2008 计算方法 A 卷参考答案

1. 解法一：埃尔米特插值法

解：①求一三次多项式 $P_3(x)$，满足 $P_3(1)=2$，$P_3(3)=12$，$P_3'(1)=1$，$P_3'(3)=-1$

则 $P_3(x)=2\alpha_0(x)+12\alpha_1(x)+\beta_0(x)-\beta_1(x)$

其中

$$\alpha_0(x)=\left[1-2(x-1)\cdot\frac{1}{1-3}\right]\cdot\left(\frac{x-3}{1-3}\right)^2=\frac{1}{4}x(x-3)^2$$

$$\alpha_1(x)=\left[1-2(x-3)\cdot\frac{1}{3-1}\right]\cdot\left(\frac{x-1}{3-1}\right)^2=\frac{1}{4}(4-x)(x-3)^2$$

$$\beta_0(x)=(x-1)\left(\frac{x-3}{1-3}\right)^2=\frac{1}{4}(x-1)(x-3)^2$$

$$\beta_1(x)=(x-3)\left(\frac{x-1}{3-1}\right)^2=\frac{1}{4}(x-3)(x-1)^2$$

$$\therefore P_3(x)=\frac{1}{2}x(x-3)^2+3(4-x)(x-1)^2+\frac{1}{4}(x-1)(x-3)^2-\frac{1}{4}(x-3)(x-1)^2$$

②

$$P_4(x)=P_3(x)+A(x-1)^2(x-3)^2$$

$$\because P_4(2)=4\ \therefore 4=P_3(2)+A(2-1)^2(2-3)^2\Rightarrow A=-\frac{7}{2}$$

$$\therefore P_4(x)=-\frac{7}{2}x^4+\frac{51}{2}x^3-\frac{125}{2}x^2+\frac{127}{2}x-21$$

解法二：求过(1,2),(2,4),(3,12)三点的 Newton 插值公式，

$$\begin{aligned}P_2(x)&=f(1)+f[1,2](x-1)+f[1,2,3](x-1)(x-2)\\&=2+2(x-1)+3(x-1)(x-2)\\&=3x^2-7x+6\end{aligned}$$

构造一四次多项式，满足五个已知条件，$P_4(x)=P_2(x)+(Ax+B)(x-1)(x-2)(x-3)$

由 $P_4'(1)=1,P_4'(3)=-1$,得 A=-7/2,B=9/2

$$\therefore P_4(x)=3x^2-7x+6+(-\frac{7}{2}x+\frac{9}{2})(x-1)(x-2)(x-3)$$

$$\therefore P_4(x)=-\frac{7}{2}x^4+\frac{51}{2}x^3-\frac{125}{2}x^2+\frac{127}{2}x-21$$

解法三：用差分表示导数的牛顿插值法

2. 解：设 $f(x)$ 的三次最佳一致逼近多项式为 $P_3(x)$，$x\in[0,1]$，令 $x=\dfrac{t+1}{2}$，$t\in[-1,1]$

$$f(x)-P_3(x)=F(t)-\overline{P}_3(t)=1-\frac{t-1}{2}+(\frac{t-1}{2})^2-(\frac{t-1}{2})^3+(\frac{t-1}{2})^4-\overline{P}_3(t)$$

$$=(\frac{1}{2})^4\cdot 2^{1-4}T_4(t)=\frac{1}{2^7}T_4(t)$$

$$\therefore \overline{P}_3(t)=1-\frac{t-1}{2}+(\frac{t-1}{2})^2-(\frac{t-1}{2})^3+(\frac{t-1}{2})^4-\frac{1}{2^7}T_4(t)$$

$$\because x=\frac{t-1}{2},\therefore t=2x-1$$

$$\therefore \overline{P_3}(2x-1)=1-x+x^2-x^3+x^4-\frac{1}{2^7}T_4(2x-1)$$

$$\therefore P_3(x)=1-x+x^2-x^3+x^4-\frac{1}{2^7}\left[8(2x-1)^4-8(2x-1)^2+1\right]$$

