

与积分相关的计算问题

一、和定积分定义相关的计算：

设 $f(x)$ 在 $[0,1]$ 上可积, 则

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right) \right] = \int_0^1 f(x) dx,$$

注: 这类题目都提出一个 $\frac{1}{n}$, 再看其余部分.

更一般地, 如果 $f(x)$ 在 $[a,b]$ 上可积, 则

$$\begin{aligned} \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left[f\left(a + \frac{1}{n}(b-a)\right) + f\left(a + \frac{2}{n}(b-a)\right) + \cdots + f\left(a + \frac{n}{n}(b-a)\right) \right] \end{aligned}$$

例 1. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln\left(1 + \frac{k}{n}\right)$; (2016—2017)

解: 这里 $f\left(\frac{k}{n}\right) = \ln\left(1 + \frac{k}{n}\right)$, 故

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln\left(1 + \frac{k}{n}\right) &= \int_0^1 \ln(1+x) dx = \int_0^1 (1+x)' \ln(1+x) dx \\ &= (1+x) \ln(1+x) \Big|_0^1 - \int_0^1 (1+x) \cdot \frac{1}{1+x} dx \\ &= 2 \ln 2 - \int_0^1 dx = 2 \ln 2 - 1. \end{aligned}$$

例 2. $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2n}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right)$; (2017—2018)

解:

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2n}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + \cdots + \frac{1}{\sqrt{1+\frac{n}{n}}} \right) \quad \left(\text{提出 } \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1+\frac{k}{n}}} = \int_0^1 \frac{1}{\sqrt{1+x}} dx \end{aligned}$$

$$= 2\sqrt{1+x}\Big|_0^1 = 2\sqrt{2} - 2.$$

二、与变上限积分相关的计算

变上限积分的导数：设 $f(x)$ 连续， $\varphi(x)$ 和 $\psi(x)$ 可导， a 和 b 是常数，则

$$\left(\int_a^x f(t)dt\right)' = f(x), \quad \left(\int_x^b f(t)dt\right)' = -f'(x);$$

$$\left(\int_a^{\psi(x)} f(t)dt\right)' = f(\psi(x))\psi'(x), \quad \left(\int_{\varphi(x)}^b f(t)dt\right)' = -f(\varphi(x))\varphi'(x);$$

$$\left(\int_{\varphi(x)}^{\psi(x)} f(t)dt\right)' = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x).$$

注：这里的被积函数 $f(t)$ 不带 x 。如果带有 x 须作变换后，再求导。

含变上限积分的极限，如果是 $\frac{0}{0}$ 型或 $\frac{\infty}{\infty}$ 型，可采用洛必达法则。

例 3. $\lim_{x \rightarrow 0} \frac{(\int_0^x e^{t^2} dt)^2}{\int_0^x (x-t) \cos t^2 dt}; \quad (2016-2017)$

分析： $x \rightarrow 0$ 时，

$$\lim_{x \rightarrow 0} (\int_0^x e^{t^2} dt)^2 = (\int_0^0 e^{t^2} dt)^2 = 0, \quad \lim_{x \rightarrow 0} \int_0^x (x-t) \cos t^2 dt = \int_0^0 (0-t) \cos t^2 dt = 0,$$

所以所求极限是 $\frac{0}{0}$ 型。

因为分母中的被积函数含有 x ，需要处理一下。

$$\begin{aligned} \int_0^x (x-t) \cos t^2 dt &= \int_0^x x \cos t^2 dt - \int_0^x t \cos t^2 dt \\ &= x \int_0^x \cos t^2 dt - \int_0^x t \cos t^2 dt \quad (\text{注：这里积分变量是 } t) \end{aligned}$$

解： $\lim_{x \rightarrow 0} \frac{(\int_0^x e^{t^2} dt)^2}{\int_0^x (x-t) \cos t^2 dt} = \lim_{x \rightarrow 0} \frac{(\int_0^x e^{t^2} dt)^2}{x \int_0^x \cos t^2 dt - \int_0^x t \cos t^2 dt}$

$$= \lim_{x \rightarrow 0} \frac{[(\int_0^x e^{t^2} dt)^2]'}{[x \int_0^x \cos t^2 dt - \int_0^x t \cos t^2 dt]'} \quad (\text{洛必达法则})$$

