + \cdots$$

All remaining terms contain Δx. As $\Delta x \to 0$, these terms vanish. These are called *higher order (junk) terms.*

Taking the limit:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Physical Interpretation: Rate of Change

Pumpkin dropped from height 400 ft.

$$y = 400 - 16t^2$$

Set $y = 0$:

$$400 - 16t^2 = 0 \Rightarrow t^2 = 25 \Rightarrow t = 5 \text{ s}$$

Average speed:

$$\frac{400}{5} = 80 \text{ ft/s}$$

Instantaneous velocity:

$$y' = -32t \Rightarrow y'(5) = -160 \text{ ft/s}$$

A negative sign indicates motion downward.

MIT Lectures: Single Variable Calculus

Lecture No. 2

1. Rate of Change

Let a function be written as

$$y = f(x)$$

If the input changes from x to $x + \Delta x$, then the output changes from $f(x)$ to $f(x + \Delta x)$.

So the change in output is

$$\Delta y = f(x + \Delta x) - f(x)$$

The ratio

$$\frac{\Delta y}{\Delta x}$$

is called the **average rate of change**.

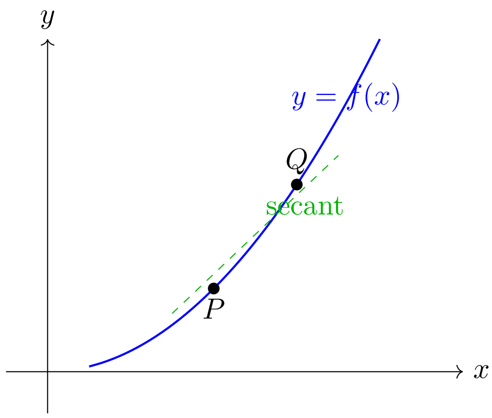
> This quantity represents the slope of the secant line between two points on the curve.

As $\Delta x \to 0$, the secant line approaches a limiting position, called the tangent line.

The slope of the tangent line is the **derivative**:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Graph: Secant Approaching Tangent



> As point Q moves closer to P, the slope of the secant approaches the slope of the tangent.

2. Limits

Definition

Let $f(x)$ be defined near $x = a$. We say

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a.

> Limit depends on nearby values, not the value at the point.

Easy Limits

An **easy limit** is one where we can get a meaningful answer simply by substituting the limiting value.

Example (from lecture):

$$\lim_{x \to 3} \frac{x^2 + x}{x + 1}$$

Step 1: Check whether direct substitution is allowed.

Substitute $x = 3$:

$$\frac{3^2 + 3}{3 + 1} = \frac{12}{4} = 3$$

Since the denominator does not become zero, the limit exists and equals:

$$\boxed{3}$$

> With an easy limit, plugging in the limiting value gives a correct and meaningful answer.

Important reminder from lecture:

The expression

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is *never* an easy limit.

Reason:

$$\Delta x = 0 \quad \text{is not allowed in the denominator.}$$

> Whenever direct substitution gives $0/0$, the limit is **not easy** and requires simplification.

Conceptual note:

The limit $x \to x_0$ is always computed under the implicit assumption that

$$x \neq x_0$$

This is why limits describe behavior *near* a point, not *at* the point itself.

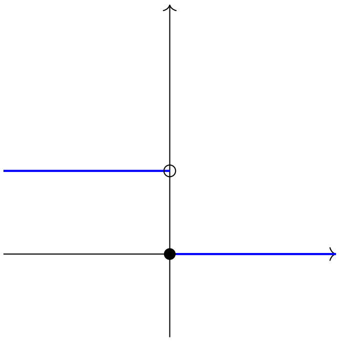
3. Continuity

A function $f(x)$ is continuous at $x = a$ if:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

> Continuity is a checklist, not a formula.

Graph: Discontinuous Function



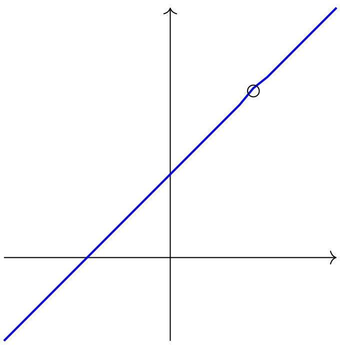
The limit exists, but the function value is different. Hence the function is not continuous.

