A Classification of (2+1)D Topological Phases with Symmetries

by

Tian Lan

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2017

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner	Arun Paramekanti Associate Professor University of Toronto
Supervisor(s)	Xiao-Gang Wen Adjunct Professor Massachusetts Institute of Technology
	Roger Melko Associate Professor Physics and Astronomy
Internal Member	Anton Burkov Associate Professor Physics and Astronomy
Internal-external Member	Florian Girelli Associate Professor Applied Mathematics
Other Member(s)	Davide Gaiotto Adjunct Professor Perimeter Institute for Theoretical Physics

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis aims at concluding the classification results for topological phases with symmetry in 2+1 dimensions. First, we know that "trivial" (i.e., not topological) phases with symmetry can be classified by Landau symmetry breaking theory. If the Hamiltonian of the system has symmetry group G_H, the symmetry of the ground state, however, can be spontaneously broken and thus a smaller group G. In other words, different symmetry breaking patterns are classified by $G \subset G_H$.

For topological phases, symmetry breaking is always a possibility. In this thesis, for simplicity we assume that there is no symmetry breaking; equivalently we always work with the symmetry group G of the ground states. We also restrict to the case that G is finite and on-site.

The classification of topological phases is far beyond symmetry breaking theory. There are two main exotic features in topological phases: (1) protected chiral, or non-chiral but still gapless, edge states; (2) fractional, or (even more wild) anyonic, quasiparticle excitations that can have non-integer internal degrees of freedom, fractional charges or spins and non-Abelian braiding statistics. In this thesis we achieved a full classification by studying the properties of these exotic quasiparticle excitations.

Firstly, we want to distinguish the exotic excitations with the ordinary ones. Here the criteria is whether excitations can be created or annihilated by *local operators*. The ordinary ones can be created by local operators, such as a spin flip in the Ising model, and will be referred to as local excitations. The exotic ones can not be created by local operators, for example a quasi-hole excitation with 1/3 charge in the $\nu = 1/3$ Laughlin state. Local operators can only create quasi-hole/quasi-electron pairs but never a single quasi-hole. They will be referred to as *topological excitations*.

Secondly, we know that local excitations always carries the representations of the symmetry group G. This constitutes the first layer of our classification, a symmetric fusion category, $\mathcal{E} = \mathrm{Rep}(G)$ for boson systems or $\mathcal{E} = \mathrm{sRep}(G^f)$ for fermion systems, consisting of the representations of the symmetry group and describing the local excitations with symmetry.

Thirdly, when we combine local excitations and topological excitations together, all the excitations in the phase must form a consistent anyon model. This constitutes the second layer of our classification, a unitary braided fusion category $\mathcal{C}$ describing all the quasiparticle excitations in the bulk. It is clear that $\mathcal{E} \subset \mathcal{C}$. Due to braiding non-degeneracy, the subset of excitations that have trivial mutual statistics with all excitations (namely the

Müger center) must coincide with the local excitations $\mathcal{E}$. Thus, $\mathcal{C}$ is a non-degenerate unitary braided fusion category over $\mathcal{E}$, or a UMTC$_{/\mathcal{E}}$.

However, it turns out that only the information of excitations in the original phase is not enough. Most importantly, we miss the information of the protected edge states. To fix this weak point, we consider the extrinsic symmetry defects, and promote them to dynamical excitations, a.k.a., "gauge the symmetry". We fully gauge the symmetry such that the gauged theory is a bosonic topological phase with no symmetry, described by a unitary modular tensor category $\mathcal{M}$, which constitutes the third layer of our classification. It is clear that $\mathcal{M}$ contains all excitations in the original phase, $\mathcal{C} \subset \mathcal{M}$, plus additional excitations coming from symmetry defects. It is a minimal modular extension of $\mathcal{C}$. $\mathcal{M}$ captures most information of the edge states and in particular fixes the chiral central charge of the edge states modulo 8.

We believe that the only thing missing is the E_8 state which has no bulk topological excitations but non-trivial edge states with chiral central charge $c = 8$. So in addition we add the central charge to complete the classification. Thus, topological phases with symmetry are classified by $(\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, c)$.

We want to emphasize that, the UBFCs $\mathcal{E}, \mathcal{C}, \mathcal{M}$ consist of large sets of data describing the excitations, and large sets of consistent conditions between these data. The data and conditions are complete and rigid in the sense that the solutions are discrete and finite at a fixed rank.

As a first application, we use a subset of data (gauge-invariant physical observables) and conditions between them to numerically search for possible topological orders and tabulate them.

We also study the stacking of topological phases with symmetry based on such classification. We recovered the known classification $H^3(G, U(1))$ for bosonic SPT phases from a different perspective, via the stacking of modular extensions of $\mathcal{E} = \mathrm{Rep}(G)$. Moreover, we predict the classification of invertible fermionic phases with symmetry, by the modular extensions of $\mathcal{E} = \mathrm{sRep}(G^f)$. We also show that the UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ determines the topological phase with symmetry up to invertible ones.

A special kind of anyon condensation is used in the study of stacking operations. We then study other kinds of anyon condensations. They allow us to group topological phases into equivalence classes and simplifies the classification. More importantly, anyon condensations reveal more relations between topological phases and correspond to certain topological phase transitions.

Acknowledgements

I would like to thank Xiao-Gang Wen for his supervision and constant support. I have benefited a lot from his deep physical intuitions, enthusiasm and encouragement. Also many thanks to my undergraduate supervisor Liang Kong who has brought me into this exciting research field. He is also a main collaborator of this work and complemented many mathematical details.

I would like to thank Zheng-Cheng Gu, Pavel Etingof, Dmitri Nikshych, Chenjie Wang and Zhenghan Wang for helpful discussions. Besides, I would like to thank my committee members Roger Melko, Davide Giotto and Anton Burkov for their valuable advice and feedback.

I am also grateful to Perimeter Institute for the wonderful work environment.

Dedication

To my parents.

Table of Contents

List of Tables

List of Abbreviations

SPT Symmetry protected topological phases.

SET Symmetry enriched topological phases. Topological phases with symmetry.

UBFC Unitary braided fusion category.

UMTC Unitary modular tensor category. Non-degenerate unitary braided fusion category.

UMTC$_{/\mathcal{E}}$ UMTC over $\mathcal{E}$. Non-degenerate unitary braided fusion category over $\mathcal{E}$.

List of Symbols

$\mathcal{E}$ Symmetric fusion category. $\mathcal{E} = \text{Rep}(G)$ or $\mathcal{E} = \text{sRep}(G^f)$.

$\text{Rep}(G)$ Representation category of bosonic symmetry group G.

$\text{sRep}(G^f)$ Fermionic variant of $\text{Rep}(G)$.

$\mathcal{C}$ $\text{UMTC}_{/\mathcal{E}}$, the main label of topological phases with symmetry $\mathcal{E}$.

$\mathcal{M}$ Modular extension of $\mathcal{C}$.

c The chiral central charge of the edge state.

$(\mathcal{C}, \mathcal{M}, c)$ The complete label of topological phases with symmetry $\mathcal{E}$.

$\boxtimes$ Deligne tensor product.

$\boxtimes_{\mathcal{E}}$ Stacking of topological phases.

$\otimes$ Fusion of anyons. Tensor product in UBFCs.

$\oplus$ Direct sum of anyons.

$\mathbb{Z}$ Integers.

$\mathbb{C}$ Complex numbers.

$\otimes_{\mathbb{C}}$ Tensor product of vector spaces, linear operators or matrices over $\mathbb{C}$.

N Rank of $\mathcal{C}$. Number of simple anyon types.

N_k^{ij} Fusion rules.

s_i Topological spin.

d_i	Quantum dimension.
$c_{A,B}$	Braiding.
X^*	Anti-particle of anyon X. Dual object of X.
$\overline{\mathcal{C}}$	Mirror conjugate of $\mathcal{C}$.
$\overline{S_{ij}}$	Complex conjugate of S_{ij}.
$H^n(G, M)$	n-th cohomology group of G with coefficients in M.
$\mathrm{Hom}_{\mathcal{C}}(A, B)$	The morphisms from A to B in $\mathcal{C}$. $\mathcal{C}$ may be omitted.
$\mathcal{D}_{\mathcal{C}}^{\mathrm{cen}}$	The centralizer of $\mathcal{D}$ in $\mathcal{C}$.
$\mathcal{M}_{ext}(\mathcal{C})$	The set of equivalence classes of modular extensions of $\mathcal{C}$.
$\mathcal{C}_{Ab}$	The full subcategory of Abelian anyons in $\mathcal{C}$ (maximal pointed subcategory).

Chapter 1

Introduction

Gapped quantum phases, or “insulators”, used to be considered boring, whose classification was solved by Landau symmetry breaking theory [1]. However, over the last few decades, many exotic phases with “topological” nature have be discovered, which are beyond the scope of symmetry breaking theory. There are two typical examples of such exotic topological phases:

- Fractional quantum Hall (FQH) states. They have anyonic quasiparticle excitations with fractional charges and fractional statistics. Also their edge states are protected gapless, i.e., not gappable by any boundary interactions. FQH states are considered to present intrinsic topological order [2, 3], which leads to different phases of matter without requiring any symmetry.

- Topological insulators. Similarly they have gapless edge states, which are protected by the symmetry, i.e., not gappable if the symmetry is not broken. Topological insulator is a special case of another large class of phases – symmetry protected topological (SPT) phases [4]. They have no intrinsic topological orders.

Topological phases of matter have drawn more and more research interest. A complete classification has been achieved in 1+1 dimensions [5]. It was proved that in 1+1D there is no topological order, thus the only two ingredients in the classification are symmetry breaking and SPT. Mathematically, symmetry breaking is described by a pair of groups $G \in G_H$, where G_H is the symmetry group of the Hamiltonian, and G is the unbroken subgroup of the ground state; 1+1D SPT phases are classified by projective representations of the symmetry group, or by the second cohomology group $H^2(G, U(1))$. The combination of the two ingredients in 1+1D is obvious.

Intrinsic topological order begins to appear from 2+1D. We need to combine symmetry and topological order into a unified framework, such that SPT phases and intrinsic topological orders with no symmetry are just two extreme special cases. This thesis aims at giving such a classification of (2+1)D topological phases of matter. The final goal is to create a "table" for all topological phases. There are two main questions for a good classification:

1. How to efficiently describe topological phases?

   It turns out the universal properties of topological phases, such as symmetry, ground state degeneracy, quasiparticle statistics, can be well organized into the mathematical framework of unitary braided fusion category. Although category theory is quite abstract, obscure and unfamiliar to most physicists, it is the right and precise language for topological phase. Alternatively, we can use a subset of universal properties to name topological phases. We expect to use as few properties as possible, as long as they are enough to distinguish different phases.

2. How to relate different topological phases?

   Apparently, one way to answer the above question is to study phase transitions between topological phases. However, a general theory for topological phase transitions is still beyond our scope.

   A simple construction to relate different phases may be stacking several layers of topological phases to obtain a new one. We will discuss such stacking operations when symmetry is taken into account.

   We will also discuss phase transitions that are driven by certain anyon condensations. Hopefully, the theory of anyon condensations can provide us a general framework for topological phase transitions.

1.1 Topological Phases and Symmetry

We take topological phase as a synonym of gapped quantum phase. Consider a physical system described by the Hilbert space $\mathcal{V}$ and Hamiltonian H. First, we require that the system admits local structures, namely, the total Hilbert space is the tensor product of local Hilbert spaces on each site, $\mathcal{V} = \otimes_i V_i$, and the Hamiltonian is the sum of local terms, $H = \sum_i H_i$, where H_i acts on only several neighbouring sites around i. Second, we require a finite energy gap $\Delta > 0$ under the thermodynamic limit. States below the gap are the ground states of the system. If the energy gap closes $\Delta \to 0$ when deforming the Hamiltonian, there is a *phase transition*. In other words, two systems belong to the same gapped quantum phase, or topological phase, if they can be deformed into each other without closing the energy gap. Equivalently, their ground states can be related by *local unitary transformations* [6].

A physical system may have some global symmetry, described by a group G. In other words, G has a faithful action on the Hilbert space $g \mapsto U_g \in GL(\mathcal{V})$, and the Hamiltonian remains invariant under such action $U_g H U_g^{-1} = H$. For rigorousness, in this thesis we will restrict to finite on-site symmetries, which means that G is finite and $U_g = \otimes_i U_{g,i}$, where $U_{g,i}$ is a local operator acting around the site i.

When a symmetry is present, the definition of topological phases must be modified a little bit. The deformation of Hamiltonian must respect the symmetry. Topological phases with symmetry are thus equivalence classes under *symmetric* local unitary transformations.

For fermion systems, we consider the fermion number parity, denoted by z, as a special element in the global symmetry group G, which is involutive and central, $z^2 = 1, zg = gz, \forall g \in G$. To distinguish with purely bosonic symmetries, we denote fermionic symmetries by (G, z) or G^f if there is no ambiguity.[1]

In this thesis, the categorical description of symmetry is more common. Roughly speaking, we use the representation category, $\mathcal{E} = \mathrm{Rep}(G)$ for boson systems, or $\mathcal{E} = \mathrm{sRep}(G^f)$[2] for fermion systems, instead of the group, to describe the symmetry.

There are universal properties, which are invariant under symmetric local unitary transformations, that can serve as "labels" of topological phases. In this chapter, topological phases will be denoted by $\mathcal{C}, \mathcal{D}, \ldots$, which can be regarded as the collection of universal properties.

1.2 Stacking Topological Phases

We can *stack* two existing topological phases to obtain a third phase, which is better visualized in (2+1)D by just constructing a two-layer system. The stacking operation is the easiest way to construct new topological phases from old ones, and is a main topic of the thesis.

[1] On the other hand, bosonic symmetry G can be viewed as $(G, z = 1)$.

[2] $\mathrm{sRep}(G^f)$ consists of the same representations as $\mathrm{Rep}(G)$, but the irreducible representations with z acting as -1 are regarded as fermions. Braiding two fermions has an extra phase factor -1.

The most simple case is when there is no symmetry, and we allow any local interactions between layers. We denote such stacking operation by $\boxtimes$. Obviously, it is commutative and associative, $\mathcal{C} \boxtimes \mathcal{D} = \mathcal{D} \boxtimes \mathcal{C}, (\mathcal{C}_1 \boxtimes \mathcal{C}_2) \boxtimes \mathcal{C}_3 = \mathcal{C}_1 \boxtimes (\mathcal{C}_2 \boxtimes \mathcal{C}_3)$. The trivial phase $\mathcal{I}$ (tensor product states) is the identity, $\mathcal{C} \boxtimes \mathcal{I} = \mathcal{I} \boxtimes \mathcal{C}$. Therefore, topological phases form a commutative monoid (a "group" that requires the existence of identity but not inverse) under stacking.

When stacking two systems $\mathcal{C}, \mathcal{D}$ with the symmetries G_1, G_2, there is a choice for the new symmetry of the two-layer system, that puts restrictions on what *symmetric* interactions between layers can be added. One natural choice is $G_1 \times G_2$, denoted by $\mathcal{C} \boxtimes \mathcal{D}$, that is to preserve the symmetry of each layer respectively.

When $G_1 = G_2 = G$, another natural choice for the new symmetry is G, denoted by $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ (recall that $\mathcal{E}$ is the representation category of G) where G is viewed as a subgroup of $G \times G$ via the embedding $g \mapsto (g, g)$. In other words, for the stacking $\boxtimes_{\mathcal{E}}$ we allow the inter-layer interactions that preserve only the subgroup G. $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ share the same symmetry G. Therefore, topological phases with symmetry G again form a commutative monoid under the stacking $\boxtimes_{\mathcal{E}}$ which preserves the symmetry.

A topological phase $\mathcal{C}$ with symmetry $\mathcal{E}$ is called invertible if there exists another phase $\mathcal{D}$ such that $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D} = \mathcal{I}$. In this case $\mathcal{C}$ and $\mathcal{D}$ are time-reversal conjugates. All invertible topological phases form an Abelian group **Inv** under stacking. The chiral central charges of the edge states add up under stacking, so taking the central charge is a group homomorphism from invertible phases **Inv** to $\mathbb{Q}$. Its image is $c_{\min}\mathbb{Z}$, where $c_{\min}$ is the smallest positive central charge. From this point of view, the non-chiral invertible phases (the kernel of the above group homomorphism) are the symmetry protected topological (SPT) phases:

$$0 \to \mathrm{SPT} \to \mathbf{Inv} \to c_{\min}\mathbb{Z} \to 0,$$

Since $H^2(\mathbb{Z}, M) = 0$ for any abelian group M, the above must be a trivial extension, namely

$$\text{Invertible topological phases with symmetry} \cong \mathrm{SPT} \times c_{\min}\mathbb{Z},$$

For boson systems, $c_{\min} = 8$ corresponding to the E_8 state. For fermion systems with symmetry $G^f = G_b \times \mathbb{Z}_2^f$, $c_{\min} = 1/2$ corresponding to the $p + \mathrm{i}p$ superconducting state. But for other fermionic symmetries it is not totally clear what $c_{\min}$ should be.

Invertible phases do not support any non-trivial quasiparticle statistics. For the non-invertible topological phases, we have to seriously study their quasiparticle excitations.

1.3 Quasiparticle Excitations

The properties of excitations play a central role in the study of topological phases. In (2+1)D, excitations whose energy density is non-zero in a local area can always be viewed as a particle-like excitation, and will be referred to as quasiparticles or anyons.

We would like to group quasiparticles into several *topological types* depending on whether they can be related by local operators. It is easy to think about the trivial topological type, i.e., quasiparticles that can be created from or annihilated to the ground state, by local operators. We also call the trivial type *local excitations*. At first glance, one may wonder, "are there exotic excitations beyond the local ones?" The answer is "Yes", and the very existence of non-trivial topological types is exactly the most important signature of *topological orders* [2, 3]. We call the non-trivial types *topological excitations*. They can not be created or annihilated by local operators. Different topological types may also be referred to as carrying different topological charges or in different superselection sectors.

When dealing with topological phases with symmetry, we need to regroup quasiparticles into different types; each type is related by *symmetric* local operators. In particular, if the degeneracy of a quasiparticle can not be lifted by any symmetric local operators, for example a local excitation carrying an irreducible representation of the symmetry group, we say that it is of a *simple type*. A generic quasiparticle of a *composite type*, for example a local excitation carrying a reducible representation, is the direct sum of several simple types. The total number of simple types in a topological phase with symmetry is called its *rank*, denoted by N.

It is natural to ask "what properties of the excitations are invariant under (symmetric) local operators". It turns out these properties can be well organized in terms of two kinds of processes, fusion and braiding. Moreover, it seems that the properties of excitations can determine all the universal properties of topological phases up to invertible ones, even those non-local properties such as the ground state degeneracies on arbitrary manifolds. This constitutes the main approach of this thesis: classify topological phases by the fusion and braiding properties of quasiparticles, or mathematically, by unitary braided fusion categories (UBFC). We find that topological phases with symmetry are classified by a sequence of UBFCs, $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$, plus a central charge c. They correspond to {local excitations}$\subset${all excitations}={local excitations plus topological excitations}$\subset${excitations in the gauged theory}={all excitations plus gauged symmetry defects} respectively.

On the other hand, unitary braided fusion category is a very rigid structure. Physically this means that the fusion of braiding of quasiparticles must satisfy a series of consistent conditions, such that for a fixed rank, there are only finite solutions [7]. This allows us to numerically search for topological phases with symmetries, using a subset of these conditions, and tabulate them [8–10].

We will also study the condensation of anyons which relates different topological phases. This includes two variants:

1. Condense a (self-)boson into the trivial state. This will produce a new topological phase with the same central charge, and a gapped domain wall between the old and the new phases. Symmetry breaking is a special case of such condensations.

2. Condense Abelian anyons into a Laughlin state. This will produce a new topological phase with the same symmetry and similar non-Abelian content.

Chapter 2

Categorical Description of Symmetry and Excitations

To describe specific physical phenomena, it is important to identify the corresponding mathematical languages. Newton invented calculus to describe the gravity and dynamics of classical world. In later developments, physics seems to fall a little behind mathematics: linear algebra for quantum mechanics, Riemann geometry for general relativity, group theory for symmetry, and so on. Now for topological phases of matter, one of the most important mathematical languages is the theory of tensor categories.

In this chapter we try to give an introduction to category theory. But, instead of listing a series of mathematical definitions, we aim at building a mathematics-physics dictionary.

2.1 The Categorical Viewpoint

Categories have two levels of structures: objects and morphisms. Morphisms are the "arrows", or operations, on the objects. It is not harmful to interpret objects as "physical objects", and morphisms as "physical measurements". The most important philosophy of category theory is to take objects as black boxes and focus on the morphisms, which is like measuring the unknown physical objects.

We denote the set of morphisms from object A to object B by $\mathrm{Hom}(A, B)$. As most valuable intuitions come from the "trivial" example, let's elaborate the idea in the simplest tensor category, the category of finite-dimensional vector spaces **Vec**. In **Vec**, objects are vector spaces, while morphisms are linear operators. Now if we pretend to know nothing about the objects in **Vec**, how can we recover their vector space structures via the morphisms, i.e., the linear operators? It is an easy observation, that any vector space V is the same as the vector space of linear operators from the base field[1] to itself, $V \cong \mathrm{Hom}(\mathbb{C}, V)$, by identifying a vector $|x\rangle$ with the linear map $f(\alpha) = \alpha|x\rangle, \alpha \in \mathbb{C}$. Thus, linear operators (morphisms) already tell us everything about **Vec**; the vector space structures of the objects are redundant information.

Why is this categorical point of view important to physicists? Because it is exactly the physical point of view. In more general cases, objects may be interpreted as particles or other physical systems, and morphisms as evolutions, interactions or other physical operations. We have to admit that all the physical systems are indeed black boxes to us; we can not know anything about them unless we try to measure or observe them. All our information about the physical world comes from the interactions between various systems and ourselves, i.e., from the "morphisms". This is just like the **Vec** example. We are like the "base field $\mathbb{C}$", and all we know is what we observe, $\mathrm{Hom}(\mathbb{C}, V)$.

2.2 Unitary Braided Fusion Category

Now let us restrict to the case of topological phases of matter and try to give a specific dictionary. We interpret as follows:

- **Object** Quasiparticle excitation, anyon
- **Morphism** Operator acting on quasiparticles, but up to symmetric local operators

If we collect all possible quasiparticle excitations in a topological phase, they naturally form a category with additional structures, which is a unitary braided fusion category. Again we interpret the additional structures term by term:

- **Unitary** A structure inherited from the inner product of Hilbert spaces (physical measurements). For the morphisms, it means operators have Hermitian conjugates. As the simplest example, **Hilb**, the category of Hilbert spaces, is just the unitary version of **Vec**, the category of vector spaces.
- **Fusion** Bring several quasiparticles together and view them as one quasiparticle. In practice, we need to fuse quasiparticles one by one along certain direction; thus

[1] In this thesis we always take the base field to be complex numbers $\mathbb{C}$

fusion is represented by a two-to-one tensor product. We denote the tensor product by $A \otimes B$. The trivial anyon (vacuum), denoted by $\mathbf{1}$, is the unit of the tensor product, $\mathbf{1} \otimes A \cong A \cong A \otimes \mathbf{1}$. Also it must be associative $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. This associative condition only holds up to some local operators, which can be represented by the F-matrix.

- **Braiding** In (2+1)D, the result of tensor product should not depend on the direction we choose. In particular, $A \otimes B$ should be naturally isomorphic to $B \otimes A$. The natural isomorphism can be generated by the braiding process: move A to the other side of B. There are two different paths to do this, clockwise or counter-clockwise. We denote the clockwise path by $c_{A,B} : A \otimes B \cong B \otimes A$, the counter-clockwise path is then given by $c^{-1}_{B,A} : A \otimes B \cong B \otimes A$. Braiding can be represented by the R-matrix. Detailed definition of F, R-matrices, also known as $6j$-symbols, F, R-symbols, can be found in, for example, Refs. [11–13].

- **Simple objects** Quasiparticles whose degeneracy can not be lifted by local operators, namely of simple types. These simple anyons are labeled by lowercase letters $i, j, k, \dots$.

- **Direct sum** A generic (composite) quasiparticle excitation can be the degenerate states of several simple quasiparticles. Such degeneracy is "accidental" and can be lifted by local operators. We can not avoid such cases because the fusion of two (even simple) anyons is in general a direct sum of simple ones. This is denoted by $i \otimes j \cong \bigoplus_k N^{ij}_k k$, where N^{ij}_k is a non-negative integer describing the multiplicity of k in $i \otimes j$. A classical example is that two spin 1/2 fuse into the direct sum of spin 0 and spin 1: $1/2 \otimes 1/2 = 0 \oplus 1$.

