

一、

$$\begin{aligned} 1. \text{ 解: } I &= \int_0^{\frac{\pi}{4}} d\theta \int_{2\cos\theta}^{\frac{2}{\cos\theta}} \frac{1}{r^4} \cdot r dr = \frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{1}{4\cos^2\theta} - \frac{\cos^2\theta}{4} \right) d\theta \\ &= \frac{1}{8} \left(\tan\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) \Big|_0^{\frac{\pi}{4}} = \frac{1}{8} \left(1 - \frac{\pi}{8} - \frac{1}{4} \right) = \frac{3}{32} - \frac{\pi}{64}. \end{aligned}$$

$$\begin{aligned} 2. \text{ 解: } I &= \int_0^2 x\sqrt{1+l^2} dx + \int_0^2 x\sqrt{1+x^2} dx \\ &= 2\sqrt{2} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} \Big|_0^2 = 2\sqrt{2} + \frac{5}{3}\sqrt{5} - \frac{1}{3}. \end{aligned}$$

$$3. \text{ 解: } L \text{ 的参数方程为 } \begin{cases} x = a \cos t \\ y = b \sin t \end{cases}, \quad t = 0 \rightarrow \pi.$$

$$\begin{aligned} I &= \int_0^\pi (a^2 \cos^2 t + 2ab \cos t \sin t)(-a \sin t) dt \\ &= -a^3 \int_0^\pi \cos^2 t \sin t dt - 2a^2 b \int_0^\pi \cos t \sin^2 t dt \\ &= \frac{a^3}{3} \cos^3 t \Big|_0^\pi - \frac{2}{3} a^2 b \sin^3 t \Big|_0^\pi = -\frac{2a^3}{3} \end{aligned}$$

$$\begin{aligned} 4. \text{ 解: } I &= \iint_{\Sigma} [(x-1+y)y+z] dS \\ &= \iint_{\Sigma} (1-y)z dS = \iint_{D_{yz}} (1-y)z \sqrt{1+l^2+l^2} dy dz \\ &= \sqrt{3} \int_0^1 dy \int_0^{1-y} (1-y)z dz = \frac{\sqrt{3}}{2} \int_0^1 (1-y)^3 dy = \frac{\sqrt{3}}{8}. \end{aligned}$$

$$\begin{aligned} 5. \text{ 解: } V &= \iint_D (x^2 + y^2) dx dy = \int_0^1 dx \int_{x^2}^x (x^2 + y^2) dy \\ &= \int_0^1 \left(x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6 \right) dx = \frac{1}{4} - \frac{1}{5} + \frac{1}{12} - \frac{1}{21} = \frac{3}{35}. \end{aligned}$$

$$\begin{aligned} 6. \text{ 解: } I &= \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^R r^2 \sin^2 \varphi \cdot r^2 \sin \varphi dr \\ &= 2\pi \times \frac{1}{5} R^5 \int_0^\pi \sin^3 \varphi d\varphi = \frac{2}{5} \pi R^5 \times \frac{4}{3} = \frac{8}{15} \pi R^5. \end{aligned}$$

二、解：作辅助线 $L_1: y=0, x: 0 \rightarrow \pi$ ，则

$$\begin{aligned} I &= \oint_{L+L_1} [\cos(x+y^2) + 2y]dx + [2y\cos(x+y^2) + 3x]dy \\ &\quad - \int_{L_1} [\cos(x+y^2) + 2y]dx + [2y\cos(x+y^2) + 3x]dy \\ &= \iint_D (3-2)dx dy - \int_0^\pi \cos x dx = \int_0^\pi dx \int_0^{\sin x} dy = \int_0^\pi \sin x dx = 2. \end{aligned}$$

三、解一： $\iiint_{\Omega} z^2 dx dy dz = \int_{-c}^c dz \iint_{D_z} z^2 dx dy$ ，其中 $D_z: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 - \frac{z^2}{c^2}$ 。

$$\text{故 } \iiint_{\Omega} z^2 dx dy dz = \int_{-c}^c z^2 \cdot \pi ab (1 - \frac{z^2}{c^2}) dz = \pi ab (\frac{2}{3}c^3 - \frac{2}{5}c^3) = \frac{4}{15} \pi abc^3.$$

$$\text{同理, } \iiint_{\Omega} x^2 dx dy dz = \frac{4}{15} \pi a^3 bc, \quad \iiint_{\Omega} y^2 dx dy dz = \frac{4}{15} \pi ab^3 c.$$

$$\text{故 } I = 3 \times \frac{4}{15} \pi abc = \frac{4}{5} \pi abc.$$

$$\text{解二：利用广义球坐标变换 } \begin{cases} x = ar \cos \theta \sin \varphi \\ y = br \sin \theta \sin \varphi, \\ z = cr \cos \varphi \end{cases}$$

$$\text{于是, } I = \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^1 r^2 \cdot abcr^2 \sin \varphi dr = \frac{4}{5} \pi abc.$$

$$\text{四、解： } I = \iint_{\Sigma} (x^2 + y^2 + z^2 - 2xy - 2yz) dS.$$

$$\text{由对称性, } I = \iint_{\Sigma} (2xy + 2yz) dS = \iint_{\Sigma} y(2x + 2z) dS = 0.$$

$$\text{则 } I = \iint_{\Sigma} (x+z) dS = (\bar{x} + \bar{z}) \cdot S_{\Sigma},$$

