

Unit 3. Introduction to Computational Fluid Dynamics (CFD)

Lecture 301: hyperbolic equations

Reference book:

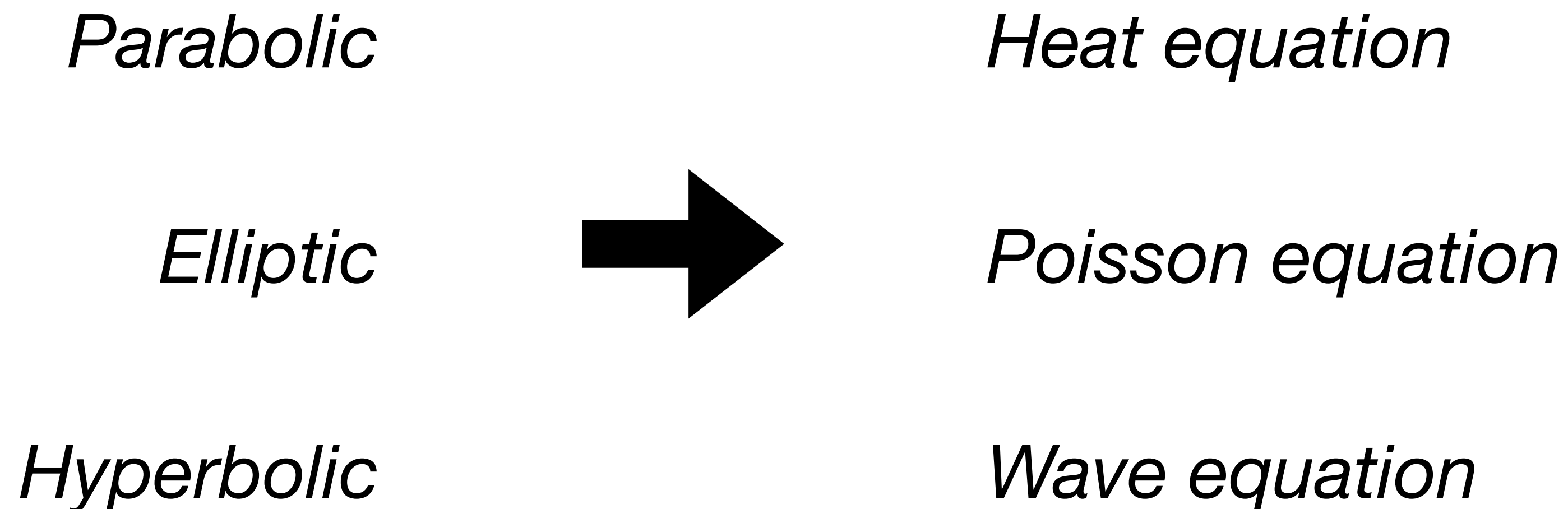
“Introduction to Computational Astrophysical Hydrodynamics” by Zingale.
http://bender.astro.sunysb.edu/hydro_by_example/CompHydroTutorial.pdf

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Partial differential equations, generalities and classification

Classification

Partial differential equations (PDEs) are usually grouped into one of three different classes:



Partial differential equations, generalities and classification

Hyperbolic PDEs

The canonical hyperbolic PDE is the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \quad (2.1)$$

The general solution to this is traveling waves in either direction:

$$\phi(x, t) = \alpha f_0(x - ct) + \beta g_0(x + ct) \quad (2.2)$$

Here f_0 and g_0 are set by the initial conditions, and the solution propagates f_0 to the right and g_0 to the left at a speed c .

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Hyperbolic PDEs -> 1D Advection equation

A simple first-order hyperbolic PDE is the linear advection equation:

$$a_t + ua_x = 0 \tag{2.3}$$

This simply propagates any initial profile to the right at the speed u . We will use linear advection as our model equation for numerical methods for hyperbolic PDEs.

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Hyperbolic PDEs -> 1D Advection equation

A system of first-order hyperbolic PDEs takes the form:

$$\mathbf{a}_t + \mathbf{A}\mathbf{a}_x = 0 \quad (2.4)$$

where $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})^\top$ and \mathbf{A} is a matrix. This system is hyperbolic if the eigenvalues of A are real

Partial differential equations, generalities and classification

Hyperbolic PDEs -> characteristics

An important concept for hyperbolic PDEs are *characteristics*—these are curves in a space-time diagram along which the solution is constant. Associated with these curves is a speed—this is the wave speed at which information on how the solution changes is communicated. For a linear PDE (or system of PDEs), these will tell you everything you need to know about the solution.