- **Dual objects** Anti-particle X^* of anyon X. $X \otimes X^*$ contains a single copy of the trivial anyon $\mathbf{1}$, $X \otimes X^* \cong \mathbf{1} \oplus \cdots$.

- **Quantum Trace** For an operator f acting on X, its quantum trace $\operatorname{Tr} f$ is the expectation value of the following process up to proper normalization: create a pair X, X^*, act f on X, and then annihilate the pair. We will pick the normalization such that $\operatorname{Tr} \mathrm{id}_{\mathbf{1}} = 1$. Then the quantum dimension is $d_i = \operatorname{Tr} \mathrm{id}_i$, and total quantum dimension is $D^2 = \sum_i d_i^2$. The topological S, T-matrix are given by the quantum trace $T_{ij} = \delta_{ij} \operatorname{Tr} c_{i,i}/d_i$, $S_{ij} = \operatorname{Tr} c_{j^*,i} c_{i,j^*}/D$.

2.3 Local Excitations and Group Representations

Next we study the local excitations and show their relations to the symmetry group G.

Let the unitary braided fusion category of all quasiparticle excitations be $\mathcal{C}$. Take the category of all local excitations, they form a full subcategory of $\mathcal{C}$, denoted by $\mathcal{E}$. Local excitations can be created by acting local operators O on the ground state $|\psi\rangle$. For any group action U_g, $U_gO|\psi\rangle = U_gOU_g^\dagger U_g|\psi\rangle = U_gOU_g^\dagger|\psi\rangle$ is an excited state with the same energy as $O|\psi\rangle$. Since we assume the symmetry to be on-site, $U_gOU_g^\dagger$ is also a local operator. Therefore, $U_gOU_g^\dagger|\psi\rangle$ and $O|\psi\rangle$ correspond to the degenerate local excitations. Thus, we should group the states $\{U_gO|\psi\rangle, \forall g \in G\}$ as a whole and view them as the same type of excitation. The local Hilbert space spanned by $\{U_gO|\psi\rangle, \forall g \in G\}$, is then a representation of G. It is in general a reducible representation but can be reduced to irreducible ones by symmetric local operators.[2] We see that local excitations "locally" carry group representations. Simple types of local excitations are labeled by irreducible representations of the symmetry group G.

As a fusion category (forget the braidings) we have $\mathcal{E} \cong \mathrm{Rep}(G)$. Further considering the braidings we find that there are two possibilities, as fermions braid with each other with an extra -1. For boson systems, $\mathcal{E} = \mathrm{Rep}(G)$ with the usual braiding for vector spaces, $c_{A,B}(a \otimes_{\mathbb{C}} b) = b \otimes_{\mathbb{C}} a$, $a \in A, b \in B$. For fermion systems $\mathcal{E} = \mathrm{sRep}(G^f)$, with a modified braiding, $c_{A,B}(a \otimes_{\mathbb{C}} b) = -b \otimes_{\mathbb{C}} a$ when $a \in A, b \in B$ are both fermionic (fermion parity z acts as -1, $za = -a, zb = -b$).

The most important thing is that the category $\mathcal{E}$ can recover the symmetry group G via Tannaka Duality. In other words, the information of morphisms in $\mathcal{E}$ alone (the data of fusion and braiding of local excitations), are enough for us to recognize the group. An easy example is that for an Abelian group, its irreducible representations form the same group under tensor product (fusion). For more general cases it is also true, by Deligne's theorem [14]. These special categories, whose braidings are either bosonic or fermionic, are named symmetric categories.

Instead of describing the symmetry by a group, it is equivalent to say finite on-site symmetry is given by a symmetric fusion category $\mathcal{E}$. A huge advantage is that in the categorical viewpoint, boson and fermion systems are treated equally.

[2] Symmetric local operators commute with group actions U_g, thus are the *intertwiners* between group representations.

2.4 Trivial Mutual Statistics, Braiding Non-degeneracy

Another important property of the local excitations is that they have *trivial mutual statistics* with all quasiparticles. By trivial mutual statistics, between quasiparticles A and B, we mean that moving A along a whole loop around B makes no difference. In terms of braidings, trivial mutual statistics means that a double braiding is the same as the identity, $c_{B,A}c_{A,B} = \mathrm{id}_{A\otimes B}$.

It is easy to see the reason why local excitations always have trivial mutual statistics. Assume that B is a local excitation. Moving A around B is the same as the following process: first annihilate B, second move A around nothing, and then create B again. This is because the operators that hopping A around, and the local operators annihilating and creating B, do not overlap and thus commute. Moving A around nothing surely makes no difference, therefore, we know that any quasiparticle A has trivial mutual statistics with a local excitation B.

On the other hand, if a quasiparticle has trivial mutual statistics with all quasiparticles, we claim that it must be a local excitation. This follows from the idea of *braiding non-degeneracy*, which means that for an *anomaly-free*[3] topological phase, everything non-local must be detectable via braidings. The subcategory of quasiparticles that have trivial mutual statistics with all quasiparticles, which is called the Müger center [15], must coincide with the subcategory of local excitations, which is determined by the symmetry of the system, a symmetric fusion category $\mathcal{E}$.

2.5 Modular Extensions and the Full Classification

By now we have outlined all the properties of the quasiparticle excitations in a topological phase with symmetry $\mathcal{E}$; they form a unitary braided fusion category $\mathcal{C}$ whose Müger center coincides with $\mathcal{E}$. But is this enough to fully characterize the phase?

Let us return to the invertible phases to check what is lacking. A topological phase is invertible under stacking if and only if it has only local excitations. So the question becomes whether $\mathcal{C} = \mathcal{E}$ describes all invertible phases. The answer is of course "No".

We know that if there is no symmetry, $\mathcal{C} = \mathcal{E} = \mathbf{Vec}$, invertible topological phases have a $\mathbb{Z}$ classification; they are generated by the E_8 state, with central charge $c = 8$, via

[3]An anomaly-free $(n+1)D$ phase can be realized by an n dimensional lattice model, without the help of an $n+1$ dimensional bulk.

stacking and time-reversal. Since the central charge is added up under stacking, it is a good quantity that complements the label of topological phases.

For boson systems with symmetry G, we have $\mathcal{C} = \mathcal{E} = \mathrm{Rep}(G)$. But they are classified by $H^3(G, U(1)) \times 8\mathbb{Z}$, where the $8\mathbb{Z}$ part is the E_8 states as explained above, and the $H^3(G, U(1))$ part is the classification of the symmetry protected topological phases [4]. We need to seek for additional structures to recover such $H^3(G, U(1))$ classification.

A good classification of invertible fermionic phases with symmetry, however, is still lacking. Since our categorical viewpoint treats bosonic and fermionic cases equally, it is promising to make predictions on the classification of invertible fermionic phases.

So besides the central charge, what should be added to the categorical description of topological phases with symmetry? Motivated by the idea of gauging the symmetry [16], we propose that the *modular extension* $\mathcal{M}$ of $\mathcal{C}$ should be included to fully characterize the phase. When gauging the symmetry, one adds extrinsic symmetry defects to the system and promotes the global symmetry to a dynamical gauge group. Extrinsic symmetry defects then become dynamical gauge flux excitations. Thus, extra quasiparticles (gauged symmetry defects) are added in the gauged theory, which can detect the local excitations in the original theory via braidings. We fully gauge the symmetry, such that the gauged theory is a topological phase with no symmetry (the Müger center of $\mathcal{M}$ is trivial). So, $\mathcal{M}$ is a unitary *modular* tensor category (UMTC), and $\mathcal{C}$ is a full subcategory of $\mathcal{M}$, hence the name modular extension. We need to avoid adding something unrelated to the original theory, therefore, the modular extension is required to be a "minimal" one, in the sense that the extra quasiparticles must have non-trivial mutual statistics with at least one quasiparticle in $\mathcal{E}$. In the following all modular extensions are assumed minimal unless specified otherwise. It is possible that certain $\mathcal{C}$ has no modular extension, which means that the symmetry is not gaugable. We consider it another anomaly-free condition that $\mathcal{C}$ must have modular extensions.

To conclude, for topological phases with a given symmetry $\mathcal{E}$, we propose that they are classified by the triple $(\mathcal{C}, \mathcal{M}, c)$, where $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$. More precisely,

- $\mathcal{E}$ is the symmetric fusion category of local excitations. It is the representation category of the symmetry group, $\mathcal{E} = \mathrm{Rep}(G)$ for boson systems and $\mathcal{E} = \mathrm{sRep}(G^f)$ for fermion systems.
- $\mathcal{C}$ is the unitary braided fusion category of all quasiparticle excitations in this phase. Its Müger center is $\mathcal{E}$, equivalently, $\mathcal{C}$ is a non-degenerate UBFC over $\mathcal{E}$, or a $\mathrm{UMTC}_{/\mathcal{E}}$.

- $\mathcal{M}$ is the unitary braider fusion category of quasiparticle excitations in the symmetry-fully-gauged theory of this phase. It is a UMTC, and a minimal modular extension of $\mathcal{C}$.
- c is the central charge, from which we can determine the layers of E_8 states.

We will give more strict definitions of the above in Section 4.2.

By now, our proposal has no experimental support yet. Our confidence mainly comes from the astonishing consistency between the mathematical framework, known examples, and physical intuitions. As introduced above, topological phases can be stacked, about which we have good physical intuitions. So it should also be possible to define a stacking for the triple $(\mathcal{C}, \mathcal{M}, c)$. Later we will seriously study such stacking, in strict mathematical language. The known classification of invertible bosonic topological phases with symmetries, obtained earlier by other approaches, is also recovered.

Chapter 3

Universal Physical Observables

We expect that UBFCs contain all the universal properties of quasiparticle excitations, however, it is usually desirable to use only a subset of universal properties to "name" topological orders. This subset consists of only the *physical observables* which do not depend on the choice of basis or gauge. In fact, if we collect all the physical observables, they should uniquely determine the topological phase. We first introduce a "physical representation" of category theory.

3.1 Fusion Space, Fusion Rules and Topological Spin

Imagine we have a two dimensional manifold M^2 with n punctures at $x_1, x_2, \ldots, x_n$. Physically, we put the system on M^2, and turn off local Hamiltonian terms around the punctures x_a,[1] such that any possible n-anyon configurations becomes degenerate with the ground state. For simplicity, we assume a boson system with no symmetry. We may further add local perturbations δH_a around x_a to lift the degeneracy. In particular, we can fix the anyon types ξ_a at x_a one by one, x_1 through x_{n-1}. The last (composite) anyon ξ_n at x_n is then fixed by the topology of M^2 and the fusion of the other $n-1$ anyons, and we may use local operators around x_n to reduce ξ_n to a simple type. After all these, we can still have a degenerate space,

$$\mathcal{V}(M^2, \xi_1, \xi_2, \ldots, \xi_n),$$

[1] Set the local term H_a to a constant which equals the ground state energy $\langle H_a \rangle$.

which will be called a fusion space. In particular,

$$\mathcal{V}(S^2,\xi_1,\xi_2,\dots,\xi_n)=\mathrm{Hom}(\mathbf{1},\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_n). \tag{3.1}$$

We then extract the basis independent properties of the fusion spaces. First consider their dimensions. The degeneracy of a sphere S^2 with no quasiparticle, or with quasiparticle and anti-quasiparticle pair, should be 1. For a torus T^2, as it can be obtained from gluing a cylinder, namely a sphere S^2 with two punctures, we have

$$\mathcal{V}(T^2)=\bigoplus_i \mathcal{V}(S^2,i,i^*). \tag{3.2}$$

Thus the *rank*, number of simple anyon types, is related to the ground state degeneracy on the torus[2]

$$N=\dim\mathcal{V}(T^2). \tag{3.3}$$

The *fusion rules* $i\otimes j=\oplus_k N^{ij}_k k$ can also be determined in terms of the dimension of a fusion space,

$$N^{ij}_k=\dim\mathrm{Hom}(k,i\otimes j)=\dim\mathcal{V}(S^2,i,j,k^*). \tag{3.4}$$

Another important observable is the *topological spin*, or simply spin, denoted by s_i. If we rotate an anyon i by 2π, the fusion space $\mathcal{V}(-,i,\dots)$ will acquire a phase factor $\mathrm{e}^{2\pi\mathrm{i}s_i}$ regardless of the background manifold and anyons. Thus s_i is just (the fractional part of) the internal angular momentum of anyon i.

In principle we should also explore other observables, while in practice it seems that (N^{ij}_k,s_i) are enough. They can distinguish all the known examples of bosonic topological orders with no symmetry, or UMTCs. When there is symmetry, we essentially use subcategories of UMTCs. Therefore, by now we use (N^{ij}_k,s_i) as a short label or name for topological phases. Alternatively, we can use the topological S,T matrices (see next section). If new examples come out whose (N^{ij}_k,s_i) conflicts and reveal new observables that we have missed, we can just append those new observables to the short label and make sure topological phases have a unique name.

[2] The ground state degeneracy on the torus is always the same as the number of simple *topological types*, which is the rank when there is no symmetry. But, in the presence of symmetry, since we count anyon types differently, there is no simple relation between the rank and the degeneracy.

In practice, the fusion rules N^{ij}_k are still too huge. So we may use the *quantum dimension* d_i to represent fusion. Physically d_i measures the "internal degrees of freedom" of anyon i. It can be extracted from the fusion space dimension via

$$\dim \mathcal{V}(M^2,\ \underbrace{i,\ldots,i}_{n\text{ copies of }i},\ldots) \sim d_i^n, \quad n\to\infty, \tag{3.5}$$

or

$$\ln d_i = \lim_{n\to\infty} \frac{\ln \dim \mathcal{V}(M^2, i^{\otimes n},\ldots)}{n}. \tag{3.6}$$

d_i gives a one-dimensional representation of fusion

$$d_i d_j = \sum_k N^{ij}_k d_k. \tag{3.7}$$

3.2 Conditions for Physical Observables

In this section we list the conditions for the physical observables [12, 17–20]. They are only necessary conditions derived from UBFCs, but in general not sufficient to describe a valid UBFC. However, they are quite useful in numerical searches for candidate topological phases [8–10]. In Refs. [9, 10] with these conditions and $\mathcal{E}\subset\mathcal{C}\subset\mathcal{M}$ we have successfully generated large tables for topological phases with symmetries, from which we selected typical ones and listed in Appendix D.

1. *Fusion ring*:
   N^{ij}_k for the UBFC $\mathcal{C}$ are non-negative integers that satisfy

   $$N^{ij}_k = N^{ji}_k,\quad N^{1i}_j=\delta_{ij},\quad \sum_{k=1}^N N^{ik}_1 N^{kj}_1 = \delta_{ij}, \tag{3.8}$$
   $$\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{in}_l N^{jk}_n \text{ or } \sum_m N^{ij}_m N_m = N_i N_j \text{ or } N_k N_i = N_i N_k.$$

   where the matrix N_i is given by $(N_i)_{kj} = N^{ij}_k$, and the indices i,j,k run from 1 to N. In fact N^{ij}_1 defines a charge conjugation $i\to i^*$:

   $$N^{ij}_1 = \delta_{ij^*}. \tag{3.9}$$

   N^{ij}_k satisfying the above conditions define a fusion ring.

2. *Rational condition*:
   N^{ij}_k and s_i for $\mathcal{C}$ satisfy [17, 21–23]

$$\sum_r V^r_{ijkl} s_r = 0 \text{ mod } 1 \tag{3.10}$$

   where

$$\begin{aligned} V^r_{ijkl} = {} & N^{ij}_r N^{kl}_{r^*} + N^{il}_r N^{jk}_{r^*} + N^{ik}_r N^{jl}_{r^*} \\ & - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{m^*} \end{aligned} \tag{3.11}$$

3. *Verlinde fusion characters*:
   The topological S-matrix is given by [see eqn. (223) in Ref. [11]]

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k \mathrm{e}^{2\pi \mathrm{i}(s_i + s_j - s_k)} d_k, \tag{3.12}$$

   where the quantum dimension d_i is the largest eigenvalue of the matrix N_i and $D = \sqrt{\sum_i d_i^2}$ is the total quantum dimension. Then [21]:

$$\frac{S_{il} S_{jl}}{S_{1l}} = \sum_k N^{ij}_k S_{kl}. \tag{3.13}$$

4. *Weak modularity*:
   The topological T-matrix is given by

$$T_{ij} = \delta_{ij} \mathrm{e}^{2\pi \mathrm{i} s_i}. \tag{3.14}$$

   Then [see eqn. (235) in Ref. [11]]

$$S^\dagger T S = \Theta T^\dagger S^\dagger T^\dagger, \quad \Theta = D^{-1} \sum_i \mathrm{e}^{2\pi \mathrm{i} s_i} d_i^2. \tag{3.15}$$

5. *Charge conjugation symmetry*:

$$S_{ij} = \overline{S_{ij^*}}, \ s_i = s_{i^*}, \text{ or } S = S^\dagger C, \ \ T = TC, \tag{3.16}$$

   where the charge conjugation matrix C is given by $C_{ij} = N^{ij}_1 = \delta_{ij^*}$.

6. Let

$$\nu_i = \frac{1}{D^2} \sum_{jk} N_i^{jk} d_j d_k \mathrm{e}^{\mathrm{i}4\pi(s_j - s_k)}, \tag{3.17}$$

then $\nu_i \in \mathbb{Z}$ if $i = i^*$ [24].

For a UMTC, we further have

1. S is a unitary matrix. In particular, this means that (3.13) can be rewritten as

$$N_k^{ij} = \sum_k \frac{S_{il} S_{jl} \overline{S_{kl}}}{S_{1l}}, \tag{3.18}$$

which is the usual Verlinde formula. This way N_k^{ij}, s_i and S, T determine each other.

2. $\Theta = \exp(2\pi \mathrm{i} \frac{c}{8})$, where c is the chiral central charge. Thus, the UMTC determines the central charge modulo 8.

3. $\nu_i = 0$ if $i \neq i^*$, and $\nu_i = \pm 1$ if $i = i^*$ [12, 18].

Chapter 4

Stacking of Topological Phases

The stacking is probably the most simple construction for topological phases. In this chapter we discuss the stacking operation in detail.

4.1 Stacking in terms of Observables

We first consider the stacking of boson systems with no symmetry, which can be easily described by the physical observables, namely (N^{ij}_k, s_i, c).

Suppose that we have two UBFC's, $\mathcal{C}$ and $\mathcal{D}$, with simple anyons labeled by $i \in \mathcal{C},\ a \in \mathcal{D}$. We can construct a new UBFC by simply stacking $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} \boxtimes \mathcal{D}$. The anyon labels of $\mathcal{C} \boxtimes \mathcal{D}$ are pairs $(i,a), i \in \mathcal{C}, a \in \mathcal{D}$, and the observables are given by

$$\begin{gathered}
(N^{\mathcal{C}\boxtimes\mathcal{D}})^{(i,a)(j,b)}_{(k,c)} = (N^{\mathcal{C}})^{ij}_k (N^{\mathcal{D}})^{ab}_c, \\
s^{\mathcal{C}\boxtimes\mathcal{D}}_{(i,a)} = s^{\mathcal{C}}_i + s^{\mathcal{D}}_a, \quad c^{\mathcal{C}\boxtimes\mathcal{D}} = c^{\mathcal{C}} + c^{\mathcal{D}}, \\
T^{\mathcal{C}\boxtimes\mathcal{D}} = T^{\mathcal{C}} \otimes_{\mathbb{C}} T^{\mathcal{D}}, \\
S^{\mathcal{C}\boxtimes\mathcal{D}} = S^{\mathcal{C}} \otimes_{\mathbb{C}} S^{\mathcal{D}}.
\end{gathered} \tag{4.1}$$

Note that if $\mathcal{C},\mathcal{D}$ has symmetry G ($\mathcal{C},\mathcal{D}$ has Müger center $\mathcal{E}$), $\mathcal{C} \boxtimes \mathcal{D}$ has symmetry $G \times G$ ($\mathcal{C} \boxtimes \mathcal{D}$ has Müger center $\mathcal{E} \boxtimes \mathcal{E}$). Only when G is trivial, i.e., $\mathcal{C},\mathcal{D}$ are UMTCs describing bosonic topological orders with no symmetry, the above is the stacking that "preserves symmetry".

For topological phases with symmetry, we need to further take a "quotient" of the observables, which corresponds to breaking the symmetry from $G \times G$ to G. But such "quotient" is non-trivial at the level of physical observables, especially for the modular extensions of $\mathcal{C}, \mathcal{D}$. Therefore, from now on, we switch to strict categorical language.

4.2 Mathematical Constructions

Below we give the strict mathematical definitions for $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$, and constructions of stacking operations. Readers who are not interested in the mathematical formulation may jump to the next section for the main results. For readers who are interested in the mathematics, a much more detailed discussion can be found in Ref. [25].

We start with the definition of UBFC. Physically it is just a generic and consistent anyon model, containing all the information on the fusion and braiding of anyons.

Definition 1. A unitary braided fusion category $\mathcal{C}$ is

- $\mathbb{C}$-linear semisimple category:
  - Simple objects i, $\mathrm{Hom}(i,i) = \mathbb{C}$.
  - General objects are direct sums of simples, $X \cong \oplus_i \dim \mathrm{Hom}(i,X)\, i$.
- $\mathrm{Hom}(X,Y)$ are finite dimensional $\mathbb{C}$-vector spaces for any objects X, Y.
- Finitely many isomorphism classes of simple objects.
- Equipped with a monoidal (tensor) structure, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the associator $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$, the tensor unit $\mathbf{1}$ and unit morphisms, satisfying pentagon and triangle equations.
- Rigid: every object X has left and right duals X^*. (With unitary structure left and right duals are automatically isomorphic, so we do not distinguish their notations.)
- The tensor unit $\mathbf{1}$ is simple $\mathrm{Hom}(\mathbf{1},\mathbf{1}) \simeq \mathbb{C}$.
  (— The above is the definition of a fusion category)
- Equipped with a braiding $c_{X,Y} : X \otimes Y \simeq Y \otimes X$ satisfying the hexagon equations.
- Equipped with a unitary structure, an anti-linear "hermitian conjugate" map $\dagger$: $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$ for all X, Y such that:
  - $(gf)^\dagger = f^\dagger g^\dagger, (\lambda f)^\dagger = \overline{\lambda} f^\dagger, (f^\dagger)^\dagger = f$.

- $ff^\dagger = 0$ implies $f = 0$.
- $\dagger$ is compatible with the tensor product and braiding, i.e., $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and the associator, unit morphisms and the braiding are unitary isomorphisms. (An isomorphism f is unitary if $f^\dagger = f^{-1}$.)

For physical applications, all categories are assumed unitary in this thesis.

Example 1. The category of Hilbert spaces, **Hilb**. It corresponds to the trivial phase with no symmetry. It has only local excitations carrying no group representations.

Example 2. The category of finite dimensional representations of a finite group G, $\mathrm{Rep}(G)$. It corresponds to the invertible bosonic phase with symmetry G.

Example 3. Given a UBFC $\mathcal{C}$, Let $\overline{\mathcal{C}}$ be its mirror (parity) conjugate, namely the same fusion category $\mathcal{C}$ with reversed braiding, and $\mathcal{C}^{tr}$ be its time-reversal conjugate, corresponding to taking the same objects with all morphisms reversed, $\mathrm{Hom}_{\mathcal{C}^{tr}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$. Then $\overline{\mathcal{C}}, \mathcal{C}^{tr}$ are both also UBFCs. We can show that $\overline{\mathcal{C}}$ is canonically braided equivalent to $\mathcal{C}^{tr}$ by taking duals (charge conjugation) $X \mapsto X^*, f \mapsto f^*$. This is the categorical version of $CP = T$.

In our classification we have local excitations being a subset, or subcategory, of all bulk excitations, which is in turn a subset of all excitations in the gauged theory. In this thesis by a subcategory or a subset of excitations, we always mean a *full* subcategory in the following sense

Definition 2. A full subcategory $\mathcal{B}$ of $\mathcal{C}$ means we take a subset of object $\mathrm{Ob}(\mathcal{B}) \subset \mathrm{Ob}(\mathcal{C})$ but all the morphisms $\mathrm{Hom}_{\mathcal{B}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$. A fusion subcategory is a full subcategory that is a fusion category itself.

Example 4. All UBFCs have a trivial fusion subcategory consists of multiples of the tensor unit $\mathbf{1}$, which is equivalent to **Hilb**. Physically, this means that anyon models always contain all the trivial anyons.

In a UBFC $\mathcal{C}$, Frobenius-Perron dimension (defined by the largest eigenvalue of the fusion matrix N_i) and quantum dimension (defined by the quantum trace of the identity morphism id_i) of object i coincide, which is physically the "number of internal degrees of freedom" of anyon i, and will be denoted by $\dim(i) = d_i$. For a general object X, $\dim(X) = \sum_i \dim \mathrm{Hom}(i,X) \dim(i)$. The total quantum dimension of a unitary fusion category is $\dim(\mathcal{C}) = \sum_i \dim(i)^2$, ($i, j, k \ldots$ labels runs over isomorphism classes of simple objects). We rely on the following lemma to identify fusion categories.

