Linear Equations Intro

Class 1 – Samyok Nepal – Week 1
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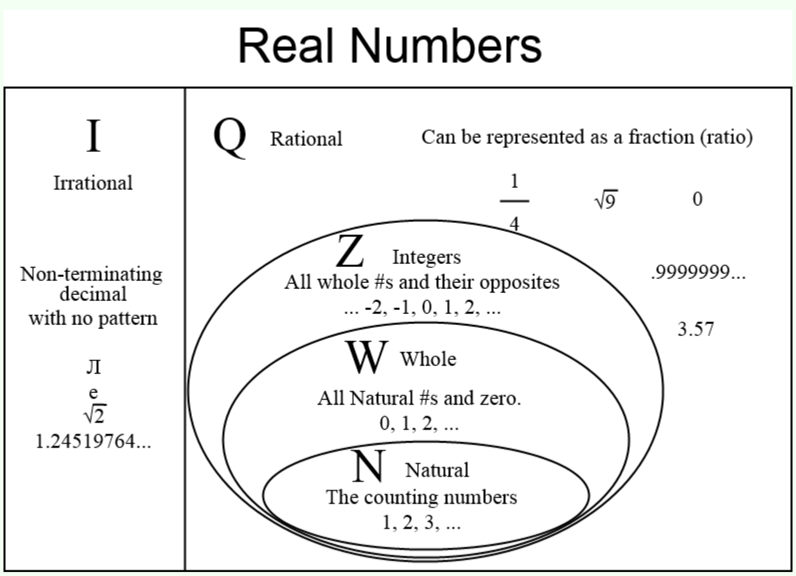
1 Assumptions

Today's class is going to be on algebra, but we have to start with the types of numbers. The numbers are going to be divided into the following categories:

Theorem 1.1: Number Sets



The class also assumes you have a good understanding of math. :)

2 Variables

What are variables? Variables are usually letters (shapes are usually represented for operations).

We are going to start with some basic word problems:

Problem 2.1:

Samantha has 31 more blue frogs than three times the number of purple frogs Tommy has. If Samantha has 49 frogs, how many frogs does Tommy have?

This is a pretty easy problem to do in your head, but lets do it the long way anyway. We have

$$S = 31 + 3T.$$

We solve for T with $S = 49$.

$$\begin{aligned} S &= 31 + 3T \\ 49 &= 31 + 3T \\ 18 &= 3T \\ 6 &= T \end{aligned}$$

You probably won't get problems this simple, so lets move on. I will leave problems **1.1-1.6 on page 2 for exercise**.

3 Tricks for Solving Specific Problems

Note that a problem about consecutive integers can be written as $x, x+1, x+2, ...$, where x is the smallest integer in the list. However, some problems can be made easier by setting x to be the *middle* number.

Let's solve the following problems both ways:

Problem 3.1

The sum of 5 consecutive integers is 120. What is the largest of these integers?

Problem 3.2

The sum of 10 consecutive integers is 25. What is the smallest of these integers?

Problem 3.3

The product of 3 consecutive odd integers is 2145. What is the smallest of these integers?

Solution 3.1: Since we have five numbers, we can set the middle number to x. We now have the list $x-2, x-1, x, x+1, x+2$ as our five consecutive numbers. Adding these up, we have

$5x = 120$. Obviously, $x = 24$. However, remember we set our *middle* number to x, so subtracting two, we have $\boxed{22}$ as our smallest number. ■

The trick to do this quickly was to notice that 5 is odd, so the middle number is an integer. We did the long way in class.

Solution 3.2: Starting off normally, except setting the fifth number to x to get the most cancellation, our 10 numbers are

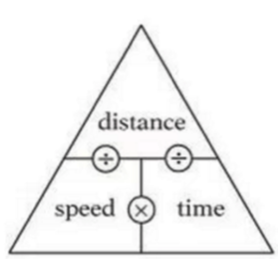
$$x-4,\ x-3,\ x-2,\ x-1,\ x,\ x+1,\ x+2,\ x+3,\ x+4,\ x+5.$$

Summing these up, we have $10x + 5 = 25$, or $x = 2$. Remembering that x is the fifth number, we can look at the list of our numbers to see that our smallest number is $\boxed{-2}$[1]. ■

Solution 3.3: Now, we have a little weird problem. Nevertheless, we go on as usual. x is again the middle number: $x-2, x, x+2$. If we define x to be odd, then all the numbers are odd. However, finding the product of these numbers is not really going to help us. We try to find the prime factorization of 2145. We notice that $\frac{2145}{3} = 715$. We also notice that $\frac{715}{5} = 143$. Finally, we know that $143 = 11 \cdot 13$. Just by prime factorizing, we see that $2145 = 11 \cdot 13 \cdot 15$, so our answer is $\boxed{11}$. ■

This is one of the few problems where not using a variable is faster than using algebra.