$$=1-x+x^2-x^3+x^4-(x-\frac{1}{2})^4+(x-\frac{1}{2})^2-\frac{1}{2^7}$$

$$=\frac{127}{128}-\frac{3}{4}x-\frac{1}{4}x^2+x^3$$

误差为$\left|f(x)-P_3(x)\right|=\left|\frac{1}{2^7}T_4(t)\right|\le\frac{1}{2^7}=\frac{1}{128}$

3. 1）证明：

因为$\int_a^b\rho(x)f(x)dx\approx\sum_{k=0}^{n}A_kf(x_k)$为高斯求积公式，所以对于不超过 2n+1 次的多项式严格成立；

因为$0\le k,l\le n,k\ne l$，所以$\varphi_k(x)\varphi_l(x)$是不超过 2n+1 次的代数多项式

所以$\int_a^b\rho(x)\varphi_k(x)\varphi_l(x)dx=\sum_{i=0}^{n}A_i\varphi_k(x_i)\varphi_l(x_i)$

又因为$\{\varphi_n(x)\}$是[a, b]上带权$\rho(x)$的正交多项式序列，所以$\int_a^b\rho(x)\varphi_k(x)\varphi_l(x)dx=0$

所以有$\sum_{i=0}^{n}A_i\varphi_k(x_i)\varphi_l(x_i)=0$，得证。

2）证明：

$\because l_k(x)=\prod_{\substack{i=0\\i\ne k}}^{n}\frac{(x-x_i)}{(x_k-x_i)},\therefore l_x^2(x)$是$2n$次代数多项式

又$\because l_k(x_j)=\delta_{kj}=\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}$，$\therefore l_k^2(x_j)=\delta_{kj}=\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}$

$$\therefore \int_a^b \rho(x) l_k^2(x)dx=\sum_{i=0}^{n} A_i l_k^2(x_i)=A_k l_k^2(x_k)=A_k$$

$$\therefore \sum_{k=0}^{n} \int_a^b \rho(x) l_k^2(x)dx=\sum_{k=0}^{n} A_k$$

令被插值函数 $f(x)\equiv 1$，由于插值型求积公式的代数精度至少为 n 次

$$\therefore \int_a^b \rho(x)\cdot f(x)dx=\int_a^b \rho(x)dx=\sum_{k=0}^{n} A_k f(x_k)=\sum_{k=0}^{n} A_k$$

所以 $\int_a^b \rho(x) l_k^2(x)dx=\int_a^b \rho(x)dx$

4. 解：

$$\begin{bmatrix}3 & 3 & 5\\ 3 & 5 & 9\\ 5 & 9 & 17\end{bmatrix}=\begin{bmatrix}1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}d_1 & & \\ & d_2 & \\ & & d_3\end{bmatrix}\begin{bmatrix}1 & l_{21} & l_{31}\\ & 1 & l_{32}\\ & & 1\end{bmatrix}$$

$$\therefore d_1=3;\ \ l_{21}d_1=3\Rightarrow l_{21}=1;\ \ l_{31}d_1=5\Rightarrow l_{31}=\frac{5}{3};\ \ l_{21}^2d_1+d_2=5\Rightarrow d_2=2$$

$$l_{31}l_{21}d_1+l_{32}d_2=9\Rightarrow l_{32}=2;\ \ l_{31}^2d_1+l_{32}^2d_2+d_3=17\Rightarrow d_3=\frac{2}{3}$$

$$\therefore \begin{bmatrix}1 & & \\ 1 & 1 & \\ \frac{5}{3} & 2 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}10\\ 16\\ 30\end{bmatrix}\Rightarrow\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}10\\ 6\\ \frac{4}{3}\end{bmatrix}$$

$$\therefore \begin{bmatrix}3 & & \\ 0 & 2 & \\ 0 & 0 & \frac{2}{3}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}10\\ 6\\ \frac{4}{3}\end{bmatrix}\Rightarrow\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}\frac{10}{3}\\ 2\\ 2\end{bmatrix}$$

$$\therefore \begin{bmatrix}1 & 1 & \frac{5}{3}\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}\frac{10}{3}\\ 2\\ 2\end{bmatrix}\Rightarrow\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 2\end{bmatrix}$$