Lemma 1 (EO [26])**.** If $\mathcal{B}$ is a fusion subcategory of $\mathcal{C}$ (or there is a fully faithful tensor embedding $\mathcal{B} \hookrightarrow \mathcal{C}$) then $\dim(\mathcal{B}) \leq \dim(\mathcal{C})$, and the equality holds iff $\mathcal{B} = \mathcal{C}$.

Below is the mathematical description of "trivial mutual statistics".

Definition 3. The objects X, Y in a UBFC $\mathcal{C}$ are said to *centralize* each other if

$$c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y},$$

where $c_{X,Y} : X \otimes Y \cong Y \otimes X$ is the braiding in $\mathcal{C}$. Equivalently, i, j centralize each other if $S_{ij} = d_i d_j / D$. Given a fusion subcategory $\mathcal{D} \subset \mathcal{C}$, its *centralizer* $\mathcal{D}_{\mathcal{C}}^{\mathrm{cen}}$ in $\mathcal{C}$ is the full subcategory of objects in $\mathcal{C}$ that centralize all the objects in $\mathcal{D}$. The centralizer is a fusion subcategory. In particular, $\mathcal{C}_{\mathcal{C}}^{\mathrm{cen}}$ is called the Müger center of $\mathcal{C}$.

Definition 4. A UBFC $\mathcal{C}$ is a unitary modular tensor category (UMTC) if $\mathcal{C}_{\mathcal{C}}^{\mathrm{cen}} = \mathbf{Hilb}$.

Lemma 2 (DGNO [27])**.** Let $\mathcal{D}$ be a fusion subcategory of a UMTC $\mathcal{C}$, then

$$(\mathcal{D}_{\mathcal{C}}^{\mathrm{cen}})_{\mathcal{C}}^{\mathrm{cen}} = \mathcal{D}, \quad \dim(\mathcal{D}) \dim(\mathcal{D}_{\mathcal{C}}^{\mathrm{cen}}) = \dim(\mathcal{C}).$$

Definition 5. A UBFC $\mathcal{E}$ is a *symmetric* fusion category if $\mathcal{E}_{\mathcal{E}}^{\mathrm{cen}} = \mathcal{E}$.

UMTC and symmetric fusion category correspond to two extreme cases, i.e., braiding is non-degenerate and maximally degenerate, respectively. Symmetric fusion categories are closely related to bosonic and fermionic symmetry groups, according to the following theorem

Theorem 1 (Deligne [14])**.** A symmetric fusion category is braided equivalent to $\mathrm{Rep}(G, z)$, where G is a finite group, and $z \in G$ is a central element such that $z^2 = 1$, and $\mathrm{Rep}(G, z)$ is the fusion category $\mathrm{Rep}(G)$ equipped with braiding c^z:

$$c^z_{X,Y}(x \otimes_{\mathbb{C}} y) = (-1)^{mn} y \otimes_{\mathbb{C}} x, \quad \forall x \in X, y \in Y, \quad zx = (-1)^m x, zy = (-1)^n y.$$

When $z = 1$ it is $\mathrm{Rep}(G)$ with the usual braiding $x \otimes_{\mathbb{C}} y \to y \otimes_{\mathbb{C}} x$. When $z \neq 1$ it is the fermion number parity. Fermions braid with each other with an extra -1. We introduce $\mathrm{sRep}(G^f) = \mathrm{Rep}(G, z)$ for $z \neq 1$ to emphasize its fermionic nature.

Example 5. $\mathrm{sRep}(\mathbb{Z}_2^f)$ is the category of super Hilbert spaces, **sHilb**, that is, $\mathbb{Z}_2$-graded Hilbert spaces with $\mathbb{Z}_2$-graded braiding. It corresponds to invertible fermionic phases with no other symmetries.

In the following we keep $\mathcal{E}$ as a (fixed) symmetric fusion category. We give the strict definition for $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$.

Definition 6. A pair $(\mathcal{C}, \iota)$, a UBFC $\mathcal{C}$ with a fully faithful embedding $\iota : \mathcal{E} \hookrightarrow \mathcal{C}_{\mathcal{C}}^{\text{cen}}$ is a UBFC *over* $\mathcal{E}$. Moreover, $\mathcal{C}$ is said a non-degenerate UBFC over $\mathcal{E}$, or UMTC$_{/\mathcal{E}}$, if $\mathcal{C}_{\mathcal{C}}^{\text{cen}} = \mathcal{E}$. Two UBFCs over $\mathcal{E}$, $(\mathcal{C}_1, \iota_1)$ and $(\mathcal{C}_2, \iota_2)$ are equivalent if there is a braided monoidal equivalence $F : \mathcal{C}_1 \to \mathcal{C}_2$ such that $F\iota_1 = \iota_2$.

We recover the usual definition of UMTC when $\mathcal{E}$ is trivial. In this case the subscript is omitted.

Definition 7. Given a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$, its (minimal) *modular extension* is a pair $(\mathcal{M}, \iota_{\mathcal{M}})$, a UMTC $\mathcal{M}$, together with a fully faithful embedding $\iota_{\mathcal{M}} : \mathcal{C} \hookrightarrow \mathcal{M}$, such that $\mathcal{E}_{\mathcal{M}}^{\text{cen}} = \mathcal{C}$. Two modular extensions $(\mathcal{M}_1, \iota_{\mathcal{M}_1}), (\mathcal{M}_2, \iota_{\mathcal{M}_2})$ are equivalent if there is a braided monoidal equivalence $F : \mathcal{M}_1 \to \mathcal{M}_2$ such that $F\iota_{\mathcal{M}_1} = \iota_{\mathcal{M}_2}$. We denote the set of equivalence classes of modular extensions of $\mathcal{C}$ by $\mathcal{M}_{ext}(\mathcal{C})$.

Remark 1. Here the condition $\mathcal{E}_{\mathcal{M}}^{\text{cen}} = \mathcal{C}$ is equivalent to $\dim(\mathcal{M}) = \dim(\mathcal{C})\dim(\mathcal{E})$, or $\mathcal{C}_{\mathcal{M}}^{\text{cen}} = \mathcal{E}$. Physically this means that the extra excitations in $\mathcal{M}$ but not in $\mathcal{C}$ all have non-trivial mutual statistics with at least one excitation in $\mathcal{E}$. In fact, let $\mathcal{M}$ be a UMTC that contains a symmetric fusion category $\mathcal{E}$ as a full subcategory, and $\mathcal{D} = \mathcal{E}_{\mathcal{M}}^{\text{cen}}$. Then, $\mathcal{E}$ is a full subcategory of $\mathcal{D}$ ($\mathcal{E}$ centralizes itself) and $\mathcal{D}_{\mathcal{M}}^{\text{cen}} = (\mathcal{E}_{\mathcal{M}}^{\text{cen}})_{\mathcal{M}}^{\text{cen}} = \mathcal{E}$. We see that $\mathcal{D}_{\mathcal{D}}^{\text{cen}} = \mathcal{D} \cap (\mathcal{D}_{\mathcal{M}}^{\text{cen}}) = \mathcal{D} \cap \mathcal{E} = \mathcal{E}$. This means that $\mathcal{D} = \mathcal{E}_{\mathcal{M}}^{\text{cen}}$ is automatically a UMTC$_{/\mathcal{E}}$, and $\mathcal{M}$ is its modular extension. This will be a useful way to construct UMTC$_{/\mathcal{E}}$'s from UMTCs.

Remark 2. For a given UMTC$_{/\mathcal{E}}$ $\mathcal{C}$, it is possible that there is no modular extension of $\mathcal{C}$. An example was constructed by Drinfeld [28]. It is a UMTC$_{/\,\text{Rep}(\mathbb{Z}_2\times\mathbb{Z}_2)}$ with rank $N = 5$ and $D^2 = 8$. The same example is also discussed in Ref. [29].

It is important to note that counting modular extensions of a fixed $\mathcal{C}$ is different from counting topological phases.

Definition 8. Two topological phases with symmetry $\mathcal{E}$, labeled by $((\mathcal{C}_1, \iota_1), (\mathcal{M}_1, \iota_{\mathcal{M}_1}), c_1)$ and $((\mathcal{C}_2, \iota_2), (\mathcal{M}_2, \iota_{\mathcal{M}_2}), c_2)$, are equivalent if $c_1 = c_2$ and there are braided monoidal equivalences $F_{\mathcal{C}} : \mathcal{C}_1 \to \mathcal{C}_2$, $F_{\mathcal{M}} : \mathcal{M}_1 \to \mathcal{M}_2$ such that $F_{\mathcal{C}}\iota_1 = \iota_2$, $\iota_{\mathcal{M}_2}F_{\mathcal{C}} = F_{\mathcal{M}}\iota_{\mathcal{M}_1}$.

Physically, when counting topological phases, we allow "relabelling" anyons in $\mathcal{C}$ and $\mathcal{M}$ together in a compatible way. But we do not allow mixing "excitations" (anyons in

$\mathcal{C}$) with "gauged symmetry defects" (anyons not in $\mathcal{C}$). Also we do not allow "relabelling" local excitations in $\mathcal{E}$, as they are related to the symmetry group which has absolute meaning. For example spin-flip $\mathbb{Z}_2$ can not be considered as the same as layer-exchange $\mathbb{Z}_2$, nor can their representations be relabelled. On the other hand, when counting modular extensions, we fix all the excitations in $\mathcal{C}$ and only allow "relabelling" "gauged symmetry defects" (anyons in $\mathcal{M}$ but not in $\mathcal{C}$).

The embeddings $\iota, \iota_{\mathcal{M}}$ are important data. However, in the following constructions, the embeddings are naturally defined, as we construct $\mathcal{E}, \mathcal{C}$ as full subcategories of $\mathcal{M}$. So we may omit the embeddings to simplify notations whenever there is no ambiguity.

Next we give the construction for the stacking of topological phases. First consider the stacking operation corresponding to the no-symmetry case. It is given by the Deligne tensor product $\boxtimes$, which defines a monoidal structure on the 2-category of UBFCs (more generally, of Abelian categories). For two UMTCs $\mathcal{C}, \mathcal{D}$, $\mathcal{C} \boxtimes \mathcal{D}$ is still a UMTC. (By construction, $\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(A \boxtimes B, X \boxtimes Y) = \mathrm{Hom}_{\mathcal{C}}(A, X) \otimes \mathrm{Hom}_{\mathcal{D}}(B, Y)$. All the structures follows component-wise.) There is a parallel story for UMTC$_{/\mathcal{E}}$, a monoidal structure $\boxtimes_{\mathcal{E}}$ such that the "stacking" of two UMTC$_{/\mathcal{E}}$s is still a UMTC$_{/\mathcal{E}}$. We introduce this construction and generalize it to modular extensions. Such stacking operation is for UMTC$_{/\mathcal{E}}$ together with their modular extensions, thus physically the stacking operations for topological phases with symmetry $\mathcal{E}$.

The basic idea is to first construct $\mathcal{C} \boxtimes \mathcal{D}$ which has symmetry $\mathcal{E} \boxtimes \mathcal{E}$, and then break the symmetry down to $\mathcal{E}$. We need to first introduce the following important concept, which controls generic Bose condensations in topological phases. Symmetry breaking $\mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$ is just a special case.

Definition 9. A *condensable algebra* in a UBFC $\mathcal{C}$ is a triple (A, m, η), $A \in \mathcal{C}$, $m : A \otimes A \to A$, $\eta : \mathbf{1} \to A$ satisfying

- Associative: $m(\mathrm{id}_A \otimes m) = m(m \otimes \mathrm{id}_A)$
- Unit: $m(\eta \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes \eta) = \mathrm{id}_A$
- Isometric: $mm^\dagger = \mathrm{id}_A$
- Connected: $\mathrm{Hom}(\mathbf{1}, A) = \mathbb{C}$
- Commutative: $mc_{A,A} = m$

Remark 3. This is an important notion that is widely studied. In the subfactor context it is called (irreducible local) *Q-system* [30]. In category literature it is also known as connected étale algebra (connected commutative separable algebra) [27, 31], or commutative special symmetric C^*-Frobenius algebra [32, 33]. The latter two are more general; they do not require the category to be unitary. In the unitary case, they are equivalent notions [30]. We follow Ref. [34] to call "condensable algebra" for its physical meaning and also simplicity.

Definition 10. A (left) *module* over a condensable algebra (A, m, η) in $\mathcal{C}$ is a pair (X, ρ), $X \in \mathcal{C}$, $\rho : A \otimes X \to X$ satisfying

$$\begin{aligned} \rho(\mathrm{id}_A \otimes \rho) &= \rho(m \otimes \mathrm{id}_M), \\ \rho(\eta \otimes \mathrm{id}_M) &= \mathrm{id}_M . \end{aligned} \tag{4.2}$$

It is further a *local* module if

$$\rho c_{M,A} c_{A,M} = \rho.$$

We denote the category of left A-modules by $\mathcal{C}_A$. A left module (X, ρ) is turned into a right module via the braiding, $(X, \rho c_{X,A})$ or $(X, \rho c^{-1}_{A,X})$, and thus a A-A-bimodule. The relative tensor functor $\otimes_A$ of bimodules then turns $\mathcal{C}_A$ into a fusion category. (This is known as α-induction in subfactor context.) In general there can be two monoidal structures on $\mathcal{C}_A$, since there are two ways to turn a left module into a bimodule (usually we pick one for definiteness when considering $\mathcal{C}_A$ as a fusion category). The two monoidal structures coincide for the fusion subcategory $\mathcal{C}_A^0$ of local A-modules. Moreover, $\mathcal{C}_A^0$ inherited the braiding from $\mathcal{C}$ and is also a UBFC; it exactly describes the excitations in the topological phase after condensing A.

Lemma 3 (DMNO [31])**.**

$$\dim(\mathcal{C}_A) = \frac{\dim(\mathcal{C})}{\dim(A)}.$$

If $\mathcal{C}$ is a UMTC, then so is $\mathcal{C}_A^0$, and

$$\dim(\mathcal{C}_A^0) = \frac{\dim(\mathcal{C})}{\dim(A)^2}.$$

Below we construct a canonical condensable algebra $L_{\mathcal{C}}$ in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ for any UBFC $\mathcal{C}$. In particular, $L_{\mathcal{E}}$ is the algebra that corresponds to the symmetry breaking $\mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$.

Definition 11. The Drinfeld center $Z(\mathcal{A})$ of a monoidal category $\mathcal{A}$ is a braided monoidal category with objects as pairs $(X \in \mathcal{A}, b_{X,-})$, where $b_{X,-} : X \otimes - \to - \otimes X$ are half-braidings that satisfy similar conditions as braidings. Morphisms are those that commute with half-braidings. The tensor product is given by

$$(X, b_{X,-}) \otimes (Y, b_{Y,-}) = (X \otimes Y, (b_{X,-} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes b_{Y,-})),$$

and the braiding is $c_{(X,b_{X,-}),(Y,b_{Y,-})} = b_{X,Y}$.

It is known that $Z(\mathcal{A})$ is a UMTC if $\mathcal{A}$ is a unitary fusion category [32]. There is a forgetful tensor functor $\mathrm{for}_{\mathcal{A}} : Z(\mathcal{A}) \to \mathcal{A}$, $(X, b_{X,-}) \mapsto X$ that forgets the half-braidings. Let $\mathcal{C}$ be a braided fusion category and $\mathcal{A}$ a fusion category, a tensor functor $F : \mathcal{C} \to \mathcal{A}$ is called a central functor if it factorizes through $Z(\mathcal{A})$, i.e., there exists a braided tensor functor $F' : \mathcal{C} \to Z(\mathcal{A})$ such that $F = \mathrm{for}_{\mathcal{A}} F'$.

Lemma 4 (DMNO [31])**.** Let $F : \mathcal{C} \to \mathcal{A}$ be a central functor, and $R : \mathcal{A} \to \mathcal{C}$ the right adjoint functor of F. Then the object $A = R(\mathbf{1}) \in \mathcal{C}$ has a canonical structure of condensable algebra. $\mathcal{C}_A$ is monoidally equivalent to the image of F, i.e. the smallest fusion subcategory of $\mathcal{A}$ containing $F(\mathcal{C})$.

If $\mathcal{C}$ is a UBFC, it is naturally embedded into $Z(\mathcal{C})$, by taking $X \mapsto (X, b_{X,-} = c_{X,-})$. So is $\overline{\mathcal{C}}$. Therefore, we have a braided tensor functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C})$. Compose it with the forgetful functor $\mathrm{for}_{\mathcal{C}} : Z(\mathcal{C}) \to \mathcal{C}$ we get a central functor

$$\begin{aligned} \otimes : \mathcal{C} \boxtimes \overline{\mathcal{C}} &\to \mathcal{C} \\ X \boxtimes Y &\mapsto X \otimes Y. \end{aligned}$$

Let R be its right adjoint functor, we obtain a condensable algebra $L_{\mathcal{C}} := R(\mathbf{1}) \cong \oplus_i (i \boxtimes i^*) \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ and $\mathcal{C} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}}$, $\dim(L_{\mathcal{C}}) = \dim(\mathcal{C})$. In particular, for a symmetric category $\mathcal{E}$, $L_{\mathcal{E}}$ is a condensable algebra in $\mathcal{E} \boxtimes \mathcal{E}$, and $\mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})_{L_{\mathcal{E}}} = (\mathcal{E} \boxtimes \mathcal{E})^0_{L_{\mathcal{E}}}$ for $\mathcal{E}$ is symmetric, all $L_{\mathcal{E}}$-modules are local.

Now, we are ready to define the stacking operation for $\mathrm{UMTC}_{/\mathcal{E}}$'s as well as their modular extensions.

Definition 12. Let $\mathcal{C}, \mathcal{D}$ be $\mathrm{UMTC}_{/\mathcal{E}}$'s, and $\mathcal{M}_{\mathcal{C}}, \mathcal{M}_{\mathcal{D}}$ their modular extensions. The stacking is defined by:

$$\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D} := (\mathcal{C} \boxtimes \mathcal{D})^0_{L_{\mathcal{E}}}, \quad \mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}} := (\mathcal{M}_{\mathcal{C}} \boxtimes \mathcal{M}_{\mathcal{D}})^0_{L_{\mathcal{E}}}$$

Theorem 2. $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ is a $\mathrm{UMTC}_{/\mathcal{E}}$, and $\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}}$ is a modular extension of $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$.

Proof. The embeddings $\mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})^0_{L_\mathcal{E}} \hookrightarrow (\mathcal{C} \boxtimes \mathcal{D})^0_{L_\mathcal{E}} = \mathcal{C} \boxtimes_\mathcal{E} \mathcal{D} \hookrightarrow \mathcal{E}^{\text{cen}}_{\mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{D}} \hookrightarrow (\mathcal{M}_\mathcal{C} \boxtimes \mathcal{M}_\mathcal{D})^0_{L_\mathcal{E}} = \mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{D}$ are obvious. So $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{D}$ is a UBFC over $\mathcal{E}$. Also

$$\dim(\mathcal{C} \boxtimes_\mathcal{E} \mathcal{D}) = \frac{\dim(\mathcal{C} \boxtimes \mathcal{D})}{\dim(L_\mathcal{E})} = \frac{\dim(\mathcal{C}) \dim(\mathcal{D})}{\dim(\mathcal{E})},$$

and $\mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{D}$ is a UMTC,

$$\dim(\mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{D}) = \frac{\dim(\mathcal{M}_\mathcal{C} \boxtimes \mathcal{M}_\mathcal{D})}{\dim(L_\mathcal{E})^2} = \dim(\mathcal{C}) \dim(\mathcal{D}).$$

Therefore, $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{D}$ and $\mathcal{E}^{\text{cen}}_{\mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{D}}$ have the same total quantum dimension, thus by Lemma 1 we know that they are the same. By Remark 1, $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{D}$ is a UMTC$_{/\mathcal{E}}$, and $\mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{D}$ is a modular extension of $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{D}$. □

Note that $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{E} = \mathcal{C}$. Therefore, for any modular extension $\mathcal{M}_\mathcal{E}$ of $\mathcal{E}$, $\mathcal{M}_\mathcal{C} \boxtimes_\mathcal{E} \mathcal{M}_\mathcal{E}$ is still a modular extension of $\mathcal{C}$. Physically this means that stacking with an invertible phase will not change the bulk excitations. In the following we want to show the inverse, that one can extract the "difference", a modular extension of $\mathcal{E}$, or an invertible phase, between two modular extensions of $\mathcal{C}$.

Lemma 5. We have $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^0_{L_\mathcal{C}} = \mathcal{C}^{\text{cen}}_\mathcal{C}$.

Proof. $(\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_\mathcal{C}}$ is equivalent to $\mathcal{C}$ (as a fusion category). Moreover, for $X \in \mathcal{C}$ the equivalence gives the free module $L_\mathcal{C} \otimes (X \boxtimes \mathbf{1}) \cong L_\mathcal{C} \otimes (\mathbf{1} \boxtimes X)$. $L_\mathcal{C} \otimes (X \boxtimes \mathbf{1})$ is a local $L_\mathcal{C}$ module if and only if $X \boxtimes \mathbf{1}$ centralize $L_\mathcal{C}$. This is the same as $X \in \mathcal{C}^{\text{cen}}_\mathcal{C}$. Therefore, we have $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^0_{L_\mathcal{C}} = \mathcal{C}^{\text{cen}}_\mathcal{C}$. □

Lemma 6 (FFRS [33])**.** For a non-commutative algebra A, we have the left center A_l of A, with algebra embedding $e_l : A_l \to A$, which is the maximal subalgebra such that $m(\mathrm{id}_A \otimes e_l) c_{A_l,A} = m(e_l \otimes \mathrm{id}_A)$. Similarly the right center A_r with $e_r : A_r \to A$, is the maximal subalgebra such that $m(e_r \otimes \mathrm{id}_A) c_{A,A_r} = m(\mathrm{id}_A \otimes e_r)$. A_l and A_r are commutative subalgebras, thus condensable. There is a canonical equivalence between the categories of local modules over the left and right centers, $\mathcal{C}^0_{A_l} = \mathcal{C}^0_{A_r}$.

Theorem 3. let $\mathcal{M}$ and $\mathcal{M}'$ be two modular extensions of the UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. There exists a unique $\mathcal{K} \in \mathcal{M}_{ext}(\mathcal{E})$ such that $\mathcal{K} \boxtimes_\mathcal{E} \mathcal{M} = \mathcal{M}'$. Such $\mathcal{K}$ is given by

$$\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}.$$

Proof. $\mathcal{K}$ is a modular extension of $\mathcal{E}$. This follows Lemma 5, that $\mathcal{E} = \mathcal{C}_{\mathcal{C}}^{\text{cen}} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})^0_{L_{\mathcal{C}}}$ is a full subcategory of $\mathcal{K}$. $\mathcal{K}$ is a UMTC by construction, and $\dim(\mathcal{K}) = \frac{\dim(\mathcal{M})\dim(\mathcal{M}')}{\dim(L_{\mathcal{C}})^2} = \dim(\mathcal{E})^2$.

To show that $\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})_{L_{\mathcal{C}}}$ satisfies $\mathcal{M}' = \mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M}$, note that $\mathcal{M}' = \mathcal{M}' \boxtimes \mathbf{Hilb} = \mathcal{M}' \boxtimes (\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\overline{\mathcal{M}}}}$. It suffices that

$$(\mathcal{M}' \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{\mathbf{1}\boxtimes L_{\overline{\mathcal{M}}}} = [(\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{C}}} \boxtimes \mathcal{M}]^0_{L_{\mathcal{E}}} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{(L_{\mathcal{C}}\boxtimes \mathbf{1})\otimes(\mathbf{1}\boxtimes L_{\mathcal{E}})}.$$

While $\mathbf{1} \boxtimes L_{\overline{\mathcal{M}}}$ and $(L_{\mathcal{C}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{E}})$ turns out to be left and right centers of the algebra $(L_{\mathcal{C}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\overline{\mathcal{M}}})$.

If $\mathcal{M}' = \mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M} = (\mathcal{K} \boxtimes \mathcal{M})^0_{L_{\mathcal{E}}}$, then

$$\mathcal{K} = (\mathcal{K} \boxtimes \mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{\mathbf{1}\boxtimes L_{\mathcal{M}}} = (\mathcal{K} \boxtimes \mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{(L_{\mathcal{E}}\boxtimes\mathbf{1})\otimes(\mathbf{1}\boxtimes L_{\mathcal{C}})} = [(\mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M}) \boxtimes \overline{\mathcal{M}}]^0_{L_{\mathcal{C}}} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{C}}}.$$

It is similar here that $\mathbf{1} \boxtimes L_{\mathcal{M}}$ and $(L_{\mathcal{E}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{C}})$ are the left and right centers of the algebra $(L_{\mathcal{E}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{M}})$. This proves the uniqueness of $\mathcal{K}$.

