Unit 3. Introduction to Computational Fluid Dynamics (CFD)

Lecture 315: Slope limiters

Reference book:

“Introduction to Computational Astrophysical Hydrodynamics” by Zingale.
http://bender.astro.sunysb.edu/hydro_by_example/CompHydroTutorial.pdf

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Introduction to CFD: Example of finite-volume scheme

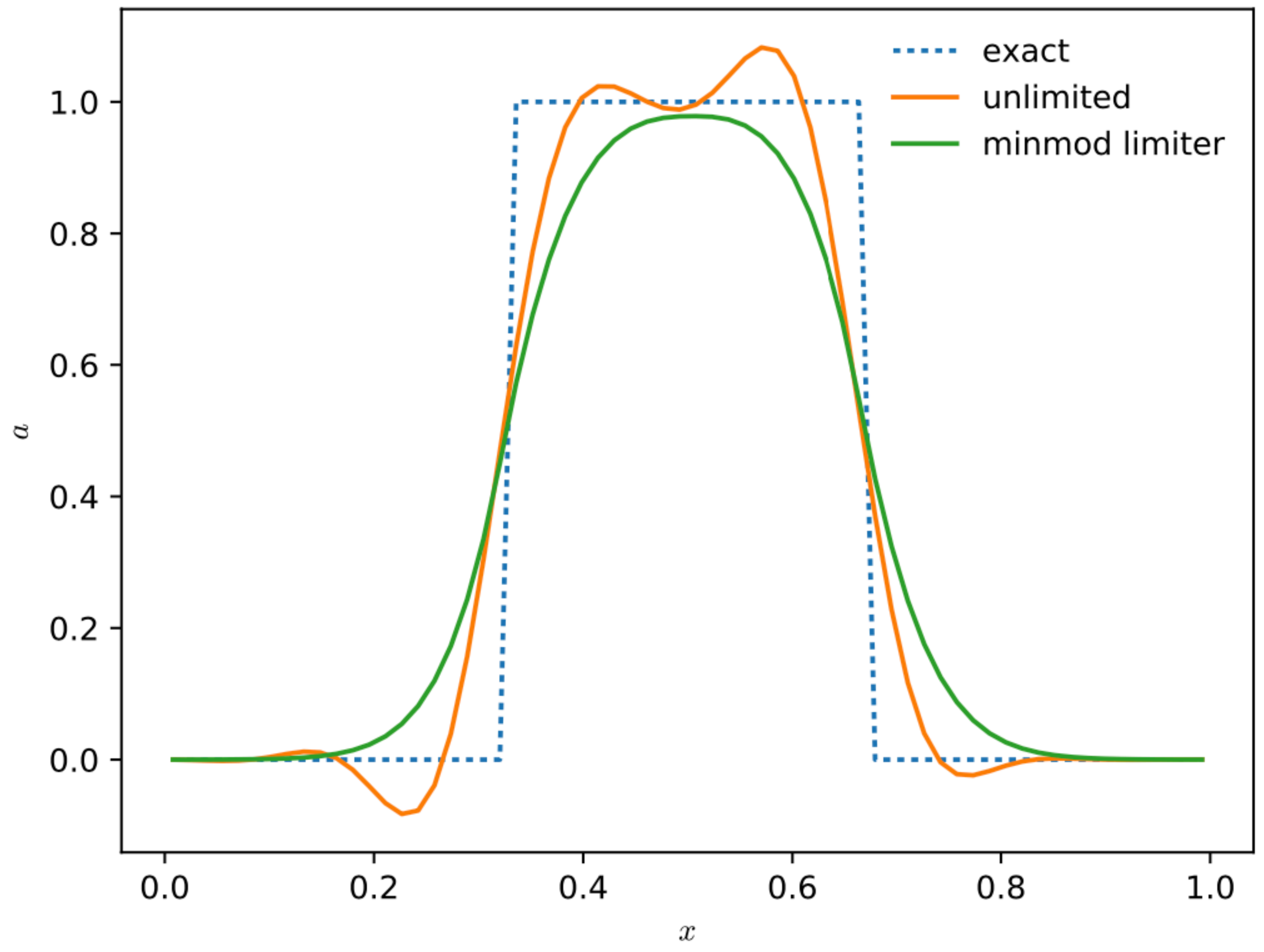
Exercise 5.1

Write a second-order solver for the linear advection equation. To mimic a real hydrodynamics code, your code should have routines for finding initializing the state, filling boundary conditions, computing the timestep, constructing the interface states, solving the Riemann problem, and doing the update. The problem flow should look like:

- *set initial conditions*
- *main evolution loop—loop until final time reached*
  - *fill boundary conditions*
  - *get timestep (Eq. 4.5)*
  - *compute interface states (Eqs. 5.6 and 5.7)*
  - *solve Riemann problem at all interfaces (Eq. 5.10)*
  - *do conservative update (Eqs. 5.4 and 5.5)*

Use both the top-hat and Gaussian initial conditions and periodic boundary conditions and compare to the first-order method.

