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OBSERVER KALMAN FILTER IDENTIFICATION (OKID)

Q1: *What is OKID?*

A1: Simply stated, it augments the input with the output data. And in the process the observer gain G turns out to be the well-known steady-state Kalman filter gain K, *if the output residual error (to be discussed later on) is white, zero-mean and and Gaussian.*

Q2: *Why would anyone use OKID?*

A2: The Kalman filter gain K can be computed from the input and output data if their noise characteristics, namely, the input noise and output noise covariances are determnined. In practice they may be difficult to obtain. Hence, OKID provides in an indirect way the steady-state kalman filter gain.

Q3: *What restrictions, if any, are imposed on the OKID-determined Kalman filter gain?*

A3: The data length must be sufficiently long and the order of observer is sufficiently largeso that the truncation error is negligible.

BASIC IDEA OF OKID

Step 1: Introduce the output as part of the input as follows!

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + G\{y(k) - y(k)\}, \qquad \textbf{G is the observer gain to be determined} \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

$$\Downarrow$$

$$x(k+1) = Ax(k) + G\{Cx(k) + Du(k)\} + \{Bu(k) - Gy(k)\}$$

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$$\Downarrow$$

$$x(k+1) = (A+GC)\, x(k) + [\, B + GD, \quad G\,] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{x}(k+1) = \bar{A}\, \mathbf{x}(k) + \bar{B}\, \mathbf{v}(k)$$

$$\bar{A} = (A + CG), \quad \bar{B} = [\, B + GD, \quad G\,], \quad \mathbf{v}(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \tag{21.1}$$

BASIC IDEA OF OKID - Cont'd

Step 2: Determine the impulse response function for the altered model system, viz., for input v and the same output y. This is done as follows by constructing the input-output matrix expressed as

$$\mathbf{y}(m,\ell) = \bar{\mathbf{Y}}(\, m, (m+r)p+r\,) * \mathbf{V}(\,(m+r)p+r, \ell)$$

$$\mathbf{y} = [\, y(0) \quad y(1) \quad y(2) \quad ... \quad y(p) \quad ... \quad y(\ell-1)\,]$$

$$\bar{\mathbf{Y}} = [\, D \quad C\bar{B} \quad C\bar{A}\bar{B} \quad ... \quad C\bar{A}^{p-1}\bar{B} \quad ... \quad C\bar{A}^{\ell-2}\bar{B}\,]$$

(*Analogy with the standard form of* $\mathbf{Y}$!) (21.2)

$$\mathbf{V} = \begin{bmatrix} u(0) & u(1) & u(2) & ... & u(p) & ... & u(\ell-1) \\ & v(0) & v(1) & ... & v(p-1) & ... & v(\ell-2) \\ & & v(0) & ... & v(p-2) & ... & v(\ell-3) \\ & & & ... & . & ... & . \\ & & & & v(0) & ... & v(\ell-p-1) \end{bmatrix}$$

where it is assumed that $C\bar{A}^k\bar{B} \approx \mathbf{0}$ **for** $k \geq p$.

BASIC IDEA OF OKID - Cont'd

Step 3: Obtain the modified impulse response function via a least-squares solver

$$\bar{\mathbf{Y}} = \mathbf{y}\mathbf{V}^{+} = \mathbf{y}\mathbf{V}^{T}[\mathbf{V}\mathbf{V}^{T}]^{-1} \tag{21.3}$$

Observe that, when $\bar{\mathbf{Y}}$ is arranged as follows,

$$\bar{\mathbf{Y}} = [\,\bar{\mathbf{Y}}_0 \quad \bar{\mathbf{Y}}_1 \quad \bar{\mathbf{Y}}_2 \quad \ldots \quad \bar{\mathbf{Y}}_p\,] \tag{21.4}$$

$\bar{\mathbf{Y}}_k$ consists of

$$\begin{aligned} \bar{\mathbf{Y}}_0 &= \mathbf{D} \\ \bar{\mathbf{Y}}_k &= \mathbf{C}\bar{\mathbf{A}}^{k-1}\bar{\mathbf{B}} \\ &= [\,C(A+GC)^{k-1}(B+GD) \quad -C(A+GC)^{k-1}G\,] \\ &= \left[\,\bar{\mathbf{Y}}_k^{(1)} \;\; -\bar{\mathbf{Y}}_k^{(2)}\,\right], \quad k = 1, 2, 3, \ldots \end{aligned} \tag{21.5}$$

