Inequalities

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1 Basics and Definitions

When two expressions are not necessarily equal, we can compare them using inequality signs. The two types of inequality signs are strict ($<$ and $>$) and nonstrict ($\leq$ and $\geq$). If we multiply or divide by a negative number, or take the reciprocal of both sides of an inequality, we "reverse" the inequality sign.

Definition 1.1: Equality Condition

The **equality condition** of a nonstrict inequality is when equality holds (the two sides are equal). The equality condition of an inequality can be very useful when solving problems.

2 The Trivial Inequality

Theorem 2.1: Trivial Inequality

The Trivial Inequality states that if x is a real number, then $x^2 \geq 0$. Equality holds if and only if x=0.

This inequality seems very trivial and obvious, and it is. However, it is also useful in proving other inequalities (like the AM-GM Inequality) and finding maximum and minimum values.

Problem 2.1

Prove that if x and y are positive, then $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$. *(Source: AoPS Intermediate Algebra)*

3 AM-GM Inequality

The Arithmetic Mean Geometric Mean Inequality states that the arithmetic mean of any n non-negative numbers is greater than or equal to the geometric mean of the numbers.

Theorem 3.1: AM-GM Inequality

As an inequality, if $a_1, a_2, ..., a_n$ are nonnegative, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Equality holds if and only if all a_i are equal.

Warning:

Remember, AM-GM will only work for **nonnegative** numbers.

Problem 3.1: AM-GM 2 Variables

Prove that $\frac{a+b}{2} \geq \sqrt{ab}$ for all a, $b \geq 0$.

Problem 3.2

Show that the sum of a number and its reciprocal is always greater than or equal to 2.

Problem 3.3

Show that $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 3$ *(Source: AoPS)*

Problem 3.4

Show that if x, y, and z are nonnegative, then $xy+yz+zx \geq x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}$ *(Source: AoPS Intermediate Algebra)*

Problem 3.5

Show that $(x+y)(y+z)(z+x) \geq 8xyz$ for all nonnegative numbers x, y, and z. *(Source: AoPS Intermediate Algebra)*

Problem 3.6: AM-GM Inequality Proof

AM-GM with 4 Variables: Prove that for all nonnegative a, b, c, and d, $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.
Prove AM-GM for any 2^n nonnegative numbers ($n > 1$).
AM-GM with 3 Variables: Prove that for all nonnegative a, b, and c, $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.
Prove AM-GM for any n nonnegative numbers ($n > 1$). What is the equality condition?

4 Cauchy-Schwarz Inequality

Theorem 4.1: Cauchy-Schwarz Inequality

Let a_1, a_2, ..., a_n, b_1, b_2, ..., b_n be real numbers. Then,

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

Equality occurs if and only if either $a_i = 0$ for all i, or there exists a constant t such that $b_i = ta_i$ for all i.

Problem 4.1

Show that $a^2 + b^2 + c^2 \geq ab + bc + ac$ for all real a, b, and c.

(a) Use Cauchy-Schwarz

(b) Use AM-GM

Problem 4.2

Show that

$$a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3}.$$

(Source: AoPS Intermediate Algebra)

Problem 4.3

What is the maximum value of

$$\frac{(2x + 3y + 4z)^2}{x^2 + y^2 + z^2}?$$

Problem 4.4

If a, b, c, x, y, and z are nonzero real numbers and $a^2 + b^2 + c^2 = 25$, $x^2 + y^2 + z^2 = 36$, and $ax + by + cz = 30$, compute

$$\frac{a + b + c}{x + y + z}.$$

(Source: ARML)

5 QM-AM-GM-HM

AM-GM is part of the **Quadratic Mean - Arithmetic Mean - Geometric Mean - Harmonic Mean** chain. AM-GM is the most often used part of the inequality chain.

Definition 5.1: Quadratic Mean

The **quadratic mean** of the *real* numbers a_1, a_2, a_3, ..., a_n is

$$\sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}{n}}$$

Definition 5.2: Harmonic Mean

The **harmonic mean** of the *nonzero* numbers a_1, a_2, a_3, ..., a_n is

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}}$$

The quadratic mean and harmonic mean are sometimes abbreviated **QM** and **HM**.

Theorem 5.1: QM-AM-GM-HM Inequality

Let a_1, a_2, ..., $a_n > 0$. Then, we have

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

$$QM \geq AM \geq GM \geq HM$$

For each inequality sign, equality holds if and only if all the a_i are equal.

Problem 5.1

Let a and b be positive real numbers. Find the minimum value of $\frac{a}{b} + \frac{b}{a}$ using
(a) AM-GM
(b) GM-HM

Problem 5.2: Nesbitt's Inequality

Prove that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$

6 Tips

- Before any manipulation, keep an eye on what your target is. Make sure you are moving in the right direction.
- The Trivial Inequality, AM-GM, and Cauchy are good tools to use, but they won't solve every problem completely. Remember other problem solving techniques like experimenting, relating the problem to similar ones, wishful thinking, and so on. Use your whole toolbox!
- AM-GM and Cauchy-Schwarz are used often, so they should be one of the first things to consider when tackling an inequality.
- Especially consider using AM-GM when you see nonnegative expressions with a constant sum or product.
- If an inequality involves a sum of squares, a product of two sums with the same number of terms, or annoying denominators, try Cauchy.
- Sum of squares can also mean QM. When you see reciprocals, try HM.
- When you see the words "minimum" or "maximum", try to use an inequality, but make sure your inequality sign is pointing the right way.

7 Problem Set

The problems are roughly ordered by difficulty, with easier problems first. Do not be discouraged if you can not solve all or most of the problems; overall, the problem set is supposed to be challenging.

1. [Mandelbrot] Determine the minimum value of the sum $\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a}$, where a, b, and c are positive real numbers.
2. Serena has 48 feet of fencing.

   (a) What is the maximum area she can enclose?

   (b) Can you generalize? If Serena has n feet of fencing, what is the maximum area she can enclose?
3. Samyok also has 48 feet of fencing. He plans to use one side of his house as a side of his enclosure, so he will need to form three sides with his fencing.

   (a) What is the maximum area he can enclose?

(b) Can you generalize? If Samyok has n feet of fencing and only needs to form three sides, what is the maximum area he can enclose?

4. Prove that $(a+b+c+d+e)(a^4+b^4+c^4+d^4+e^4) \geq 25abcde$

5. [AoPS] Let a,b,c be nonnegative real numbers. Prove that

$$(ax^2+bx+c)(cx^2+bx+a) \geq (a+b+c)^2x^2$$

6. [AoPS Intermediate Algebra] Let x, y, and z be positive real numbers. Show that

$$(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9.$$

   (a) Use Cauchy-Schwarz

   (b) Use AM-HM

7. [AoPS Intermediate Algebra] Find the maximum of x^2y if $x+y=6$ and $x \geq 0$.

8. Let a,b,c be nonnegative real numbers. If $a+2b+3c=12$, what is the maximum value of ab^2c^3?

9. [AM-GM Inequalities by tkhalid] Prove $(x+y)(x^2+y^2)(x^3+y^3) \geq 8x^3y^3$

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