Introduction to CFD: Limiting



The second-order method likely showed some oscillations in the solution, especially for the top-hat initial conditions.

Godunov's theorem says that any monotonic linear method for advection is first-order accurate.

In this context, monotonic means that no new minima or maxima are introduced.

The converse is true too, which suggests that in order to have a second-order accurate method for this linear equation, the algorithm itself must be nonlinear.

Introduction to CFD: the minmod limiter

Slope:

$$\left.\frac{\partial a}{\partial x}\right|_i = \frac{a_{i+1} - a_{i-1}}{2\Delta x} \tag{5.8}$$

Exercise 5.2

To remove the oscillations in practice, we limit the slopes to ensure that no new minima or maxima are introduced during the advection process. There are many choices for limited slopes. A popular one is the minmod *limiter. Here, we construct the slopes in the interface states as:*

$$\left.\frac{\partial a}{\partial x}\right|_i = \texttt{minmod}\left(\frac{a_i - a_{i-1}}{\Delta x}, \frac{a_{i+1} - a_i}{\Delta x}\right) \tag{5.11}$$

instead of Eq. 5.8. *with*

$$\texttt{minmod}(a, b) = \begin{cases} a & \textit{if } |a| < |b| \text{ and } a \cdot b > 0 \\ b & \textit{if } |b| < |a| \text{ and } a \cdot b > 0 \\ 0 & \textit{otherwise} \end{cases} \tag{5.12}$$

Use this slope in your second-order advection code and notice that the oscillations go away—see Figure 5.4.

Introduction to CFD: the minmod limiter

We can get a feel for what happens with and without limiting pictorially.

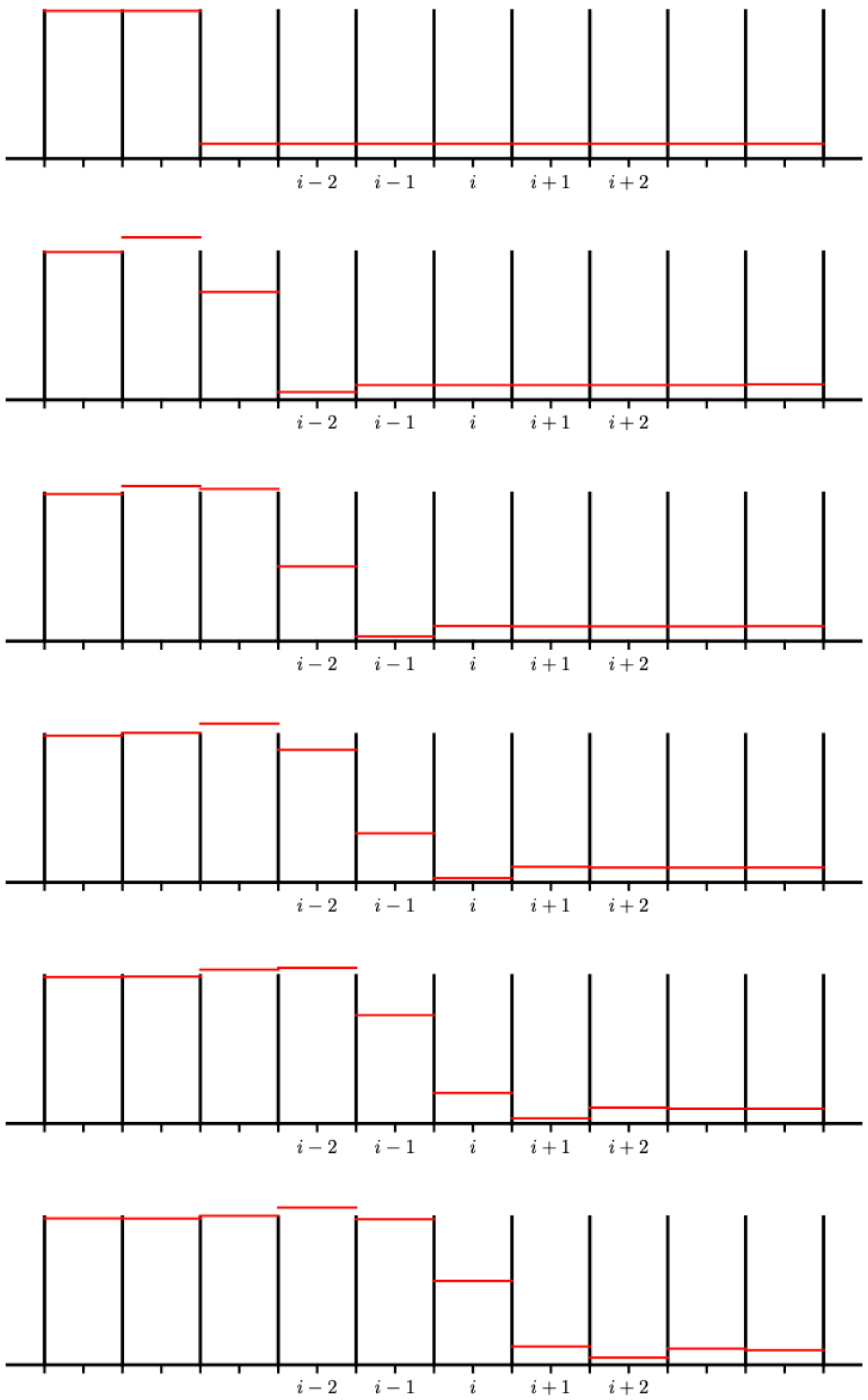
Figures on the next slide show the evolution of an initial discontinuity with and without limiting.

