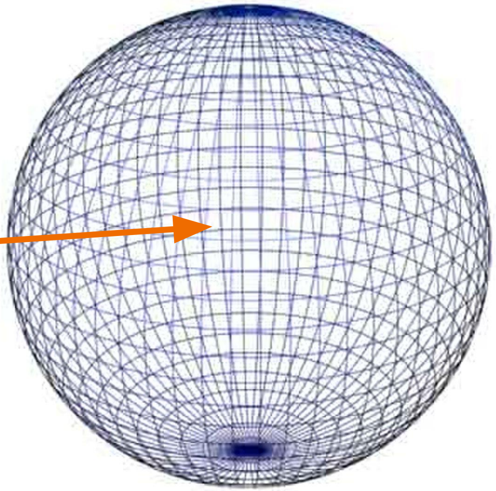
Neutron Stars

Lecture 2: stationary configurations

Ignacio F. Ranea-Sandoval (Argentina)

An isolated NS: a theoretical model

Stelar configuration: interior solution to Einstein field equations. Matter is assumed to be described by a perfect fluid



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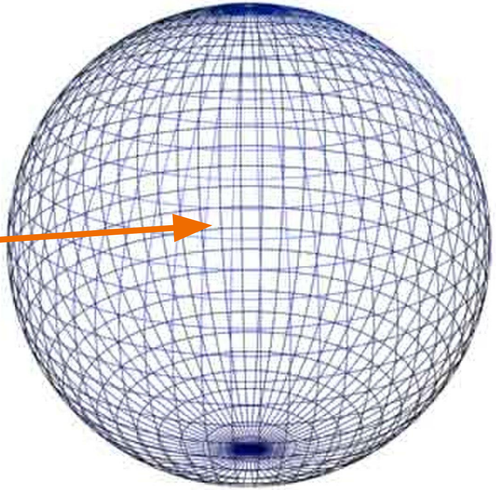
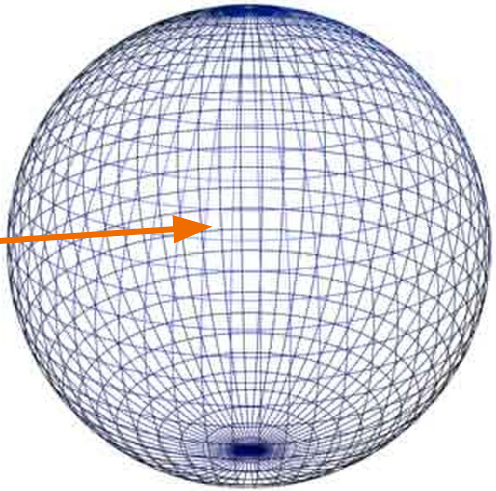


Exterior: vacuum solution, asymptotically flat

An isolated NS: a theoretical model



Stelar configuration: interior solution to Einstein field equations. Matter is assumed to be described by a perfect fluid

Match at the surface!!

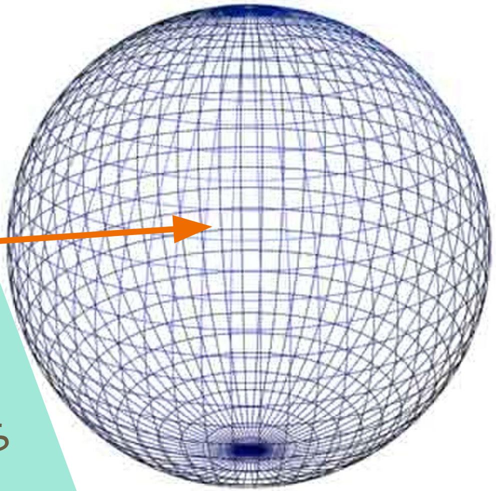
Exterior: vacuum solution, asymptotically flat

An isolated NS: a theoretical model

Stelar configuration: interior solution to Einstein field equations. Matter is assumed to be describe

Match at th

Geometry and matter are related via de Einstein field equations that relate matter content described by the stress-energy tensor and the Einstein tensor that describes the geometry



Exterior: vacuum solution, asymptotically flat

An isolated NS: a theoretical model

The fluid part:

Perfect fluid. Do not include viscosity. The stress-energy tensor is

$$T_{ik} = (p + \epsilon)u_i u_k - p g_{ik}$$

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pressure energy density

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4-velocity

metric tensor

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An isolated NS: a theoretical model

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Perfect fluid. Do not include viscosity. The stress-energy tensor is

4-velocity   metric tensor

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pressure   energy density

We assume a static configuration, so only the temporal component of the 4-velocity is non-zero and can be obtained from the 4-velocity normalization to -1.

An isolated NS: a theoretical model

Geometry of a spherically symmetric system:

We start simple!

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

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We use Schwarzschild coordinates.
The metric functions depend only of r and t.

Diagonal metric tensor

$$g_{00} = e^{\nu} \qquad g_{11} = -e^{\lambda} \qquad g_{22} = -r^2 \qquad g_{33} = -r^2\sin^2\theta$$

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Reduces to Minkowsky flat solution when the metric functions vanish.

An isolated NS: a theoretical model

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The Einstein field equations for this system reduces to three differential equations

$$\frac{d}{dr}[r(1 - e^{-\lambda})] = \frac{8\pi G}{c^4} r^2 \epsilon$$

$$\frac{e^{-\lambda}}{r}\frac{d\nu}{dr} = \frac{1}{r^2}\left(1 - e^{-\lambda}\right) + \frac{8\pi G}{c^4} p$$

$$\frac{dp}{dr} = -\tfrac{1}{2}(p + \epsilon)\frac{d\nu}{dr}$$

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$$\xrightarrow{\frac{2Gm(r)}{c^2} \equiv r\left(1-e^{-\lambda(r)}\right)}$$

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \epsilon$$

mass energy inside radius r

An isolated NS: a theoretical model

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mass energy inside radius r

$$\frac{e^{-\lambda}}{r}\frac{d\nu}{dr} = \frac{1}{r^2}\left(1-e^{-\lambda}\right) + \frac{8\pi G}{c^4} p \quad \longrightarrow \quad \frac{d\nu}{dr} = \frac{2G}{rc^2}\left[m + \frac{4\pi G}{c^2} p r^3\right]\left(r - \frac{2mG}{c^2}\right)^{-1}$$

$$\frac{dp}{dr} = -\frac{1}{2}(p+\epsilon)\frac{d\nu}{dr}$$

An isolated NS: a theoretical model

The Tolman-Oppenheimer-Volkoff (TOV) equations

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \epsilon$$

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Potential exercise!
Write the equation for dp/dr so that it resembles the Newtonian version + relativistic corrections

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Write the equation for dp/dr so that it resembles the Newtonian version + relativistic corrections

$$\frac{dP}{dr} = -\frac{G\,m(r)\,\rho(r)}{r^2}$$

An isolated NS: a theoretical model

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What are we missing to be able to solve this equations?

An isolated NS: a theoretical model

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What are we missing to be able to solve this equations?

We need **boundary conditions** and a relationship between pressure and energy density (called equation of state **EOS**) to be able to solve them!