4 Distance, Rate, and Time



The formula $d = rt$ is quite well known. I usually remember d=rt and all the other stuff by this triangle above. Just remember it. Often it is easier to remember just $d = rt$ and solve from there.

Theorem 4.1: Average Speed

Average speed can be represented as

$$r = \frac{\text{Total Distance}}{\text{Total Time}}.$$

[1] Notice that nowhere in the problem did it say that they were positive integers.

Problem 4.1: ***Source: CMMS***

Marlon jogs 2 miles to the park in 25 minutes, turns around, and takes another 55 minutes to walk the same path back to his house. What is his average speed for the round trip? (page 4)

Problem 4.2: ***Source: CMMS***

On a bike ride, Calvin starts at home and goes up a long hill for thirty minutes at just 6 mph. At the top, he turns around and rides home along the same path at a speed of 18 mph. What is his average speed for the round trip? (page 4)

Solution 4.1: (Straight from CMMS) The round trip is four miles and takes 80 minutes or 4/3 hours:

$$4 = \frac{4}{3}r \text{ gives us } r = 3\text{mph}.$$

■

Solution 4.2: (Straight from CMMS) The solution is in your book so we will not go over it. ■

Problem 4.3: ***Source: CMMS***

The water from a swimming pool evaporates at a rate of 6 gallons per hour in the shade and 19 gallons per hour in the sun. For several weeks in August, the amount of water lost to evaporation in the shade was equal to the amount lost in the sun. What was the average rate of evaporation from the pool?

Solution 4.3: (Straight from CMMS; sketch) Find that the LCM of both numbers is 114, total amount is 228. Time for shade is 114/6=19 and time for sun is 114/19=6. Therefore the average is 228/(6+19)=9.12. ■

Problem 4.4: ***Source: CMMS***

Calvin bikes again up a hil and then back down. He averages a miles per hour on the way up, and on the way down he averages b miles per hour. What is his average speed for the entire trip?
Hint: Use ab as the length of the hill.

Solution 4.4: If we use ab as the length of the hill, it will take Calvin $ab/a = b$ hours to ride up the hill and $ab/b = a$ hours to ride back down. The total distance is $2ab$ and the total time is $a + b$. This gives us the average speed of $\frac{2ab}{a+b}$.■

Theorem 4.2: Harmonic Mean

The Harmonic Mean of a and b is also the average speed of a and b. It is represented like this:

$$\text{Average Speed of } a \text{ and } b = \frac{2ab}{a+b}.$$

Warning

Remember when you use the Harmonic Mean both distances travelled must be the same. For example, you could use the Harmonic Mean when you are going up and down a hill, but not when you go up halfway once and three times the next run.

5 Moles and Holes

Have you ever heard of a mole digging a hole? It's basically a different form of $d = rt$. We begin with the well known brain teaser:

Problem 5.1: ***Source: CMMS***

It takes 7 minutes for 7 moles to dig 7 holes. How long will it take for 8 moles to dig 8 holes?

Solution 5.1: If you have 1 mole digging 1 hole, it will take 7 minutes for the mole to dig that hole (think about this for a bit). Now imagine 8 moles lining up to dig 8 holes. They all start at the same time, so it will take $\boxed{\text{7 min.}}$ for 8 moles to dig 8 holes. ■

There are some more difficult "moles digging holes" problems, so lets formalize our attack:

Problem 5.2: ***Source: CMMS***

5 moles can dig 4 holes in 3 minutes. How many minutes will it take for 9 moles to dig 6 holes?

Solution 5.2: We introduce a new formula:

Theorem 5.1: Work With Moles

The work w done by m moles in time t at a rate r is represented by

$$w = rt(m).$$

Does this look familiar?

We plug in values into Theorem 5.1 and solve for r:

$$4 = r \cdot 3 \cdot 5 \implies r = \frac{4}{15} \text{ (holes/min./mole)}$$

We now want to solve for the time t with 9 moles digging 6 holes at a rate of $\frac{4}{15}$ minutes per hole. Since we know all the other variables, we can do this:

$$6 = \frac{4}{15} \cdot t \cdot 9 \implies t = \frac{15}{36} \cdot 6 = \frac{15}{6} = \boxed{2.5 \text{ minutes}}$$

■

Last Moles Digging Holes Problem:

Problem 5.3

Alice and Bilbo are mowing lawns together. Alice can mow a 5 acre lawn in 2 hours, but Bilbo is a little faster and can mow a 3 acre lawn in 1.5 hours. If they work together, how many minutes will it take them to mow:

1. 1 acre?
2. 15 acres?
3. 5 acres?