4. Types of Discontinuities

(a) Removable Discontinuity

This is for all functions such that:

- the limit exists
- but the function value is missing or incorrect

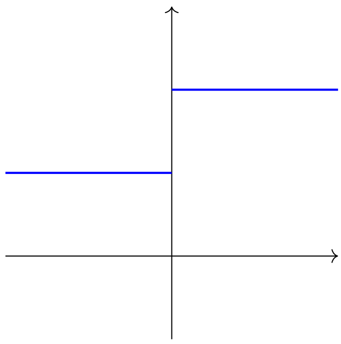


There is a hole in the graph. The discontinuity can be fixed.

(b) Jump Discontinuity

This is for all functions such that:

- left-hand limit exists
- right-hand limit exists
- both limits are not equal

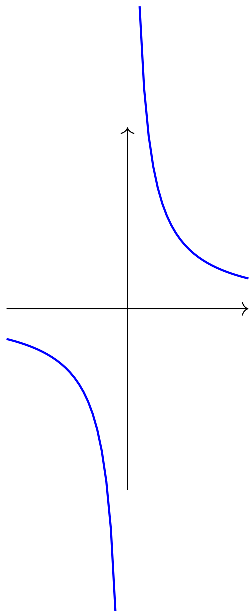


The graph jumps from one height to another.

(c) Infinite Discontinuity

This is for all functions such that:

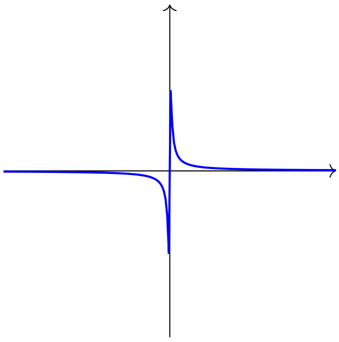
- function grows without bound near a point
- vertical asymptote exists



(d) Oscillatory Discontinuity

This is for all functions such that:

- the function oscillates infinitely
- no single value is approached



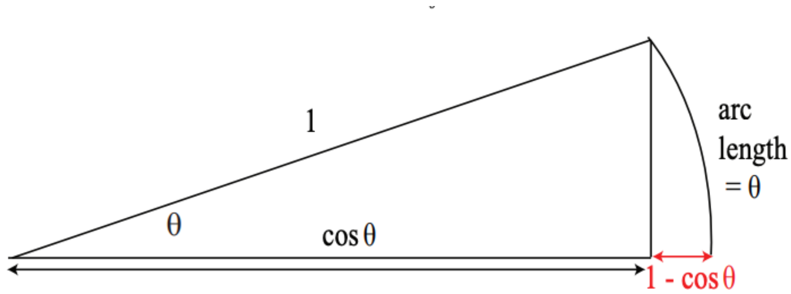
Bounded function, but limit still does not exist.

5. Trigonometric Limits (Angles in Radians)

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$$

These limits are proved using geometry and identities, not by direct substitution.

Graphical Explanation of $1 - \cos\theta$



What geometry is being used:

The diagram is drawn using the **unit circle**, i.e., a circle of radius 1.

- The slanted line has length 1 (radius of the unit circle).
- The horizontal projection of this radius is $\cos\theta$.
- The small **horizontal gap** between 1 and $\cos\theta$ is

$$1 - \cos\theta$$

- The curved arc on the circle has length θ.

Why arc length equals θ:
In a circle of radius r, arc length $= r\theta$. Since we are using a unit circle ($r = 1$),

$$\text{arc length} = \theta.$$

Key geometric observation:

- As $\theta \to 0$, the arc becomes shorter and flatter.

- Short arcs of a circle look almost like straight line segments.
- The horizontal gap $1 - \cos\theta$ shrinks much faster than the arc length θ.

Short arcs are nearly straight lines. Hence $1 - \cos\theta$ becomes much smaller than θ.