$$= \lim_{x \rightarrow 0} \frac{2(\int_0^x e^{t^2} dt)(\int_0^x e^{t^2} dt)'}{\int_0^x \cos t^2 dt + x(\int_0^x \cos t^2 dt)' - (\int_0^x t \cos t^2 dt)'}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{2(\int_0^x e^{t^2} dt)e^{x^2}}{\int_0^x \cos t^2 dt + x \cos x^2 - x \cos x^2} \\
&= \lim_{x \rightarrow 0} \frac{2(\int_0^x e^{t^2} dt)e^{x^2}}{\int_0^x \cos t^2 dt} = \lim_{x \rightarrow 0} 2e^{x^2} \cdot \lim_{x \rightarrow 0} \frac{\int_0^x e^{t^2} dt}{\int_0^x \cos t^2 dt} \\
&= 2 \lim_{x \rightarrow 0} \frac{\int_0^x e^{t^2} dt}{\int_0^x \cos t^2 dt} \quad (\text{还是 } \frac{0}{0} \text{ 型}) \\
&= 2 \lim_{x \rightarrow 0} \frac{(\int_0^x e^{t^2} dt)'}{(\int_0^x \cos t^2 dt)'} \quad (\text{洛必达法则}) \\
&= 2 \lim_{x \rightarrow 0} \frac{e^{x^2}}{\cos x^2} = 2.
\end{aligned}$$

例 4. $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} e^{-t^2} dt}{\sin^2 x}$; (2017—2018)

分析: $\lim_{x \rightarrow 0} \int_0^{x^2} e^{-t^2} dt = \int_0^0 e^{-t^2} dt = 0$, $\lim_{x \rightarrow 0} \sin^2 x = 0$, 所以是 $\frac{0}{0}$ 型, 其中 $\sin^2 x \sim x^2$, ($x \rightarrow 0$).

解:

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\int_0^{x^2} e^{-t^2} dt}{\sin^2 x} &= \lim_{x \rightarrow 0} \frac{\int_0^{x^2} e^{-t^2} dt}{x^2} \quad (\text{等价无穷小代换: } \sin^2 x \sim x^2, (x \rightarrow 0)) \\
&= \lim_{x \rightarrow 0} \frac{(\int_0^{x^2} e^{-t^2} dt)'}{(x^2)'} \quad (\text{洛必达法则}) \\
&= \lim_{x \rightarrow 0} \frac{e^{-x^2} (x^2)'}{2x} = \lim_{x \rightarrow 0} e^{-x^2} = 1.
\end{aligned}$$

例 5. $\lim_{x \rightarrow 0} \frac{\int_0^x t(e^{(x-t)^2} - 1) dt}{\cos x - e^{-\frac{x^2}{2}}}$; (2019—2020)

分析: 本题也是 $\frac{0}{0}$ 型.

(1) 分子的被积函数含有 x , 为了求导数, 分子须做一些处理.

令 $x-t=u$, 则

$$\begin{aligned}
\int_0^x t(e^{(x-t)^2} - 1)dt &= \int_x^0 (x-u)(e^{u^2} - 1)(-du) \\
&= \int_0^x (x-u)(e^{u^2} - 1)du \\
&= \int_0^x x(e^{u^2} - 1)du - \int_0^x u(e^{u^2} - 1)du \\
&= x \int_0^x (e^{u^2} - 1)du - \int_0^x u(e^{u^2} - 1)du ;
\end{aligned}$$

(2) 分母需用麦克劳林公式找到等价无穷小.

解: 令 $x-t=u$, 则

$$\int_0^x t(e^{(x-t)^2} - 1)dt = x \int_0^x (e^{u^2} - 1)du - \int_0^x u(e^{u^2} - 1)du .$$

由麦克劳林公式, 有

$$\begin{aligned}
\cos x - e^{-\frac{x^2}{2}} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) - (1 - \frac{x^2}{2} + \frac{1}{2!}(-\frac{x^2}{2})^2 + o(x^4)) \\
&= (\frac{1}{4!} - \frac{1}{4 \cdot 2!})x^4 + o(x^4) \\
&= -\frac{1}{12}x^4 + o(x^4) .
\end{aligned}$$

因此, $\lim_{x \rightarrow 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{-\frac{1}{12}x^4} = \lim_{x \rightarrow 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{12}x^4} = \lim_{x \rightarrow 0} (1 + \frac{o(x^4)}{-\frac{1}{12}x^4}) = 1$, 即

当 $x \rightarrow 0$ 时, $\cos x - e^{-\frac{x^2}{2}} \sim -\frac{1}{12}x^4$.

