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# Multiscaling in Turbulence through the Lens of Shell Models.

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Abstract: Abstract...

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## Contents



## 1 Introduction To Shell Model:

The Navier–Stokes equations mathematically express momentum balance for Newtonian fluids and make use of conservation of mass. They arise from applying Isaac Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term—hence describing viscous flow.
The Equation is given by,

$$\partial_t \mathbf{v} + \mathbf{v}.\nabla\mathbf{v} = \nu\partial_x^2\mathbf{v} - \nabla P + \mathbf{f} \tag{1.1}$$

The Non-linear term on the L.H.S. when taken Fourier Transform returns a convoluted term that has a summation over all Fourier modes that is impossible for a computer to calculate. So, we usually introduce a UV cutoff in the system which can introduce simulation errors. To solve this problem up to some extent Shell Models were introduced.
The GOY shell Model is defined as follows:

$$\left[\frac{d}{dt} + \nu_0 k_n^2\right] u_n = \iota[a_n u_{n+1} u_{n+2} + b_n u_{n-1} u_{n+1} + c_n u_{n-1} u_{n-2}]^* \tag{1.2}$$

where, $k_n = 2^{n-4}$ , $a_n = k_n$ , $b_n = -\frac{1}{2}k_{n-1}$ , $b_n = -\frac{1}{2}k_{n-2}$ and viscosity $\nu = 10^{-7}$.The constants in the above equations are calculated using energy, enstrophy, and helicity conservational laws. Since the GOY model does not have direct sweeping in this sense, it is sometimes thought of as a highly simplified quasi-Lagrangian version of the Navier-Stokes equation. Thus we might expect nontrivial dynamic multiscaling for GOY-model structure functions. The equal-time structure functions for this model are defined as:

$$S_p(k_n) \equiv \langle [u_n u_n^*]^{p/2} \rangle \sim k_n^{-\zeta_p} \tag{1.3}$$

We begin solving the coupled ODEs using the Rk-4 method for time marching and we have averaged over 20,000 independent realizations. The initial velocities were chosen to be random normal numbers which were generated using the Box-Muller method, to speed up the process Numba's just-in-time compiler was used, and from the multiprocessing library was used to parallelize the function calls. Further curve fitting was done using SciPy's in-built function.