Conclusion:

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

This result follows directly from the geometry of the unit circle.

Graphical Explanation of $\frac{\sin\theta}{\theta}$

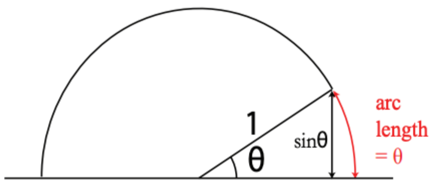
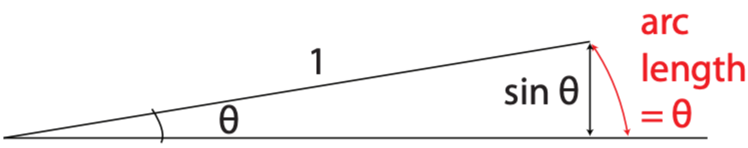


Figure 12: A circle of radius 1 with an arc of angle θ



Explanation:

- θ is the arc length on the unit circle
- $\sin\theta$ is the vertical height
- For small angles, arc length $\approx$ height
- Therefore the ratio approaches 1

This argument is valid only in radians.

Differentiable Implies Continuous

If it f is differentiable at $x = a$, then it must be continuous at $x = a$.

Differentiability is stronger than continuity.

MIT Lectures: Single Variable Calculus

Lecture No. 4

Chain rule, Higher derivatives

Lecture 04

Product Rule

If $u = u(x)$ and $v = v(x)$, then

$$\frac{d}{dx}(uv) = u'v + uv'$$

Example

$$\frac{d}{dx}(x^n \sin x)$$

$$= x^n \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^n)$$

$$= x^n \cos x + nx^{n-1} \sin x$$

Proof of Product Rule

Let $u = u(x)$ and $v = v(x)$.

After a small change Δx, the functions become $u(x + \Delta x)$ and $v(x + \Delta x)$.

$$\Delta(uv) = u(x + \Delta x)v(x + \Delta x) - u(x)v(x)$$

Adding and subtracting $u(x)v(x + \Delta x)$ on RHS

$$= u(x + \Delta x)v(x + \Delta x) - u(x)v(x) + u(x)v(x + \Delta x) - u(x)v(x + \Delta x)$$

$$= [u(x + \Delta x) - u(x)]v(x + \Delta x) + u(x)[v(x + \Delta x) - v(x)]$$

Divide both sides by Δx.

$$\frac{\Delta(uv)}{\Delta x} = \frac{\Delta u}{\Delta x}v(x + \Delta x) + u(x)\frac{\Delta v}{\Delta x}$$

Taking limit as $\Delta x \to 0$, we get

$$\frac{d}{dx}(uv) = u'v + uv'$$

Quotient Rule

If $v \neq 0$, the Quotient Rule id defined as

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Proof of Quotient Rule:

Let $u = u(x)$ and $v = v(x)$.

After a small change Δx, the functions become Δu and Δv.

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

Divide by Δx.

$$\frac{\Delta(u/v)}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

Taking limit as $\Delta x \to 0$,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Example

$$f(x) = \frac{1}{x}$$

Let $u = 1$ and $v = x$. Using the quotient rule:

$$f'(x) = \frac{x \cdot 0 - 1 \cdot 1}{x^2} = -\frac{1}{x^2}$$

Chain Rule

Let $y = f(x)$ and $x = g(t)$. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Example

$$y = (\sin t)^{10}$$

Let $x = \sin t$.

$$\frac{dy}{dt} = 10x^9 \frac{dx}{dt}$$
$$= 10(\sin t)^9 \cos t$$

0.0.1 Higher-Order Derivatives

- **First Derivative:** $f'(x) = \frac{dy}{dx}$
- **Second Derivative:** $f''(x) = \frac{d^2y}{dx^2}$
- **Third Derivative:** $f'''(x) = \frac{d^3y}{dx^3}$
- **n-th Derivative:** $f^{(n)}(x) = \frac{d^ny}{dx^n}$

