Unit 3. Introduction to Computational Fluid Dynamics (CFD)

Lecture 311: Finite-volume methods

Reference book:

"Introduction to Computational Astrophysical Hydrodynamics" by Zingale.
http://bender.astro.sunysb.edu/hydro_by_example/CompHydroTutorial.pdf

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Introduction to CFD: Numerical grids

The grid is the fundamental object for representing continuous functions in a discretised fashion, making them amenable to computation.

In physics, we typically use structured grids—these are logically Cartesian, meaning that the position of a quantity on the grid can be specified by a single integer index in each dimension.

This works for our types of problems because we don't have irregular geometries— we typically use boxes, disks, or spheres.

Introduction to CFD: grid discretisation

We represent derivatives numerically by discretising the domain into a finite number of zones (a numerical grid).

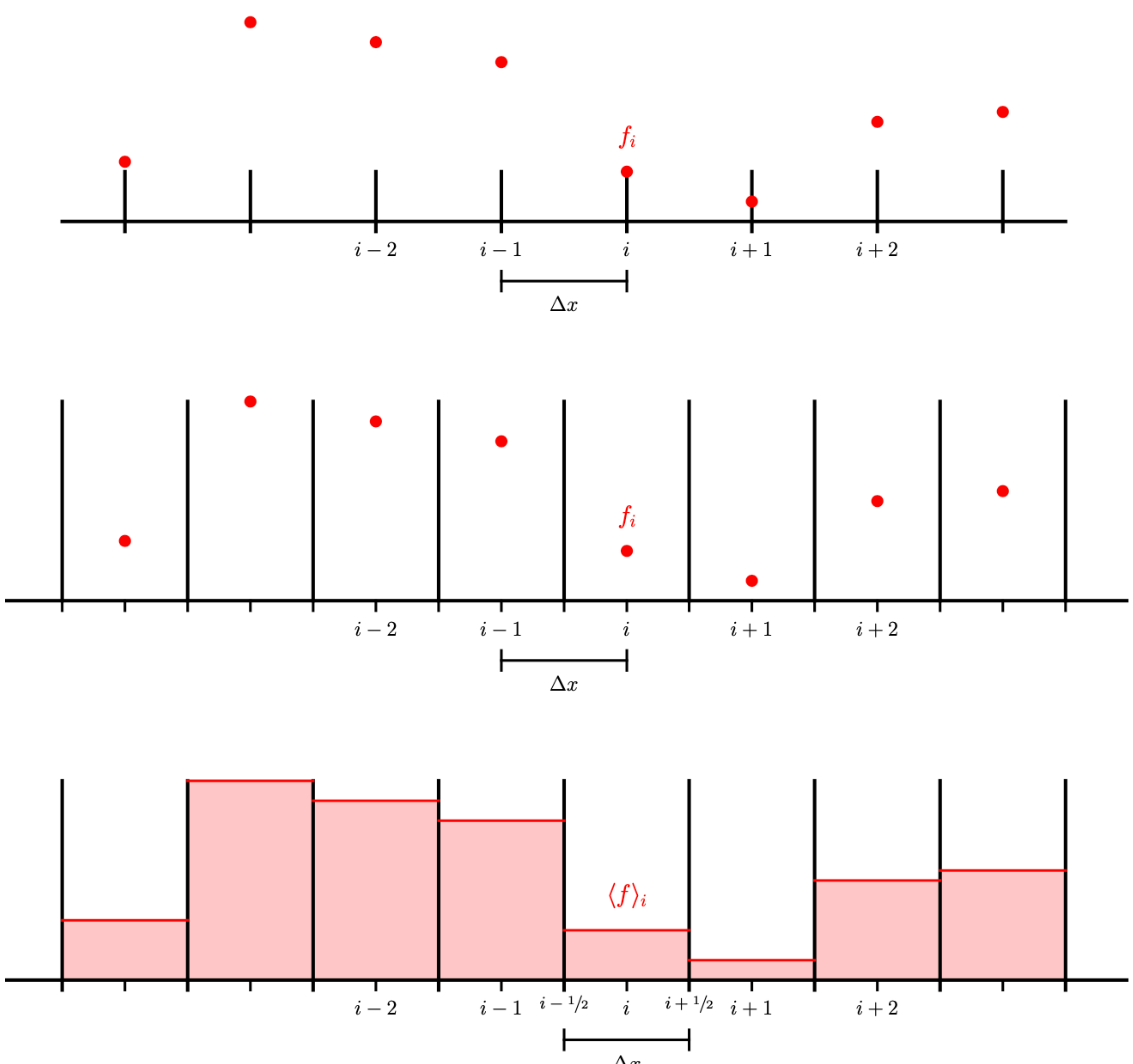
This converts a continuous derivative into a difference of discrete data. Different approximations have different levels of accuracy.