An isolated NS: a theoretical model

The Tolman-Oppenheimer-Volkoff (TOV) equations

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \epsilon \qquad \text{with} \qquad m(0) = 0$$

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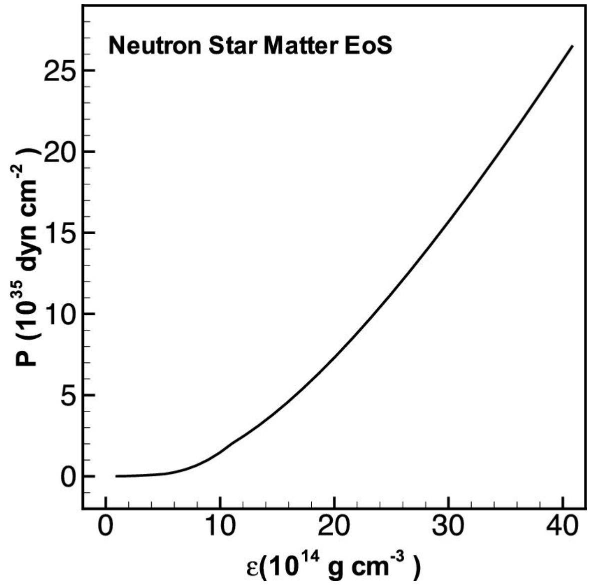
Geometry



Microphysics

An isolated NS: a theoretical model

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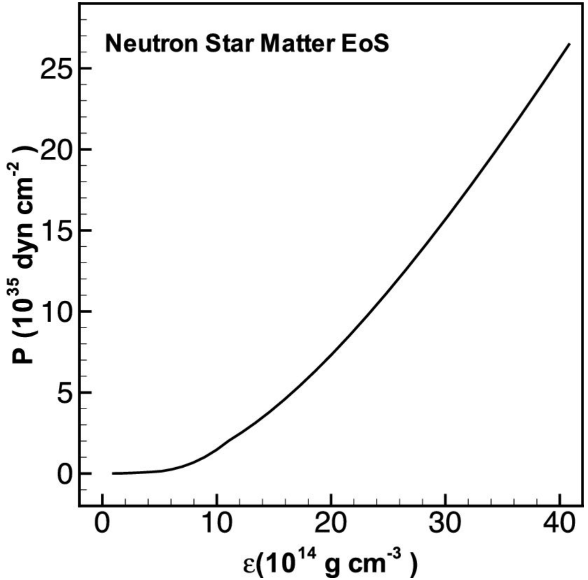
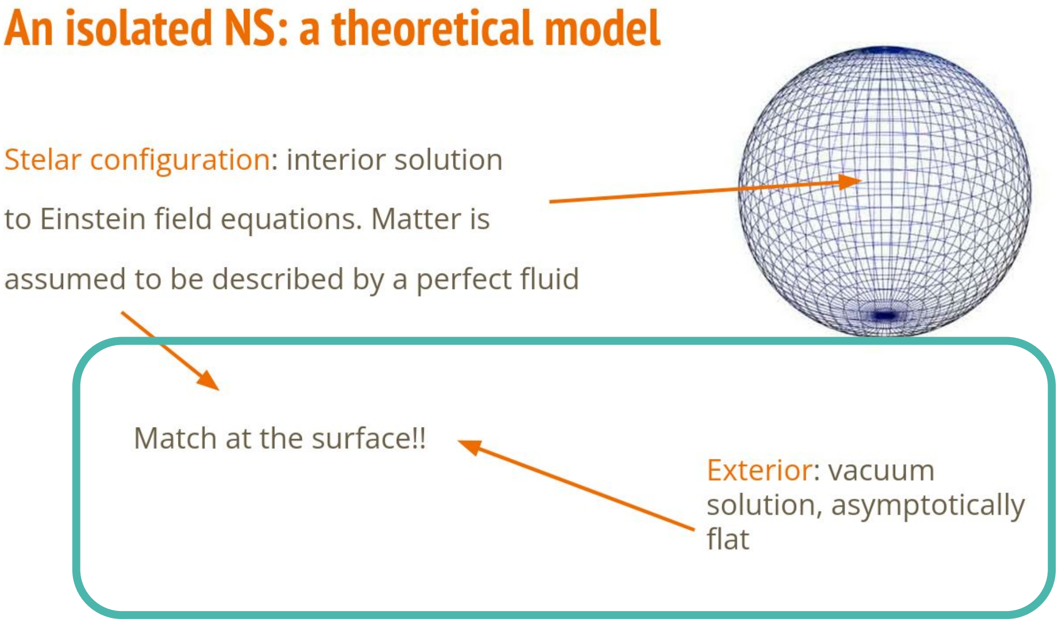
Geome



Microphysics

Are we missing something?

An isolated NS: a theoretical model



The exterior solution

What happens with the p(r), ε(r) and m(r) when we are outside the star and r>R?

An isolated NS: a theoretical model

The exterior solution

What happens with the $p(r)$, $\varepsilon(r)$ and $m(r)$ when we are outside the star and $r>R$?

$p(r) = \varepsilon(r) = 0$ as we are in vacuum

$m(r) = M$ total gravitational mass

An isolated NS: a theoretical model

The exterior solution

What happens with the p(r), ε(r) and m(r) when we are outside the star and r>R?

p(r) = ε(r) = 0 as we are in vacuum   m(r) = M total gravitational mass

Imposing flat spacetime for large values of coordinate r

It is possible to obtain λ(r) analytically

$$e^{\lambda} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \qquad \text{for} \qquad r \geq R$$

More or less the same for ν(r)

$$e^{\nu} = 1 - \frac{2GM}{rc^2} \qquad \text{for} \qquad r \geq R$$

An isolated NS: a theoretical model

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Imp

It is

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We obtain the Schwarzschild solution to the Einstein equations

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $r_S = \frac{2GM}{c^2}$ is the Schwarzschild radius.

$$e^{\nu} = 1 - \frac{2GM}{rc^2} \qquad \text{for} \qquad r \geq R$$

An isolated NS: a theoretical model

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More details
in the black hole lecture
in a few days

$$e^{\nu} = 1 - \frac{2GM}{rc^2} \qquad \text{for} \qquad r \geq R$$

An isolated NS: a theoretical model

Takeaway

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \epsilon \qquad \text{with} \qquad m(0) = 0$$

$$\frac{d\nu}{dr} = \frac{2Gm}{r^2c^2}\left[1 + \frac{4\pi G p r^3}{mc^2}\right]\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \qquad \text{with} \qquad \nu(R) = \ln\left(1 - \frac{2GM}{Rc^2}\right)$$

$$\frac{dp}{dr} = -\frac{1}{2}(p+\epsilon)\frac{d\nu}{dr} \qquad \text{with} \qquad p(R) = 0$$

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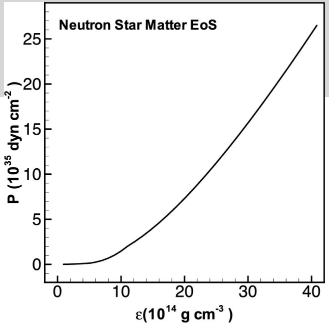
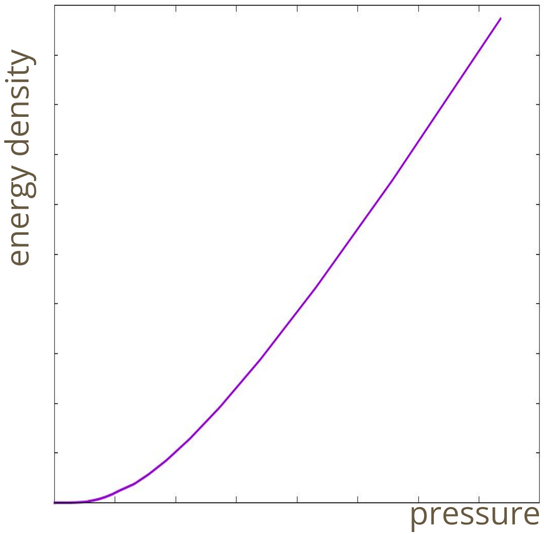
as we mention, an EOS is needed to close the system!