Example 1

If $u = \sin x$,

$$u' = \cos x,$$
$$u'' = -\sin x,$$
$$u''' = -\cos x$$

Example 2

If $f(x) = x^n$,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d^2}{dx^2}(x^n) = \frac{d}{dx}nx^{n-1} = n(n-1)x^{n-2}$$

Continuing,

$$\frac{d^n}{dx^n}(x^n) = n(n-1)(n-2)(n-3)....1 = n!$$

$$\frac{d^{n+1}}{dx^{n+1}}(x^n) = 0,$$

MIT Lectures: Single Variable Calculus

Lecture No. 7

Derivative of x^r a Real number r

Let $r \in \mathbb{R}$. The power rule for differentiation is defined as:

$$\frac{d}{dx}(x^r) = rx^{r-1}, \quad \text{for all } r \in \mathbb{R}$$

Proof by Method-1: Change Base

Let $r \in \mathbb{R}$. Write

$$x^r = e^{r \ln x}$$

Differentiating using the chain rule,

$$\frac{d}{dx}(x^r) = \frac{d}{dx}\left(e^{r \ln x}\right) = e^{r \ln x} \cdot \frac{r}{x} = rx^{r-1}$$

This shows that the power rule holds for all real exponents.

Proof by Method-2: Logarithmic Differentiation

Let

$$u = x^r$$

Taking the natural logarithm,

$$\ln u = r \ln x$$

Differentiate both sides,

$$\frac{u'}{u} = \frac{r}{x}$$

$$u' = u \cdot \frac{r}{x}$$

Hence,

$$\frac{d}{dx}(x^r) = rx^{r-1}$$

Interpretation of Natural Logarithm

For a quantity P that changes over time, the relative rate of change is:

$$\frac{change\ in\ Price}{Price} = \frac{\Delta P}{P}$$

For continuous changes, this is expressed as:

$$\frac{P'}{P} = \frac{d}{dt}(\ln P)$$

This explains why natural logarithms are widely used in economics and growth models.

Chain Rule Interpretation

Let

$$y = 10x + a \quad \text{and} \quad x = 5t + b$$

Then

$$\frac{dy}{dx} = 10,$$

This shows y is 10 times changing faster than x

$$\frac{dx}{dt} = 5$$

This shows x is 10 times changing faster than t.

Now,

$$y = 10x + a = 10(5t + b) + a = 50t + 10b + a$$

By the chain rule, y is 50 times changing faster than t.

$$\frac{dy}{dt} = 50$$

This shows how rates multiply when variables depend on each other.

Example

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\ln(\sec x)) = \tan x$$

$$\frac{d}{dx}(x^{10} + 8x)^6 = 6(x^{10} + 8x)^5(10x^9 + 8)$$

Geometric Interpretation of $y'(x)$ and $y(x)$

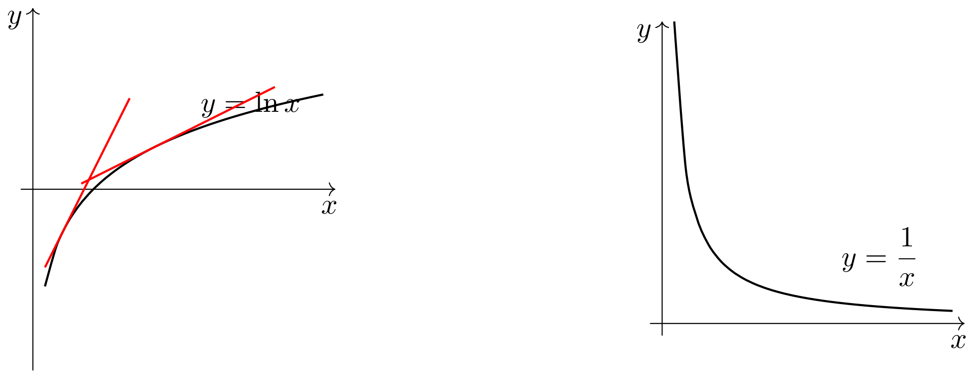


Figure 1: Left. The function $y = \ln x$ with tangent lines. Right. Its derivative $y = \dfrac{1}{x}$.