The above established the equivalences between UMTCs. To further show that they are equivalences between modular extensions, one need to check the embeddings of $\mathcal{E}, \mathcal{C}$. Here the only non-trivial braided tensor equivalences are those between the categories of local modules over left and right centers. By the detailed construction given in Ref. [33], one can check that they indeed preserve the embeddings of $\mathcal{E}, \mathcal{C}$. $\square$

Let us list several consequences of Theorem 3.

Corollary 4. $\mathcal{M}_{ext}(\mathcal{E})$ forms a finite Abelian group. The identity is $Z(\mathcal{E})$ and the inverse of $\mathcal{M}$ is $\overline{\mathcal{M}}$.

Proof. It is easy to verify that the stacking $\boxtimes_{\mathcal{E}}$ for modular extensions is associative and commutative. To show that they form a group we only need to find out the identity and inverse. In this case $\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{E}}} = \mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}}$, Theorem 3 becomes $\mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M} = \mathcal{M}'$, for any modular extensions $\mathcal{M}, \mathcal{M}'$ of $\mathcal{E}$. Thus, $\overline{\mathcal{M}'} \boxtimes_{\mathcal{E}} \mathcal{M}' = \overline{\mathcal{M}'} \boxtimes_{\mathcal{E}} \mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M} = \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M}$, i.e. $\mathcal{Z}_{\mathcal{E}} := \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M}$ is the same category for any extension $\mathcal{M}$, which is exactly the identity element. It is then obvious that the inverse of $\mathcal{M}$ is $\overline{\mathcal{M}}$. The finiteness follows from Ref. [7].

In fact, the identity $\mathcal{Z}_{\mathcal{E}}$ should be $Z(\mathcal{E})$, the Drinfeld center of $\mathcal{E}$. (This is Theorem 6. The embedding $\mathcal{E} \hookrightarrow Z(\mathcal{E})$ is given by the lift of the identity functor on $\mathcal{E}$, i.e., $\mathcal{E} \hookrightarrow Z(\mathcal{E}) \to \mathcal{E}$ equals $\mathrm{id}_{\mathcal{E}}$.) $\square$

Example 6. $\mathcal{M}_{ext}(\mathrm{sRep}(\mathbb{Z}_2^f)) \cong \mathbb{Z}_{16}$, with central charge $c = n/2 \bmod 8, n = 0, 1, 2, \ldots, 15$. This is the 16-fold way [11].

Example 7 (LKW [25])**.** $\mathcal{M}_{ext}(\mathrm{Rep}(G)) \cong H^3(G, U(1))$, all with central charge $c = 0 \bmod 8$. This agrees with the classification of bosonic SPT phases [4].

Corollary 5. For a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, $\mathcal{M}_{ext}(\mathcal{C})$, if exists, forms a $\mathcal{M}_{ext}(\mathcal{E})$-torsor. The action of $\mathcal{M}_{ext}(\mathcal{E})$ on $\mathcal{M}_{ext}(\mathcal{C})$ is given by the stacking $\boxtimes_{\mathcal{E}}$.

Below is a standalone theorem that fixes the unit element in the Abelian group of modular extensions.

Theorem 6. Let $\mathcal{M}$ be a modular extension of a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$:

$$(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}} = Z(\mathcal{E}).$$

In particular, this means that $\mathcal{Z}_{\mathcal{E}} = Z(\mathcal{E})$.

Proof. There is a Lagrangian algebra $L_{\mathcal{M}}$ in $\overline{\mathcal{M}} \boxtimes \mathcal{M}$, such that the category of $L_{\mathcal{M}}$-modules in $\overline{\mathcal{M}} \boxtimes \mathcal{M}$ is $(\overline{\mathcal{M}} \boxtimes \mathcal{M})_{L_{\mathcal{M}}} = \mathcal{M}$, via the functor $L_{\mathcal{M}} \otimes (i \boxtimes \mathbf{1}) \mapsto i$. $L_{\mathcal{M}}$ is a condensable algebra over $L_{\mathcal{C}}$, and also a condensable algebra in $(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}$. We would like to show that $[(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}]_{L_{\mathcal{M}}} = \mathcal{E}$. To see this, note that $\mathcal{E} \hookrightarrow (\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}$, the image of $\mathcal{E}$ identifies with the free $L_{\mathcal{C}}$-modules $L_{\mathcal{C}} \otimes (i \boxtimes \mathbf{1}) \cong L_{\mathcal{C}} \otimes (\mathbf{1} \boxtimes i), i \in \mathcal{E}$. Further check the free $L_{\mathcal{M}}$-modules in $(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}$ generated by these objects, and we find that $L_{\mathcal{M}} \otimes_{L_{\mathcal{C}}} [L_{\mathcal{C}} \otimes (i \boxtimes \mathbf{1})] \cong L_{\mathcal{M}} \otimes (i \boxtimes \mathbf{1}) \mapsto i$. This means that $\mathcal{E} \subset [(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}]_{L_{\mathcal{M}}}$. Since they have the same total quantum dimension, we must have $[(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}]_{L_{\mathcal{M}}} = \mathcal{E}$. Since $L_{\mathcal{M}}$ is Lagrangian in $(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}$, $(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}} = Z([(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}]_{L_{\mathcal{M}}}) = Z(\mathcal{E})$. Moreover, $L_{\mathcal{M}} \otimes_{L_{\mathcal{C}}} - : (\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}} \to [(\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}]_{L_{\mathcal{M}}}$ coincides with the forgetful functor $Z(\mathcal{E}) \to \mathcal{E}$. Thus the embedding $\mathcal{E} \hookrightarrow (\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\mathcal{C}}}$ composed with the forgetful functor $Z(\mathcal{E}) \to \mathcal{E}$ gives the identity functor on $\mathcal{E}$. □

4.3 Main Results of the Stacking Operation

We conclude the main results in the previous section. Topological phase with symmetry $\mathcal{E}$ are classified by the triple $(\mathcal{C}, \mathcal{M}, c)$. We mathematically constructed the stacking operation between them,

$$(\mathcal{C}_1, \mathcal{M}_1, c_1) \boxtimes_{\mathcal{E}} (\mathcal{C}_2, \mathcal{M}_2, c_2) = (\mathcal{C}_1 \boxtimes_{\mathcal{E}} \mathcal{C}_2, \mathcal{M}_1 \boxtimes_{\mathcal{E}} \mathcal{M}_2, c_1 + c_2). \tag{4.3}$$

In particular, the trivial phase with symmetry $\mathcal{E}$ is given by $(\mathcal{E}, Z(\mathcal{E}), c=0)$, and invertible topological phases with symmetry $\mathcal{E}$ are described by $(\mathcal{E}, \mathcal{M}, c)$, where $\mathcal{M}$ is a modular extension of $\mathcal{E}$, $\mathcal{M} \in \mathcal{M}_{ext}(\mathcal{E})$. They indeed form a Abelian group under the stacking operation defined above. For boson systems, $\mathcal{E} = \mathrm{Rep}(G)$, $\mathcal{M}_{ext}(\mathrm{Rep}(G)) \cong H^3(G, U(1))$, and they all have central charge $c = 0 \mod 8$. The group structure $H^3(G, U(1)) \times 8\mathbb{Z}$ is recovered. For fermion systems, we expect that $\mathcal{M}_{ext}(\mathrm{sRep}(G^f))$ gives a full classification of invertible phases. We can obtain both the fermionic SPT, namely the $c = 0$ part in $\mathcal{M}_{ext}(\mathrm{sRep}(G^f))$, and the smallest positive central charge $c_{\min}$ of the chiral invertible phases. Thus, invertible topological phases with symmetry $\mathcal{E}$ are classified by

$$\mathrm{SPT} \times c_{\min}\mathbb{Z}, \quad \mathrm{SPT} \times c_{\min}\mathbb{Z}/8\mathbb{Z} \cong \mathcal{M}_{ext}(\mathcal{E}). \tag{4.4}$$

By now we do not have a general formula for $\mathcal{M}_{ext}(\mathrm{sRep}(G^f))$, so we do not know $c_{\min}$ for generic G^f. Also we have checked the form $\mathrm{SPT} \times c_{\min}\mathbb{Z}/8\mathbb{Z} \cong \mathcal{M}_{ext}(\mathrm{sRep}(G^f))$ only for $G^f = G_b \times \mathbb{Z}_2^f$, or small G^f that is not of the form $G_b \times \mathbb{Z}_2^f$, but not for generic G^f; it remains a conjecture to be proven.

Also if we stack an invertible phase $(\mathcal{E}, \mathcal{M}_\mathcal{E}, c_1)$ onto $(\mathcal{C}, \mathcal{M}, c_2)$, it only changes the modular extension part,

$$(\mathcal{E}, \mathcal{M}_\mathcal{E}, c_1) \boxtimes_\mathcal{E} (\mathcal{C}, \mathcal{M}, c_2) = (\mathcal{C}, \mathcal{M}_\mathcal{E} \boxtimes_\mathcal{E} \mathcal{M}, c_1 + c_2). \tag{4.5}$$

By stacking all invertible phases (all modular extensions of $\mathcal{E}$), all modular extensions of $\mathcal{C}$ can be generated. Moreover, the "difference" between two modular extensions is a unique invertible phase (unique modular extension of $\mathcal{E}$). In short, the modular extensions of $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$ form a torsor over the Abelian group $\mathcal{M}_{ext}(\mathcal{E})$.

Therefore, a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, if its modular extension exists, already fixed the topological phase up to invertible ones. Appending the modular extension to the label further fixes the invertible ones up to E_8 states[1], and appending the central charge c totally fixes the topological phase. On the other hand, if a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$ has no modular extension, namely the symmetry can not be gauged, it is anomalous and can only be realized on the boundary of (3+1)D topological phases [29].

[1] UMTC fixes central charge c modulo 8.

Chapter 5

Anyon Condensation

In this chapter we discuss other constructions that relate topological phases. The general pattern is starting from a phase $\mathcal{C}$, condensing certain anyons into certain states, and driving a transition into a new phase $\mathcal{D}$. The other anyons that are not condensed become anyons in the new phase $\mathcal{D}$.

To do this, let us first consider constructing an effective theory for anyons. For simplicity, assume that we are going to condense only one type of anyon, denoted by A. We then try to turn off the energy cost of A. In terms of the Hamiltonian $H_{\mathcal{C}}$ of the old phase $\mathcal{C}$, we assume that there is certain parameter g controlling the interactions of the underlying system, such that at $H_{\mathcal{C}}(g=1)$ we have the original phase $\mathcal{C}$ and at $H_{\mathcal{C}}(g=0)$ the anyon A becomes gapless. In other words, when $g=0$ we arrive at a critical point, where any many-A-anyon state becomes degenerate with the ground state. Next we further tune the Hamiltonian, by adding an effective Hamiltonian H_A on many-A-anyon states. H_A describes the state that we want the anyon A to condense into. The new phase $\mathcal{D}$ is then

$$H_{\mathcal{D}} = H_{\mathcal{C}}(g=0) + H_A. \tag{5.1}$$

We can also describe this state in terms of effective many-A-anyon wavefunction $\langle \{z_a\} | \Psi \rangle$, where $\{z_a\}$ denote the positions of A anyons, and Ψ is the effective ground state of H_A. Such effective wavefunction allows us to write down the ground state wavefunction of the new phase $\mathcal{D}$. The ground state of phase $\mathcal{D}$ is given by

$$|0_{\mathcal{D}}\rangle = \sum_{\{z_a\}} |\{z_a\}\rangle \langle \{z_a\} | \Psi \rangle. \tag{5.2}$$

Let $\{z_i\}$ denote the degrees of freedom of the underlying system (such as spins, electron positions), we have the following wavefunction:

$$\langle\{z_i\}|0_{\mathcal{D}}\rangle = \sum_{\{z_a\}}\langle\{z_i\}|\{z_a\}\rangle\langle\{z_a\}|\Psi\rangle, \tag{5.3}$$

where $\langle\{z_i\}|\{z_a\}\rangle$ is the many-A-anyon wavefunction in the old phase $\mathcal{C}$.[1]

In order to perform such condensation, it is obvious that the properties of anyon A and the target state H_A, Ψ must satisfy non-trivial consistent conditions. We analyse two variants in the following. In concrete physical systems, the above ideas may not be easy to be realized precisely. But for the two variants to be discuss below, we do have precise mathematical (categorical) constructions from phase $\mathcal{C}$ to phase $\mathcal{D}$.

5.1 Type I: Bose Condensation

The first variant is condensing a boson A into a trivial state, or a A-condensate. This means that we want the effective wavefunction to be $\langle\{z_a\}|\Psi\rangle = 1$ for any configurations $\{z_a\}$. The underlying mathematics is introduced in the last chapter. The condensable algebra (A, m, η) is a self boson to be condensed, and the morphisms m, η exactly describes the condensation process. Roughly speaking, recall that m is a "multiplication morphism", an operator mapping from two copies of A to a single A. The isometric condition $mm^\dagger = \mathrm{id}_A$ means that $m^\dagger m$ is a projector acting on two copies of A. We can consider that $m^\dagger m$ projects a pair of A onto a "singlet" state. The unit, associative and commutative conditions ensures that such projectors can be consistently applied to any numbers of A-anyons and leads to a singlet state, which is exactly the A-condensate. So m is related to the effective theory $H_A \sim -\sum m^\dagger m$ for the new phase $\mathcal{D}$.

The special algebra $L_{\mathcal{E}}$ discussed in the last chapter corresponds to breaking the symmetry of the two-layer system from $\mathcal{E} \boxtimes \mathcal{E}$ to $\mathcal{E}$. It is global symmetry for $\mathrm{UMTC}_{/\mathcal{E}}$'s, but gauge symmetry for the gauged theory, or modular extensions. Other algebras can be considered as inducing general "topological symmetry breaking".

[1] For example in the $\nu = 1/m$ Laughlin state:

$$\langle\{z_i\}|\{z_a\}\rangle = \prod(z_a - z_i)\prod_{i<j}(z_i - z_j)^m \times \mathrm{e}^{-\frac{1}{4}\sum|z_i|^2},$$

where $\{z_a\}$ are the positions of quasi-hole excitations.

In general, given a topological phase $(\mathcal{C}, \mathcal{M}, c)$ with symmetry $\mathcal{E}$, condensing a condensable algebra $A \in \mathcal{C}$, gives rise to a new topological phase described by $(\mathcal{C}_A^0, \mathcal{M}_A^0, c)$. If $A \cap \mathcal{E}$ (the largest subalgebra of A in $\mathcal{E}$) is non-trivial, such condensation will break the symmetry to a smaller one, otherwise the symmetry is preserved. At the same time, the condensation creates a gapped domain wall between the old and the new phases. Point-like excitations on the domain wall should be described by $\mathcal{C}_A$ (and $\mathcal{M}_A$ for the gauged theories).

Recall that $\mathcal{C}_A^0, \mathcal{C}_A$ are just categories of (local) A-modules in $\mathcal{C}$, so the Bose condensation is essentially the representation theory of algebras in unitary braided fusion categories, generalizing that in usual vector spaces. It is possible to spell out all the data in terms of tensors and write down a concrete representation theory [35]. However, this way is not efficient when A is a large algebra.

Again we can study the Bose condensation at the level of physical observables. This will lead to some necessary conditions.

First we restrict to bosonic topological orders with no symmetry. Assume that by condensing A in phase $\mathcal{C}$, we obtain a new phase $\mathcal{D} = \mathcal{C}_A^0$, and a gapped domain wall $\mathcal{W} = \mathcal{C}_A$. Let the topological S, T-matrices for $\mathcal{C}, \mathcal{D}$ be $(S^{\mathcal{C}}, T^{\mathcal{C}}), (S^{\mathcal{D}}, T^{\mathcal{D}})$. We consider the following fusion space: Put phase $\mathcal{C}$ and phase $\mathcal{D}$ on a sphere S^2, separated by the gapped domain wall $\mathcal{W}$, and an anyon a^* in phase $\mathcal{C}$, an anyon i in phase $\mathcal{D}$. We denote the corresponding fusion space by $\mathcal{V}(S^2, i, \mathcal{W}, a^*)$. Its dimension

$$W_{ia} := \dim[\mathcal{V}(S^2, i, W, a^*)], \tag{5.4}$$

is an important physical observable. The matrix W satisfies the following necessary conditions [36, 37],

$$\begin{aligned} S^{\mathcal{D}} W = W S^{\mathcal{C}}, T^{\mathcal{D}} W = W T^{\mathcal{C}}, \\ W_{ia} W_{jb} \leq \sum_{kc} (N^{\mathcal{D}})_{ij}^k W_{kc} (N^{\mathcal{C}})_{ab}^c. \end{aligned} \tag{5.5}$$

We can compute the dimension of the fusion space $\mathcal{V}(S^2, i, W, a^*)$ by first creating a pair aa^* in phase $\mathcal{C}$, then tunneling a through the domain wall. In the channel where the tunneling does not leave any topological quasiparticle on the domain wall, a in phase $\mathcal{C}$ will become a composite anyon $q_{\mathcal{W},a}$ in phase $\mathcal{D}$,

$$q_{\mathcal{W},a} = \oplus_i W_{ia} i. \tag{5.6}$$

Thus the fusion-space dimension W_{ia} is also the *number* of tunneling channels from, a of phase $\mathcal{C}$, to, i of phase $\mathcal{D}$. So we also refer to W as the "tunneling matrix".

We may as well create a pair ii^* in phase $\mathcal{D}$ and tunnel i^* to a^*. $W^\dagger$ describes such tunneling in the opposite direction (i.e., $W : A \to B$, $W^\dagger : B \to A$). $W^\dagger$ and W contains the same physical data. To be consistent, tunneling i^* to a^* should give rise to the same fusion-space dimension, $(W^\dagger)_{a^*i^*} = W_{i^*a^*} = W_{ia}$. This is guaranteed by $W(S^{\mathcal{C}})^2 = (S^{\mathcal{D}})^2 W$.

In particular, since the algebra A condenses, it becomes the vacuum in phase $\mathcal{D}$, thus tunnelling the trivial particle $\mathbf{1}$ in $\mathcal{D}$ to $\mathcal{C}$ should give the algebra A,

$$A \cong \oplus_a W_{\mathbf{1}a} a. \tag{5.7}$$

And W_{ia} itself gives a Lagrangian condensable algebra L in the folded phase $\mathcal{C} \boxtimes \overline{\mathcal{D}}$,

$$L \cong \oplus_{a\in\mathcal{C}, i\in\mathcal{D}} W_{ia}\, a \boxtimes i^*. \tag{5.8}$$

We conclude that the tunnelling matrix W satisfying (5.5) can fix the object in the triple (A, m, η) of an condensable algebra, but the data of m, η are missing. Indeed, there are known examples that W does not correspond to any valid algebra [38]. However, the conditions (5.5) are good enough to exclude impossible Bose condensation and pick up a few candidates of condensable algebras.

For a topological phase $(\mathcal{C}, \mathcal{M}, c)$ with symmetry $\mathcal{E}$, the above still works for the modular extension $\mathcal{M}$. We require that the condensed boson A is in $\mathcal{C}$,

$$W_{\mathbf{1}a} = 0, \quad \text{if } a \notin \mathcal{C}. \tag{5.9}$$

And if $W_{\mathbf{1}a} = \delta_{\mathbf{1}a}$ for $a \in \mathcal{E}$, equivalently $A \cap \mathcal{E} = \mathbf{1}$, the symmetry is preserved; otherwise the symmetry is broken to a smaller one.

Note that in the presence of symmetry, a condensable algebra can be *anomalous*. Let $(\mathcal{C}, \mathcal{M}, c)$ be a topological phase with symmetry $\mathcal{E}$, and A a condensable algebra in $\mathcal{C}$ such that $A \cap \mathcal{E} = \mathbf{1}$ (A does not break symmetry). Let $(\mathcal{D}, \mathcal{N}, c) = (\mathcal{C}_A^0, \mathcal{M}_A^0, c)$ be the phase after Bose-condensing A. The corresponding domain wall is $(\mathcal{C}_A, \mathcal{M}_A)$ (before gauging $\mathcal{C}_A$ and after gauging $\mathcal{M}_A$).

The following is true, which can be thought as boundary-bulk duality (the domain wall is the boundary of the folded two-layer phase),

1. $Z(\mathcal{M}_A) = \mathcal{M} \boxtimes \overline{\mathcal{N}}$.

2. Similarly, $Z(\mathcal{C}_A)$ is a modular extension of $\mathcal{C} \boxtimes_{\mathcal{E}} \overline{\mathcal{D}}$. Here the embedding is determined as follows. Firstly, $\mathcal{E}$ is a full subcategory of $\mathcal{D} = \mathcal{C}_A^0 \subset \mathcal{C}_A$ with the embedding $\mathcal{E} \hookrightarrow \mathcal{C}_A$ as first embedding $\mathcal{E}$ into $\mathcal{C}$ and then take the free modules. Thus $\mathcal{E} \hookrightarrow \mathcal{C}_A$ is a central functor and lifts to an embedding $\mathcal{E} \hookrightarrow Z(\mathcal{C}_A)$. Then $\mathcal{C} \boxtimes_{\mathcal{E}} \overline{\mathcal{D}} = \mathcal{E}^{\text{cen}}_{Z(\mathcal{C}_A)}$.

However, it may not be true that $Z(\mathcal{C}_A) = \mathcal{M} \boxtimes_{\mathcal{E}} \overline{\mathcal{N}}$. The difference between $Z(\mathcal{C}_A)$ and $\mathcal{M} \boxtimes_{\mathcal{E}} \overline{\mathcal{N}}$ is a modular extension $\mathcal{K}$ of $\mathcal{E}$, namely a (2+1)D $\mathcal{E}$-SPT phase, $(\mathcal{E}, \mathcal{K}, c = 0)$, $Z(\mathcal{C}_A) = \mathcal{M} \boxtimes_{\mathcal{E}} \overline{\mathcal{N}} \boxtimes_{\mathcal{E}} \mathcal{K}$. Its physical meaning is that the domain wall $(\mathcal{C}_A, \mathcal{M}_A)$ must have a "bulk": a (2+1)D domain wall in the (3+1)D bulk, and the (2+1)D domain wall hosts the corresponding SPT phase. When $\mathcal{K}$ is non-trivial, $\mathcal{K} \neq Z(\mathcal{E})$, we say that the algebra A is anomalous. We will discuss an example of this in Appendix B.

If two topological phases with symmetry $\mathcal{E}$ are related by anomaly-free Bose condensations that preserve the symmetry, we say that they are *Witt equivalent* over $\mathcal{E}$. The equivalence classes are called Witt classes, denoted by $\mathrm{Witt}_{\mathcal{E}}$. Taking Witt classes is compatible with the stacking $\boxtimes_{\mathcal{E}}$, namely Witt classes still form a commutative monoid under the stacking $\boxtimes_{\mathcal{E}}$. Moreover, due to Lemma 5 and Theorem 6, the inverse Witt class always exists, given by the mirror conjugate. Thus Witt classes $\mathrm{Witt}_{\mathcal{E}}$ actually form an Abelian group, called the Witt group. Note that we take into account modular extensions and central charges in our definition of Witt classes. The Witt group defined in Ref. [39], the equivalence classes of $\mathrm{UMTC}_{/\mathcal{E}}$'s alone under Bose condensations, which does not exclude anomalous $\mathrm{UMTC}_{/\mathcal{E}}$'s that have no modular extension, or anomalous Bose condensations discussed above, can be different from our definition; when all these anomalies vanish, it is $(\mathrm{Witt}_{\mathcal{E}}/8\mathbb{Z})/\mathcal{M}_{ext}(\mathcal{E})$. Various constructions, such as symmetry breaking $\mathcal{E} \to \mathcal{E}'$ or stacking $- \boxtimes \mathcal{E}'$, can induce group homomorphisms from $\mathrm{Witt}_{\mathcal{E}}$ to $\mathrm{Witt}_{\mathcal{E}'}$ or $\mathrm{Witt}_{\mathcal{E}\boxtimes\mathcal{E}'}$.

Topological phases in the same Witt class have the same central charge, similar topological spins and mutual statistics as their S, T-matrices are related via (5.5). This is one way to "group" topological phases.

5.2 Type II: Abelian Condensation

The second variant is condensing Abelian anyons[2] into a Laughlin-like state. This idea dates back to Haldane and Halperin, known as "hierarchy" construction [40, 41]. But below we discuss it at a more general level.