Alice and Bilbo start working at 10. First, Alice mows for 2 hours. Then, Bilbo works with Alice until all n acres are mowed. How many hours has Alice been working if n is:

4. n =5?
5. n =20?
6. n =45?

Solution 5.3: We let Alice's speed be a and Bilbo's speed b. Alice can mow $\frac{5}{2}$ acres in 1 hour. Bilbo can mow 6 acres in 3 hours or 2 acres every hour. Adding these up, combined, Alice and Bilbo can mow $\frac{15}{2} = 7.5$ acres an hour.

1. This one is easy. Because they can do $\frac{15}{2}$ acres an hour, they can do 1 acre in

$$\frac{1}{\frac{15}{2}} = \frac{2}{15}$$

hours or $\frac{2}{15} \cdot 60 = \boxed{8 \text{ minutes}}$. ■

2. Their combined speed is $\frac{15}{2}$ acres an hour. Therefore, it will take them 2 hours or $\boxed{\text{120 minutes}}$ to mow 15 acres. ■

Theorem 5.2: Speed Trick

If someone is mowing $\dfrac{k}{j}$ of the work, it will take them $\dfrac{k}{j}$ the time.

If someone is mowing at $\dfrac{k}{j}$ the speed, it will take them $\dfrac{j}{k}$ the time. (proved in class)

3. Because it took them 2 hours to mow 15 acres, we "multiply" both "sides" by $\frac{1}{3}$ to get $\frac{2}{3}$ of an hour or $\boxed{\text{40 minutes}}$. ■

Now for the second part, we note that when Bilbo starts, 5 acres have already been completed. We only need to figure out how long it takes for them together to mow $n - 5$ acres, and then add 2 hours at the end. We use the $d = rt$ formula:

$$n - 5 = \frac{15}{2} \cdot (t - 2)$$

Now we only need to solve for t (we know n:

1. $n = 5$:

$$\begin{aligned}
n - 5 &= \frac{15}{2} \cdot (t - 2) \\
5 - 5 &= \frac{15}{2} \cdot (t - 2) \\
0 &= \frac{15}{2} \cdot (t - 2) \\
0 &= t - 2 \\
\boxed{t = 2}
\end{aligned}$$

and we are done. ■

2. Similarly: $(n = 20)$

$$\begin{aligned} n - 5 &= \frac{15}{2} \cdot (t-2) \\ 20 - 5 &= \frac{15}{2} \cdot (t-2) \\ 15 &= \frac{15}{2} \cdot (t-2) \\ 2 &= t - 2 \\ \boxed{t = 4} \end{aligned}$$

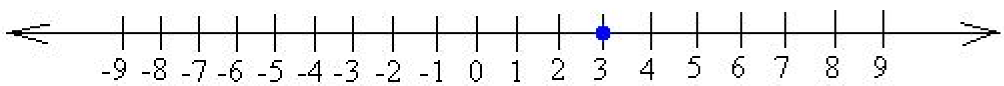
   and we are done. ■

3. $(n = 45)$ Now we use Theorem 5.2. We know that it takes 2 hours for them together to do 15 acres. Multiplying "both sides" by $\frac{40}{15} = \frac{8}{3}$, it takes them $\frac{16}{3}$ hours to do 40 acres. Add another 2 hours on top of that and you have our final answer of $\boxed{\frac{22}{3} \text{ hours}}$. ■