BASIC IDEA OF OKID - Cont'd

Step 4: Obtain the original Markov parameters or the original Hankel matrix

$\mathbf{Y}_1 = CB$ **of the original system can be obtained from**

$$\mathbf{Y}_1 = CB = C(B + GD) - (CG)D = \bar{\mathbf{Y}}_1^{(1)} - \bar{\mathbf{Y}}_1^{(2)} D \tag{21.6}$$

$$\begin{aligned} \bar{\mathbf{Y}}_2^{(1)} &= C(A + GC)(B + GD) \\ &= CAB + CGCB + C(A + GC)GD \\ &= \mathbf{Y}_2 + \bar{\mathbf{Y}}_1^{(2)}\, \mathbf{Y}_1 + \bar{\mathbf{Y}}_2^{(2)} D \\ &\Downarrow \\ \mathbf{Y}_2 = CAB &= \bar{\mathbf{Y}}_2^{(1)} - \bar{\mathbf{Y}}_1^{(2)}\, \mathbf{Y}_1 - \bar{\mathbf{Y}}_2^{(2)} D \end{aligned} \tag{21.7}$$

By induction we obtain the following recurrence formulas:

$$\boxed{\begin{aligned} &D = \mathbf{Y}_0 = \bar{\mathbf{Y}}_0 \\ &\mathbf{Y}_k = \mathbf{Y}_k^{(1)} - \sum_{i=1}^{k} \bar{\mathbf{Y}}_i^{(2)} \mathbf{Y}_{k-i}, \quad \textbf{for} \quad k = 1, 2, ..., p \\ &\mathbf{Y}_k = -\sum_{i=1}^{p} \bar{\mathbf{Y}}_i^{(2)} \mathbf{Y}_{k-i}, \quad \textbf{for} \quad k = (p+1), ..., \infty. \end{aligned}} \tag{21.8}$$

BASIC IDEA OF OKID - Cont'd

Step 4: Obtain the original Markov parameters or the original Hankel matrix - cont'd

Rearrange (21.8) **in the form of**

$$
\begin{aligned}
\bar{\mathbf{Y}}^{(2)}\,\mathbf{H} &= \underline{\mathbf{Y}} \\
\bar{\mathbf{Y}}^{(2)} &= \left[-\bar{\mathbf{Y}}_p^{(2)} \quad -\bar{\mathbf{Y}}_{p-1}^{(2)} \quad \ldots \quad -\bar{\mathbf{Y}}_1^{(2)} \right] \\
\mathbf{H} &= \begin{bmatrix} Y_2 & Y_3 & Y_4 & \ldots & Y_{N+1} \\ Y_3 & Y_4 & Y_5 & \ldots & Y_{N+2} \\ . & . & . & \ldots & . \\ Y_{p+1} & Y_{p+2} & Y_{p+3} & \ldots & Y_{N+p} \end{bmatrix} \\
\underline{\mathbf{Y}} &= [\, Y_{p+2} \quad Y_{p+3} \quad Y_{p+4} \quad \ldots \quad Y_{N+p+1} \,]
\end{aligned}
\tag{21.9}
$$

Note that the Hankel matrix H can be expressed as before

$$
\mathbf{H} = \mathbf{V}_p\,\mathbf{A}\,\mathbf{W}_N = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{p-1} \end{bmatrix} \mathbf{A}\ [\, B \quad AB \quad A^2B \quad \ldots \quad A^{N-1}B \,] \tag{21.10}
$$

BASIC IDEA OF OKID - Cont'd

Step 4: Obtain the original Markov parameters or the original Hankel matrix - cont'd

Solution of the Hankel matrix from (21.9)**:**

$$\begin{bmatrix} \mathbf{I} & & & & \\ \bar{\mathbf{Y}}_1^{(2)} & \mathbf{I} & & & \\ \bar{\mathbf{Y}}_2^{(2)} & \bar{\mathbf{Y}}_1^{(2)} & \mathbf{I} & & \\ \cdot & \cdot & \cdot & \cdot & \\ \bar{\mathbf{Y}}_3^{(2)} & \bar{\mathbf{Y}}_2^{(2)} & \bar{\mathbf{Y}}_1^{(2)} & \ldots & \mathbf{I} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \cdot \\ Y_{k+1} \end{bmatrix} = \begin{bmatrix} \bar{Y}_1 - \bar{Y}_1^{(2)} D \\ \bar{Y}_2 - \bar{Y}_2^{(2)} D \\ \bar{Y}_3 - \bar{Y}_1^{(2)} D \\ \cdot \\ \bar{Y}_{k+1} - \bar{Y}_{k+1}^{(2)} D \end{bmatrix} \tag{21.11}$$

One must pivote for the solution of this equation.