故

$$\begin{aligned}
&\lim_{x \rightarrow 0} \frac{\int_0^x t(e^{(x-t)^2} - 1)dt}{\cos x - e^{-\frac{x^2}{2}}} \\
&= \lim_{x \rightarrow 0} \frac{x \int_0^x (e^{u^2} - 1)du - \int_0^x u(e^{u^2} - 1)du}{-\frac{1}{12}x^4} \\
&= \lim_{x \rightarrow 0} \frac{[x \int_0^x (e^{u^2} - 1)du - \int_0^x u(e^{u^2} - 1)du]'}{-\frac{1}{12}(x^4)'} \quad (\text{洛必达法则})
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{\int_0^x (e^{u^2} - 1) du + x(e^{x^2} - 1) - x(e^{x^2} - 1)}{-\frac{1}{12}4x^3} \\
&= -3 \lim_{x \rightarrow 0} \frac{\int_0^x (e^{u^2} - 1) du}{x^3} = -3 \lim_{x \rightarrow 0} \frac{[\int_0^x (e^{u^2} - 1) du]'}{(x^3)'} \quad (\text{洛必达法则}) \\
&= -3 \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{3x^2} = -\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x^2} = -1. \quad (x \rightarrow 0 \text{ 时, } e^{x^2} - 1 \sim x^2)
\end{aligned}$$

例 6. 设函数 $y = y(x)$ 由方程 $\int_1^{y^3} e^{-t^2} dt + \int_x^0 \cos^6(x-t) dt = 0$ 所确定, 求 $\left. \frac{dy}{dx} \right|_{x=0}$. (2020—2021)

解: 取 $x=0$, 由 $\int_1^{y^3} e^{-t^2} dt + \int_x^0 \cos^6(x-t) dt = 0$ 得

$$\int_1^{y^3} e^{-t^2} dt + \int_0^0 \cos^6(x-t) dt = 0 \Rightarrow \int_1^{y^3} e^{-t^2} dt = 0.$$

因为 e^{-t^2} 大于 0, 故 $\int_1^{y^3} e^{-t^2} dt = 0 \Rightarrow y^3 = 1 \Rightarrow y = 1$, 即 $y|_{x=0} = 1$.

令 $u = x - t$, 则 $\int_x^0 \cos^6(x-t) dt = \int_0^x \cos^6 u (-du) = -\int_0^x \cos^6 u du$.

故已知等式化为

$$\int_1^{y^3} e^{-t^2} dt - \int_0^x \cos^6 u du = 0.$$

方程两边对 x 求导, 则

$$e^{-y^3} \cdot 3y^2 y' - \cos^6 x = 0, \quad (\text{注: } y \text{ 是 } x \text{ 的函数})$$

即
$$y' = \frac{\cos^6 x}{e^{-y^3} \cdot 3y^2}.$$

将 $x=0, y=1$ 代入, 得 $y'|_{x=0} = \frac{1}{e^{-1} \cdot 3 \cdot 1^2} = \frac{e}{3}.$

三、函数方程:

若给定 $f(x) = g(x) + h(x) \int_a^b \varphi(x) f(x) dx$, 求 $f(x)$, 这里 a 和 b 是常数, $g(x), h(x)$ 为已知函数.

注意到 $\int_a^b \varphi(x) f(x) dx$ 是常数, 记 $A = \int_a^b \varphi(x) f(x) dx$, 只要把 A 算出就可以.

由 $f(x) = g(x) + Ah(x)$ 两边同乘 $\varphi(x)$, 然后积分

$$A = \int_a^b f(x)\varphi(x)dx = \int_a^b g(x)\varphi(x)dx + A \int_a^b h(x)\varphi(x)dx,$$

解方程, 可求得 A .

例 7. 设函数 $f(x)$ 在 $[0, \pi]$ 上连续, 且满足 $f(x) = e^x + \int_0^\pi f(x) \sin x dx$, 试求 $f(x)$.