An isolated NS: a theoretical model

How do we solve the TOV equations?

a. Select your favourite EOS $\varepsilon(p)$



An isolated NS: a theoretical model

How do we solve the TOV equations?

a. Select your favourite EOS $\varepsilon(p)$
b. For a given central energy density, fix every other variable at the center
c. Start to numerically integrate the TOV equations (RK of 4th order is a good option, but others might work as well) from these initial conditions
d. Continue the integration until the boundary condition $p(r)=0$ is fulfilled
e. At this point integration is stopped and you get the mass M and radius R of the stellar configuration with a given central energy density

An isolated NS: a theoretical model

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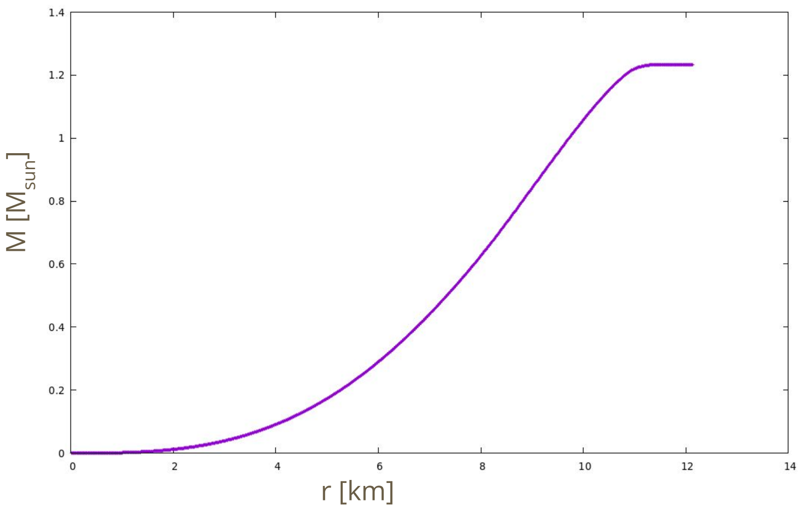

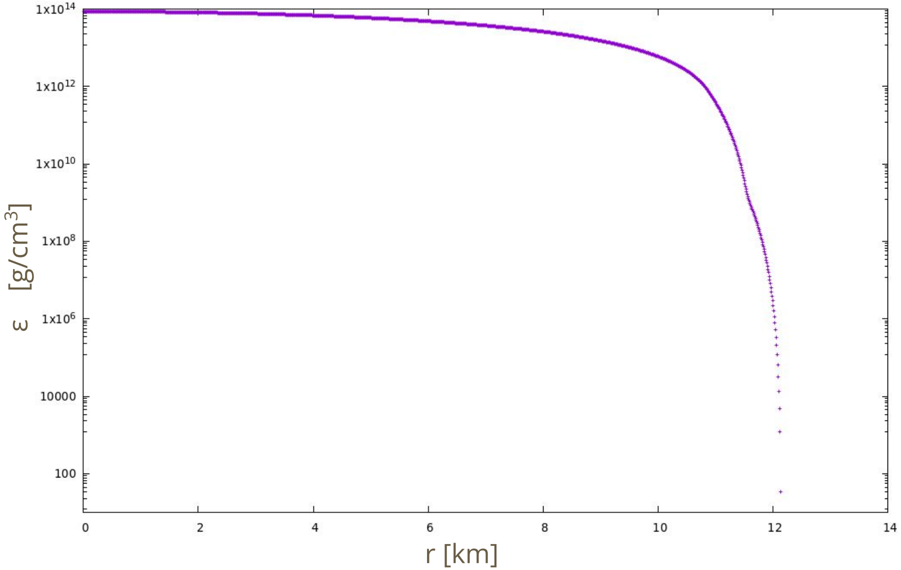

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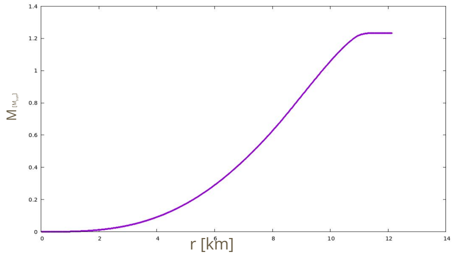

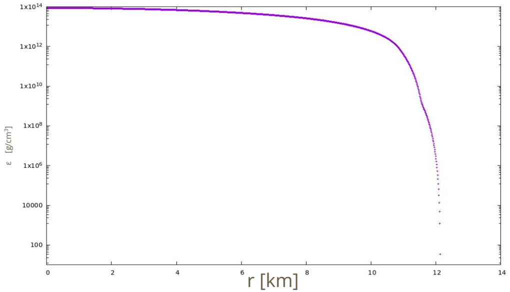

e. At this point integration is stopped and you get the mass M and radius R of the stellar configuration with a given central energy density

f. Increase the value of the central pressure to get (at the end) the so-called mass-radius relationship for a given EOS

An isolated NS: a theoretical model

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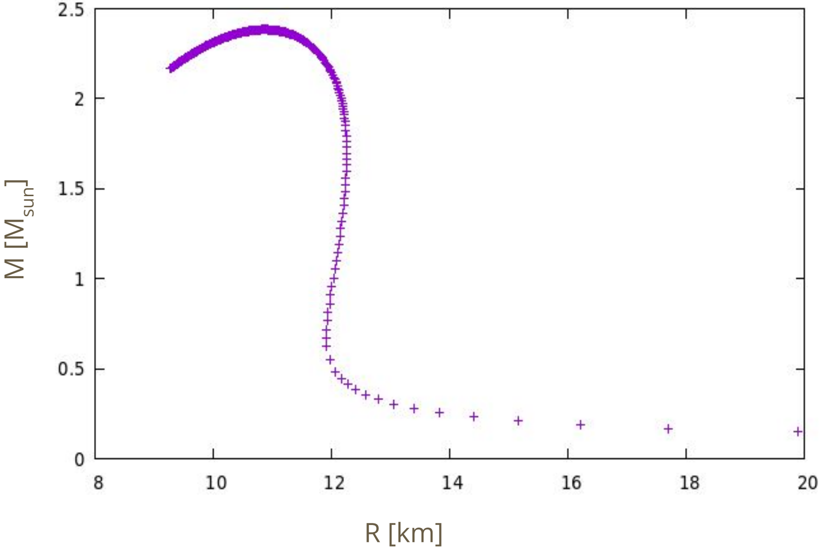