We start with a topological phase $\mathcal{C}$. The anyons in $\mathcal{C}$ are labeled by $i, j, k, \cdots$. Let a_c

[2] Abelian anyons are anyons with quantum dimension 1. Here "Abelian" means that the braiding processes between Abelian anyons commute with each other, as they are just phases factors. On the contrary, braiding processes between non-Abelian anyons in general do not commute, and must be represented by matrices.

be an Abelian anyon in $\mathcal{C}$ with spin s_c. We try to condense a_c into the Laughlin state,

$$\langle\{z_a\}|\Psi\rangle = \prod_{a<b}(z_a - z_b)^{M_c} \times \mathrm{e}^{-\frac{1}{4}\sum|z_a|^2}. \tag{5.10}$$

The resulting topological phase is described by $\mathcal{D}$, determined by $\mathcal{C}$, a_c and M_c. Here z_a, z_b are the positions of a_c anyons. M_c must be consistent with anyon statistics. Consider exchanging two a_c anyons, we obtain: a phase factor $\mathrm{e}^{2\pi\mathrm{i}\frac{M_c}{2}}$ from the wave function and a phase factor $\mathrm{e}^{2\pi\mathrm{i}s_{a_c}}$ from anyonic statistics. To be consistent, total phase factor must be 1:

$$\frac{M_c}{2} + s_{a_c} \in \mathbb{Z}. \tag{5.11}$$

So we need to take $M_c = m_c - 2s_{a_c}$, where m_c is an even integer.

Anyon i in the phase $\mathcal{C}$ may be dressed with a flux M_i in the new phase $\mathcal{D}$.

$$\Psi(i, M_i) = \prod_b(\xi_i - z_b)^{M_i} \prod_{a<b}(z_a - z_b)^{M_c} \times \mathrm{e}^{-\frac{1}{4}\sum|z_a|^2}. \tag{5.12}$$

ξ_i is the position of anyon i. Thus an anyon in the new phase is represented by a pair (i, M_i). Again, M_i can not be arbitrary. If a_c has trivial mutual statistics with i, M_i can be any integer. Otherwise, consider moving a_c around (i, M_i) and we obtain: a phase factor $\mathrm{e}^{2\pi\mathrm{i}M_i}$ from the flux M_i and a phase factor $\mathrm{e}^{2\pi\mathrm{i}t_i}$ from the mutual statistics between a_c and i. The mutual statistics can be extracted from the S matrix, $\mathrm{e}^{2\pi\mathrm{i}t_i} = DS_{ia_c^*}/d_i$, $t_{a_c} = 2s_{a_c}$. To be consistent, total phase factor must be 1:

$$M_i + t_i \in \mathbb{Z}. \tag{5.13}$$

Next we compute the fusion rules and spins in the new phase $\mathcal{D}$. The spin of (i, M_i) is given by the spin of i plus the "spin" of the flux M_i:

$$s_{(i,M_i)} = s_i + \frac{M_i^2}{2M_c}. \tag{5.14}$$

To fuse anyons $(i, M_i), (j, M_j)$ in the new phase, just fuse i, j as in the old phase, and add up the flux:

$$(i, M_i) \otimes (j, M_j) = \bigoplus_k N_k^{ij}(k, M_i + M_j). \tag{5.15}$$

But note that this is not the final fusion rules, because anyons (i, M_i) in the new phase are subject to the equivalence relation

$$(i, M_i) \sim (i \otimes a_c, M_i + M_c). \tag{5.16}$$

This is because the anyon a_c dressed with a flux M_c is a "trivial excitation" in the new phase:

$$\Psi(a_c, M_c) \sim \prod_b^n (\xi_{a_c} - z_b)^{M_c} \prod_{a<b}^n (z_a - z_b)^{M_c} = \prod_{a<b}^{n+1} (z_a - z_b)^{M_c},$$
$$(a_c, M_c) \sim (\mathbf{1}, 0). \tag{5.17}$$

The anyon types in $\mathcal{D}$ actually correspond to the equivalence classes. After imposing the equivalence relation one obtains the final fusion rules in the new phase.

Applying the equivalence relation (5.16) q times, we obtain

$$(i, M_i) \sim (i \otimes a_c^{\otimes q}, M_i + qM_c). \tag{5.18}$$

Let q_c be the "period" of a_c, i.e., the smallest positive integer such that $a_c^{\otimes q_c} = \mathbf{1}$. We see that

$$(i, M_i) \sim (i, M_i + q_c M_c). \tag{5.19}$$

Thus, we can focus on the reduced range of $M_i + t_i \in \{0, 1, 2, \cdots, q_c|M_c| - 1\}$. Let $N^{\mathcal{C}}, N^{\mathcal{D}}$ denote the rank of $\mathcal{C}, \mathcal{D}$ respectively. Within the reduced range, we have $q_c|M_c|N^{\mathcal{C}}$ different labels, and we want to show that the orbits generated by the equivalence relation (5.18) all have the same length, which is q_c. To see this, just note that for $0 < q < q_c$, either $i \neq i \otimes a_c^{\otimes q}$, or if $i = i \otimes a_c^{\otimes q}$, $M_i \neq M_i + qM_c$; in other words, the labels (i, M_i) are all different within q_c steps. It follows that the rank of $\mathcal{D}$ is $N^{\mathcal{D}} = |M_c|N^{\mathcal{C}}$.

The above enables us to extend the construction to categorical level, which goes down to the level of F, R matrices.

The first step is to construct a unitary braided fusion category $\tilde{\mathcal{D}}$, based on the observation that the range of the second flux label can be reduced to $q_c|M_c|$. Such $\tilde{\mathcal{D}}$ can be viewed as an "extension" of $\mathcal{C}$ by $\mathbb{Z}_{q_c|M_c|}$. The anyons are labeled by the pair (i, M_i) where $i \in \mathcal{C}$ and $M_i + t_i \in \mathbb{Z}_{q_c|M_c|}$. Fusion is then given by addition

$$(i, M_i) \otimes (j, M_j) = \oplus_k N_k^{ij} (k, [M_i + M_j]_{q_c|M_c|}), \tag{5.20}$$

where $[\cdots]_{q_c|M_c|}$ denotes the residue modulo $q_c|M_c|$. The F, R-matrices in $\tilde{\mathcal{D}}$ are given by those in $\mathcal{C}$ modified by appropriate phase factors. More precisely, let $F^{i_1 i_2 i_3}_{i_4}$ and $R^{i_1 i_2}_{i_3}$ be the F, R-matrices in $\mathcal{C}$; then in $\tilde{\mathcal{D}}$ we take

$$\begin{aligned}\tilde{F}^{(i_1,M_1)(i_2,M_2)(i_3,M_3)}_{(i_4,M_4)} &= F^{i_1 i_2 i_3}_{i_4} \mathrm{e}^{\frac{\pi \mathrm{i}}{M_c} M_1 (M_2+M_3-[M_2+M_3]_{q_c|M_c|})}, \\ \tilde{R}^{(i_1,M_1)(i_2,M_2)}_{(i_3,M_3)} &= R^{i_1 i_2}_{i_3} \mathrm{e}^{\frac{\pi \mathrm{i}}{M_c} M_1 M_2}.\end{aligned} \tag{5.21}$$

It is straightforward to check that they satisfy the pentagon and hexagon equations, and $\tilde{\mathcal{D}}$ is a valid unitary braided fusion category. Moreover, the modified R matrices do give us the desired modified spin. The S matrix is

$$\begin{aligned}S^{\tilde{\mathcal{D}}}_{(i,M_i),(j,M_j)} &= \sum_k \frac{N^{ij}_k}{D_{\tilde{\mathcal{D}}}} d_k \mathrm{e}^{2\pi \mathrm{i}[s_{(i,M_i)}+s_{(j,M_j)}-s_{(k,M_i+M_j)}]} \\ &= \sqrt{\frac{q_c}{|M_c|}} S^{\mathcal{C}}_{ij} \mathrm{e}^{-2\pi \mathrm{i} \frac{M_i M_j}{M_c}}.\end{aligned} \tag{5.22}$$

The second step is to reduce $\tilde{\mathcal{D}}$ to $\mathcal{D}$. Categorically, just note that $\{(a_c, M_c)^{\otimes q}, q = 0, \ldots, q_c - 1\}$ forms a symmetric fusion subcategory of $\tilde{\mathcal{D}}^{\mathrm{cen}}_{\tilde{\mathcal{D}}}$, which can be identified with $\mathrm{Rep}(\mathbb{Z}_{q_c})$; by condensing this $\mathrm{Rep}(\mathbb{Z}_{q_c})$, i.e., condensing the regular algebra $\mathrm{Fun}(\mathbb{Z}_{q_c})$ in $\mathrm{Rep}(\mathbb{Z}_{q_c})$, we obtain the desired $\mathcal{D}$. Put it simply, we just further impose the equivalence relation (5.18) in $\tilde{\mathcal{D}}$, such that one orbit of length q_c is viewed as one type of anyon instead of q_c different types. This way we complete the construction of Abelian anyon condensation at full categorical level.

Below we will study the properties of $\mathcal{D}$ in detail. It would be more convenient to use (i, M_i) directly, which is the same as working in a $\tilde{\mathcal{D}}$. Then we can further impose the equivalence relation. For example, when we need to sum over anyons in $\mathcal{D}$, we can instead do

$$\sum_{I \in \mathcal{D}} \to \frac{1}{q_c} \sum_{i \in \mathcal{C}} \sum_{m=0}^{q_c|M_c|-1}. \tag{5.23}$$

Now we are ready to calculate other quantities of the new phase $\mathcal{D}$. First, it is easy to see that the quantum dimensions remain the same $d_{(i,m)} = d_i$. The total quantum dimension is then

$$D^2_{\mathcal{D}} = \frac{1}{q_c} \sum_{i \in \mathcal{C}} \sum_{m=0}^{q_c|M_c|-1} d^2_{(i,m)} = |M_c| D^2_{\mathcal{C}}. \tag{5.24}$$

The S matrix is

$$S^{\mathcal{D}}_{(i,M_i),(j,M_j)} = \frac{1}{\sqrt{q_c}} S^{\tilde{\mathcal{D}}}_{(i,M_i),(j,M_j)} = \frac{1}{\sqrt{|M_c|}} S^{\mathcal{C}}_{ij} \mathrm{e}^{-2\pi\mathrm{i}\frac{M_i M_j}{M_c}}. \tag{5.25}$$

From the above it is easy to check that $\mathcal{C}^{\mathrm{cen}}_{\mathcal{C}} = \mathcal{D}^{\mathrm{cen}}_{\mathcal{D}}$. This means that the symmetry $\mathcal{E} = \mathcal{C}^{\mathrm{cen}}_{\mathcal{C}} = \mathcal{D}^{\mathrm{cen}}_{\mathcal{D}}$ is preserved.

If both $\mathcal{C}, \mathcal{D}$ are UMTCs, the new $S^{\mathcal{D}}, T^{\mathcal{D}}$ matrices, as well as $S^{\mathcal{C}}, T^{\mathcal{C}}$, should both obey the modular relations $STS = \mathrm{e}^{2\pi\mathrm{i}\frac{c}{8}} T^\dagger S T^\dagger$, from which we can extract the central charge of $\mathcal{D}$. Firstly, using the modular relation for both $\mathcal{C}$ and $\mathcal{D}$, we find that

$$\begin{aligned} &\frac{1}{q_c\sqrt{|M_c|}} \sum_{i,j,k\in\mathcal{C}} \sum_{p=0}^{q_c|M_c|-1} \left\{ \overline{S^{\mathcal{C}}_{xi}} S^{\mathcal{C}}_{ik} T^{\mathcal{C}}_{kk} S^{\mathcal{C}}_{kj} \overline{S^{\mathcal{C}}_{jy}} \exp\left[\frac{2\pi\mathrm{i}}{2M_c}(t_i + t_j - t_k + p)^2\right] \right\} \\ &\quad = \exp\left(2\pi\mathrm{i}\frac{c^{\mathcal{D}} - c^{\mathcal{C}}}{8}\right) T^{\mathcal{C}}_{xx} \delta_{xy}. \end{aligned} \tag{5.26}$$

We can show that $c^{\mathcal{D}} - c^{\mathcal{C}} = \mathrm{sgn}(M_c) \mod 8$. Using the reciprocity theorem for generalized Gauss sums [42]:

$$\sum_{n=0}^{|c|-1} \mathrm{e}^{\pi\mathrm{i}\frac{an^2+bn}{c}} = \sqrt{|c/a|}\, \mathrm{e}^{\frac{\pi\mathrm{i}}{4}[\mathrm{sgn}(ac) - \frac{b^2}{ac}]} \sum_{n=0}^{|a|-1} \mathrm{e}^{-\pi\mathrm{i}\frac{cn^2+bn}{a}}, \tag{5.27}$$

where a, b, c are integers, $ac \neq 0$ and $ac + b$ even. Thus,

$$\begin{aligned} &\sum_{p=0}^{q_c|M_c|-1} \exp\left[\frac{2\pi\mathrm{i}}{2M_c}(t_i + t_j - t_k + p)^2\right] \\ &= \frac{1}{q_c} \mathrm{e}^{\frac{\pi\mathrm{i}(t_i+t_j-t_k)^2}{M_c}} \sum_{p=0}^{q_c^2|M_c|-1} \mathrm{e}^{\frac{\pi\mathrm{i}}{M_c q_c^2}\left[q_c^2 p^2 + 2q_c^2(t_i+t_j-t_k)p\right]} = \frac{\sqrt{|M_c|}}{q_c} \mathrm{e}^{\frac{\pi\mathrm{i}}{4}\mathrm{sgn}(M_c)} \sum_{p=0}^{q_c^2-1} \mathrm{e}^{-\pi\mathrm{i}[M_c p^2 + 2(t_i+t_j-t_k)p]} \\ &= \frac{\sqrt{|M_c|}}{q_c} \mathrm{e}^{\frac{\pi\mathrm{i}}{4}\mathrm{sgn}(M_c)} \sum_{p=0}^{q_c^2-1} \mathrm{e}^{-\pi\mathrm{i}(m_c - 2s_c)p^2} \frac{S^{\mathcal{C}}_{ia_c^{\otimes p}}}{S^{\mathcal{C}}_{i\mathbf{1}}} \frac{S^{\mathcal{C}}_{ja_c^{\otimes p}}}{S^{\mathcal{C}}_{j\mathbf{1}}} \frac{\overline{S^{\mathcal{C}}_{ka_c^{\otimes p}}}}{S^{\mathcal{C}}_{k\mathbf{1}}} \\ &= \frac{\sqrt{|M_c|}}{q_c} \mathrm{e}^{\frac{\pi\mathrm{i}}{4}\mathrm{sgn}(M_c)} \sum_{p=0}^{q_c^2-1} T^{\mathcal{C}}_{a_c^{\otimes p}, a_c^{\otimes p}} \frac{S^{\mathcal{C}}_{ia_c^{\otimes p}}}{S^{\mathcal{C}}_{i\mathbf{1}}} \frac{S^{\mathcal{C}}_{ja_c^{\otimes p}}}{S^{\mathcal{C}}_{j\mathbf{1}}} \frac{\overline{S^{\mathcal{C}}_{ka_c^{\otimes p}}}}{S^{\mathcal{C}}_{k\mathbf{1}}}. \end{aligned} \tag{5.28}$$

Substituting the above result into (5.26), we have

$$
\begin{aligned}
&\frac{1}{q_c\sqrt{|M_c|}}\sum_{i,j,k\in\mathcal{C}}\sum_{p=0}^{q_c|M_c|-1}\left\{\overline{S^{\mathcal{C}}_{xi}}S^{\mathcal{C}}_{ik}T^{\mathcal{C}}_{kk}S^{\mathcal{C}}_{kj}\overline{S^{\mathcal{C}}_{jy}}\exp\left[\frac{2\pi\mathrm{i}}{2M_c}(t_i+t_j-t_k+p)^2\right]\right\}\\
&=\frac{1}{q_c^2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\operatorname{sgn}(M_c)}\sum_{p=0}^{q_c^2-1}\sum_k T^{\mathcal{C}}_{kk}T^{\mathcal{C}}_{a_c^{\otimes p},a_c^{\otimes p}}\frac{\overline{S^{\mathcal{C}}_{ka_c^{\otimes p}}}}{S^{\mathcal{C}}_{k\mathbf{1}}}\sum_i\frac{\overline{S^{\mathcal{C}}_{xi}}S^{\mathcal{C}}_{ik}S^{\mathcal{C}}_{ia_c^{\otimes p}}}{S^{\mathcal{C}}_{i\mathbf{1}}}\sum_j\frac{S^{\mathcal{C}}_{kj}\overline{S^{\mathcal{C}}_{jy}}S^{\mathcal{C}}_{ja_c^{\otimes p}}}{S^{\mathcal{C}}_{j\mathbf{1}}}\\
&=\frac{1}{q_c^2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\operatorname{sgn}(M_c)}\sum_{p=0}^{q_c^2-1}\sum_k T^{\mathcal{C}}_{k\otimes a_c^{\otimes p},k\otimes a_c^{\otimes p}}N^{k,a_c^{\otimes p}}_{x}N^{k,a_c^{\otimes p}}_{y}\\
&=\frac{1}{q_c^2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\operatorname{sgn}(M_c)}\sum_{p=0}^{q_c^2-1}\sum_k T^{\mathcal{C}}_{k\otimes a_c^{\otimes p},k\otimes a_c^{\otimes p}}\delta_{k\otimes a_c^{\otimes p},x}\delta_{xy}\\
&=\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\operatorname{sgn}(M_c)}T^{\mathcal{C}}_{xx}\delta_{xy},
\end{aligned}
\tag{5.29}
$$

as desired. In fact, based on the physical picture, we have a stronger result

$$c^{\mathcal{D}}=c^{\mathcal{C}}+\operatorname{sgn}(M_c). \tag{5.30}$$

In the following we refer to the above construction from $\mathcal{C}$ to $\mathcal{D}$ as the one-step condensation. It is always reversible. In $\mathcal{D}$, choosing $a_c'=(\mathbf{1},1),\ s_c'=\frac{1}{2M_c},\ m_c'=0,\ M_c'=-1/M_c$, and repeating the construction, we will go back to $\mathcal{C}$. To see this we perform the construction for a $\tilde{\mathcal{D}}$. Taking $(j,M_j)=(a_c')^*=(\mathbf{1},-1)$ in (5.25) we find that the mutual statistics between (i,M_i) and $a_c'=(\mathbf{1},1)$ is $t'_{(i,M_i)}=\frac{M_i}{M_c}$. Let $((i,M_i),P_i)$ label the anyons after the above one-step condensation. We have two equivalence relations

$$((i,M_i),P_i)\sim((i\otimes a_c,M_i+M_c),P_i), \tag{5.31}$$

which reduces $\tilde{\mathcal{D}}$ to $\mathcal{D}$ and

$$((i,M_i),P_i)\sim((i,M_i)\otimes(\mathbf{1},1),P_i+M_c')=((i,M_i+1),P_i-1/M_c), \tag{5.32}$$

which arises from the second one-step condensation. Combining them we can eliminate the flux labels such that every label is equivalent to a representative of the following canonical form

$$((i,-t_i),t_i/M_c), \tag{5.33}$$

which can then be identified with the anyon i in $\mathcal{C}$. It is easy to check that the F, R-matrices for the representatives are the same as the original ones in $\mathcal{C}$. We also need to show that it is true for the whole equivalence class. Note that imposing the equivalence relations is nothing but condensing $\mathrm{Rep}(\mathbb{Z}_{q_c})$ and $\mathrm{Rep}(\mathbb{Z}_{q_c|M_c|})$, and the equivalence classes correspond to the free modules over the regular algebras. Since taking free modules is a braided tensor functor, we know that the resulting F, R-matrices are equivalent to those of the representatives. Thus, we indeed come back to the original phase $\mathcal{C}$.

Therefore, Abelian anyon condensations are reversible, which defines an equivalence relation between topological phases. We call the corresponding equivalence classes the “non-Abelian families”.

Now we examine the important quantity $M_c = m_c - 2s_c$ which relates the ranks before and after the one-step condensation, $N^{\mathcal{D}} = |M_c|N^{\mathcal{C}}$. Since m_c is a freely chosen even integer, when a_c is not a boson or fermion ($s_c \neq 0$ or $1/2 \mod 1$), we can always make $0 < |M_c| < 1$, which means that the rank is reduced after one-step condensation. We then have the important conclusion: Each non-Abelian family have “root” phases with the smallest rank. Abelian anyons in the root phases must be bosons or fermions.

We can further show that the Abelian bosons or fermions in the root phases have trivial mutual statistics among them. To see this, assuming that a, b are Abelian anyons in a root phase. Since the mutual statistics is given by $DS_{ab} = \exp[2\pi \mathrm{i}(s_a + s_b - s_{a\otimes b})]$, and $a, b, a \otimes b$ are all bosons or fermions, non-trivial mutual statistics can only be $DS_{ab} = -1$. Now consider two cases: (1) one of a, b, say a, is a fermion, then by condensing a (choosing $a_c = a$, $m_c = 2$, $s_c = 1/2$, $t_b = 1/2$), in the new phase, the rank remains the same but $s_{(b,0)} = s_b + \frac{t_b^2}{2M_c} = s_b + 1/8$, which means $(b, 0)$ is an Abelian anyon but neither a boson nor a fermion. By condensing $(b, 0)$ again we can reduce the rank, which conflicts with the root phase assumption. (2) a, b are all bosons. Still we condense a with $m_c = 2, s_c = 0, t_b = 1/2$. In the new phase the rank is doubled but $s_{(b,0)} = s_b + \frac{t_b^2}{2M_c} = 1/16$, which means further condensing $(b, 0)$ with $m_c' = 0$ the rank is reduced to $1/8$, which is again, smaller than the rank of the beginning root phase, thus contradictory.

Therefore, in the root phases, Abelian anyons are bosons or fermions with trivial mutual statistics among them. If we denote the full subcategory of Abelian anyons in a UBFC $\mathcal{C}$ by $\mathcal{C}_{Ab}$, the above means that in a root phase $\mathcal{C}$, $\mathcal{C}_{Ab}$ is a symmetric fusion category. We also have a straightforward corollary: All Abelian topological orders have the same unique root, which is the trivial topological order. In other words, all Abelian topological orders are in the same trivial non-Abelian family.

To easily determine if two phases belong to the same non-Abelian family, it is very helpful to introduce some *non-Abelian invariants*:

1. The fractional part of the central charge, c mod 1. Since the one-step condensation changes the central charge by $\mathrm{sgn}(M_c)$ (see (5.30)), we know that central charges in the same non-Abelian family can only differ by integers.

2. It is not hard to check that, in the one-step condensation, the number of simple anyon types with the same quantum dimension d, denoted by $N(d_i = d)$, is also multiplied by $|M_c|$. Thus the ratio $N(d_i = d)/N$ is a constant within one non-Abelian family.

3. The third invariant is a bit involved. Note that in the one-step condensation, if i has trivial mutual statistics with a_c, $t_i = 0$, then $(i, 0)$ in $\mathcal{D}$ have the same spin as i in $\mathcal{C}$ and the same mutual statistics with $(j, M_j), \forall M_j$ as that between i and j in $\mathcal{C}$. Therefore, the centralizer of Abelian anyons, $(\mathcal{C}_{Ab})^{\text{cen}}_{\mathcal{C}}$, namely, the subset of anyons that have trivial mutual statistics with all Abelian anyons, is the same within one non-Abelian family.

With these we can show the condition that $\mathcal{C}_{Ab}$ is symmetric fusion category (Abelian anyons are bosons or fermions with trivial mutual statistics among them) is also sufficient for a topological phase $\mathcal{C}$ to be a root state with the smallest rank among a non-Abelian family. First note that the rank of $\mathcal{C}_{Ab}$, $N^{\mathcal{C}_{Ab}}$ is just the number of simple anyon types with quantum dimension 1, thus $N^{\mathcal{C}_{Ab}}/N^{\mathcal{C}} = N(d_i = 1)/N$ is a constant. $\mathcal{C}$ has the smallest rank if and only if $\mathcal{C}_{Ab}$ has the smallest rank. On the other hand, $\mathcal{N} := (\mathcal{C}_{Ab})^{\text{cen}}_{\mathcal{C}}$ is also an invariant. We then have $(\mathcal{C}_{Ab})^{\text{cen}}_{\mathcal{C}_{Ab}} = (\mathcal{C}_{Ab})^{\text{cen}}_{\mathcal{C}} \cap \mathcal{C}_{Ab} = \mathcal{N}_{Ab} \subset \mathcal{C}_{Ab}$. As $\mathcal{N}_{Ab}$ is invariant, when $\mathcal{C}_{Ab}$ is symmetric, $(\mathcal{C}_{Ab})^{\text{cen}}_{\mathcal{C}_{Ab}} = \mathcal{C}_{Ab}$, it has the smallest rank which is the same as $\mathcal{N}_{Ab}$.