There are two main types of structured grids used in astrophysics: *finite-difference* and *finite-volume*. These differ in way the data is represented:

1. On a **finite-difference grid**, the discrete data is associated with a specific point in space.

2. On a **finite-volume grid**, the discrete data is represented by averages over a control volume. Nevertheless, these methods can often lead to very similar looking discrete equations.

Introduction to CFD: grid discretisation



Different types of structured grids showing the same data.

Top: a finite- difference grid—the discrete data are associated with a specific point in space.

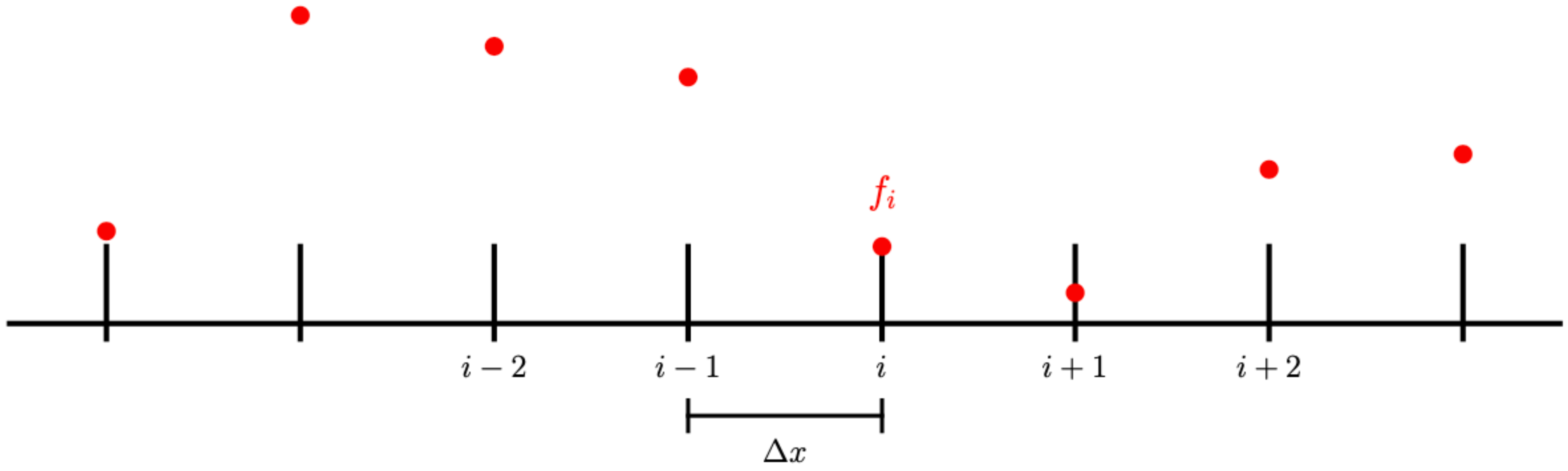
Middle: a cell-centered finite-difference grid—again the data is at a specific point, but now we imagine the domain divided into zone with the data living at the center of each zone.

Bottom: a finite-volume grid—here the domain is divided into zones and we store the average value of the function within each zone.

Introduction to CFD: grid discretisation

Classic finite-difference grid.