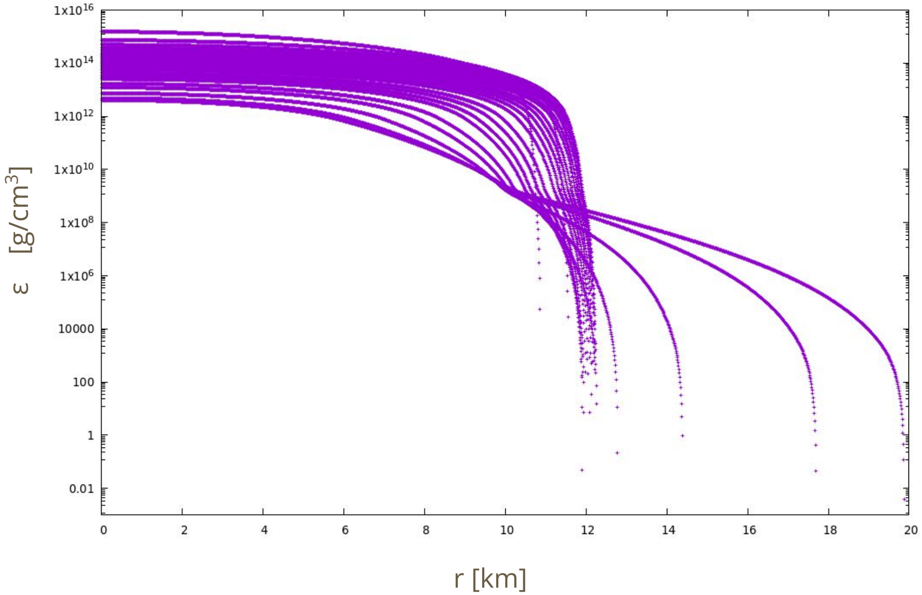

An isolated NS: a theoretical model

A potential (kind of hard) exercise:

Perform the series expansion at the center of the energy density, the pressure and the mass needed in point b) and that is needed for the numerical integration of the TOV equations. Perform these calculations up to second non-vanishing order and have in mind that in the pressure and energy density only even powers of r are involved while in the mass appear only odds.

An isolated NS: a theoretical model

Baryonic mass

The total mass of non-interacting baryons inside a stellar configuration is oftenly called baryonic (or rest) mass.

$$M_b \equiv A_b m_b = m_b \int_0^R n_b(r) 4\pi r^2 \left(1 - \frac{2Gm}{rc^2}\right)^{-1/2} dr$$

arbitrary constant usually taken to be the nucleon mass

baryon numerical density

An isolated NS: a theoretical model

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the nucleon mass

It is constant during the evolution of an isolated NS

baryon numerical density

An isolated NS: a theoretical model

After solving with a given EOS

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \epsilon \qquad \text{with} \qquad m(0) = 0$$

$$\frac{d\nu}{dr} = \frac{2Gm}{r^2c^2}\left[1 + \frac{4\pi G p r^3}{mc^2}\right]\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \qquad \text{with} \qquad \nu(R) = \ln\left(1 - \frac{2GM}{Rc^2}\right)$$

$$\frac{dp}{dr} = -\frac{1}{2}(p + \epsilon)\frac{d\nu}{dr} \qquad \text{with} \qquad p(R) = 0$$

for the complete range of central pressures we get the mass-radius relationship

An isolated NS: a theoretical model

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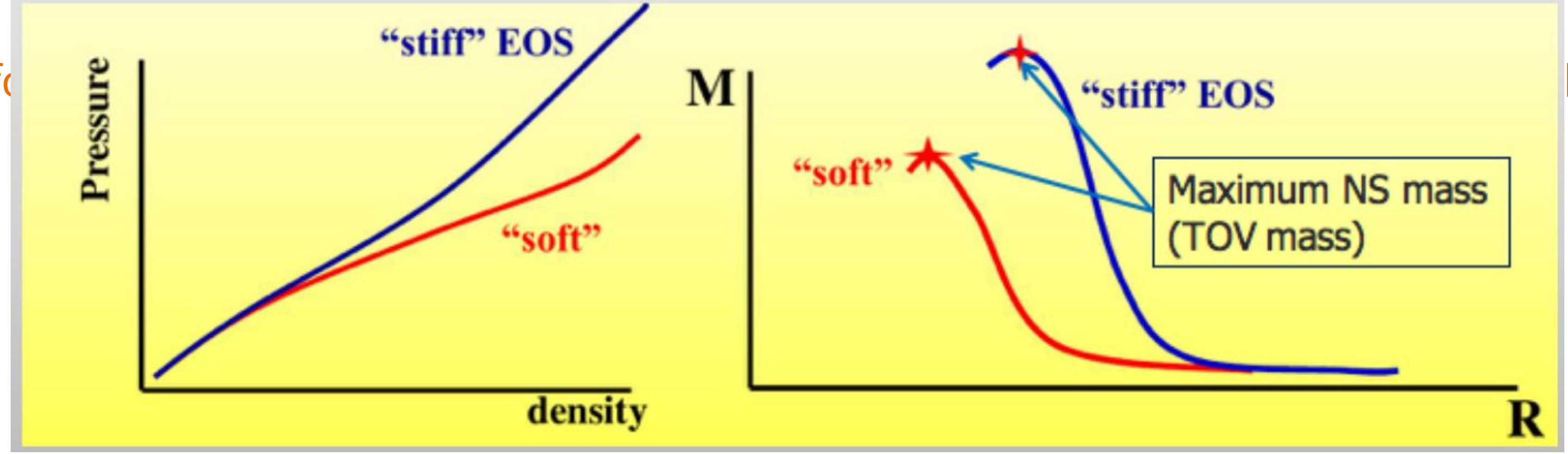
fo hip