- The slope of the tangent is positive. The graph of y' lies above the x axis.
- From left to right, the graph of y becomes steeper. The graph of y' decreases.

Remark

The graph of y' and y can be different.
We take

$$y = \ln x$$

then

$$\frac{dy}{dx} = \frac{1}{x}$$

MIT Lectures: Single Variable Calculus

Lecture No. 9

Linear Approximation

For a differentiable function f near $x = x_0$,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

This approximation represents the tangent line at x_0.

Example

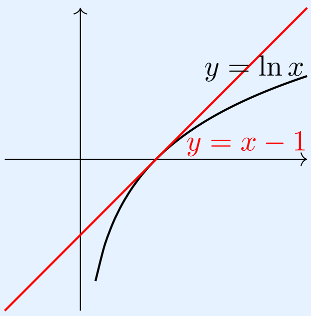
Let $f(x) = \ln x$ and choose $x_0 = 1$.

$$f(1) = 0, \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

Thus, applying the approximation equation, we get

$$\ln x \approx 0 + 1(x - 1)$$

$\ln x \approx x - 1 \quad \text{near } x = 1$



From the figure, when the base point $x = 1$, then both the black line ($y = lnx$) and red line ($y = x - 1$) are almost similar.

Proof of the Linear Approximation

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

This represents the instantaneous rate of change.

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

For small Δx,

$$\frac{\Delta f}{\Delta x} \approx f'(x_0)$$

This represents the average rate of change.

$$\Delta f \approx f'(x_0)\Delta x$$

Since

$$\Delta f = f(x) - f(x_0), \quad \Delta x = x - x_0,$$

we obtain

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$$

Hence,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0), \quad x \approx x_0$$

This is the linear approximation formula.

Special Case

When $x_0 = 0$,

$$f(x) \approx f(0) + f'(0)x, \quad x \approx 0$$

Example

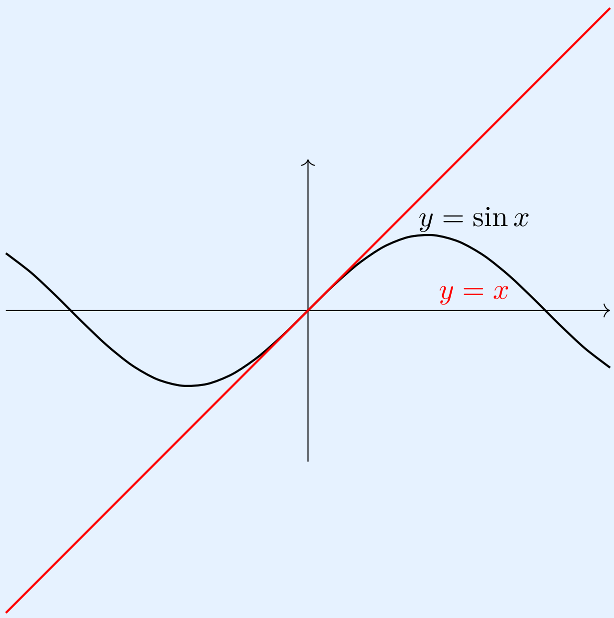
Let $f(x) = sinx$ and choose $x_0 = 0$.

$$f(0) = 0, \quad f'(x) = cosx, \quad f'(1) = 1$$

Thus, applying the approximation near $x_0 = 0$, we get

$$\sin x \approx sin(0) + cos(0)(x - 0)$$

$$\ln x \approx x \quad near \quad x = 0$$



From the figure, when the base point $x = 0$, then both the black line ($y = sinx$) and red line ($y = x$) are almost similar.

