Unit 2. Partial differential equations (PDEs) in physics

Lecture 209: Elliptic equations

Reference book:

“Introduction to Computational Astrophysical Hydrodynamics” by Zingale.
http://bender.astro.sunysb.edu/hydro_by_example/CompHydroTutorial.pdf

W. Banda-Barragán, 2022

Elliptic equations

The simplest elliptic PDE is *Laplace's equation*:

$$\nabla^2\phi = 0 \tag{9.1}$$

Only slightly more complex is *Poisson's equation* (Laplace + a source term):

$$\nabla^2\phi = f \tag{9.2}$$

These equations can arise in electrostatics (for the electric potential), solving for the gravitational potential from a mass distribution, or enforcing a divergence constraint on a vector field (we'll see this when we consider incompressible flow).

Elliptic equations

Another common elliptic equation is the *Helmholtz equation*:

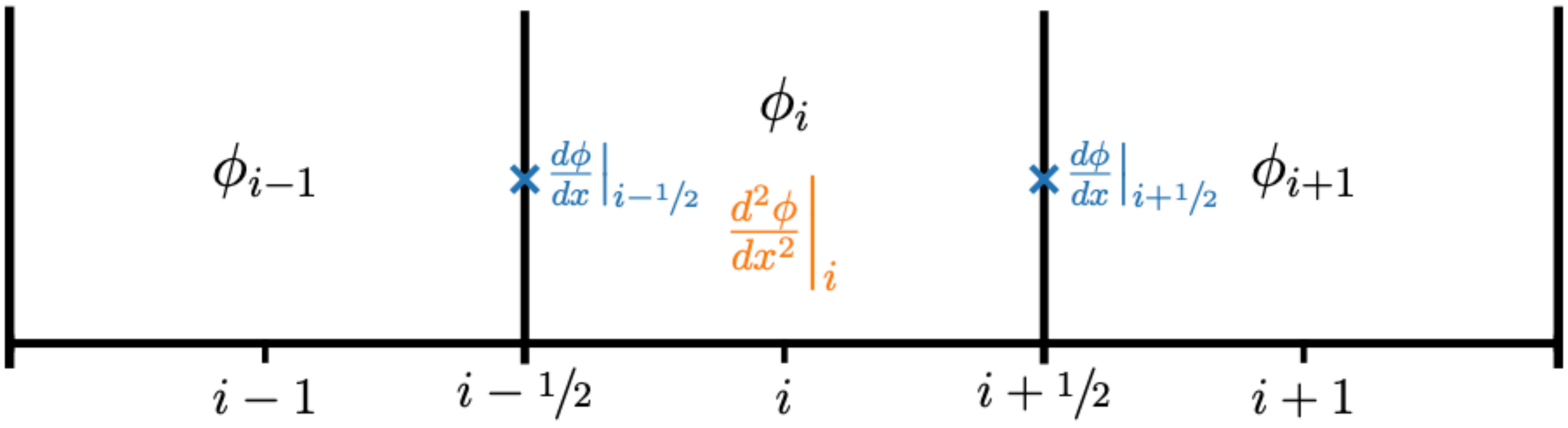
$$(\alpha - \nabla \cdot \beta \nabla)\phi = f \tag{9.3}$$

A Helmholtz equation can arise, for example, from a time-dependent equation (like diffusion) by discretizing in time.

Fourier Method for Elliptic equations

A direct way of solving a constant-coefficient elliptic equation is using Fourier transforms. Using a general Fourier transform (which we consider here) works only for

periodic boundary conditions, but other basis functions (e.g., all sines or all cosines) can be used for other boundary conditions.



Poisson equation: FFT method

Consider the Poisson equation:

$$\nabla^2 \phi = f \tag{9.4}$$

We will difference this in a second-order accurate fashion—see Figure 9.1. In 1-d, the Laplacian is just the second-derivative. If our solution is defined at cell-centers, then we first compute the first-derivative on cell edges:

$$\left.\frac{d\phi}{dx}\right|_{i-1/2} = \frac{\phi_i - \phi_{i-1}}{\Delta x} \tag{9.5}$$

$$\left.\frac{d\phi}{dx}\right|_{i+1/2} = \frac{\phi_{i+1} - \phi_i}{\Delta x} \tag{9.6}$$

These are second-order accurate on the interface. We can then compute the second-derivative at the cell-center by differencing these edge values:

$$\left.\frac{d^2\phi}{dx^2}\right|_i = \frac{d\phi/dx|_{i+1/2} - d\phi/dx|_{i-1/2}}{\Delta x} \tag{9.7}$$

Poisson equation: FFT method

The extension to 2-d is straightforward. Thinking of the Laplacian as $\nabla^2\phi = \nabla \cdot \nabla\phi$, we first compute the gradient of ϕ on edges:

$$[\nabla\phi \cdot \hat{x}]_{i+1/2,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} \tag{9.8}$$

$$[\nabla\phi \cdot \hat{y}]_{i,j+1/2} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} \tag{9.9}$$