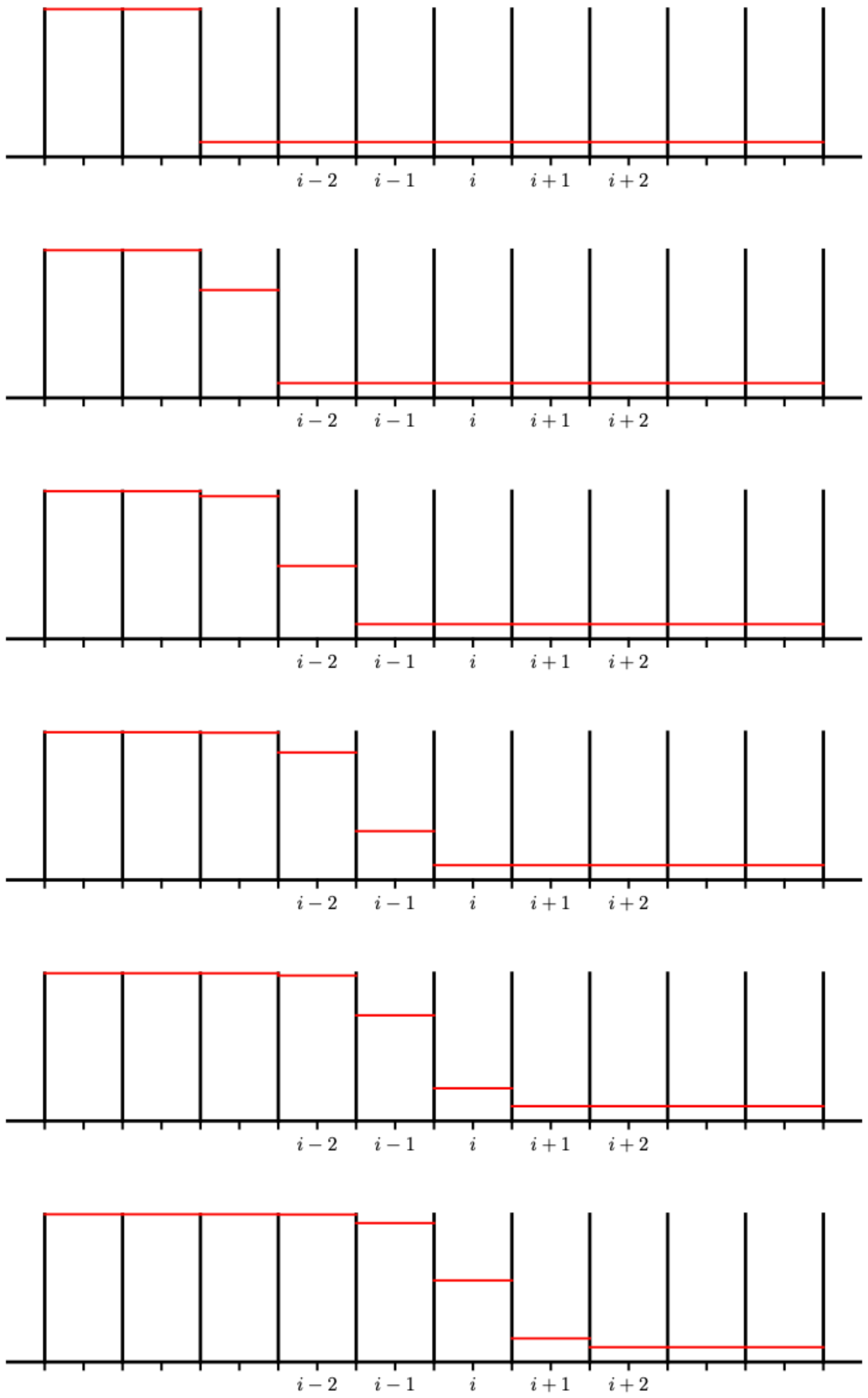
Without limiting, we see an overshoot behind the discontinuity and an undershoot ahead of it—these develop after only a single step.