The non-Abelian family is yet another way to "group" topological phases. We see that its invariants are quite different from those of Witt classes. The Bose condensation preserves central charges and spins, but changes quantum dimensions, while Abelian condensation changes central charges and spins, but preserves quantum dimensions. So far we know that some Abelian condensation can be mimicked by stacking with an auxiliary state and then perform Bose condensation. For example, Abelian-condensing a $\mathbb{Z}_2$ fermion with $M_c = 1$, is the same as stacking with a $\mathbb{Z}_4$-fusion-rule state whose $s_i = 0, 1/8, 1/2, 1/8$, $c = 1$, and then Bose-condensing the fermion pair. However, it is not clear if every a_c, M_c has such an auxiliary state. By now we consider the two types of anyon condensations to be independent.

Similar to the roots in a non-Abelian family, in a Witt class there are topological phases that no longer admit Bose condensations; they are good representatives of the family/class. However, unlike the Abelian condensation, Bose condensation are *not* reversible. So a huge advantage of non-Abelian families over Witt classes is that from a root one can reconstruct the whole family. Classifying the root phases is the same as classifying all topological phases. We have listed the low rank roots and non-Abelian families of topological phases with no symmetry in Ref. [43].

Chapter 6

Examples

In this chapter we introduce several simple examples with the toric code UMTC and Ising UMTCs. Besides directly describing bosonic topological orders, they can also be viewed as the gauged theories and describe topological phases with $\mathbb{Z}_2^f$ or $\mathbb{Z}_2$ symmetries. There are also non-trivial anyon condensations between these phases. This way with two simple UMTCs we can illustrate the general structures discussed above.

We want to mention that there are far more examples studied in the literature than mentioned in this thesis. There are several systematic approaches to realize intrinsic topological orders, for example, the K-matrix formulation for all Abelian topological orders [44] (see Appendix C for a brief introduction), the Levin-Wen string-net model [45] for non-chiral topological orders with gapped boundaries. Besides, conformal field theory and Kac-Moody algebras are also very powerful in constructing chiral topological phases, but less systematic than the previous two approaches. They all give rise to concrete wavefunctions or lattice models for topological orders. However, most of them are limited to the realization of intrinsic topological orders, not considering the symmetries. So far only the string-net models are systematically extended to include bosonic symmetries [46, 47].

6.1 Toric Code UMTC

As the first example we describe the toric code [48] UMTC. There are 4 types of anyons, labeled by $\mathbf{1}, e, m, f$. Their fusion rules and spins are given in Table 6.1.

Table 6.1: Fusion rules and topological spins of toric code UMTC

$i \otimes j$	$\mathbf{1}$	e	m	f
$\mathbf{1}$	$\mathbf{1}$	e	m	f
e	e	$\mathbf{1}$	f	m
m	m	f	$\mathbf{1}$	e
f	f	m	e	$\mathbf{1}$
s_i	0	0	0	1/2

For convenience, we also list its S, T-matrices

$$T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{6.1}$$

It can be realized by the toric code model, $\mathbb{Z}_2$ quantum double model, $\mathbb{Z}_2$ gauge theory, or Levin-Wen string-net model [45] with $\mathrm{Rep}(\mathbb{Z}_2)$ as the input fusion category. On a lattice of spin 1/2 (on links), the fixed-point Hamiltonian reads

$$H = -\sum_{\text{vertices}} A_v - \sum_{\text{plaquettes}} B_p, \tag{6.2}$$

where A_v is the product of σ_z on the links around the vertex, and B_p is the product of σ_x on the links around the plaquette. In the string-net picture, we interpret $\sigma_z = -1$ as the presence of (non-trivial) string, and $\sigma_z = 1$ as the absence of string (or presence of the trivial string). Thus A_v enforces that $\mathbb{Z}_2$ fusion rules of string (non-trivial strings fuse to the trivial one; in other words, string cannot break at the vertex), and B_p creates a closed string loop in the plaquette and fuse it to the edges of the plaquette.

The ground state is the equal weight superposition of all closed loop configurations. The e anyons are created/annihilated/hopped by the string operators $W_e = \prod \sigma_x$, flipping spins along the links. The m anyons are created/annihilated/hopped by the string operators $W_m = \prod \sigma_z$ acting on the dual lattice, along paths that cross links. The f anyons are created/annihilated/hopped by the product of W_e, W_m.

6.2 Ising UMTC

The Ising fusion rules is given in Table 6.2, with three types of anyons $\mathbf{1}, \sigma, \psi$.

Table 6.2: The Ising fusion rules

$i \otimes j$	$\mathbf{1}$	σ	ψ
$\mathbf{1}$	$\mathbf{1}$	σ	ψ
σ	σ	$\mathbf{1} \oplus \psi$	σ
ψ	ψ	σ	$\mathbf{1}$

Such fusion rules allow 8 different sets of solutions for pentagon and hexagon equations, corresponding to 8 Ising-type UMTCs. They have spins $s_{\mathbf{1}} = 0, s_\psi = 1/2, s_\sigma = 1/16 + n/8$, and central charge $c = 1/2 + n$, where $n = 0, 1, \ldots, 7$. Usually by the Ising UMTC we mean the one with $n = 0$, $s_\sigma = 1/16$, $c = 1/2$, and its mirror conjugate $\overline{\text{Ising}}$ the one with $n = 7$, $s_\sigma = -1/16$, $c = -1/2$. The one with $n = 1$, $c = 3/2$ can be realized by the $SU(2)_2$ Chern-Simons theory, the Moore-Read (or Pfaffian) state [49]

$$\Psi_{\text{MR}} = \text{Pf}(\frac{1}{z_a - z_b}) \prod_{a<b} (z_a - z_b) \times \mathrm{e}^{-\frac{1}{4}\sum |z_a|^2}. \tag{6.3}$$

All the 8 Ising-type UMTCs have the same S-matrix

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}. \tag{6.4}$$

Since the Ising UMTC is chiral, it cannot be realized by a commuting projector lattice model. But Ising $\boxtimes$ $\overline{\text{Ising}}$ can be realized by the Levin-Wen model with Ising itself as the input fusion category. Now there are three types of strings, $\mathbf{1}, \sigma, \psi$ (One may imagine a lattice with spin 1 on the links). Again the Hamiltonian has the following form,

$$H = -\sum_{\text{vertices}} A_v - \sum_{\text{plaquettes}} B_p. \tag{6.5}$$

A_v enforces the Ising fusion rule, such that the preferred string configuration is: σ strings form closed loops; ψ strings either form closed loops, or end on σ strings. B_p still creates σ, ψ loops in the plaquette and fuse them to the edges. The detailed fusion process is described by the data of Ising fusion category (mainly the F-matrix). The quasiparticle excitations are described by the Ising $\boxtimes$ $\overline{\text{Ising}}$ UMTC.

6.3 Bose Condensation

In the above examples, Ising-type UMTCs do not allow any Bose condensation, but there are several possible Bose condensations in the toric code UMTC or Ising $\boxtimes \overline{\text{Ising}}$ UMTC.

In the toric code UMTC, one can take the condensable algebra to be either $\mathbf{1} \oplus e$ or $\mathbf{1} \oplus m$. After Bose condensation, the trivial topological phase is obtained, also there is a gapped boundary whose excitations are described by the $\text{Rep}(\mathbb{Z}_2)$ fusion category. Namely there is only one non-trivial type of particle-like excitation on the gapped boundary that has a $\mathbb{Z}_2$ fusion rule.

In the Ising $\boxtimes \overline{\text{Ising}}$ UMTC, one can Bose-condenses the algebra $\mathbf{1} \oplus \sigma\overline{\sigma} \oplus \psi\overline{\psi}$ and obtain the trivial phase. The corresponding gapped boundary is described by the Ising fusion category.

The above Bose condensation is a general feature of topological phase $\mathcal{C}$ that can be realized by Levin-Wen models. There must a Lagrangian algebra A in $\mathcal{C}$ such that $\mathcal{C}_A^0$ is the trivial phase, and $\mathcal{C}_A$ is the fusion category describing the corresponding gapped boundary. Also $\mathcal{C}_A$ is a input fusion category of the Levin-Wen model and $\mathcal{C} = Z(\mathcal{C}_A)$.

The other Bose condensation in the Ising $\boxtimes \overline{\text{Ising}}$ UMTC is more interesting. Condense the algebra $\mathbf{1} \oplus \psi\overline{\psi}$[1] and we will obtain exactly the toric code UMTC. If we order the anyons in the Ising $\boxtimes \overline{\text{Ising}}$ UMTC as $\mathbf{1}\overline{\mathbf{1}}, \mathbf{1}\overline{\sigma}, \mathbf{1}\overline{\psi}, \sigma\overline{\mathbf{1}}, \sigma\overline{\sigma}, \sigma\overline{\psi}, \psi\overline{\mathbf{1}}, \psi\overline{\sigma}, \psi\overline{\psi}$, such Bose condensation corresponds to the following tunneling matrix

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{array}{c} \mathbf{1}\overline{\mathbf{1}} \to \mathbf{1}, \quad \psi\overline{\psi} \to \mathbf{1}, \\ \mathbf{1}\overline{\psi} \to f, \quad \psi\overline{\mathbf{1}} \to f, \\ \sigma\overline{\sigma} \to e \oplus m. \end{array} \tag{6.6}$$

6.4 As Invertible Fermionic Phases

Let's consider the invertible fermionic phases with no other symmetry. We have $\mathcal{E} = \mathcal{C} = \text{sRep}(\mathbb{Z}_2^f) = \{\mathbf{1}, f\}$. It is easy to see that the above toric code UMTC and 8 Ising-type UMTCs are all modular extensions of $\text{sRep}(\mathbb{Z}_2^f)$. In fact in $\mathcal{M}_{ext}(\text{sRep}(\mathbb{Z}_2^f))$ there are also 7 other Abelian rank 4 UMTCs with central charge $c = 1, 2, \ldots, 7$, constituting the 16-fold way [11].

[1] In the literature this is usually called condensing the fermion pair $\psi\overline{\psi}$.

Viewed as fermionic topological orders, $(\mathrm{sRep}(\mathbb{Z}_2^f), \text{toric code UMTC}, c = 0)$ is the trivial fermion product state. $(\mathrm{sRep}(\mathbb{Z}_2^f), \mathrm{Ising}, c = 1/2)$, $(\mathrm{sRep}(\mathbb{Z}_2^f), \overline{\mathrm{Ising}}, c = -1/2)$ correspond to $p \pm \mathrm{i}p$ superconductors, where the Ising anyon σ corresponds to the vortex in the $p \pm \mathrm{i}p$ superconductors.

The Bose condensation (6.6) introduced in the last section is also the stacking $\boxtimes_{\mathrm{sRep}(\mathbb{Z}_2^f)}$ for modular extensions. Physically it means that stacking $p + \mathrm{i}p$ with $p - \mathrm{i}p$ produces the trivial fermion product state.

6.5 As Topological Phases with $\mathbb{Z}_2$ symmetry

First consider the invertible phases with $\mathbb{Z}_2$ symmetry, $\mathcal{E} = \mathcal{C} = \mathrm{Rep}(\mathbb{Z}_2)$. It is easy to see the toric code UMTC is a modular extension of $\mathrm{Rep}(\mathbb{Z}_2)$. The other modular extension of $\mathrm{Rep}(\mathbb{Z}_2)$ is the double-semion UMTC. This is consistent with the fact that (2+1)D SPT phases with on-site unitary $\mathbb{Z}_2$ symmetry is classifies by $H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$.

Here we consider a non-trivial example, the toric code model with $\mathbb{Z}_2$ symmetry that exchanges e, m anyons. In this case the original description of toric code UMTC is no longer symmetric. The correct $\mathrm{UMTC}_{/\mathrm{Rep}(\mathbb{Z}_2)}$ $\mathcal{C}$ turns out to have 5 types of anyons $\mathbf{1}_+, \mathbf{1}_-, f_+, f_-, \tau$. The first four are the original anyons $\mathbf{1}, f$ carrying even/odd $\mathbb{Z}_2$ charge. The last one τ is the composite of e and m, $\tau \sim e \oplus m$. As the $\mathbb{Z}_2$ symmetry exchanges e, m, they together form a new anyon type τ with quantum dimension $d_\tau = 2$. This degeneracy cannot be lifted by symmetric local perturbations.

One of its modular extension is $\mathrm{Ising} \boxtimes \overline{\mathrm{Ising}}$, with the embedding

$$\mathbf{1}_+ \mapsto \mathbf{1}\overline{\mathbf{1}}, \quad \mathbf{1}_- \mapsto \psi\overline{\psi}, \quad f_+ \mapsto \psi\overline{\mathbf{1}}, \quad f_- \mapsto \mathbf{1}\overline{\psi}, \quad \tau \mapsto \sigma\overline{\sigma}. \tag{6.7}$$

The other modular extension is then the stacking $\boxtimes_{\mathrm{Rep}(\mathbb{Z}_2)}$ of $\mathrm{Ising} \boxtimes \overline{\mathrm{Ising}}$ with the double-semion UMTC, which turns out to be $SU(2)_2 \boxtimes \overline{SU(2)_2}$.

As $\mathrm{Ising} \boxtimes \overline{\mathrm{Ising}}$ is the gauged theory, the toric code model with on-site e, m exchange symmetry can be realized by "ungauging" the Ising string-net model. Roughly speaking, this is done by making the σ strings in the string-net model into $\mathbb{Z}_2$ symmetry defects rather than fluctuating strings [46, 47].

Conclusion and Outlook

In this thesis, we gave a classification of (2+1)D bosonic or fermionic topological phases with finite on-site symmetries. We first introduced the underlying mathematics, the theory of unitary braided fusion categories, which describes the fusion and braiding of quasiparticle excitations. Then topological phases with symmetry are classified by a sequence of UBFCs, $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$, plus a central charge c. Here $\mathcal{E}$ is the symmetric fusion category describing the local excitations, which carry representations of the symmetry group. Thus, $\mathcal{E}$ is also the categorical description of the symmetry. $\mathcal{E} = \mathrm{Rep}(G)$ corresponds to bosonic phases while $\mathcal{E} = \mathrm{sRep}(G^f)$ corresponds to fermionic phases. $\mathcal{C}$ is the UBFC describing all the excitations, whose Müger center coincides with $\mathcal{E}$. $\mathcal{M}$ is a minimal modular extension of $\mathcal{C}$ that describes the excitations in the gauged theory. $\mathcal{M}$ encodes some information of the invertible topological phases, in particular the SPT phases. In the end, as $\mathcal{M}$ only fixes the central charge c modulo 8, the E_8 state which has no symmetry, no bulk excitations, but edge state with central charge 8, is totally undetectable by the categorical approach. One can stack multiple layers of E_8 states or its time-reversals without changing $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$. To fix this ambiguity we appended the total central charge c to $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$ to complete the classification.

We also studied the stacking of topological phases and two types of anyon condensations. They allow us to construct new topological phases from known ones, and "group" them into suitable equivalence classes or families, which simplifies the classification of topological phases.

We have been focused on finite on-site unitary symmetries in the thesis. This is mainly due to the technical difficulty dealing with the case where there are infinitely many irreducible representations of the symmetry group. In Appendix B we briefly discussed how to include anti-unitary symmetries. It should be possible to overcome the technical difficulties, and extend the basic idea of the thesis to include also continuous and space-time symmetries in the future.

Combined with previous results, a complete classification of topological phases with symmetry in below 2+1D is almost at hand. It is then interesting to investigate topological phases in 3+1D. The first step is to figure out 3+1D topological orders, which may require higher category theory as the natural underlying mathematical language. Next we also need to combine topological orders with symmetries. But recall that topological order appears as a new mechanism for phases of matter starting from 2+1D, in 3+1D similarly we can have new mechanisms that are even beyond topological order. Some examples are Haah's code [50], and the stacking of infinite layers (extending to the 3rd dimension) of 2+1D topological orders. A clear understanding of the new mechanisms in 3+1D is still beyond our scope and will be an intriguing future project.

References

[1] L Landau. Zur Theorie der Phasenumwandlungen II. *Phys. Z. Sowjet*, 11:26–35, 1937.

[2] X. G. Wen. Vacuum degeneracy of chiral spin states in compactified space. *Phys. Rev. B*, 40(10):7387–7390, October 1989.

[3] X. G. Wen. Topological orders in rigid states. *Int. J. Mod. Phys. B*, 04(02):239–271, February 1990.

[4] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen. Symmetry protected topological orders and the group cohomology of their symmetry group. *Phys. Rev. B*, 87(15):155114, April 2013.

[5] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. Complete classification of one-dimensional gapped quantum phases in interacting spin systems. *Phys. Rev. B*, 84(23):235128, December 2011.

[6] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. *Phys. Rev. B*, 82(15):155138, October 2010.

[7] Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang. Rank-finiteness for modular categories. *J. Am. Math. Soc.*, 29(3):857–881, July 2015.

[8] Xiao-Gang Wen. A theory of 2+1D bosonic topological orders. *Natl. Sci. Rev.*, 3(1):68–106, March 2016.

[9] Tian Lan, Liang Kong, and Xiao-Gang Wen. A theory of 2+1D fermionic topological orders and fermionic/bosonic topological orders with symmetries. *Phys. Rev. B*, 94(15):155113, July 2015.

[10] Tian Lan, Liang Kong, and Xiao-Gang Wen. Classification of (2+1)-dimensional topological order and symmetry-protected topological order for bosonic and fermionic systems with on-site symmetries. *Phys. Rev. B*, 95(23):235140, June 2017.

[11] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Ann. Phys.*, 321(1):2–111, January 2006.

[12] Zhenghan Wang. *Topological Quantum Computation.* Number 112 in CBMS Regional Conference Ser. in Mathematics Series. Conference Board of the Mathematical Sciences, 2010.

[13] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*, volume 205 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, Rhode Island, July 2015.

[14] P Deligne. Catégories tensorielles. (Tensor categories). *Mosc. Math. J.*, 2(2):227–248, 2002.

[15] Michael Müger. Galois Theory for Braided Tensor Categories and the Modular Closure. *Adv. Math.*, 150(2):151–201, March 2000.

[16] Michael Levin and Zheng-Cheng Gu. Braiding statistics approach to symmetry-protected topological phases. *Phys. Rev. B*, 86(11):115109, September 2012.

[17] B Bakalov and Alexander Kirillov. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series.* American Mathematical Society, Providence, Rhode Island, November 2001.

[18] Eric Rowell, Richard Stong, and Zhenghan Wang. On Classification of Modular Tensor Categories. *Commun. Math. Phys.*, 292(2):343–389, December 2009.

[19] Edward Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.*, 121(3):351–399, September 1989.

[20] Doron Gepner and Anton Kapustin. On the classification of fusion rings. *Phys. Lett. B*, 349(1-2):71–75, April 1995.

[21] Cumrun Vafa. Toward classification of conformal theories. *Phys. Lett. B*, 206(3):421–426, May 1988.

[22] Greg Anderson and Greg Moore. Rationality in conformal field theory. *Commun. Math. Phys.*, 117(3):441–450, September 1988.

[23] Pavel Etingof. On Vafa's theorem for tensor categories. *Math. Res. Lett.*, 9:651–657, July 2002.

[24] Paul Bruillard. Rank 4 Premodular Categories. *New York J. Math.*, 22, April 2016.

[25] Tian Lan, Liang Kong, and Xiao-Gang Wen. Modular Extensions of Unitary Braided Fusion Categories and 2+1D Topological/SPT Orders with Symmetries. *Commun. Math. Phys.*, 351(2):709–739, April 2017.

[26] Pavel Etingof and Viktor Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4(3):627–654,782–783, January 2004.

[27] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. On braided fusion categories I. *Sel. Math.*, 16(1):1–119, June 2010.

[28] Vladimir Drinfeld. unpublished note.

[29] Xie Chen, F. J. Burnell, Ashvin Vishwanath, and Lukasz Fidkowski. Anomalous Symmetry Fractionalization and Surface Topological Order. *Phys. Rev. X*, 5(4):041013, October 2015.

[30] Roberto Longo and John E. Roberts. A Theory of Dimension. *K-Theory*, 11(2):103–159, April 1996.

[31] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik. The Witt group of non-degenerate braided fusion categories. *J. für die reine und Angew. Math. (Crelles Journal)*, 2013(677):135–177, January 2013.

[32] Michael Müger. From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebr.*, 180(1-2):81–157, November 2003.

[33] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. Correspondences of ribbon categories. *Adv. Math.*, 199(1):192–329, January 2003.

[34] Liang Kong. Anyon condensation and tensor categories. *Nucl. Phys. B*, 886:436–482, September 2014.

[35] I. S. Eliëns, J. C. Romers, and F. A. Bais. Diagrammatics for Bose condensation in anyon theories. *Phys. Rev. B*, 90(19):195130, October 2013.

[36] Tian Lan, Juven C. Wang, and Xiao Gang Wen. Gapped domain walls, gapped boundaries, and topological degeneracy. *Phys. Rev. Lett.*, 114(7):76402, February 2015.

[37] Yasuyuki Kawahigashi. A Remark on Gapped Domain Walls Between Topological Phases. *Lett. Math. Phys.*, 105(7):893–899, July 2015.

[38] Alexei Davydov. Unphysical diagonal modular invariants. *J. Algebr.*, 446:1–18, January 2016.

[39] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik. On the structure of the Witt group of braided fusion categories. *Sel. Math.*, 19(1):237–269, March 2013.

[40] F. D. M. Haldane. Fractional Quantization of the Hall Effect: A Hierarchy of Incompressible Quantum Fluid States. *Phys. Rev. Lett.*, 51(7):605–608, August 1983.

[41] B. I. Halperin. Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States. *Phys. Rev. Lett.*, 52(18):1583–1586, April 1984.

[42] B. Berndt, R. Evans, and K. Williams. *Gauss and Jacobi Sums.* 1998.

[43] Tian Lan and Xiao-Gang Wen. Hierarchy Construction and Non-Abelian Families of Generic Topological Orders. *Phys. Rev. Lett.*, 119(4):040403, July 2017.

[44] X. G. Wen and A. Zee. Classification of Abelian quantum Hall states and matrix formulation of topological fluids. *Phys. Rev. B*, 46(4):2290–2301, 1992.

[45] Michael A. Levin and Xiao-Gang Wen. String-net condensation: A physical mechanism for topological phases. *Phys. Rev. B*, 71(4):045110, January 2005.

[46] Chris Heinrich, Fiona Burnell, Lukasz Fidkowski, and Michael Levin. Symmetry-enriched string nets: Exactly solvable models for SET phases. *Phys. Rev. B*, 94(23):235136, December 2016.

[47] Meng Cheng, Zheng-Cheng Gu, Shenghan Jiang, and Yang Qi. Exactly Solvable Models for Symmetry-Enriched Topological Phases. June 2016.

[48] A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Phys.*, 303(1):2–30, January 2003.

[49] Gregory Moore and Nicholas Read. Nonabelions in the fractional quantum hall effect. *Nucl. Phys. B*, 360(2-3):362–396, August 1991.

[50] Jeongwan Haah. Local stabilizer codes in three dimensions without string logical operators. *Phys. Rev. A*, 83(4):042330, April 2011.

[51] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. Symmetry, Defects, and Gauging of Topological Phases. *ArXiv e-prints*, October 2014.

[52] Hao Song, Sheng-Jie Huang, Liang Fu, and Michael Hermele. Topological Phases Protected by Point Group Symmetry. *Phys. Rev. X*, 7(1):011020, February 2017.

[53] Ethan Lake. Anomalies and symmetry fractionalization in reflection-symmetric topological order. *Phys. Rev. B*, 94(20):205149, November 2016.

[54] Yang Qi, Chao-Ming Jian, and Chenjie Wang. Folding approach to topological orders enriched by mirror symmetry. October 2017.

[55] Shlomo Gelaki, Deepak Naidu, and Dmitri Nikshych. Centers of graded fusion categories. *Algebr. Number Theory*, 3(8):959–990, December 2009.

APPENDICES

Appendix A

Relation to the G-crossed UMTC approach

In this appendix we discuss the relation between our approach and the G-crossed UMTC approach for bosonic symmetry enriched topological (SET) phases [51]. The latter may be a bit more familiar to physicists. It fixes the underlying intrinsic topological order, or a UMTC, and try to define the action of a symmetry group G on it. Besides, one also needs to consider the symmetry G-defects. The G-defects can not be freely braided like the quasiparticles; they leave defect lines behind. But, there is a "G-crossed braiding" for them. The UMTC plus the G-defects together with the action of the symmetry group G, forms the so called G-crossed UMTC.