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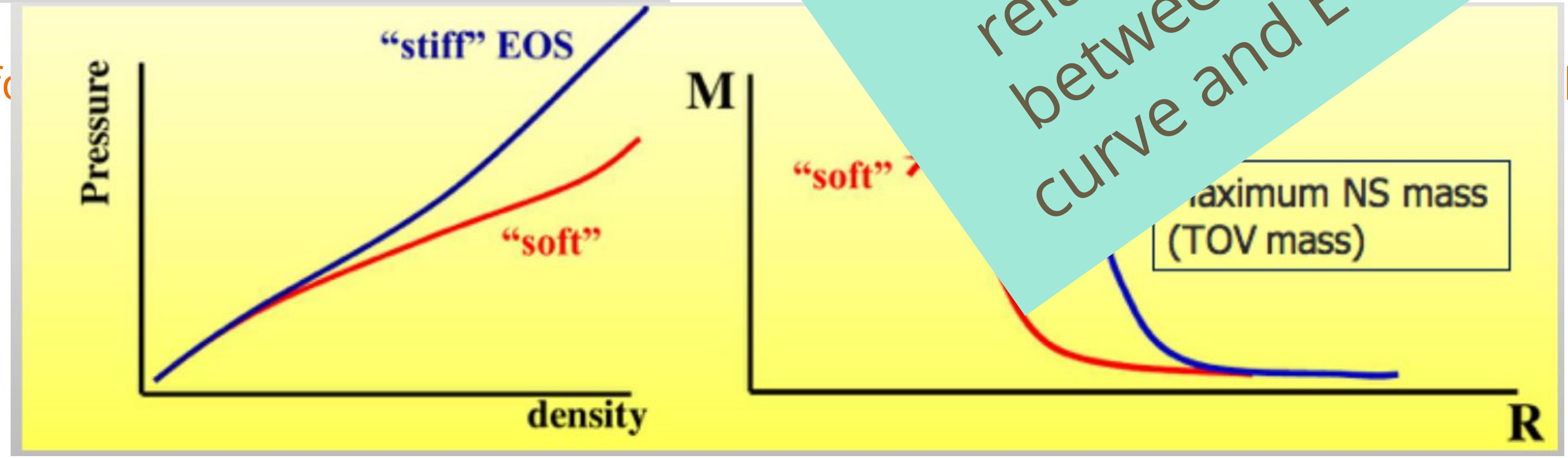
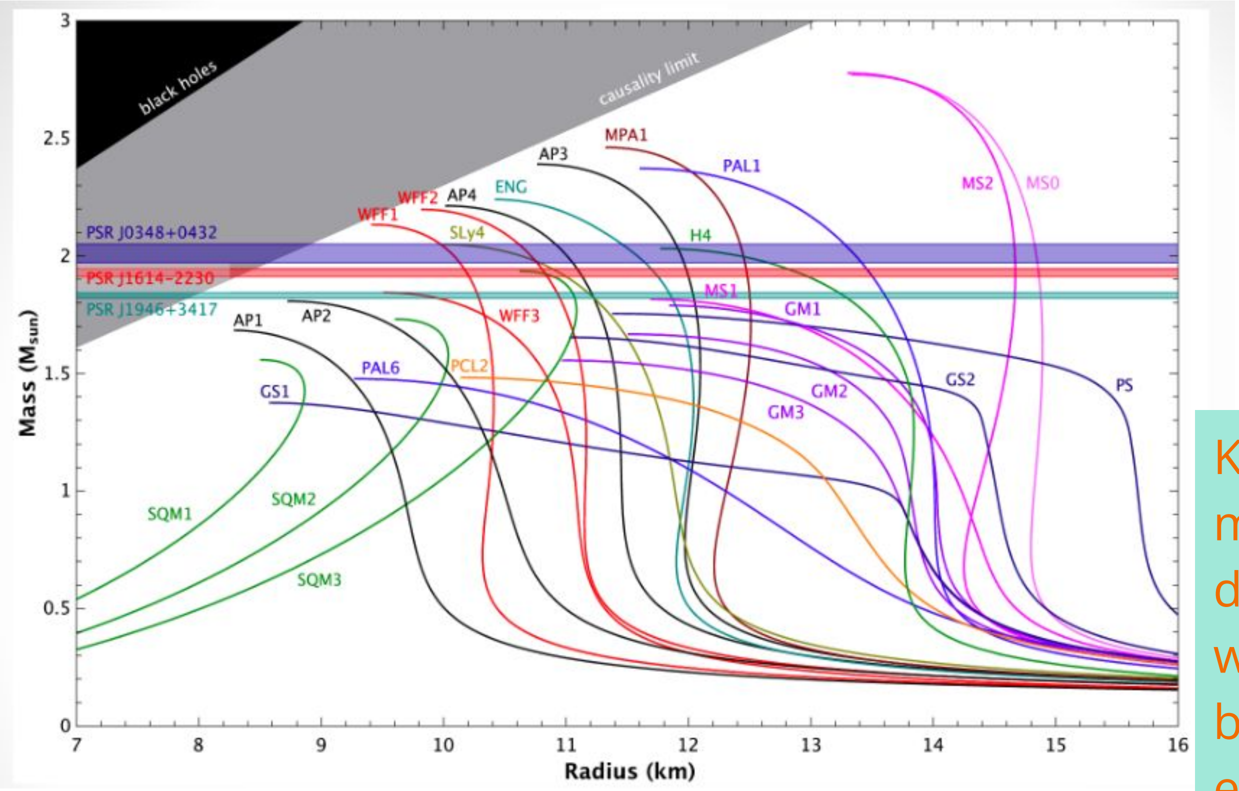


Unique relationship between MR curve and EOS!

Neutron Stars as Astronomical High Energy Laboratories



Knowing radius and masses of NSs allow to discard EOS and in this way learn about the behaviour of matter at extreme densities

Some general restrictions

An upper bound for the maximum mass of a NS

The possible maximum mass of any NS is an unknown quantity. We will present an upper bound for such quantity based on basic general physical principles

The speed sound of the fluid can not be larger than the speed of sound in vacuum $dp/d\varepsilon \leq c^2$

The thermodynamic stability criteria $dp/d\varepsilon > 0$ has to be satisfied

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$$P(\epsilon) = \begin{cases} (\epsilon - \epsilon_s)c^2 & \text{for} \quad \epsilon \geq \epsilon_s \\ 0 & \text{for} \quad \epsilon < \epsilon_s \end{cases}$$

causal limit EOS

Some general restrictions

An upper bound for the maximum mass of a NS

The possible maximum ... ass of any

an upper bound fo ... antity ba

The s ... larg

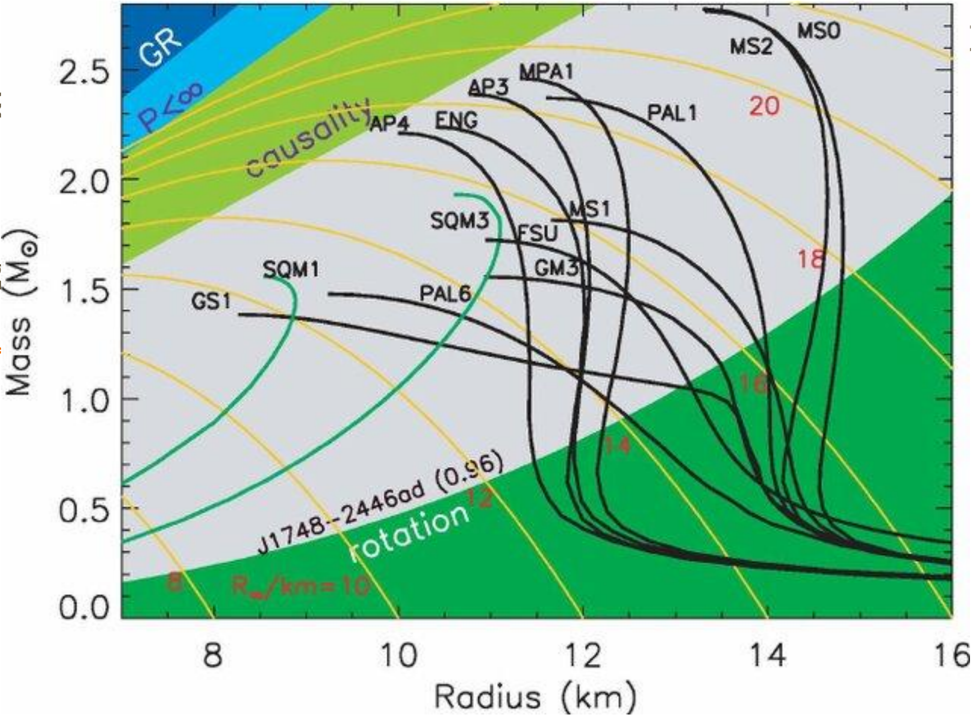
After solving TOV equations one gets that

$$M_{max} = 3.0\left(\frac{5\times10^{14}\,gcm^{-3}}{\epsilon_s}\right)^{1/2} M_\odot$$

Haensel, Potekhin and Yakovlev 2007

$\geq \epsilon_s$

or $\quad \epsilon < \epsilon_s$



Some general restrictions

Buchdahl theorem

Under these reasonable hypothesis for a stellar configuration

- Finite central pressure and energy density
- Density decreases from the center of the star to its surface but it is always positive
- Density is null outside the stellar configuration
- Metric coefficients are positive

Some general restrictions

Buchdahl theorem

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- Metric coefficients are positive