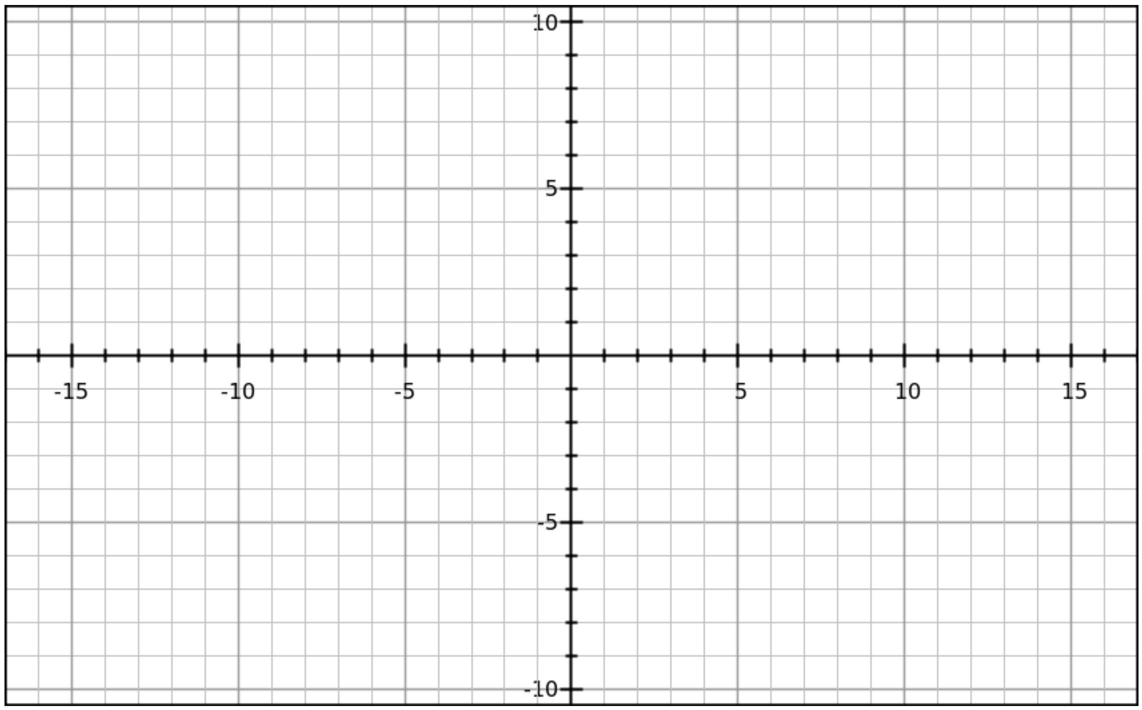
6 Lines, Linear Equations, and Graphs

We will go a little further than in CMMS.

Imagine the number line. We can do a lot of things on this number line. For example, the figure below shows the number line for $x = 3$.



Well, after a while, this gets boring. What happens if we stack one number line on top of another? Preferably, we let the 0s intersect. Lets call the place where the 0 is the origin, because everything starts there:



We call the "x axis" the one that is horizontal, and the "y axis" the one that is vertical[2]. Now, a graph like this one is used when we have an equation (or equations) that have two variables[3].

Head over to `desmos.com/calculator`.

Problem 6.1: Graph Points

Graph points based on the following equations:

1. $y = 2x + 5$
2. $y = -\frac{1}{2}x + 5$
3. $y = 4$
4. $x = 2$

[2]Eventually you will hit more letters, such as z, the third dimension. We probably will not touch this in this class.

[3]You can also use it with three, but any additional letters will not use graphs. It is also good to know that only the most complex graphs show up on math contests.

Theorem 6.1: Slope Intercept Form

The simplest equations with two unknowns are in the form

$$y = mx + b.$$

This is called **slope intercept form**. m is called the *slope* of the line, and b is the *y-intercept*, or the place the line intersects the y axis.

The graph of a linear equation is always a straight line.

Theorem 6.2: Slope

The slope of a line can be represented in many ways. It is denoted usually by the letter m and is represented as

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1},$$

with coordinates (x_1, y_1) and (x_2, y_2).

Lets apply slope to one problem:

Problem 6.2: ***Source: CMMS***

Find the slope of the line passing through the coordinates $(7, -2)$ and $(-4, 1)$.

Now intercepts:

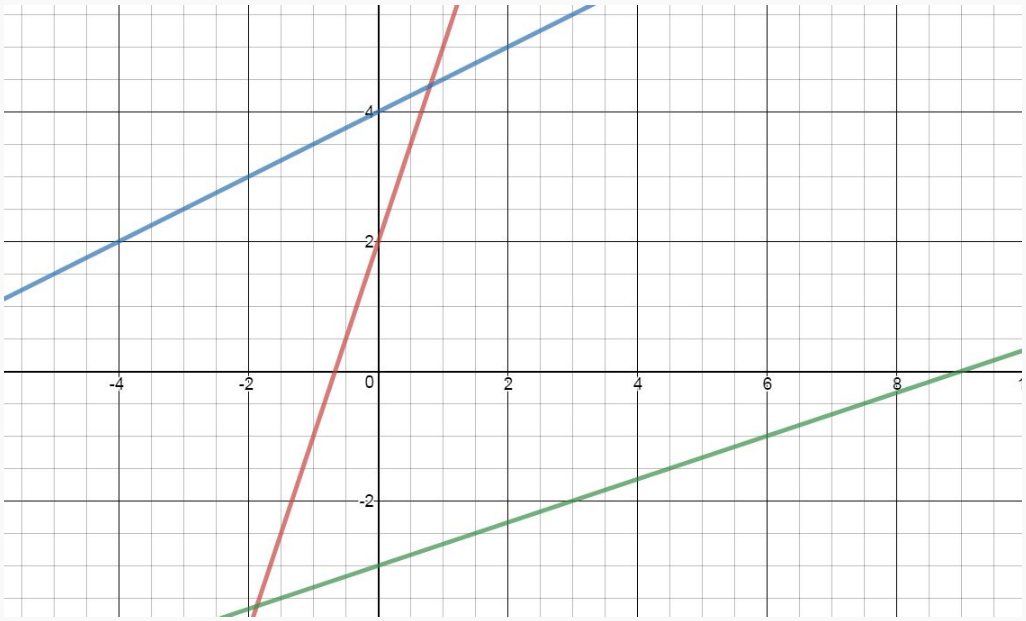
Theorem 6.3: Intercepts

Intercepts are where the line crosses the two axes. The y intercept is where $x = 0$, and vice-versa. The y intercept is the ordered pair when $x = 0$. It is represented by b in slope intercept form.

Lets go from lines to equations:

Problem 6.3

Find the equations of these lines: (in slope intercept form)



Solution 6.3:

$$\text{Green: } y = \frac{1}{3}x - 3 \text{ — Red: } y = 3x + 2 \text{ — Blue: } y = \frac{1}{2}x + 4$$

7 Other Forms of Lines

Theorem 7.1: Point Slope Form

The from

$$y - y_1 = m(x - x_1)$$

is known as **Point Slope Form**.

Let's Practice:

Problem 7.1: Point Slope

Find the equation of the line that passes through (10,3) and (-5,9) in point slope form.

Solution 7.1: $m = \frac{-2}{5} \implies y - 3 = -\frac{2}{5}(x - 10)$.

Theorem 7.2: Standard Form

Standard Form of a linear equation is in the form:

$$Ax + By = C$$

where A is positive, and A, B, and C are integers.

- $-\frac{A}{B}$ =Slope
- $\frac{C}{B}$ =y-intercept

Lastly, Parallel and Perpendicular lines:

Theorem 7.3: Parallel and Perpendicular Slopes

Parallel lines always have the same slope.
Congruent lines have the same equation.
Perpendicular lines have opposite reciprocal slopes.

One "troll" problem we did in class: (This problem was solved under 10 seconds. You try!)

Exercise 7.1: ***Source: 2017 MATHCOUNTS National Countdown Round***

Right Triangle $\triangle$ PQR has vertices at P(1294,104) and R(-1394, 1123). If $\angle$ Q = 90°, what is the product of the slopes of the legs?

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Hint: the legs of a right triangle are the sides that form a right angle. (not the hypotenuse)
(Samyok also changed the numbers *slightly*)

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Original Problem: `https://youtu.be/vFTeN17Z4rc?t=1759`.

8 Harder Practice Problems

There are a few problems in CMMS, but here are some for those looking for a challenge:

Exercise 8.1

Joseph the ant was practicing for the triathlon. He could swim at a rate of 5 feet per second, run at 7 feet per second, and bike at a rate of 13 miles per hour. If all three lengths were the same, what was his average speed in yards per seconds?

Exercise 8.2: The Four Corners Journey

There exists an ant walking on a square. He follows these rules:

- He looks up at the sky if and only if he is on a vertex of a square. He looks up once and then starts walking to the next vertex (clockwise) based on this:
  - If the ant sees the sun, he walks at a rate of 12 feet per minute.
  - If he sees the moon, he walks at the leisurely rate of 7 feet per minute.

If he sees the moon, the sun, the sun, and then the sun again, what was his average speed throughout the journey of the four corners?

Exercise 8.3

Jack ran every morning for a week. On the first day, he ran at a rate of 1 mile per hour. On each day after the first, he ran 1 mile per hour greater than the day before. What was his average speed over all 7 days?

Exercise 8.4: *Source: MATHCOUNTS*

Jack and Jill drove in separate cars to their favorite hill, leaving from the same place at the same time. Jill drove 20% faster than Jack and arrived half an hour earlier. How many hours did Jack drive?

Exercise 8.5: *Source: 2017 AMC 10B*

Samia set off on her bicycle to visit her friend, traveling at an average speed of 13 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she ran the rest of the way at 8 kilometers per hour. In all it took her 42 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia run?