Comparing to the $\mathrm{UMTC}_{/\mathrm{Rep}(G)}$ (UMTC over $\mathcal{E} = \mathrm{Rep}(G)$) approach introduced in the main text, this is just an equivalent perspective. $\mathrm{UMTC}_{/\mathrm{Rep}(G)}$ is the "symmetric" perspective while G-crossed UMTC is the "symmetry-broken" perspective. From a G-crossed UMTC, by taking representations (equivariantization) we obtain $\mathrm{Rep}(G) = \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$. More precisely,

- The tensor unit $\mathbf{1}$ (which spans the category of Hilbert spaces **Hilb**) becomes the representation category $\mathrm{Rep}(G)$. In other words, local excitations acquire symmetry charges.

- The UMTC (trivial component in the G-crossed UMTC, trivial G-defects) becomes the $\mathrm{UMTC}_{/\mathrm{Rep}(G)}$ $\mathcal{C}$. The topological excitations can carry usual group representations or projective representations when the group actions do not permute topological charges, but more general "representations" when the group actions permute topological charges.

- The G-crossed UMTC becomes the modular extension $\mathcal{M}$. G-defects are promoted to gauge fluxes, dynamical excitations in the gauged theory.

On the other hand, from $\mathrm{Rep}(G) = \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$, by breaking the symmetry (condensing $\mathrm{Rep}(G)$ or the regular algebra $\mathrm{Fun}(G)$ in $\mathrm{Rep}(G)$, de-equivariantization), we go back to the G-crossed UMTC and explicit G-actions are recovered.

To illustrate this idea, let's consider the example, trivial topological order with $\mathbb{Z}_2$ symmetry. In the G-crossed UMTC perspective, we consider all local Hilbert spaces, the UMTC **Hilb** with a $\mathbb{Z}_2$ action. In particular we allow the $\mathbb{Z}_2$ action to change local quantum states

$$|0\rangle \to |1\rangle. \tag{A.1}$$

We are not forbidden from describing the symmetry with its action on this $|0\rangle, |1\rangle$ basis. But in a real physical system with $\mathbb{Z}_2$ symmetry, $|0\rangle$ alone can not be stable, i.e., can not be an energy eigenstate, since it is not a representation of $\mathbb{Z}_2$. To really observe the state $|0\rangle$, our probe has to somehow break the $\mathbb{Z}_2$ symmetry. On the other hand, the even/odd irreducible representations

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{A.2}$$

can be stable excited states without breaking the $\mathbb{Z}_2$ symmetry and correspond to the categorical $\mathcal{E} = \mathrm{Rep}(G)$ way to describe the symmetry.

Through the main text we use the term "topological phases with symmetry" instead of the term "symmetry enriched topological phases", or SET, which is more common in the literature. This also reflects the different philosophies in the two perspectives. For us, we fix the symmetry at first, and try to classify all topological phases with this symmetry and study the stacking that preserves the symmetry. In the G-crossed approach, one at first fixes a bosonic topological order with no symmetry, and tries to add consistent G-actions and G-defects. Thus the topological order is "enriched" by the symmetry.

There are two main differences between the two approaches. The first is that in G-crossed UMTC approach, when trying to define the G-action on the underlying UMTC $\mathcal{C}$, since they are many layers of structures, such as anyon types, local operators, fusion, braiding and so on, there can be obstructions in $H^3(G, \mathcal{C}_{Ab})$ for certain choice of the G-action. Only when the obstruction vanishes, one can consistently define the G-action on the UMTC. However, a $\text{UMTC}_{/\,\text{Rep}(G)}$ is automatically free of such obstructions. Similarly, when trying to consistently add G-defects, there can be obstructions in $H^4(G, U(1))$, which prevent from extending a UMTC with G-action to a G-crossed UMTC. Again, existence of modular extensions implies that such obstructions vanish. But there are indeed examples that certain UBFCs have no minimal modular extensions [28, 29]. Non-vanishing $H^4(G, U(1))$ means that the corresponding topological phase is anomalous and can only exist on the (2+1)D surface of a (3+1)D SPT phase described by the obstruction in $H^4(G, U(1))$ [29]. These obstructions are explicit in the G-crossed UMTC approach, but implicit in our approach. It is not clear how to read out the obstructions directly from $\text{UMTC}_{/\mathcal{E}}$'s without using the mathematical equivalence with G-crossed UMTCs.

The second difference is more fundamental which forces us to take our new perspective. Although the two approaches are equivalent for boson systems, the G-crossed approach can not be applied to $\mathbb{Z}_2^f$, the fermion number parity. We think that the underlying physical reason is that $\mathbb{Z}_2^f$ can not be broken, not only for the system but also for all our probes. Thus, only the "symmetric" perspective works. Again let's use the example of trivial fermion topological order with no other symmetry, to illustrate this. The even irreducible representation of $\mathbb{Z}_2^f$, $z|+\rangle = |+\rangle$, is now physically a boson, and the odd irreducible representation $z|-\rangle = -|-\rangle$ is a fermion. To "observe" the $\mathbb{Z}_2^f$ action

$$z|0\rangle = |1\rangle, \tag{A.3}$$

we must have the states

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle), \tag{A.4}$$

which are the superpositions of bosons with fermions. This is impossible. Therefore, we can only have "representations" but no "actions" of $\mathbb{Z}_2^f$. For fermionic topological phases, we have to use $\text{sRep}(G^f) = \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$. Surely one can break the bosonic part $G_b = G^f/\mathbb{Z}_2^f$, and obtain, similar to G-crossed UMTCs, a theory of G_b-crossed $\text{UMTC}_{/\,\text{sRep}(\mathbb{Z}_2^f)}$'s. This is possible but has not been very well developed comparing to its bosonic companion. Nonetheless, the $\text{sRep}(\mathbb{Z}_2^f)$ part can never be broken.

In the end, we want to mention that partially breaking the symmetry is also of interest. This leads a theory that mixes the G-crossed part and the over $\mathcal{E}$ part, where G_b-crossed $\text{UMTC}_{/\,\text{sRep}(\mathbb{Z}_2^f)}$ is just a special case. It may reveal more structures and provide us with explicit formulas on group-cohomological classifications and obstructions for fermionic SPT and SET phases.

Appendix B

Mirror and Time-reversal Symmetry

In this Appendix we briefly discuss the categorical description for mirror and time-reversal symmetries. Recall that the mirror conjugate $\overline{\mathcal{C}}$ of a UBFC $\mathcal{C}$ is canonically braided equivalent to the time-reversal conjugate $\mathcal{C}^{tr}$. Therefore, the classification of time-reversal SETs should be the same as mirror SETs. We start by considering UMTCs with mirror symmetry action:

Definition 13. A topological phase (UMTC) $\mathcal{C}$ has mirror symmetry (potentially anomalous) if there is a braided tensor equivalence $T : \mathcal{C} \to \overline{\mathcal{C}}$. More precisely, there are two braided tensor functors $T : \mathcal{C} \to \overline{\mathcal{C}}$, $\overline{T} : \overline{\mathcal{C}} \to \overline{\overline{\mathcal{C}}} = \mathcal{C}$, such that $\overline{T}T \cong \mathrm{id}_{\mathcal{C}}$, $T\overline{T} \cong \mathrm{id}_{\overline{\mathcal{C}}}$.

Here the data $\overline{T}T \cong \mathrm{id}_{\mathcal{C}}$, $T\overline{T} \cong \mathrm{id}_{\overline{\mathcal{C}}}$ encode the mirror symmetry fractionalization.

The existence of $T : \mathcal{C} \cong \overline{\mathcal{C}}$ implies that the central charge of $\mathcal{C}$ is $c = 0$ or 4 mod 8. A first anomaly-free condition is that $\mathcal{C}$ has exactly zero central charge. To study the other anomalies, since such equivalence T is *not* a braided tensor equivalence (automorphism) from $\mathcal{C}$ to $\mathcal{C}$ itself, we can not directly apply the techniques developed for on-site symmetries (G-crossed UMTC or UMTC$_{/\mathcal{E}}$ with modular extensions). So we use the folding trick [52, 53] to turn mirror symmetry into an on-site $\mathbb{Z}_2$ symmetry. Folding the topological phase $\mathcal{C}$ along the mirror axis, and we obtain a double-layer phase $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ together with a canonical gapped boundary $\mathcal{C}$. Alternatively, such folding can be encoded in a canonical Lagrangian condensable algebra $L_{\mathcal{C}} \cong \bigoplus_{i \in \mathcal{C}} i \boxtimes i^*$ in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. Condensing $L_{\mathcal{C}}$ one obtains the trivial phase $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^0_{L_{\mathcal{C}}} = \mathbf{Hilb}$ and the gapped boundary $(\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}} = \mathcal{C}$ as a fusion category (forget braidings on $\mathcal{C}$).

Now the mirror symmetry is turned into the on-site $\mathbb{Z}_2$ action on $\mathcal{C} \boxtimes \overline{\mathcal{C}}$, on the gapped boundary $\mathcal{C}$ and on $L_{\mathcal{C}}$.

1. **Action on $\mathcal{C} \boxtimes \overline{\mathcal{C}}$**

   Let $\tilde{T} : \mathcal{C} \boxtimes \overline{\mathcal{C}} \xrightarrow{T \boxtimes \overline{T}} \overline{\mathcal{C}} \boxtimes \mathcal{C} \cong \mathcal{C} \boxtimes \overline{\mathcal{C}}$, where the second equivalence is just exchanging two layers. It is clear $\tilde{T}^2 \cong \mathrm{id}_{\mathcal{C} \boxtimes \overline{\mathcal{C}}}$. Moreover, it "fractionalization" $H^2(\mathbb{Z}_2, (\mathcal{C} \boxtimes \overline{\mathcal{C}})_{Ab})$ is always trivial. This means that we can take $\tilde{T}^2 = \mathrm{id}_{\mathcal{C} \boxtimes \overline{\mathcal{C}}}$.

2. **Action on the gapped boundary $\mathcal{C}$**

   We have the bulk to boundary map

$$\begin{aligned} \mathcal{C} \boxtimes \overline{\mathcal{C}} &\to \mathcal{C} \\ X \boxtimes Y &\mapsto X \otimes Y \end{aligned} \tag{B.1}$$

   As $T, \overline{T}$ are essentially the same tensor functor, the action on the gapped boundary is just $T : \mathcal{C} \cong \mathcal{C}$ viewed as a tensor functor. (Note that braided tensor functor and tensor functor contain the *same* data. Being braided or not is just a *property* of a tensor functor.) $\tilde{T}$ is naturally the induced action in the bulk on $Z(\mathcal{C}) = \mathcal{C} \boxtimes \overline{\mathcal{C}}$ from the action T on the boundary $\mathcal{C}$.

3. **Action on $L_{\mathcal{C}}$**

   We see that the data $\overline{T}T \cong \mathrm{id}_{\mathcal{C}}, T\overline{T} \cong \mathrm{id}_{\overline{\mathcal{C}}}$ are combined together in $\tilde{T}^2 \cong \mathrm{id}_{\mathcal{C} \boxtimes \overline{\mathcal{C}}}$. To see the mirror symmetry fractionalization, we consider the action on $L_{\mathcal{C}}$, which is an *algebra isomorphism* $\alpha : \tilde{T}(L_{\mathcal{C}}) \cong L_{\mathcal{C}}$ satisfying

$$\begin{tikzcd} \tilde{T}^2(L_{\mathcal{C}}) \arrow[r, "\tilde{T}(\alpha)"] \arrow[d, "\cong"] & \tilde{T}(L_{\mathcal{C}}) \arrow[d, "\alpha"] \\ L_{\mathcal{C}} \arrow[r, "="] & L_{\mathcal{C}} \end{tikzcd} \tag{B.2}$$

   Here algebra isomorphism means that

$$\begin{tikzcd} \tilde{T}(L_{\mathcal{C}}) \otimes \tilde{T}(L_{\mathcal{C}}) \arrow[r, "\alpha \otimes \alpha"] \arrow[d, "\cong"] & L_{\mathcal{C}} \otimes L_{\mathcal{C}} \arrow[dd, "m"] \\ \tilde{T}(L_{\mathcal{C}} \otimes L_{\mathcal{C}}) \arrow[d, "\tilde{T}(m)"] & \\ \tilde{T}(L_{\mathcal{C}}) \arrow[r, "\alpha"] & L_{\mathcal{C}} \end{tikzcd} \tag{B.3}$$

   where m is the multiplication morphism.

Note that the action α in general is *not* an automorphism of $L_{\mathcal{C}}$, but an isomorphism from $L_{\mathcal{C}}$ to $\tilde{T}(L_{\mathcal{C}})$. We like to prove that α is the same data as $\overline{T}T \cong \mathrm{id}_{\mathcal{C}}$. Say $\alpha = \oplus \alpha_i$ where $\alpha_i : \tilde{T}(i \boxtimes i^*) = \overline{T}(i^*) \boxtimes T(i) \to j^* \boxtimes j$, $j = T(i)$. Set $\overline{T}T(i) \xrightarrow{\alpha_i \, \mathrm{id}_i} i$. To be consistent with $T\overline{T}T(i) \xrightarrow{\alpha_i \, \mathrm{id}_{T(i)}} T(i)$, we should set $T\overline{T}(i) \xrightarrow{\alpha_{T(i)} \, \mathrm{id}_i} i$. Under such identification, (B.2) which now reads "$\alpha_i \alpha_{T(i^*)}$ equals $\tilde{T}^2(i \boxtimes i^*) \to i \boxtimes i^*$", is the same as "$\overline{T}T \boxtimes T\overline{T}(i \boxtimes i^*) \to i \boxtimes i^*$ equals $\tilde{T}^2(i \boxtimes i^*) \to i \boxtimes i^*$". Also after writing the multiplication m out explicitly, we can show that "α is an algebra isomorphism (B.3)" is equivalent to "$\overline{T}T(i) \xrightarrow{\alpha_i \, \mathrm{id}_i} i$ is a monoidal natural isomorphism".

Now we have a two-layer system with an on-site $\mathbb{Z}_2$ symmetry. We turn to the "symmetric perspective", UMTC$_{/\mathcal{E}}$'s with modular extensions. We first "take representations" (equivariantization) of the on-site $\mathbb{Z}_2$ action $\tilde{T}$, denoted by $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$, which is a UMTC$_{/\operatorname{Rep}(\mathbb{Z}_2)}$. Similar to representations in the category of vector spaces, which is a pair (V, ρ), a vector space V with actions $\rho : G \to \operatorname{Aut}(V)$, an object ("representation") in $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$ is a pair $(X, \mu_{\tilde{T}})$, an object $X \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ with an action of $\tilde{T}$, $\mu_{\tilde{T}} : X \cong X$ satisfying similar condition as (B.2).

Thus, $(L_{\mathcal{C}}, \alpha)$ is an object in $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$. In $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$, choosing the action α is just assigning "symmetry charges" to components of $L_{\mathcal{C}}$. Since α is an algebra isomorphism, $(L_{\mathcal{C}}, \alpha)$ is also an algebra in $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$. This means that the assignment of "symmetry charges" must make the new object in $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$ still an algebra. Condensing $(L_{\mathcal{C}}, \alpha)$ we get $[(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}]_{(L_{\mathcal{C}}, \alpha)} = \mathcal{C}^{\mathbb{Z}_2}$ and $[(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}]^0_{(L_{\mathcal{C}}, \alpha)} = \operatorname{Rep}(\mathbb{Z}_2)$, where $\mathcal{C}^{\mathbb{Z}_2}$ is the equivariantization of $\mathcal{C}$ as fusion categories (i.e., the gapped boundary).

$\mathcal{C}^{\mathbb{Z}_2}$ is almost the same construction as $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$. The only thing need to be noted here is that we view $\mathcal{C}$ as only a fusion category without braidings, so the "symmetry fractionalization on the boundary" is given by $H^2(\mathbb{Z}_2, Z(\mathcal{C})_{Ab})$, valued in Abelian anyons in the center of $\mathcal{C}$, $Z(\mathcal{C}) = \mathcal{C} \boxtimes \overline{\mathcal{C}}$, rather than $\mathcal{C}$ itself. This makes $H^2(\mathbb{Z}_2, Z(\mathcal{C})_{Ab})$ trivial so we can always set $T^2 = \mathrm{id}_{\mathcal{C}}$ as tensor functors. This is consistent with the fact that symmetry fractionalization of $\tilde{T}$ for the two-layer system is trivial. On the other hand $\overline{T}T \cong \mathrm{id}_{\mathcal{C}}, T\overline{T} \cong \mathrm{id}_{\overline{\mathcal{C}}}$ as braided tensor functors is independent data, which is "mirror symmetry fractionalization" of the single-layer system.

Now we are ready to discuss the anomaly of mirror symmetry fractionalization. As it is now encoded in the action α on the algebra $L_{\mathcal{C}}$, the anomaly of mirror symmetry fractionalization is the same as the anomaly of the condensable algebra $(L_{\mathcal{C}}, \alpha)$. Recall the discussions in Section 5.1. We can just take $Z(\mathcal{C}^{\mathbb{Z}_2})$ as a modular extension of $(\mathcal{C} \boxtimes \overline{\mathcal{C}})^{\mathbb{Z}_2}$. Condensing $(L_{\mathcal{C}}, \alpha)$ in the gauged theory $Z(\mathcal{C}^{\mathbb{Z}_2})$ we get the gauged theory of a $\mathbb{Z}_2$-SPT phase, which reveals the anomaly of the action α (or mirror symmetry fractionalization).

If $Z(\mathcal{C}^{\mathbb{Z}_2})^0_{(L_\mathcal{C},\alpha)} = Z(\mathrm{Rep}(\mathbb{Z}_2))$, α is anomaly free; if $Z(\mathcal{C}^{\mathbb{Z}_2})^0_{(L_\mathcal{C},\alpha)} \neq Z(\mathrm{Rep}(\mathbb{Z}_2))$ (and it can only be the double-semion phase), α is anomalous.

Next we discuss an example. In the toric code model, $\mathcal{C} = Z(\mathrm{Rep}(\mathbb{Z}_2))$. One possible mirror action is that the functor T that acts as identity on objects but maps to reversed braiding. Then $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ is just the double-layer toric code, and $Z(\mathcal{C}^{\mathbb{Z}_2})$ should be $D(D_4) \equiv Z(\mathbf{Vec}_{D_4}) = Z(\mathrm{Rep}(D_4))$, where $\mathbf{Vec}_G$ denotes the category of G-graded vector spaces. The algebra in $D(D_4)$ corresponding to $eTmT$ (both e, m has $T^2 = -1$, or $\alpha_{\mathbf{1}} = \alpha_f = 1, \alpha_e = \alpha_m = -1$) gives a condensation to the double-semion phase [54]. Thus $eTmT$ is anomalous.

It would be beneficial to go through the equivariantization process in detail and show the similarities and differences between equivariantization and taking usual representations. Let's calculate how $\mathcal{C}^{\mathbb{Z}_2}$ gives $\mathbf{Vec}_{D_4}$ with $\mathcal{C} = Z(\mathrm{Rep}(\mathbb{Z}_2)) = \{\mathbf{1}, e, m, f\}$ and T the functor that does not permute any objects. Naively we have 8 simple objects in $\mathcal{C}^{\mathbb{Z}_2}$, of the form (i, x), $i \in \mathcal{C}$, $x = \pm 1$ corresponding to the morphism $T(i) \xrightarrow{x\,\mathrm{id}_i} i$. But now one can not simply add up the $\mathbb{Z}_2$ charge x. Fusion in $\mathcal{C}^{\mathbb{Z}_2}$ is given by $(i, x) \otimes (j, y) = (i \otimes j, w)$, where w is the morphism $T(i \otimes j) \cong T(i) \otimes T(j) \xrightarrow{x\,\mathrm{id}_i \otimes y\,\mathrm{id}_j} i \otimes j$.

In order for T to be a braided functor between $\mathcal{C}$ and $\overline{\mathcal{C}}$, we can not take $T(i \otimes j) \cong T(i) \otimes T(j)$ to be simply identity morphisms. More precisely, T is braided if

$$\begin{tikzcd} T(i \otimes j) \arrow[r, "\cong"] \arrow[d, "T(c_{ij})"'] & T(i) \otimes T(j) \arrow[d, "c_{T(i),T(j)}"] \\ T(j \otimes i) \arrow[r, "\cong"] & T(j) \otimes T(i) \end{tikzcd} \tag{B.4}$$

where c_{ij} is the braiding. For toric code, in certain gauge we have for example $c_{e,m} = 1, c_{m,e} = -1$, so in mirror conjugate $\overline{\mathcal{C}}$ we have $c_{\overline{e},\overline{m}} = c^{-1}_{m,e} = -1, c_{\overline{m},\overline{e}} = 1$. It is then clear that the difference between $T(e \otimes m) \cong T(e) \otimes T(m)$ and $T(m \otimes e) \cong T(m) \otimes T(e)$ must be -1. A good choice happens to be $T(i \otimes j) \xrightarrow{c_{ij}} T(i) \otimes T(j)$. We take

$$c_{ee} = c_{mm} = c_{em} = c_{fm} = c_{ef} = 1, \quad c_{ff} = c_{me} = c_{mf} = c_{fe} = -1. \tag{B.5}$$

Thus the fusion rules are

$$(i, x) \otimes (j, y) = (i \otimes j, c_{ij} xy). \tag{B.6}$$

which is just the extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$ with "2-cocycle" c_{ij}. To see that it is the same as D_4, just check the following

- $\mathbb{Z}_4$ subgroup $\{(\mathbf{1},1),(f,1),(\mathbf{1},-1),(f,-1)\}$.
- $\mathbb{Z}_2$ subgroup $\{(\mathbf{1},1),(e,1)\}$.
- $(e,1),(f,1)$ generate the group, and $(e,1)\otimes(f,1)\otimes(e,1)=(f,-1)$.

Thus the fusion is isomorphic to $\mathbb{Z}_4 \rtimes \mathbb{Z}_2 = D_4$. Since $\mathcal{C}=Z(\mathrm{Rep}(\mathbb{Z}_2))$ has trivial associator or F-matrices, $\mathcal{C}^{\mathbb{Z}_2}$ also has trivial F-matrices. Thus $\mathcal{C}^{\mathbb{Z}_2}$ is the fusion category with D_4 fusion rules and no additional 3-cocycle twists, which means that $\mathcal{C}^{\mathbb{Z}_2}=\mathbf{Vec}_{D_4}$, $Z(\mathcal{C}^{\mathbb{Z}_2}) = D(D_4)$.

It is also interesting to calculate the case when T permutes e,m. We have 5 simple objects in $\mathcal{C}^{\mathbb{Z}_2}$, $\{(\mathbf{1},1),(\mathbf{1},-1),(f,1),(f,-1),(e\oplus m,1)\}$, where $(e\oplus m,1)$ is of quantum dimension 2. In this case $T(i\otimes j)\cong T(i)\otimes T(j)$ can be chosen to be just identify morphisms (for $T:\mathcal{C}\to\overline{\mathcal{C}}$, but for unitary on-site e,m exchange they can not be identities). Its fusion rules are the same as $\mathrm{Rep}(D_4)$ or $\mathrm{Rep}(Q_8)$. By calculating the F-matrices in $\mathcal{C}^{\mathbb{Z}_2}$ it should be possible to explicitly identify $\mathcal{C}^{\mathbb{Z}_2}$ with $\mathrm{Rep}(D_4)$, but this way is too involved. We use a result in Ref. [55] to bypass it. The result says that $Z(\mathcal{C}\rtimes G)=Z(\mathcal{C}^G)$. As the only non-trivial structure of T acting on $\mathcal{C}$ is exchanging e,m, the corresponding $\mathcal{C}\rtimes\mathbb{Z}_2 = \mathbf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}\rtimes\mathbb{Z}_2 = \mathbf{Vec}_{(\mathbb{Z}_2\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2} = \mathbf{Vec}_{D_4}$. Thus we also have $Z(\mathcal{C}^{\mathbb{Z}_2}) = Z(\mathbf{Vec}_{D_4}) = D(D_4) = Z(\mathrm{Rep}(D_4))$. This also implies that $\mathcal{C}^{\mathbb{Z}_2}$ must be $\mathrm{Rep}(D_4)$. (Note that when T does not permute e,m, we showed that T has other non-trivial structures. In this case $\mathcal{C}\rtimes\mathbb{Z}_2 = \mathbf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}\rtimes\mathbb{Z}_2 = \mathbf{Vec}^{\omega_3}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}$ with some 3-cocycle ω_3 twist.)