It can be shown that INDEPENDENTLY of the EOS

$R \geq 9/8\, R_S$ where R_S is the Schwarzschild radius $R_S = 2GM/c^2$

NS+NS merger and gravitational wave emission

Chirp mass

The early stage of the inspiral phase is determined by the chirp mass

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{-1/5}}$$

Tidal Love number

In a binary system the tidal field of one NS induces a mass-quadrupole moment on the companion. This can be quantified by the induced quadrupole moment to the external tidal field which is proportional to the tidal deformability

$$\Lambda = \frac{2}{3} k_2 \left(\frac{c^2 R}{G m} \right)^5$$

k_2 is the second Love number

NS+NS merger and gravitational wave emission

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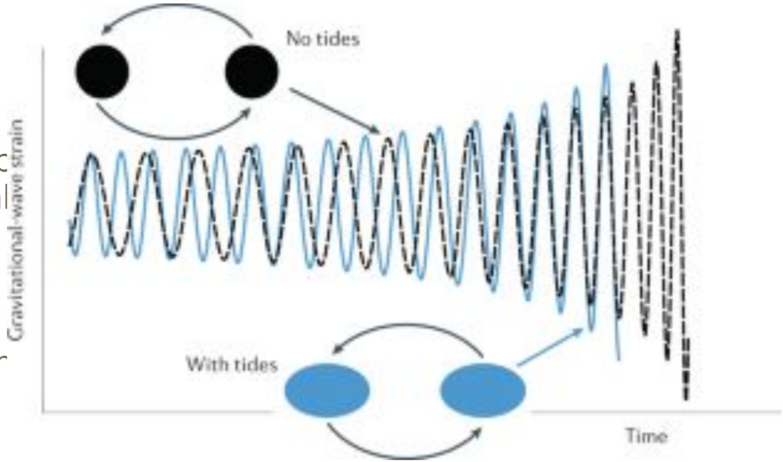

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quantified by the induced quadrupole moment to the external tidal
deformability

$$\Lambda = \frac{2}{3} k_2 \left(\frac{c^2 R}{Gm} \right)^5$$

k_2 is the second Love nur

This quantity is of paramount importance as the gravitational wave signal closer to the moment of the merge depends on it. We would not enter into this details in this lectures

NS+NS merger and gravitational wave emission

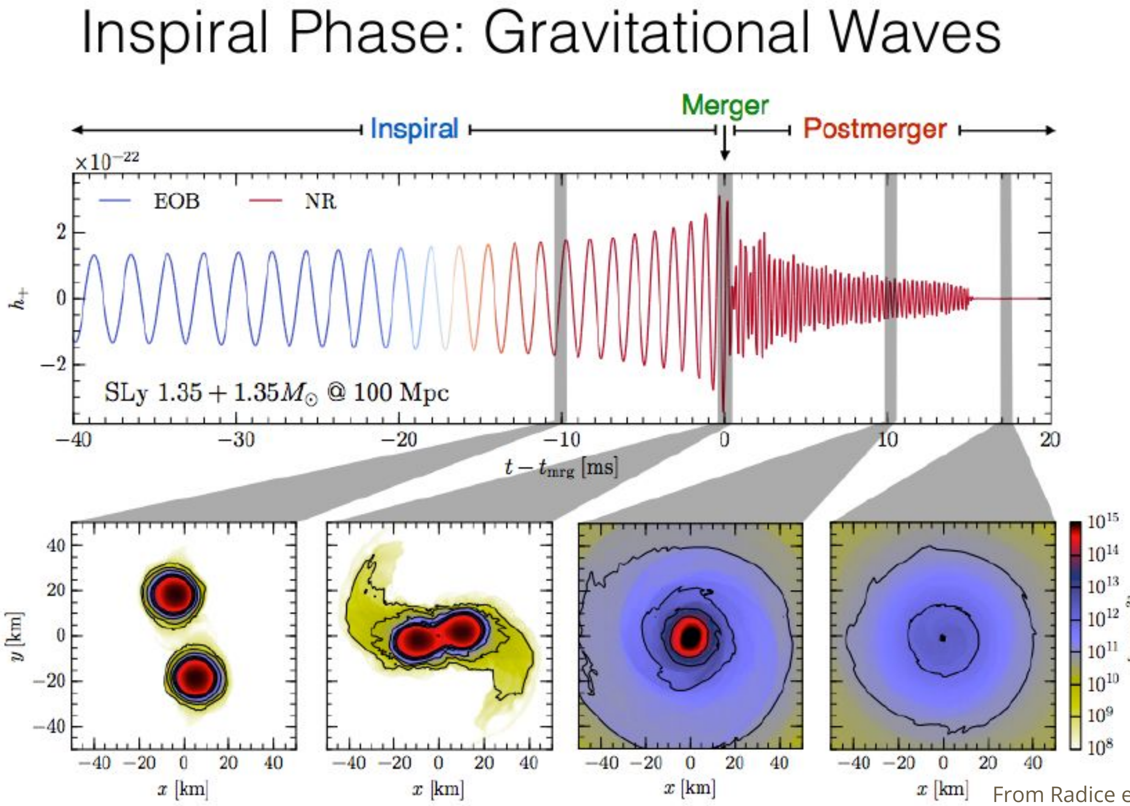
Chirp mass

The early stage of the inspiral pha

Tidal Love number

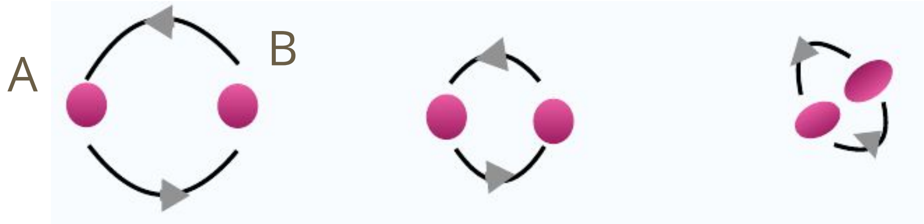
In a binary system the tidal field c
quantified by the induced quadru
deformability

This quantity is
closer to the mo



From Radice et al.2016

The tidal Love number



In binaries, NSs are deformed by the gravitational field of their companion

In Newtonian theory

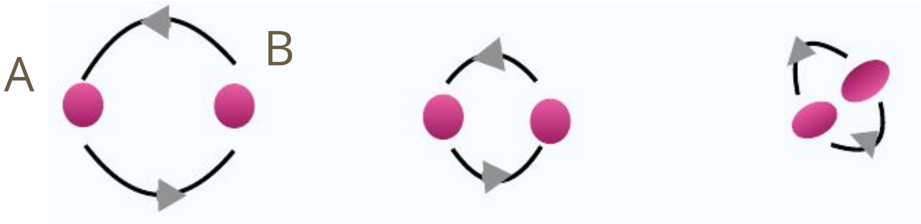
$$\mathcal{E}_{ij} = -\left.\frac{\partial^2 \Phi_{\text{ext}}}{\partial x_i \partial x_j}\right|_{\vec{x}=\vec{x}_c}$$

Tidal momentum

- i and j from 1 to 3
- Φ_{ext} is the potential of the external object
- x_c position of the center of mass of the body subject to the tidal field

We will consider a non-rotating spherical NS, A, under the point-like gravitational field of object B.

The tidal Love number



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Tidal momentum

- i and j from 1 to 3
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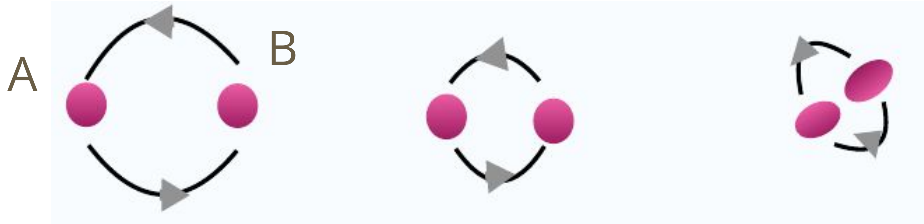
Focus on a point P with mass m_P at the surface of object A

We will consider a non-rotating spherical NS, A, under the point-like gravitational field of object B.