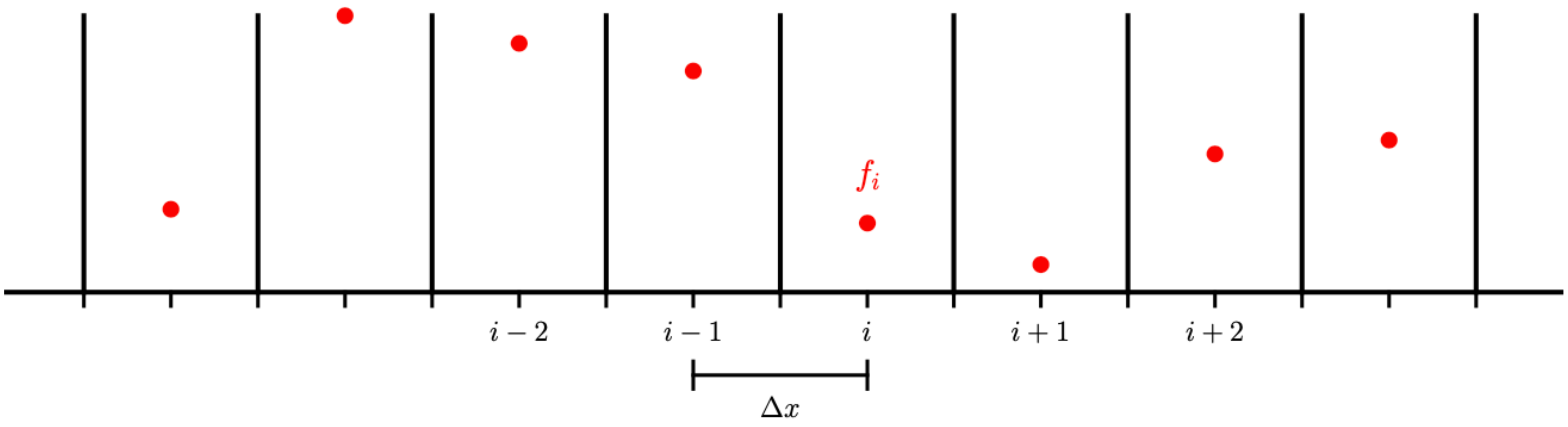
The discrete data, f_i, are stored as points regularly spaced in x. With this discretisation, the spatial locations of the points are simply $x_i = i\,\Delta x$, where $i = 0, \ldots, N^*$. Note that for a finite-sized domain, we would put a grid point precisely on the physical boundary at each end.



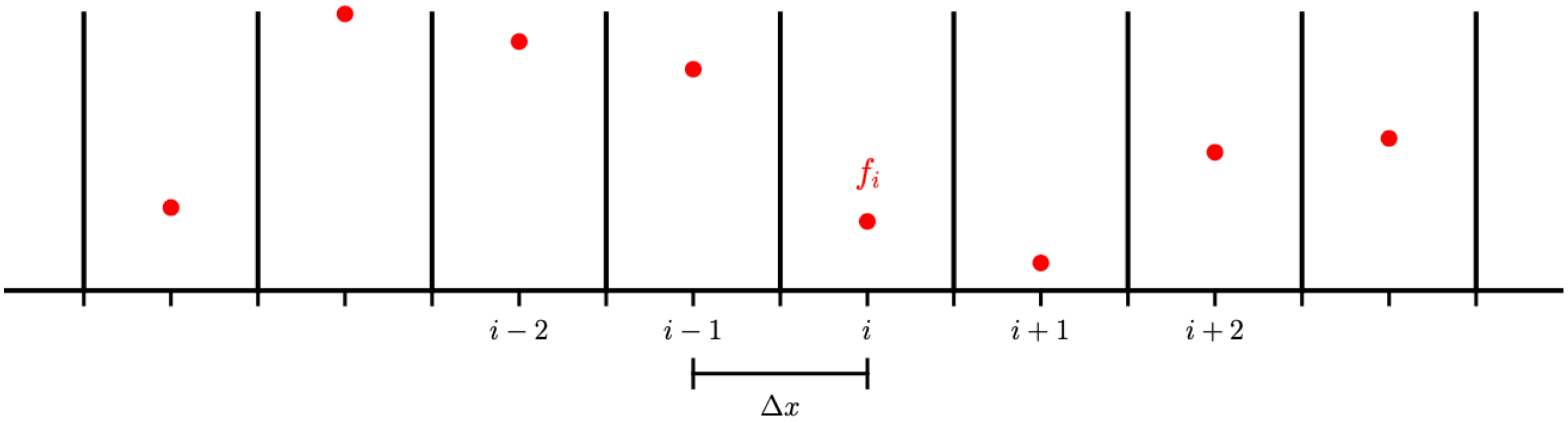
Introduction to CFD: grid discretisation

The middle grid is also finite-difference, but now we imagine first dividing the domain into N cells or zones, and we store the discrete data, f_i, at the center of the zone. This is often called a ***cell-centred finite-difference* grid.**

The physical coordinate of the zone centers (where the data lives) are:
$x_i = (i + 1/2)\Delta x$, where $i = 0,\ldots,N-1$.