其中 $(\bar{x}, \bar{y}, \bar{z}) = (\frac{1}{2}, 0, \frac{1}{2})$ 为形心坐标， S_{Σ} 为 Σ 的面积，则 $S_{\Sigma} = 4\pi \times \frac{1}{2} = 2\pi$ 。

$$\text{故 } I = 2\pi.$$

五、解:
$$\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})^p = \sum_{n=1}^{\infty} \frac{(-1)^n}{(\sqrt{n+1} + \sqrt{n})^p},$$

因为 $p > 0$, $\frac{1}{(\sqrt{n+1} + \sqrt{n})^p}$ 单调减少, 且 $\lim_{n \rightarrow \infty} \frac{1}{(\sqrt{n+1} + \sqrt{n})^p} = 0$, 由莱布尼茨判

别法, 级数 $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})^p$ 收敛的.

又因为
$$\lim_{n \rightarrow \infty} \frac{|(-1)^n (\sqrt{n+1} - \sqrt{n})^p|}{\frac{1}{n^{p/2}}} = \lim_{n \rightarrow \infty} \frac{1}{(\sqrt{1 + \frac{1}{n}} + 1)^p} = \frac{1}{2^p},$$

由 p 级数的敛散性知, 当 $p/2 > 1$, 即 $p > 2$ 时, 级数 $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})^p$ 绝对

收敛; 当 $0 < p \leq 2$ 时, 级数 $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})^p$ 条件收敛.

六、解: 做辅助面 $\Sigma_1: z = 0, x^2 + \frac{y^2}{4} \leq 1$, 取下侧.

于是
$$\begin{aligned} I &= \oiint_{\Sigma + \Sigma_1} xz dy dz + 2zy dz dx + 3xy dx dy - \iint_{\Sigma_1} xz dy dz + 2zy dz dx + 3xy dx dy \\ &= \iiint_{\Omega} (z + 2z) dx dy dz + \iint_D 3xy dx dy \\ &= 3 \int_0^1 z dz \iint_{D_z} dx dy + 0 \quad (\text{由对称性}) \\ &= 3 \int_0^1 z 2\pi(1-z) dz = \pi. \end{aligned}$$

七、解: $f(x)$ 是以 2π 为周期的偶函数, 且在不连续点处满足 $f(x) = \frac{f(x^+) + f(x^-)}{2}$.

因此 $b_n = 0$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$$

$$\begin{aligned}
&= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} -\cos nx dx \right) \\
&= \frac{4}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 0, & n=2k \\ (-1)^k \frac{4}{(2k+1)\pi}, & n=2k+1, \quad k=0,1,2,\dots \end{cases}
\end{aligned}$$

于是, $f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos(2k+1)x}{2k+1}, -\infty < x < +\infty.$

特别, 令 $x=0$, 得 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}.$

八、解: 记 $a_n = (-1)^n \frac{n^2+1}{n}$, 易知 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, 故收敛半径为 1.

又因为 $x = \pm 1$ 时, 通项不趋于 0, 故级数发散. 因此, 级数 $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n} x^n$ 的收敛域为 $(-1, 1).$

当 $|x| < 1$ 时, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n} x^n = \sum_{n=1}^{\infty} (-1)^n n x^n + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}.$

记 $s_1(x) = \sum_{n=1}^{\infty} (-1)^n n x^n = x \sum_{n=1}^{\infty} (-1)^n (x^n)' = x \left(\sum_{n=1}^{\infty} (-1)^n x^n \right)'$

$$= x \left(\frac{-x}{1+x} \right)' = x \frac{-1-x+x}{(1+x)^2} = -\frac{x}{(1+x)^2};$$

$s_2(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$, 则 $s_2'(x) = \sum_{n=1}^{\infty} (-1)^n x^{n-1} = -\frac{1}{1+x},$

于是, $s_2(x) = -\int_0^x \frac{1}{1+x} dx = -\ln(1+x),$

故 $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n} x^n = -\frac{x}{1+x} - \ln(1+x), -1 < x < 1.$

九、解: 曲面 $\Sigma: z = \sqrt{2Rx - x^2 - y^2}$ 在 xy 平面投影区域为 $x^2 + y^2 \leq 2rx$, 且

$$z_x = -\frac{x-R}{z}, \quad z_y = -\frac{y}{z}$$

$$\text{则 } I = \iint_{\Sigma} \left[(y-z) \frac{x-R}{z} + (z-x) \frac{y}{z} + (x-y) \right] dx dy$$

$$= \iint_D \left[R - \frac{Ry}{z} \right] dx dy = \iint_D R dx dy = \pi R r^2.$$