The tidal Love number



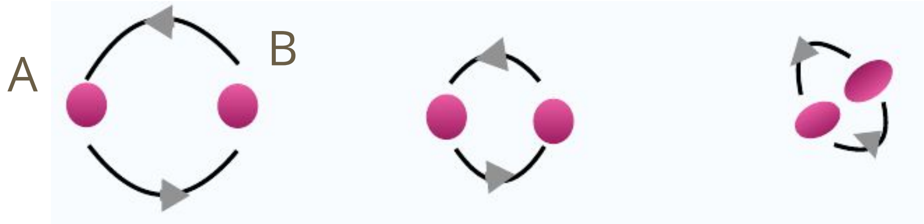
$$m_P \vec{a}_P = -g_A m_P \cdot \hat{u}_P + \vec{F}_{AB} = -g_A m_P \cdot \hat{u}_P - m_P \vec{\nabla} \Phi(\vec{r}_P)$$

surface gravity at point P due to object A

$$\hat{u}_P = \frac{\vec{r}_P - \vec{r}_c}{|\vec{r}_P - \vec{r}_c|}$$

gravitational potential of body B at point P

The tidal Love number



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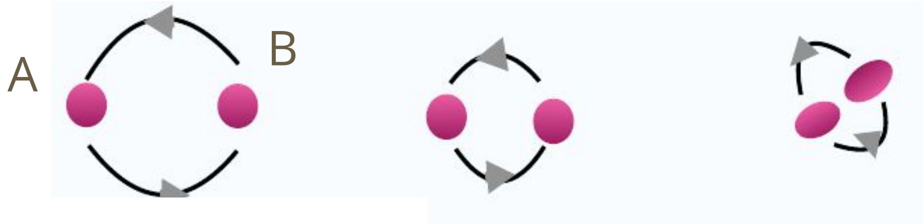
$$\frac{\partial \Phi}{\partial x_i} \approx \left.\frac{\partial \Phi}{\partial x_i}\right|_{r_c} + \left.\frac{\partial^2 \Phi}{\partial x_j \partial x_i}\right|_{r_c} (\vec{x} - \vec{r_c})_j = \left.\frac{\partial \Phi}{\partial x_i}\right|_{r_c} - \mathcal{E}_{ji}\,(\vec{x} - \vec{r_c})_j$$

The tidal Love number



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- We place at mass-centered reference frame of object A
- Neglect all rotational effects

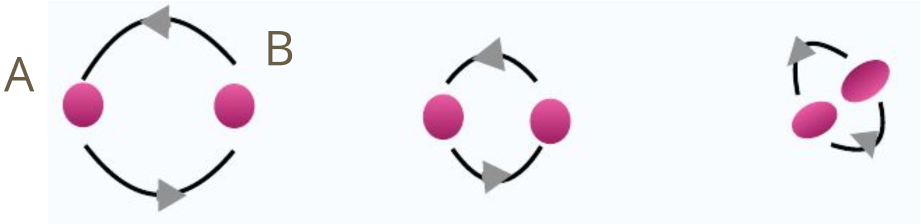
$$F_i^{\text{tidal}} = m_P x_j \mathcal{E}_{ji} = -m_P \frac{\partial \Phi^{\text{tidal}}}{\partial x_i}$$

mass-centered coordinates

$$\Phi^{\text{tidal}} = \frac{1}{2} x_i x_j \mathcal{E}_{ij}$$

We have a quadrupolar tidal potential, we expect that it will induce a quadrupolar deformation on body A

The tidal Love number



$$Q_{ij} = \int d^3x\, \rho(\vec{x}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right)$$

quadrupole moment

mass density

$$Q_{ij} = -\boxed{\lambda}\mathcal{E}_{ij}$$

dimensional analysis

$$\lambda = \frac{2}{3} k_2 R^5$$

Love number

Tidal deformability

The tidal Love number

A

$$Q_{ij} = \int d^3x\, \rho(\vec{x}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right)$$

udrupole moment

mass density

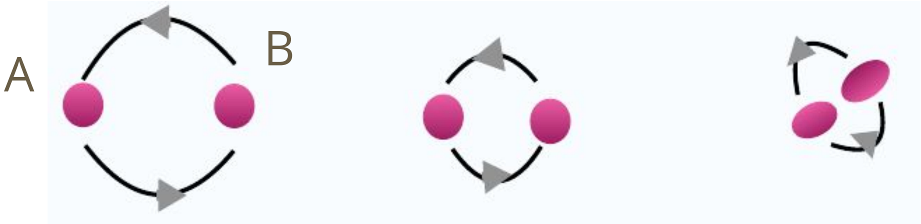
Large tidal deformability means that a body is easily deformed under the influence of external tidal forces

$$Q_{ij} = -\boxed{\lambda}\mathcal{E}_{ij} \qquad \lambda = \frac{2}{3} k_2 R^5$$

Tidal deformability

Love number

The tidal Love number



$$Q_{ij} = \int d^3x\, \rho(\vec{x}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right)$$

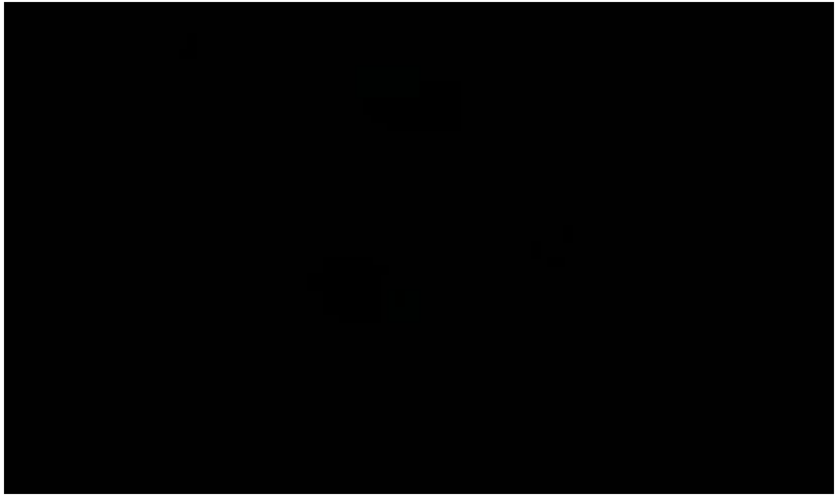
quadrupole moment

mass

$$\Phi_{\rm tot} = -\frac{M}{r} - \frac{3}{r^5} Q_{ij} x_i x_j + \frac{1}{2} \mathcal{E}_{ij} x_i x_j = -\frac{M}{r} + \left[\frac{2k_2}{G} \left(\frac{R}{r} \right)^5 + \frac{1}{2} \right] \mathcal{E}_{ij} x_i x_j$$

Love number

The tidal Love number



Corrections from General Relativity

(using geometric units c = G = 1)

In the weak field approximation

$$\Phi = -\frac{(1+g_{00})}{2} = -\frac{M}{r} - \frac{3}{r^5}Q_{ij}x_ix_j + O\left(\frac{1}{r^3}\right) + \frac{1}{2}\mathcal{E}_{ij}x_ix_j + O\left(r^3\right)$$

After some much harder algebra it can be shown that to obtain the Love number one has to solve together with the TOV equations an extra differential equation

The tidal Love number

Corrections from General Relativity

$$r y'(r) + y(r)^2 + y(r) e^{\lambda} \left[1 + 4\pi r^2 (P - \epsilon)\right] + r^2 Q(r) = 0$$

$$Q(r) = 4\pi e^{\lambda} \left[9P + 5\epsilon + \frac{P + \epsilon}{dP/d\epsilon}\right] - 6\frac{e^{\lambda}}{r^2} - \nu'^2 ,$$