Introduction to CFD: grid discretisation



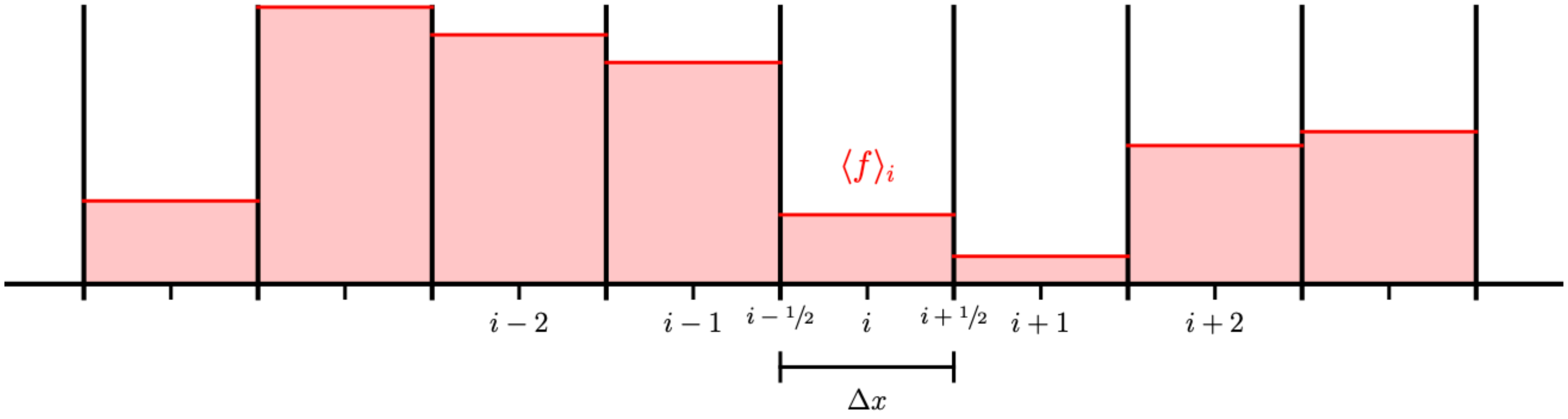
Note that now for a finite-sized domain, the left edge of the first cell will be on the boundary and the first data value will be associated at a point $\Delta x/2$ inside the boundary.

A similar situation arises at the right physical boundary. Some finite-difference schemes stagger the variables, e.g., putting velocity on the boundaries and density at the center.

Introduction to CFD: finite volume grid

Finite-volume grid:

The layout looks identical to the cell-centred finite difference grid, except now instead of the discrete data being associated at a single point in space, **we keep track of the total amount of *f* in the zone (indicated as the shaded regions).**



Introduction to CFD: finite volume grid

Since we generally don't know how *f* varies in the zone, **we will typically talk about the average of *f*, $\langle f \rangle_i$, over the zone,** and represent this by a horizontal line. **The total amount of *f* in the zone is then simply $\Delta x \langle f \rangle_i$.**

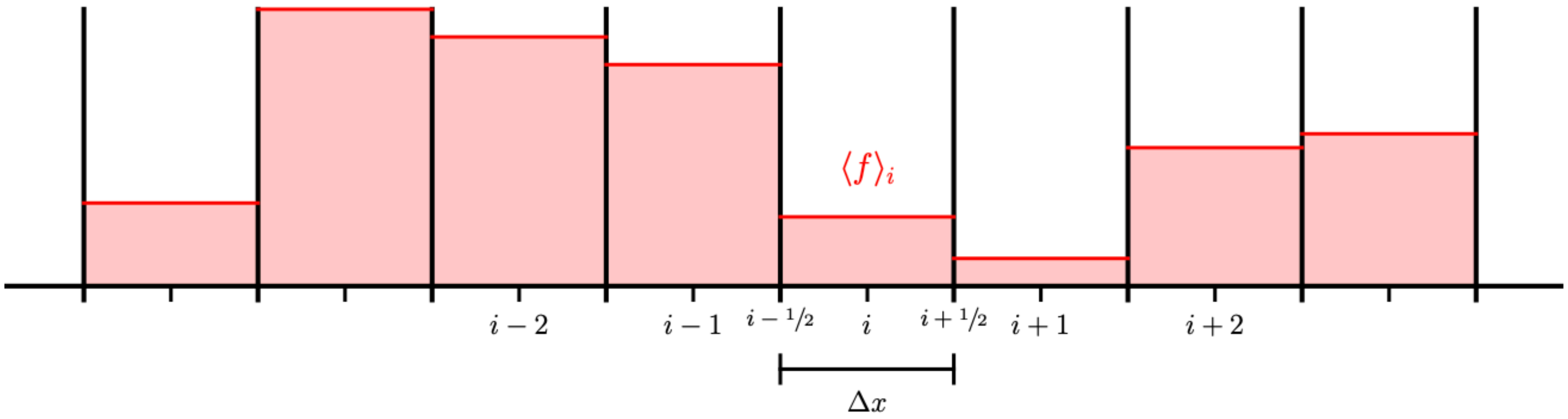
We label the left and right edges of a zone with half-integer indices $i - 1/2$ and $i + 1/2$. The physical coordinate of the centre of the zone is the same as in the cell-centred finite-difference case.