5. 解：$x^{(k+1)}=x^{(k)}+a(b-Ax^{(k)})=(I-aA)x^{(k)}+ab$

迭代矩阵 $B = I - aA = \begin{bmatrix} 1-3a & -2a \\ -a & 1-2a \end{bmatrix}$

要使迭代法收敛，须使迭代阵 B 的谱半径小于 1

$\det(\lambda I - B) = 0 \Rightarrow \lambda_1 = 1-a$；$\lambda_2 = 1-4a$

$$\therefore \begin{cases} |1-a| < 1 \\ |1-4a| < 1 \end{cases} \Rightarrow \begin{cases} 0 < a < 2 \\ 0 < a < \dfrac{1}{2} \end{cases} \Rightarrow 0 < a < \frac{1}{2}$$

所以，当 $0 < a < \dfrac{1}{2}$ 时，迭代法收敛

要使迭代收敛最快，须使 $\rho(B) = \max\{|1-a|, |1-4a|\}$ 最小，画图确定

1-a=4a-1 的时候，即 $a = \dfrac{2}{5}$，$\rho(B) = \dfrac{3}{5}$ 时，迭代法收敛速度最快。

6. 证：令 $\varphi(x) = \sqrt{2+x}$，则 $x_{k+1} = \sqrt{2+x_k}$。取初值 $x_0 = \sqrt{2}$，则有

$x_1 = \sqrt{2+x_0} = \sqrt{2+\sqrt{2}}$；$x_2 = \sqrt{2+x_1} = \sqrt{2+\sqrt{2+\sqrt{2}}}$；……；

$$x_{k+1} = \sqrt{2+x_k} = \sqrt{\underbrace{2+\sqrt{2+\cdots+\sqrt{2}}}_{k}}$$

因为 $\varphi(2) = \sqrt{2+2} = 2$，且 $\varphi'(x) = \dfrac{1}{2\sqrt{2+x}}$

当 x=2 时，$|\varphi'(x)| = \dfrac{1}{4} < 1$

所以迭代公式 $x_{k+1} = \sqrt{2+x_k}$ 收敛，且收敛到 $x^* = 2$

7. 令 $x = \sqrt{99}$，$x^* = 9.9499$，所以 $|x - x^*| \le \dfrac{1}{2} \times 10^{-4}$

令 $U = (10-x)^{10}$，$V = \dfrac{1}{(10+x)^{10}}$

则根据 $\varepsilon(f(x^*)) \approx |f'(x^*)|\varepsilon(x^*)$ 有：

$\varepsilon(U) = 10(10-x)^9 \varepsilon(x)$

所以 $|\varepsilon(U)| \le \dfrac{1}{2} \times 10^{-1} \times 10^{-13}$，

$\varepsilon(V) = -10 \times (10-U)^{-11} \varepsilon(U)$

所以 $|\varepsilon(V)| \le \dfrac{1}{2} \times 10^{-5} \times 10^{-12}$

由 $U^*=V^*$, $U-U^*=U-V+V-V^*$

$|U-V|-|V-V^*|\leqslant|U-U^*|\leqslant|U-V|+|V-V^*|$

所以，$|U-U^*|\in[0.50953\times10^{-15}-0.25097\times10^{-17}, 0.50953\times10^{-15}+0.25097\times10^{-17}]$

所以 U 具有 1 位有效数字

所以 V 具有 5 位有效数字