With each additional step, the overshoots and undershoots move further away from the discontinuity.

In contrast, the case with limiting shows no over- or undershoots around the initial discontinuity. By the end of the evolution, we see that the profile is much narrower in the limiting case than in the case without limiting.

Introduction to CFD: No Limiting versus Limiting





Introduction to CFD: Other limiters?

Note:
Most limiters will have some sort of test on the product of a left-sided and right-sided difference ($a \cdot b$ above)—this is < 0 at an extremum, which is precisely where we want to limit.

A slightly more complex limiter is the MC limiter (monotonized central difference). First we define an extrema test,

$$\xi = (a_{i+1} - a_i) \cdot (a_i - a_{i-1}) \tag{5.13}$$

Then the limited difference is:

$$\left.\frac{\partial a}{\partial x}\right|_i = \begin{cases} \min\left[\frac{|a_{i+1}-a_{i-1}|}{2\Delta x}, 2\frac{|a_{i+1}-a_i|}{\Delta x}, 2\frac{|a_i-a_{i-1}|}{\Delta x}\right] \operatorname{sign}(a_{i+1} - a_{i-1}) & \xi > 0 \\ 0 & \textit{otherwise} \end{cases} \tag{5.14}$$

Introduction to CFD: The MC limiter

The main goal of a limiter is to reduce the slope near extrema. Figure 5.7 shows a finite-volume grid with the original data, cell-centered slopes, and MC limited slopes.

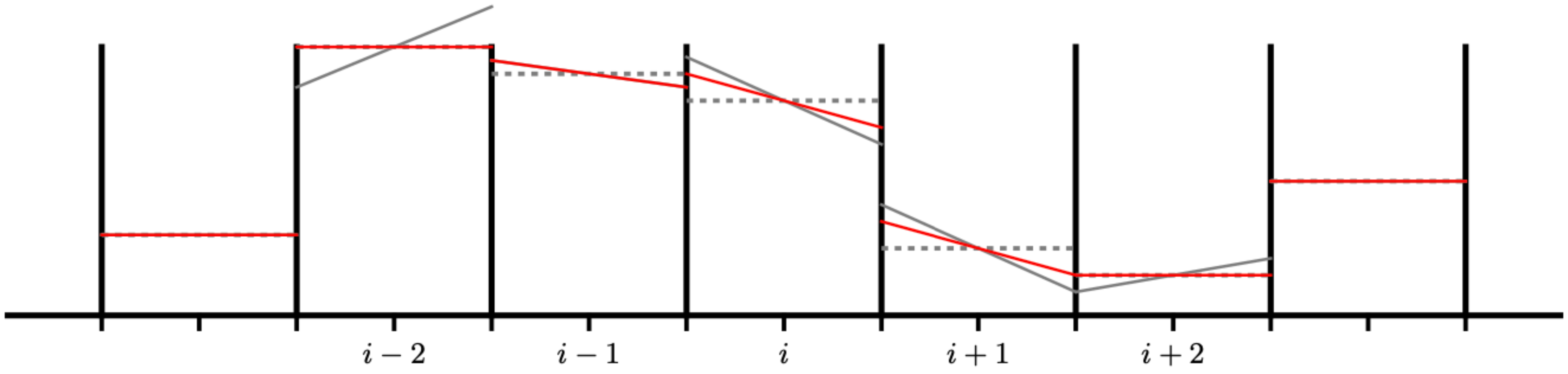


Figure 5.7: A finite-volume grid showing the cell averages (gray, dotted, horizontal lines), unlimited center-difference slopes (gray, solid) and MC limited slopes (red). Note that in zones i and $i+1$, the slopes are limited slightly, so as not to overshoot or undershoot the neighboring cell value. Cell $i-1$ is not limited at all, whereas cells $i-2$, and $i+2$ are fully limited—the slope is set to 0—these are extrema.

Introduction to CFD: The MC limiter

Note that near the strong gradients is where the limiting kicks in. The different limiters are all constructed by enforcing a condition requiring the method to be *total variation diminishing*, or TVD.

It is common to express the slope simply as the change in the state variable:

$$\Delta a_i = \left.\frac{\partial a}{\partial x}\right|_i \Delta x \tag{5.15}$$

and to indicate the limited slope as $\overline{\Delta a_i}$.