In all cases, for a *regular* structured grid, we take Δx to be constant. For the finite difference grids, the discrete value at each point is obtained from the continuous function $f(x)$ as: $f_i = f(x_i)$

3.3 Finite-volume grids

In the finite-volume discretization, f_i represents the average of $f(x,t)$ over the interval $x_{i-1/2}$ to $x_{i+1/2}$, where the half-integer indices denote the zone edges (i.e. $x_{i-1/2} = x_i - \Delta x/2$):

$$\langle f \rangle_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x) dx \tag{3.2}$$



Introduction to CFD: finite volume methods

The lower panel of Figure 3.1 shows a finite-volume grid, with the half-integer indices bounding zone i marked. Here we've drawn $\langle f \rangle_i$ as a horizontal line spanning the entire zone—this is to represent that it is an average within the volume defined by the zone edges. To second-order accuracy,

$$\langle f \rangle_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x) dx \sim f(x_i) \tag{3.3}$$

This means that we can treat the average of f over a zone as simply $f(x)$ evaluated at the zone center if we only want second-order accuracy. Using the subscript notation, we can express the average of the zone to the right as $\langle f \rangle_{i+1}$.

Exercise 3.1

Show that Eq. 3.3 is true to $O(\Delta x^2)$ by expanding $f(x)$ as a Taylor series in the integral, e.g., as:

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \frac{1}{2} f''(x_i)(x - x_i)^2 + \frac{1}{6} f'''(x_i)(x - x_i)^3 + \ldots \quad (3.4)$$

When we want to interpolate data on a finite-volume grid, we want to construct an interpolating polynomial that is conservative. A *conservative interpolant* reconstructs a continuous functional form, $f(x)$, from a collection of cell-averages subject to the requirement that when $f(x)$ is averaged over a cell, it returns the original cell-average.

Introduction to CFD: finite volume methods

Exercise 3.2

Consider three cell averages: $\langle f \rangle_{i-1}$, $\langle f \rangle_i$, $\langle f \rangle_{i+1}$. Fit a quadratic polynomial through these points,

$$f(x) = A(x - x_i)^2 + B(x - x_i) + C \tag{3.5}$$

where the coefficients, A, B, and C are found by applying the constraints:

$$\langle f \rangle_{i-1} = \frac{1}{\Delta x} \int_{x_{i-3/2}}^{x_{i-1/2}} f(x) dx \tag{3.6}$$

$$\langle f \rangle_{i} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x) dx \tag{3.7}$$

$$\langle f \rangle_{i+1} = \frac{1}{\Delta x} \int_{x_{i+1/2}}^{x_{i+3/2}} f(x) dx \tag{3.8}$$

Show that the conservative interpolant is:

$$\begin{aligned} f(x) = & \frac{\langle f \rangle_{i-1} - 2\langle f \rangle_i + \langle f \rangle_{i+1}}{2\Delta x^2}(x - x_i)^2 + \\ & \frac{\langle f \rangle_{i+1} - \langle f \rangle_{i-1}}{2\Delta x}(x - x_i) + \\ & \frac{-\langle f \rangle_{i-1} + 26\langle f \rangle_i - \langle f \rangle_{i+1}}{24} \end{aligned} \tag{3.9}$$