Again, since this is defined on edges, this represents a centered difference, and is therefore second-order accurate. We then difference the edge-centered gradients to the center to get the Laplacian at cell-centers:

$$\begin{aligned}
[\nabla^2\phi]_{i,j} &= \frac{[\nabla\phi \cdot \hat{x}]_{i+1/2,j} - [\nabla\phi \cdot \hat{x}]_{i-1/2,j}}{\Delta x} + \frac{[\nabla\phi \cdot \hat{y}]_{i,j+1/2} - [\nabla\phi \cdot \hat{y}]_{i,j-1/2}}{\Delta y} \\
&= \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = f_{i,j} \qquad (9.10)
\end{aligned}$$

Poisson equation: FFT method

We now assume that we have an FFT subroutine (see § $\boxed{1.2.6}$) that can take our discrete real-space data, $\phi_{i,j}$ and return the discrete Fourier coefficients, Φ_{k_x,k_y}, and likewise for the source term:

$$\Phi_{k_x,k_y} = \mathcal{F}(\phi_{i,j}) \quad F_{k_x,k_y} = \mathcal{F}(f_{i,j}) \tag{9.11}$$

The power of the Fourier method is that derivatives in real space are multiplications in Fourier space, which makes the solution process in Fourier space straightforward.

Poisson equation: FFT method

We now express $\phi_{i,j}$ and $f_{i,j}$ as sums over their Fourier components. Here we define M as the number of grid points in the x-direction and N as the number of grid points in the y-direction. As before, we are using i as the grid index, we will use I as the imaginary unit:

$$\phi_{i,j} = \frac{1}{MN} \sum_{k_x=0}^{M-1} \sum_{k_y=0}^{N-1} \Phi_{k_x,k_y} e^{2\pi I i k_x / M} e^{2\pi I j k_y / N} \tag{9.12}$$

$$f_{i,j} = \frac{1}{MN} \sum_{k_x=0}^{M-1} \sum_{k_y=0}^{N-1} F_{k_x,k_y} e^{2\pi I i k_x / M} e^{2\pi I j k_y / N} \tag{9.13}$$

Poisson equation: FFT method

Inserting these into the differenced equation, we have:

$$\frac{1}{MN}\sum_{k_x=0}^{M-1}\sum_{k_y=0}^{N-1}\left\{\frac{\Phi_{k_x,k_y}}{\Delta x^2}e^{2\pi Ijk_y/N}\left[e^{2\pi I(i+1)k_x/M}-2e^{2\pi Iik_x/M}+e^{2\pi I(i-1)k_x/M}\right]+\right.$$
$$\left.\frac{\Phi_{k_x,k_y}}{\Delta y^2}e^{2\pi Iik_x/M}\left[e^{2\pi I(j+1)k_y/N}-2e^{2\pi Ijk_y/N}+e^{2\pi I(j-1)k_y/N}\right]\right\}=$$
$$\frac{1}{MN}\sum_{k_x=0}^{M-1}\sum_{k_y=0}^{N-1}F_{k_x,k_y}e^{2\pi Iik_x/M}e^{2\pi Ijk_y/N} \tag{9.14}$$

We can bring the righthand side into the sums on the left, and we can then look at just a single (k_x, k_y) term in the series:

Poisson equation: FFT method

$$e^{2\pi Iik_x/M}e^{2\pi Ijk_y/N}\left\{\frac{\Phi_{k_x,k_x}}{\Delta x^2}\left[e^{2\pi Ik_x/M}+e^{-2\pi Ik_x/M}-2\right]+\right.$$
$$\left.\frac{\Phi_{k_x,k_x}}{\Delta y^2}\left[e^{2\pi Ik_y/N}+e^{-2\pi Ik_y/N}-2\right]-F_{k_x,k_y}\right\}=0 \tag{9.15}$$

Simplifying, we have:

$$\Phi_{k_x,k_y}=\frac{1}{2}\frac{F_{k_x,k_y}}{\left[\cos(2\pi k_x/M)-1\right]\Delta x^{-2}+\left[\cos(2\pi k_y/N)-1\right]\Delta y^{-2}} \tag{9.16}$$

This is the algebraic solution to the Poisson equation in Fourier (frequency) space. Once we evaluate this, we can get the real-space solution by doing the inverse transform:

$$\phi_{i,j}=\mathcal{F}^{-1}(\Phi_{k_x,k_y}) \tag{9.17}$$

Poisson equation: FFT method

We can test this technique with the source term:

$$f = 8\pi^2 \cos(4\pi y)\left[\cos(4\pi x) - \sin(4\pi x)\right] - \\ 16\pi^2 \left[\sin(4\pi x)\cos(2\pi y)^2 + \sin(2\pi x)^2 \cos(4\pi y)\right] \quad (9.18)$$

which has the analytic solution[†]:

$$\phi = \sin(2\pi x)^2 \cos(4\pi y) + \sin(4\pi x)\cos(2\pi y)^2 \quad (9.19)$$

Note that this solution has the required periodic behavior. Figure 9.2 shows the solution.

The main downside of this approach is that, because we solve for a single component independently (Eq. 9.16), this only works for linear problems with constant coefficients. This makes it an excellent choice for cosmological problems solving the gravitational Poisson equation with periodic boundaries on all sides of the domain (see, e.g., [41]).