## 1.1 Simulations:

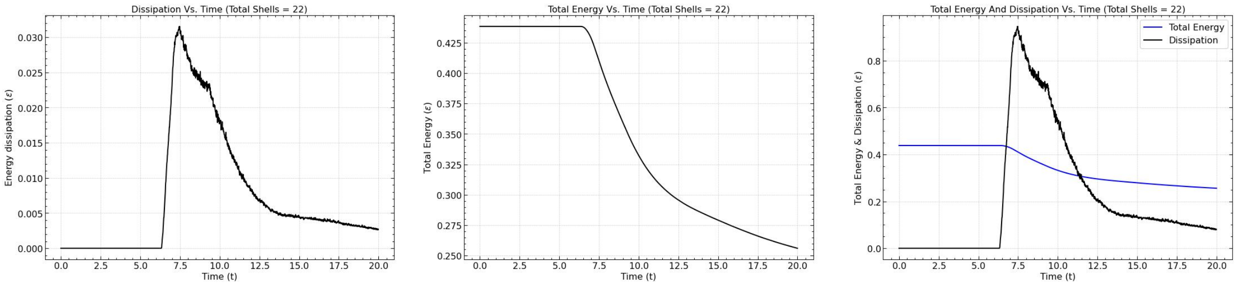



**Figure 1**. Dissiapation and Total Energy for 22 Shells simulation

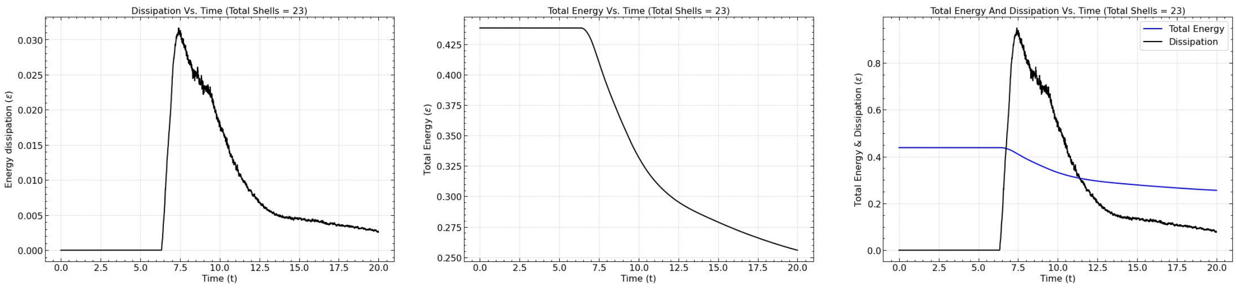



**Figure 2**. Dissiapation and Total Energy for 23 Shells simulation

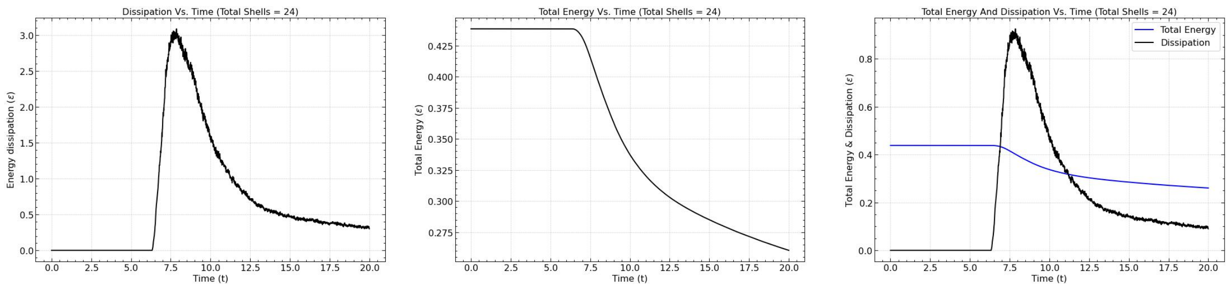



**Figure 3**. Dissiapation and Total Energy for 24 Shells simulation

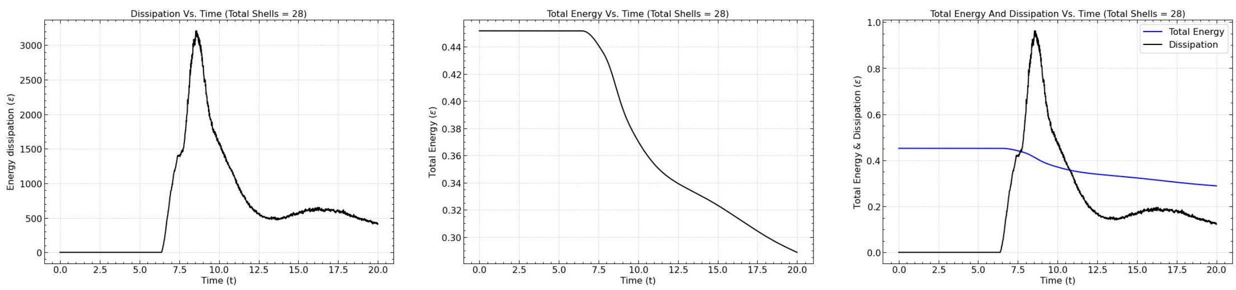



**Figure 4**. Dissiapation and Total Energy for 28 Shells simulation

We further check for the -5/3 scaling predicted by Kolmogorov-41 for various shell simulations and we get the following:

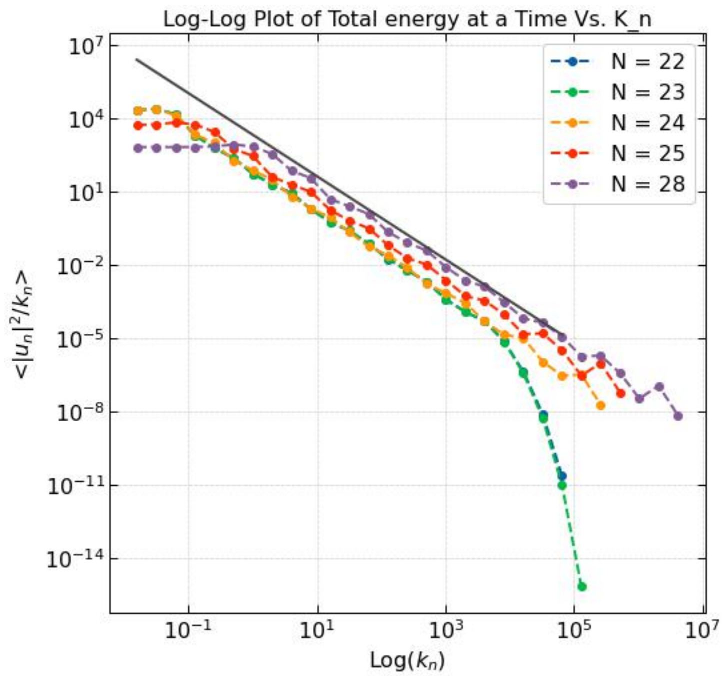



**Figure 5**. 2nd order Structure Function showing -5/3rd scaling

Higher structure functions were calculated and are plotted below:

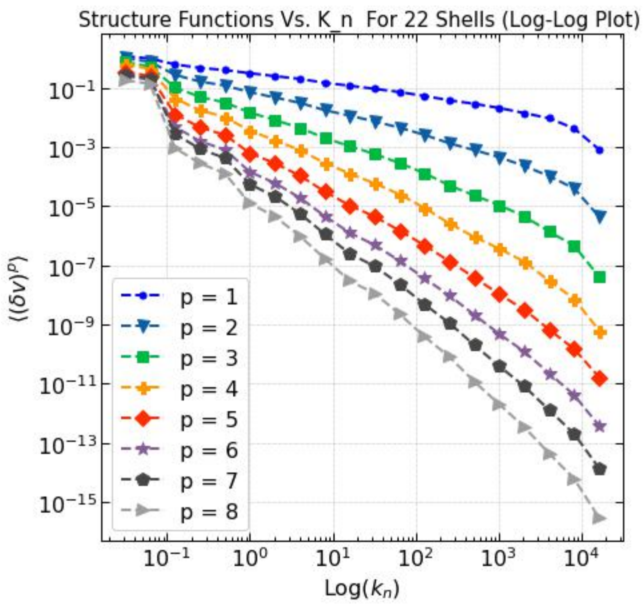
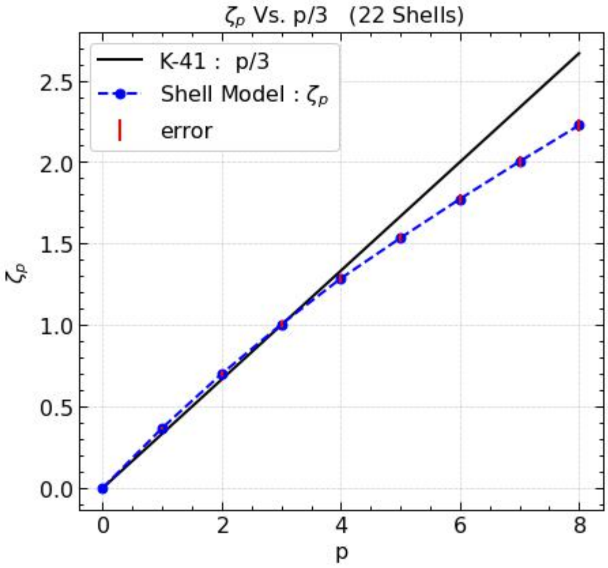



**Figure 6**. Caption

Now, finally, we calculate the slopes of each structure function and compare them with the prediction by Kolmogorov in his 1941 paper.




| | p/3 | $\zeta_p$ |
|---|---|---|
| 0 | 0.0 | 0 |
| 1 | 0.3333 | 0.3687 ∓ 0.0075 |
| 2 | 0.6667 | 0.7009 ∓ 0.0137 |
| 3 | 1.0 | 1.0047 ∓ 0.0197 |
| 4 | 1.3333 | 1.2858 ∓ 0.0261 |
| 5 | 1.6667 | 1.5339 ∓ 0.0233 |
| 6 | 2.0 | 1.7734 ∓ 0.0263 |
| 7 | 2.3333 | 2.0025 ∓ 0.0304 |
| 8 | 2.6667 | 2.2255 ∓ 0.0354 |

**Figure 7**. Caption

### 1.2 Result Interpretations:

- For dissipation, we see a peak that marks the transfers of energy from small k to large k values, and at the same time, we see total energy start to decrease which is intuitive because we are dealing with decaying turbulance.
- Second-order structure functions that have been plotted for various numbers of shells show the existence of a clear power law that marks the Inertial Range.
- Further, our simulation shows a clear linear scaling for the third structure function which is also consistent with K-41's result.

### 1.3 Conclusions:

We see that Shell Models can reproduce major results belonging to the theory of decaying turbulence, even though we are just using the nearest shell interaction. There is a clear deviation from the K-41 prediction of linear 'p/3' scaling of structure functions which implies the multifractal nature of turbulence.