Appendix C

Abelian Topological Orders, K-matrix and Abelian Condensation

Consider a bosonic Abelian topological order, which can always be described by an even K-matrix K_0 of dimension κ. Anyons are labeled by κ-dimensional integer vectors $\boldsymbol{l}_0$. Two integer vectors $\boldsymbol{l}_0$ and $\boldsymbol{l}'_0$ are equivalent (i.e., describe the same type of topological excitation) if they are related by

$$\boldsymbol{l}'_0 = \boldsymbol{l}_0 + K_0 \boldsymbol{k}, \tag{C.1}$$

where $\boldsymbol{k}$ is an arbitrary integer vector. Fusion of anyons are done by first adding up vectors and then imposing the above equivalence relation. The mutual statistical angle between two anyons, $\boldsymbol{l}_0$ and $\boldsymbol{k}_0$, is given by

$$\theta_{\boldsymbol{l}_0,\boldsymbol{k}_0} = 2\pi \boldsymbol{k}_0^T K_0^{-1} \boldsymbol{l}_0. \tag{C.2}$$

The spin of the anyon $\boldsymbol{l}_0$ is given by

$$s_{\boldsymbol{l}_0} = \frac{1}{2} \boldsymbol{l}_0^T K_0^{-1} \boldsymbol{l}_0. \tag{C.3}$$

Next we discuss Abelian condensation in Abelian topological orders in the K-matrix formulation. Let us construct a new topological order from the K_0 topological order by assuming Abelian anyons labeled by $\boldsymbol{l}_c$ condense. Here we treat the anyon as a bound state between a boson and flux. We then smear the flux such that it behaves like an additional uniform magnetic field, and condense the boson into $\nu = 1/m_c$ Laughlin state (where

m_c = even). The resulting new topological order is described by the $(\kappa+1)$-dimensional K-matrix

$$K_1 = \begin{pmatrix} K_0 & \boldsymbol{l}_c \\ \boldsymbol{l}_c^T & m_c \end{pmatrix} \tag{C.4}$$

In the following, we are going to show that, to describe the result of the $\boldsymbol{l}_c$ anyon condensation, we do not need to know K_0 directly. We only need to know the spin of the condensing particle $\boldsymbol{l}_c$

$$s_c = \frac{1}{2}\boldsymbol{l}_c^T K_0^{-1}\boldsymbol{l}_c, \tag{C.5}$$

and the mutual statistics

$$\theta_{\boldsymbol{l}_0,\boldsymbol{l}_c} \equiv 2\pi t_{\boldsymbol{l}_0}, \quad t_{\boldsymbol{l}_0} = \boldsymbol{l}_c^T K_0^{-1}\boldsymbol{l}_0 \tag{C.6}$$

between $\boldsymbol{l}_0$ and $\boldsymbol{l}_c$.

First, we find that, as long as $m_c - 2s_c \neq 0$, K_1 is invertible with

$$K_1^{-1} = \begin{pmatrix} K_0^{-1} + \frac{K_0^{-1}\boldsymbol{l}_c\boldsymbol{l}_c^T K_0^{-1}}{m_c-2s_c} & -\frac{K_0^{-1}\boldsymbol{l}_c}{m_c-2s_c} \\ -\frac{\boldsymbol{l}_c^T K_0^{-1}}{m_c-2s_c} & \frac{1}{m_c-2s_c} \end{pmatrix} \tag{C.7}$$

The anyons in the new K_1 topological order are labeled by $\kappa+1$-dimensional integer vectors $\boldsymbol{l}^T = (\boldsymbol{l}_0^T, m)$. The spin of $\boldsymbol{l}$ is

$$\begin{aligned} s_{\boldsymbol{l}} &= \frac{1}{2}\boldsymbol{l}^T K_1^{-1}\boldsymbol{l} = \frac{1}{2}\Big(2s_0 + \frac{m^2 + t_{\boldsymbol{l}_0}^2 - 2mt_{\boldsymbol{l}_0}}{m_c - 2s_c}\Big) \\ &= s_{\boldsymbol{l}_0} + \frac{1}{2}\frac{(m - t_{\boldsymbol{l}_0})^2}{m_c - 2s_c} \end{aligned} \tag{C.8}$$

The vectors $\boldsymbol{l}^T = (\boldsymbol{l}_0^T, m)$ and $\boldsymbol{l}'^T = (\boldsymbol{l}_0'^T, m')$ are equivalent if they are related by

$$\boldsymbol{l}_0' - \boldsymbol{l}_0 = K_0\boldsymbol{k}_0 + k\boldsymbol{l}_c, \quad m' - m = \boldsymbol{l}_c^T \cdot \boldsymbol{k}_0 + m_c k, \tag{C.9}$$

for any κ-dimensional integer vector $\boldsymbol{k}_0$ and integer k. To avoid the gauge ambiguity, for the integer vectors $\boldsymbol{l}_0$, we pick a representative for each equivalence class (by (C.1), fixing the gauge). Taking $k = 1$ and appropriate $\boldsymbol{k}_0$ such that $\boldsymbol{l}_0'$ and $\boldsymbol{l}_0$ are the pre-fixed representatives, we see that

$$(\boldsymbol{l}_0^T, m) \sim (\boldsymbol{l}_0'^T \sim \boldsymbol{l}_0^T + \boldsymbol{l}_c^T, m + t_{\boldsymbol{l}_0'} - t_{\boldsymbol{l}_0} + m_c - 2s_c). \tag{C.10}$$

We also want to express the fusion in the new phase in terms of the pre-fixed representatives $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3$. Assuming that $(\boldsymbol{l}_3^T, m_3) \sim (\boldsymbol{l}_1^T + \boldsymbol{l}_2^T, m_1 + m_2)$, and taking $k = 0$ and appropriate $\boldsymbol{k}_0$ in (C.9) (the cases of non-zero k can be generated via (C.10)), we find that

$$\begin{aligned}
&(\boldsymbol{l}_1^T, m_1) + (\boldsymbol{l}_2^T, m_2) \\
&\sim (\boldsymbol{l}_3^T \sim \boldsymbol{l}_1^T + \boldsymbol{l}_2^T, m_3 = m_1 + m_2 + t_{\boldsymbol{l}_3} - t_{\boldsymbol{l}_1} - t_{\boldsymbol{l}_2}).
\end{aligned} \tag{C.11}$$

We can easily calculate the determinant of K_1 whose absolute value is the rank of the new phase:

$$\begin{aligned}
\det(K_1) &= \det \begin{pmatrix} K_0 & \boldsymbol{l}_c \\ \boldsymbol{l}_c^T & m_c \end{pmatrix} = \det(K_0)(m_c - \boldsymbol{l}_c^T K_0^{-1} \boldsymbol{l}_c) \\
&= (m_c - 2s_c) \det(K_0)
\end{aligned} \tag{C.12}$$

Let $M_c = m_c - 2s_c$. It is an important gauge invariant quantity relating the ranks of the two phases. If we perform the condensation with a different anyon $\boldsymbol{l}_c'$ and a different even integer m_c', but make sure that $\boldsymbol{l}_c' \sim \boldsymbol{l}_c$ and $M_c' = m_c' - 2s_c' = m_c - 2s_c = M_c$, the new topological order will be the same.

It is worth mentioning that such construction is reversible: for the K_1 state, take $\boldsymbol{l}_c'^T = (\boldsymbol{0}^T, 1), m_c' = 0$, and repeat the construction:

$$K_2 = \begin{pmatrix} K_0 & \boldsymbol{l}_c & 0 \\ \boldsymbol{l}_c^T & m_c & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} K_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim K_0. \tag{C.13}$$

We return to the original K_0 state.

Appendix D

Selected Tables for Topological Phases with Symmetries

In this Appendix we give several tables of UMTC$_{/\mathcal{E}}$'s for various $\mathcal{E}$, in terms of quantum dimensions d_i and topological spins s_i [9, 10]. Since we are not able to calculate modular extensions for all entries, some of them may be invalid. However, within certain numerical search bound, we have given all possible candidates. Also as we showed in Chapter 4, as long as the UMTC$_{/\mathcal{E}}$ is valid and has modular extensions, it determines the topological phases up to invertible ones. So these tables can be viewed as listing candidates for topological phases with symmetries, up to invertible ones. As the classification of invertible bosonic phases is clear, we will only mention the classification of invertible fermionic phases in the following examples. The numerical search was done by my supervisor Xiao-Gang Wen based on his algorithm searching for bosonic topological orders (UMTCs) [8].

Table D.1 lists all bosonic topological phases with $\mathbb{Z}_2$ symmetry for $N = 3, 4$ and $D^2 \leq 100$. All the topological orders in this list have modular extensions, and are realizable by (2+1)D boson systems. We use $N_c^{|\Theta|}$ to label UMTC$_{/\mathcal{E}}$'s, where $\Theta = D^{-1}\sum_i \mathrm{e}^{2\pi \mathrm{i} s_i} d_i^2 = |\Theta|\mathrm{e}^{2\pi \mathrm{i} c/8}$ and $D^2 = \sum_i d_i^2$. In the "comment/K-matrix" column, SB means the state after symmetry breaking, $\boxtimes^t$ indicates a twisted fusion rule (from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_4$), N_c^B means a bosonic topological order with central charge c (see Refs. [8, 9]) and K-matrix describes an Abelian topological order (see Appendix C for a brief introduction). Here $\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$.

Table D.2 lists all bosonic topological phases with $\mathbb{Z}_3$ symmetry, for $N = 4, 5, 6$ and $D^2 \leq 100$, $N = 7$ and $D^2 \leq 60$, $N = 8$ and $D^2 \leq 40$.

Table D.3 lists all bosonic topological phases with S_3 symmetry, for $N = 4, 5, 6$ and

Table D.1: Bosonic topological phases with $\mathbb{Z}_2$ symmetry.

$N_c^{\vert\Theta\vert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment/K-matrix
$2_0^{\zeta_2^1}$	2	$1, 1$	$0, 0$	$\mathcal{E} = \mathrm{Rep}(\mathbb{Z}_2)$
$3_2^{\zeta_2^1}$	6	$1, 1, 2$	$0, 0, \frac{1}{3}$	SB:$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
$3_{-2}^{\zeta_2^1}$	6	$1, 1, 2$	$0, 0, \frac{2}{3}$	SB:$K = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$
$4_1^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{1}{4}$	$2_1^B \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$
$4_1^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{1}{4}$	$2_1^B \boxtimes^t \mathrm{Rep}(\mathbb{Z}_2)$
$4_{-1}^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{3}{4}, \frac{3}{4}$	$2_{-1}^B \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$
$4_{-1}^{\zeta_2^1}$	4	$1, 1, 1, 1$	$0, 0, \frac{3}{4}, \frac{3}{4}$	$2_{-1}^B \boxtimes^t \mathrm{Rep}(\mathbb{Z}_2)$
$4_{14/5}^{\zeta_2^1}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{2}{5}, \frac{2}{5}$	$2_{14/5}^B \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$
$4_{-14/5}^{\zeta_2^1}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{3}{5}, \frac{3}{5}$	$2_{-14/5}^B \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$
$4_0^{\zeta_2^1}$	10	$1, 1, 2, 2$	$0, 0, \frac{1}{5}, \frac{4}{5}$	SB:$K = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$
$4_4^{\zeta_2^1}$	10	$1, 1, 2, 2$	$0, 0, \frac{2}{5}, \frac{3}{5}$	SB:$K = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

$D^2 \leq 100$, $N = 7$ and $D^2 \leq 60$, $N = 8$ and $D^2 \leq 40$.

Table D.4 lists fermionic topological phases with $\mathbb{Z}_2^f$ symmetry. The corresponding SPT is trivial and $c_{\text{min}} = 1/2$. In the labels we use c mod c_{min}. For fermionic phases we always have $\Theta = 0$. Thus we add additional labels to distinguish entries. Θ_2 in the table is defined as $\Theta_2 \equiv D^{-1} \sum_i \mathrm{e}^{\mathrm{i}4\pi s_i} d_i^2$. Also $\angle\Theta_2 := \mathrm{Im} \ln \Theta_2$. The table contains all fermionic topological orders with $N = 2$, $N = 4$ and $D^2 \leq 600$, $N = 6$ and $D^2 \leq 400$. They all have modular extensions and are all realizable by (2+1)D fermion systems.

Table D.5 lists fermionic topological phases with $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ symmetry. The corresponding SPT is classified by $\mathbb{Z}_8$ and $c_{\text{min}} = 1/2$. The list contains all topological orders with $N = 6$ and $D^2 \leq 300$, $N = 8$ and $D^2 \leq 60$, $N = 10$ and $D^2 \leq 20$.

Table D.6 lists fermionic topological phases with $\mathbb{Z}_4^f$ symmetry. The corresponding SPT is trivial and $c_{\text{min}} = 1$. The list contains all topological orders with $N = 6$ and $D^2 \leq 100$, $N = 8$ and $D^2 \leq 60$, $N = 10$ and $D^2 \leq 20$.

Table D.2: Bosonic topological phases with $\mathbb{Z}_3$ symmetry.

$N_c^{\|\Theta\|}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment/K-matrix
$3_0^{\zeta_4^1}$	3	$1, 1, 1$	$0, 0, 0$	$\mathcal{E} = \text{Rep}(\mathbb{Z}_3)$
$4_4^{\zeta_4^1}$	12	$1, 1, 1, 3$	$0, 0, 0, \frac{1}{2}$	SB:$K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$
$6_1^{\zeta_4^1}$	6	$1, 1, 1, 1, 1, 1$	$0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	$2_1^B \boxtimes \text{Rep}(\mathbb{Z}_3)$
$6_{-1}^{\zeta_4^1}$	6	$1, 1, 1, 1, 1, 1$	$0, 0, 0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}$	$2_{-1}^B \boxtimes \text{Rep}(\mathbb{Z}_3)$
$6_{14/5}^{\zeta_4^1}$	10.854	$1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}$	$2_{14/5}^B \boxtimes \text{Rep}(\mathbb{Z}_3)$
$6_{-14/5}^{\zeta_4^1}$	10.854	$1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, 0, 0, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}$	$2_{-14/5}^B \boxtimes \text{Rep}(\mathbb{Z}_3)$
$8_3^{\zeta_4^1}$	24	$1, 1, 1, 1, 1, 1, 3, 3$	$0, 0, 0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{2}$	$2_{-1}^B \boxtimes 4_4^{\zeta_4^1}$
$8_{-3}^{\zeta_4^1}$	24	$1, 1, 1, 1, 1, 1, 3, 3$	$0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$2_1^B \boxtimes 4_4^{\zeta_4^1}$
$8_{6/5}^{\zeta_4^1}$	43.416	$1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, 3, \frac{3+\sqrt{45}}{2}$	$0, 0, 0, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{2}, \frac{1}{10}$	$2_{-14/5}^B \boxtimes 4_4^{\zeta_4^1}$
$8_{-6/5}^{\zeta_4^1}$	43.416	$1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, 3, \frac{3+\sqrt{45}}{2}$	$0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{2}, \frac{9}{10}$	$2_{14/5}^B \boxtimes 4_4^{\zeta_4^1}$

Table D.3: Bosonic topological phases with S_3 symmetry.

$N_c^{\vert\Theta\vert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment/K-matrix
$3_0^{\sqrt{6}}$	6	$1, 1, 2$	$0, 0, 0$	$\mathcal{E} = \mathrm{Rep}(S_3)$
$5_4^{\sqrt{6}}$	24	$1, 1, 2, 3, 3$	$0, 0, 0, \frac{1}{2}, \frac{1}{2}$	SB:4_4^B
$5_4^{\sqrt{6}}$	24	$1, 1, 2, 3, 3$	$0, 0, 0, \frac{1}{2}, \frac{1}{2}$	SB:4_4^B $K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$
$5_4^{\sqrt{6}}$	24	$1, 1, 2, 3, 3$	$0, 0, 0, \frac{1}{2}, \frac{1}{2}$	SB:4_4^B
$6_1^{\sqrt{6}}$	12	$1, 1, 2, 1, 1, 2$	$0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	$2_1^B \boxtimes \mathrm{Rep}(S_3)$
$6_1^{\sqrt{6}}$	12	$1, 1, 2, 1, 1, 2$	$0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	SB:2_1^B
$6_{-1}^{\sqrt{6}}$	12	$1, 1, 2, 1, 1, 2$	$0, 0, 0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}$	$2_{-1}^B \boxtimes \mathrm{Rep}(S_3)$
$6_{-1}^{\sqrt{6}}$	12	$1, 1, 2, 1, 1, 2$	$0, 0, 0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}$	SB:2_{-1}^B
$6_2^{\sqrt{6}}$	18	$1, 1, 2, 2, 2, 2$	$0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	SB:3_2^B
$6_2^{\sqrt{6}}$	18	$1, 1, 2, 2, 2, 2$	$0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	SB:3_2^B
$6_{-2}^{\sqrt{6}}$	18	$1, 1, 2, 2, 2, 2$	$0, 0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$	SB:3_{-2}^B
$6_{-2}^{\sqrt{6}}$	18	$1, 1, 2, 2, 2, 2$	$0, 0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$	SB:3_{-2}^B
$6_{14/5}^{\sqrt{6}}$	21.708	$1, 1, 2, \zeta_3^1, \zeta_3^1, \zeta_8^4$	$0, 0, 0, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}$	$2_{14/5}^B \boxtimes \mathrm{Rep}(S_3)$
$6_{-14/5}^{\sqrt{6}}$	21.708	$1, 1, 2, \zeta_3^1, \zeta_3^1, \zeta_8^4$	$0, 0, 0, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}$	$2_{-14/5}^B \boxtimes \mathrm{Rep}(S_3)$

Table D.4: Fermionic topological phases with $\mathbb{Z}_2^f$ symmetry.

$N_c^F\binom{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment/K-matrix
$2_0^F\binom{\zeta_2^1}{0}$	2	$1, 1$	$0, \frac{1}{2}$	trivial $\mathcal{F}_0 = \mathrm{sRep}(\mathbb{Z}_2^f)$
$4_0^F\binom{0}{0}$	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2_1^B$, $K = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$
$4_{1/5}^F\binom{\zeta_2^1\zeta_3^1}{3/20}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2_{-14/5}^B$
$4_{-1/5}^F\binom{\zeta_2^1\zeta_3^1}{-3/20}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2_{14/5}^B$
$4_{1/4}^F\binom{\zeta_6^3}{1/2}$	13.656	$1, 1, \zeta_6^2, \zeta_6^2 = 1 + \sqrt{2}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	Modular extension $SU(2)_6$
$6_0^F\binom{\zeta_2^1}{1/4}$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3_{-2}^B$, $K = (3)$
$6_0^F\binom{\zeta_2^1}{-1/4}$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3_2^B$, $K = (-3)$
$6_0^F\binom{\zeta_6^3}{1/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{1/2}^B$
$6_0^F\binom{\zeta_6^3}{-1/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{-1/2}^B$
$6_0^F\binom{1.0823}{3/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{3/2}^B$
$6_0^F\binom{1.0823}{-3/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{-3/2}^B$
$6_{1/7}^F\binom{\zeta_2^1\zeta_5^2}{-5/14}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3_{8/7}^B$
$6_{-1/7}^F\binom{\zeta_2^1\zeta_5^2}{5/14}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3_{-8/7}^B$
$6_0^F\binom{\zeta_{10}^5}{-1/12}$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$	Modular extension $\overline{SU(2)_{10}}$
$6_0^F\binom{\zeta_{10}^5}{1/12}$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$	Modular extension $SU(2)_{10}$

Table D.5: Fermionic topological phases with $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ symmetry.

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment/K-matrix
$4^0_0(\begin{smallmatrix}2\\0\end{smallmatrix})$	4	$1,1,1,1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\mathcal{E} = \mathrm{sRep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$
6^0_0	12	$1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{2}{3}$	SB:$K = \begin{pmatrix}-1 & -2\\-2 & -1\end{pmatrix}$
6^0_0	12	$1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{5}{6}$	SB:$K = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$
$8^0_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	8	$1,1,1,1,1,1,1,1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$2^B_1 \boxtimes \mathrm{sRep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$
$8^0_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	8	$1,1,1,1,1,1,1,1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	SB:$4^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$
$8^0_{-14/5}(\begin{smallmatrix}\zeta_8^4\\3/20\end{smallmatrix})$	14.472	$1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{1}{10},\frac{3}{5},\frac{3}{5}$	$2^B_{-14/5} \boxtimes \mathrm{sRep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$
$8^0_{14/5}(\begin{smallmatrix}\zeta_8^4\\-3/20\end{smallmatrix})$	14.472	$1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{2}{5},\frac{2}{5},\frac{9}{10},\frac{9}{10}$	$2^B_{14/5} \boxtimes \mathrm{sRep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$
$8^0_0(\begin{smallmatrix}2\\0\end{smallmatrix})$	20	$1,1,1,1,2,2,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{2}{5},\frac{3}{5},\frac{9}{10}$	SB:$10^F_0(\begin{smallmatrix}\zeta_2^1\\0\end{smallmatrix})$
$8^0_0(\begin{smallmatrix}2\\1/2\end{smallmatrix})$	20	$1,1,1,1,2,2,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{5},\frac{3}{10},\frac{7}{10},\frac{4}{5}$	SB:$10^F_0(\begin{smallmatrix}\zeta_2^1\\1/2\end{smallmatrix})$
$8^0_{1/4}(\begin{smallmatrix}\zeta_2^1\zeta_6^3\\1/2\end{smallmatrix})$	27.313	$1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	SB:$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix})$
$8^0_{1/4}(\begin{smallmatrix}\zeta_2^1\zeta_6^3\\1/2\end{smallmatrix})$	27.313	$1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	SB:$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}4\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$	SB:$8^F_0(\begin{smallmatrix}\sqrt{8}\\0\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}4\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$	SB:$8^F_0(\begin{smallmatrix}\sqrt{8}\\0\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}\sqrt{8}\\1/8\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$	SB:$8^F_0(\begin{smallmatrix}2\\1/8\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}\sqrt{8}\\1/8\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$	SB:$8^F_0(\begin{smallmatrix}2\\1/8\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	SB:$8^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	SB:$8^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}\sqrt{8}\\-1/8\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$	SB:$8^F_0(\begin{smallmatrix}2\\-1/8\end{smallmatrix})$
$10^0_0(\begin{smallmatrix}\sqrt{8}\\-1/8\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$	SB:$8^F_0(\begin{smallmatrix}2\\-1/8\end{smallmatrix})$

Table D.6: Fermionic topological phases with $\mathbb{Z}_4^f$ symmetry.

$N_c^{\lvert\Theta\rvert}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	comment/K-matrix
4_0^0	4	$1,1,1,1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\mathcal{E}=\text{sRep}(\mathbb{Z}_4^f)$
6_0^0	12	$1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{2}{3}$	$K=-\begin{pmatrix}1&2\\2&1\end{pmatrix}$
6_0^0	12	$1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{5}{6}$	$K=\begin{pmatrix}1&2\\2&1\end{pmatrix}$
8_0^0	8	$1,1,1,1,1,1,1,1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$2_{-1}^B \boxtimes \text{sRep}(\mathbb{Z}_4^f)$
8_0^0	8	$1,1,1,1,1,1,1,1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$2_1^B \boxtimes \text{sRep}(\mathbb{Z}_4^f)$
$8_{-14/5}^0$	14.472	$1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{1}{10},\frac{3}{5},\frac{3}{5}$	$2_{-14/5}^B \boxtimes \text{sRep}(\mathbb{Z}_4^f)$
$8_{14/5}^0$	14.472	$1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,0,\frac{1}{2},\frac{1}{2},\frac{2}{5},\frac{2}{5},\frac{9}{10},\frac{9}{10}$	$2_{14/5}^B \boxtimes \text{sRep}(\mathbb{Z}_4^f)$
8_0^0	20	$1,1,1,1,2,2,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{2}{5},\frac{3}{5},\frac{9}{10}$	SB:$10_0^F(\substack{\zeta_2^1\\0})$
8_0^0	20	$1,1,1,1,2,2,2,2$	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{5},\frac{3}{10},\frac{7}{10},\frac{4}{5}$	SB:$10_0^F(\substack{\zeta_2^1\\1/2})$
$10_0^0(\substack{4\\0})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$	SB:$8_0^F(\substack{\sqrt{8}\\0})$
$10_0^0(\substack{4\\0})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$	SB:$8_0^F(\substack{\sqrt{8}\\0})$
$10_0^0(\substack{\sqrt{8}\\1/8})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$	SB:$8_0^F(\substack{2\\1/8})$
$10_0^0(\substack{\sqrt{8}\\1/8})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$	SB:$8_0^F(\substack{2\\1/8})$
$10_0^0(\substack{0\\0})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	SB:$8_0^F(\substack{0\\0})$
$10_0^0(\substack{0\\0})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	SB:$8_0^F(\substack{0\\0})$
$10_0^0(\substack{\sqrt{8}\\-1/8})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$	SB:$8_0^F(\substack{2\\-1/8})$
$10_0^0(\substack{\sqrt{8}\\-1/8})$	16	$1,1,1,1,1,1,1,1,2,2$	$0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$	SB:$8_0^F(\substack{2\\-1/8})$