Unit 3. Introduction to Computational Fluid Dynamics (CFD)

Lecture 301: hyperbolic equations

Reference book:

“Introduction to Computational Astrophysical Hydrodynamics” by Zingale.
http://bender.astro.sunysb.edu/hydro_by_example/CompHydroTutorial.pdf

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Classification

Partial differential equations (PDEs) are usually grouped into one of three different classes:

Parabolic		*Heat equation*
Elliptic	$\Rightarrow$	*Poisson equation*
Hyperbolic		*Wave equation*

Hyperbolic PDEs

The canonical hyperbolic PDE is the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \tag{2.1}$$

The general solution to this is traveling waves in either direction:

$$\phi(x,t) = \alpha f_0(x - ct) + \beta g_0(x + ct) \tag{2.2}$$

Here f_0 and g_0 are set by the initial conditions, and the solution propagates f_0 to the right and g_0 to the left at a speed c.

Hyperbolic PDEs -> 1D Advection equation

A simple first-order hyperbolic PDE is the linear advection equation:

$$a_t + u a_x = 0 \tag{2.3}$$

This simply propagates any initial profile to the right at the speed u. We will use linear advection as our model equation for numerical methods for hyperbolic PDEs.

Hyperbolic PDEs -> 1D Advection equation

A system of first-order hyperbolic PDEs takes the form:

$$\mathbf{a}_t + \mathbf{A}\mathbf{a}_x = 0 \tag{2.4}$$

where $\mathbf{a} = (a_0, a_1, \ldots a_{N-1})^\intercal$ and $\mathbf{A}$ is a matrix. This system is hyperbolic if the eigenvalues of A are real

Hyperbolic PDEs -> characteristics

An important concept for hyperbolic PDEs are *characteristics*—these are curves in a space-time diagram along which the solution is constant. Associated with these curves is a speed—this is the wave speed at which information on how the solution changes is communicated. For a linear PDE (or system of PDEs), these will tell you everything you need to know about the solution.