Example: 1

Approximation of $\ln(1+x)$
Let $f(x) = \ln(1+x)$ at $x_0 = 0$.

$$f(0) = 0, \quad f'(0) = 1$$

Now,

$$\ln(1+x) \approx f(0) + f'(0)(x-0)$$

Hence,

$$\ln(1+x) \approx x$$

Example: 2

Find the linear approximation f(x) at to $x = 0$

$$f(x) = \frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}.$$

Using approximations,

$$e^{-3x} \approx 1 - 3x, \quad (1+x)^{-1/2} \approx 1 - \frac{1}{2}x.$$

Therefore,

$$f(x) \approx (1-3x)\left(1 - \frac{1}{2}x\right) = 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2.$$

Dropping the x^2 term and higher order terms,

$$f(x) \approx 1 - \frac{7}{2}x.$$

Remark

Let

$$f(x) = \ln x.$$

We cannot take the base point $x_0 = 0$ because the function is undefined at $x = 0$. Hence, we cannot have a tangent at $x_0 = 0$.
Since

$$f'(x) = \frac{1}{x},$$

The derivative is also undefined at $x = 0$.
Therefore, we use a change of variable.

$$\ln u \approx u - 1$$

Let

$$u = 1 + x \quad \Rightarrow \quad u - 1 = x.$$

Using this,

$$\ln(1+x) \approx x \quad \text{for } x \approx 0.$$

Application of Linear Approximation

1) ln(1.1)

Using $\ln(1+x) \approx x$,

$$\ln(1.1) = \ln\left(1 + \frac{1}{10}\right) \approx \frac{1}{10}$$

$$\ln(1.1) \approx \frac{1}{10}$$

This allows fast numerical estimation.

2) Example from Real Life: GPS System

We can approximate my time and the time measured by a satellite. Time dilation is given by

$$T' = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Here v is the velocity.

This gives a rough idea of how my time is related to the time in the satellite.

Using the approximation,

$$T' \approx T\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right).$$

Let

$$u = \frac{v^2}{c^2}.$$

Then,

$$(1-u)^{-1/2} \approx 1 + \frac{1}{2}u \approx 1 + \frac{1}{2}\frac{v^2}{c^2}.$$

If we apply this to the GPS, the GPS satellite will have a different error, and the difference between my time and satellite time can be calculated.

Usually, the velocity of a GPS satellite is

$$v \approx 4 \text{ km/s}.$$

The speed of light is

$$c \approx 3 \times 10^5 \text{ km/s}.$$

Hence,

$$u = \frac{v^2}{c^2} \approx 10^{-10}.$$

Thus, the error in the GPS location is of the order of millimeters, which does not matter practically.

Higher-order terms are ignored because

$$u^2 \approx 10^{-20},$$

which is totally negligible.

Quadratic Approximation

Including the second derivative,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

Example

Consider

$$f(x) = \ln(1 + x).$$

The quadratic approximation about the base point $x_0 = 0$ is

$$f(x) = f(0) + f'(0)(x - 0) + \frac{1}{2}f''(0)(x - 0)^2.$$

We compute the derivatives.

$$f'(x) = \frac{1}{1 + x}, \quad f''(x) = -\frac{1}{(1 + x)^2}.$$

Evaluating at $x = 0$,

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1.$$

Substituting into the quadratic approximation formula,

$$\ln(1 + x) = 0 + 1 \cdot x + \frac{1}{2}(-1)x^2.$$

Hence,

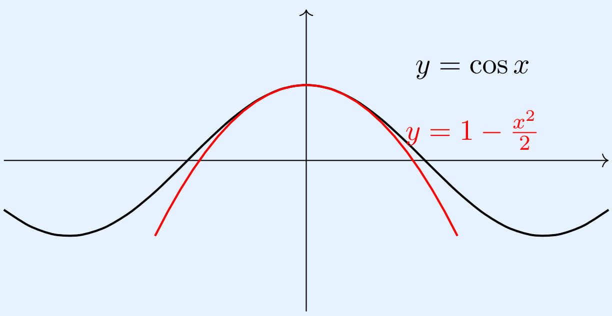
$$\ln(1 + x) \approx x - \frac{1}{2}x^2, \quad x \approx 0.$$

Example Approximations Near Zero

$$\sin x \approx x$$

$$\cos x \approx 1 - \frac{x^2}{2}$$

$$e^x \approx 1 + x + \frac{x^2}{2}$$



These approximations explain why polynomial expressions can closely model smooth curves near a point $x = 0$.