Step 5: Compute the observer gain, G

Let us introduce the following definition

$$Y_k^0 = CA^{k-1}G, \quad k = 1, 2, 3, \ldots \tag{21.12}$$

Then we find

$$\boxed{\begin{aligned}
&Y_1^0 = CG = \bar{Y}_1^{(2)} \\
&Y_2^0 = \bar{Y}_1^{(2)} - \bar{Y}_1^{(2)} Y_1^0 \\
&\quad \Downarrow \\
&Y_1^0 = \bar{Y}_1^{(2)} \\
&Y_k^0 = \bar{Y}_k^{(2)} - \sum_{i=1}^{k-1} \bar{Y}_i^{(2)} Y_{k-i}^0 \quad \textbf{for} \quad k = 2, 3, \ldots, p \\
&Y_k^0 = -\sum_{i=1}^{p} \bar{Y}_i^{(2)} Y_{k-i}^0 \quad \textbf{for} \quad k = p+1, p+2, \ldots, \infty
\end{aligned}} \tag{21.13}$$

Hence, the observer gain G is obtained in an analoguous way to obtain B as follows:

$$G = (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \mathbf{Y}^0 \tag{21.14}$$

BASIC IDEA OF OKID - Cont'd

Step 5: Compute the observer gain, **G** *- cont'd*

where $\mathbf{Y}^0$ in (21.13) **is obtained as follows**

$$\mathbf{Y}^0 = \begin{bmatrix} CG \\ CAG \\ CA^2G \\ \cdot \\ \cdot \\ CA^kG \end{bmatrix} \tag{21.15}$$

which is obtained from

$$\begin{bmatrix} \mathbf{I} & & & & \\ \bar{\mathbf{Y}}_1^{(2)} & \mathbf{I} & & & \\ \bar{\mathbf{Y}}_2^{(2)} & \bar{\mathbf{Y}}_1^{(2)} & \mathbf{I} & & \\ \cdot & \cdot & \cdot & \cdot & \\ \bar{\mathbf{Y}}_3^{(2)} & \bar{\mathbf{Y}}_2^{(2)} & \bar{\mathbf{Y}}_1^{(2)} & \dots & \mathbf{I} \end{bmatrix} \begin{bmatrix} Y_1^0 \\ Y_2^0 \\ Y_3^0 \\ \cdot \\ Y_{k+1}^0 \end{bmatrix} = \begin{bmatrix} \bar{Y}_1^{(2)} \\ \bar{Y}_2^{(2)} \\ \bar{Y}_3^{(2)} \\ \cdot \\ \bar{Y}_{k+1}^{(2)} \end{bmatrix} \tag{21.16}$$

BASIC IDEA OF OKID - Cont'd

Q: Can we obtain [A, B, C, G] in a single realization?

A: Yes, if you employ the following:

$$\begin{aligned} \mathbf{P}_k &= [\, Y_k \quad Y_k^0 \,] = [\, C A^{k-1} B \quad C A^{k-1} G \,] = C A^{k-1} [\, B \quad G \,] \\ &= [\, \bar{Y}_k^{(1)} - \bar{Y}_k^{(1)} D \quad \bar{Y}_k^{(2)} \,] - \sum_{i=1}^{k-1} \bar{Y}_i^{(2)} [\, Y_{k-i} \quad Y_{k-i}^0 \,] \quad k = 1, 2, ..., \ell \end{aligned} \tag{21.17}$$

We would not prove but state the following:

If the data elngth is sufficiently long and the order of the observer is sufficiently large so taht the truncation error is negligible, then the observe gain is negative of the steady-state kalman filter gain. **In other words we have**

$$\boxed{G = -K_{steadt \quad state}} \tag{21.18}$$