(2016—2017)

解: 记 $A = \int_0^\pi f(x) \sin x dx$. 由 $f(x) = e^x + \int_0^\pi f(x) \sin x dx$ 可得 $f(x) = e^x + A$, 于是,

$$f(x) \sin x = e^x \sin x + A \sin x,$$

两边积分, 得

$$\int_0^\pi f(x) \sin x dx = \int_0^\pi e^x \sin x dx + A \int_0^\pi \sin x dx$$

即
$$A = \int_0^\pi e^x \sin x dx + 2A.$$

移项后, 得

$$A = -\int_0^\pi e^x \sin x dx = e^x \cos x \Big|_0^\pi - \int_0^\pi e^x \cos x dx \quad (\text{分部积分})$$

$$= -e^\pi - 1 - (e^x \sin x \Big|_0^\pi - \int_0^\pi e^x \sin x dx) \quad (\text{分部积分})$$

$$= -e^\pi - 1 + \int_0^\pi e^x \sin x dx$$

$$= -e^\pi - 1 - A,$$

故
$$A = -\frac{1}{2}(e^\pi + 1).$$

所以,
$$f(x) = e^x - \frac{1}{2}(e^\pi + 1).$$

例 8. 设函数 $f(x)$ 在区间 $[0, \frac{\pi}{2}]$ 上连续, 且满足 $f(x) = \sin^3 x + 2 \int_0^{\frac{\pi}{2}} f(x) \sin x dx$, 试求

$f(x)$. (2017—2018)

解: 令 $A = \int_0^{\frac{\pi}{2}} f(x) \sin x dx$, 则 $f(x) = \sin^3 x + 2A$, 即 $f(x) \sin x = \sin^4 x + 2A \sin x$.

于是,
$$A = \int_0^{\frac{\pi}{2}} f(x) \sin x dx = \int_0^{\frac{\pi}{2}} \sin^4 x dx + 2A \int_0^{\frac{\pi}{2}} \sin x dx$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 2A$$

解得 $A = -\frac{3}{16}\pi$.

注: $\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3}, & n \text{ 为奇数时,} \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ 为偶数时.} \end{cases}$

四、被积函数为积分上限函数的积分计算

例 9. 设 $f(x) = \int_x^{\frac{\pi}{2}} \frac{\sin t}{t} dt$, 求定积分 $\int_0^{\frac{\pi}{2}} (x+1)e^x \cdot f(x) dx$. (2018—2019)

分析: 本题求不出 $f(x)$, 但 $f'(x) = -\frac{\sin x}{x}$.

解:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} (x+1)e^x \cdot f(x) dx &= \int_0^{\frac{\pi}{2}} (xe^x)' \cdot f(x) dx \\ &= (xe^x) \cdot f(x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} xe^x \cdot f'(x) dx \\ &= \frac{\pi}{2} e^{\frac{\pi}{2}} f\left(\frac{\pi}{2}\right) - 0 + \int_0^{\frac{\pi}{2}} e^x \sin x dx \\ &= \int_0^{\frac{\pi}{2}} e^x \sin x dx. \end{aligned}$$

因为

$$\begin{aligned} \int_0^{\frac{\pi}{2}} e^x \sin x dx &= e^x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x dx \\ &= e^{\frac{\pi}{2}} - (e^x \cos x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x (-\sin x) dx) \\ &= e^{\frac{\pi}{2}} - (-1 + \int_0^{\frac{\pi}{2}} e^x \sin x dx) \\ &= e^{\frac{\pi}{2}} + 1 - \int_0^{\frac{\pi}{2}} e^x \sin x dx, \end{aligned}$$

故 $\int_0^{\frac{\pi}{2}} e^x \sin x dx = \frac{1}{2}(e^{\frac{\pi}{2}} + 1)$.

所以, $\int_0^{\frac{\pi}{2}} (x+1)e^x \cdot f(x) dx = \frac{1}{2}(e^{\frac{\pi}{2}} + 1)$.

例 10. 设 $f(x) = \int_x^1 \cos t^2 dt$, 则 $\int_0^1 f(x) dx =$ _____. (2021—2022)

解: 注意到, $f(1) = \int_1^1 \cos t^2 dt = 0$, 于是,

$$\begin{aligned}
\int_0^1 f(x) \mathrm{d} x &= \int_0^1 (x)' f(x) \mathrm{d} x = x f(x) \Big|_0^1 - \int_0^1 x f'(x) \mathrm{d} x \\
&= f(1) - \int_0^1 x(-\cos x^2) \mathrm{d} x \\
&= \int_0^1 x \cos x^2 \mathrm{d} x = \frac{1}{2} \int_0^1 \cos x^2 \mathrm{d} x^2 \\
&= \frac{1}{2} \sin x^2 \Big|_0^1 = \frac{1}{2} \sin 1 .
\end{aligned}$$