$y(0) = 2$

Easily solved using the same numerical scheme presented for the structure equations

The tidal Love number

Corrections from General Relativity

$ry'($

$Q(r$

$$k_2 = \frac{8\beta^5}{5}(1-2\beta)^2[2+2\beta(y-1)-y] \times \{2\beta[6-3y+3\beta(5y-8)]$$
$$+ 4\beta^3[13-11y+\beta(3y-2)+2\beta^2(1+y)]$$
$$+ 3(1-2\beta)^2[2-y+2\beta(y-1)]\ln(1-2\beta)\}^{-1},$$

$\beta = M/R$ is called compacticity

The tidal Love number

Corrections from General Relativity

$ry'($

$Q(r$

$$k_2 = \frac{8\beta^5}{5}(1-2\beta)^2[2+2\beta(y-1) \ldots 3y + 3\beta(5y-8)]$$

$$+ 4\beta^3[13 - 11y + \beta(3 \ldots$$

$$+ 3(1-2\beta)^2[2 \ldots \beta))\}^{-1},$$

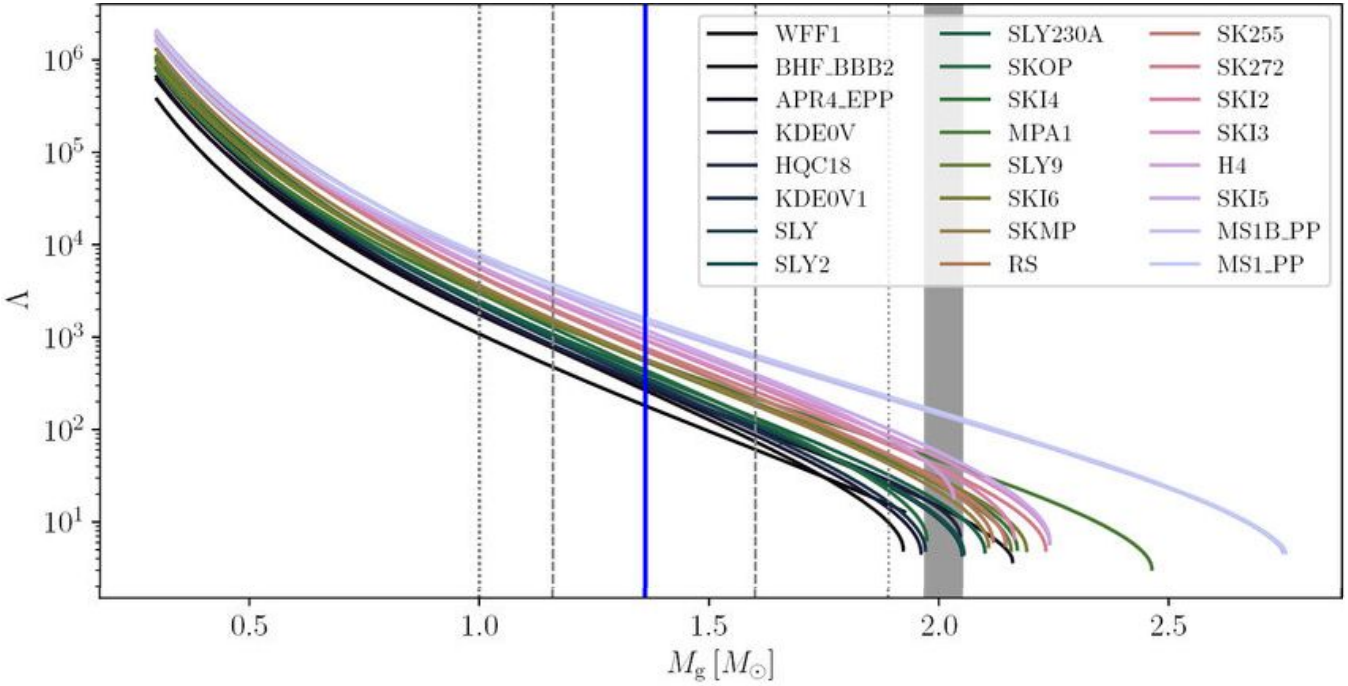
$\beta = M/R$

...ticity

For NSs, the differences between Newtonian and relativistic approaches give differences up to ~25%

The tidal Love number

Theoretical models

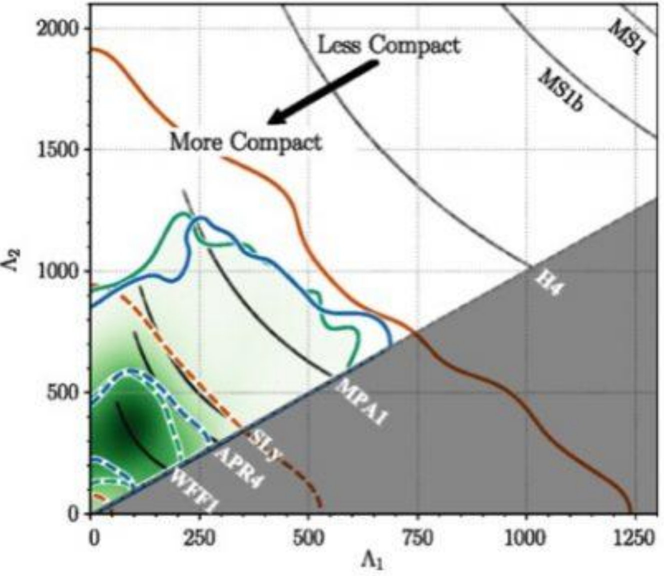


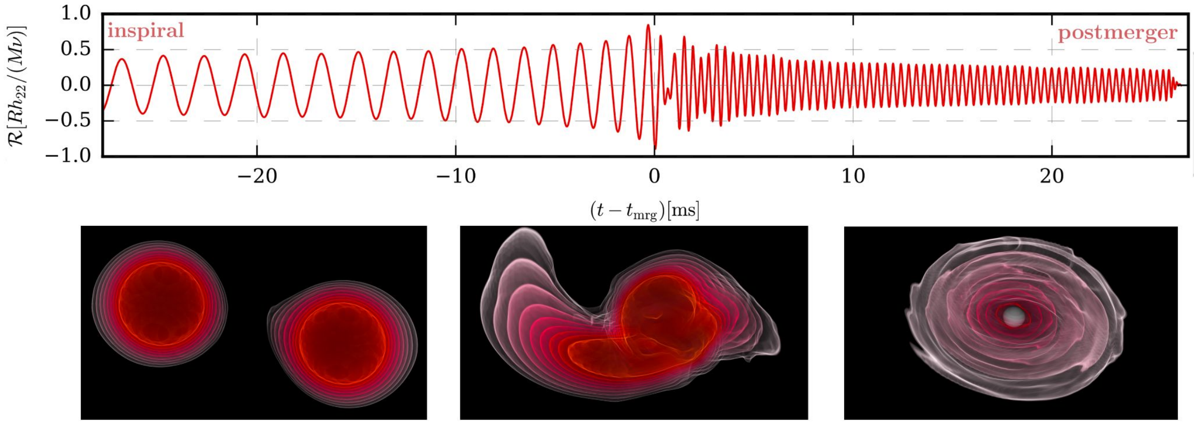
dimensionless tidal deformability

$\Lambda = \lambda/M^5$

The tidal Love number

GW170817 event!





The dimensionless tidal deformability of NSs has been observationally constrained!

Next lecture...

How do we describe from a theoretical point of view matter at extreme conditions?