

P8160 Group Project 3: Bayesian Modeling of Hurricane Trajectories

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Statistical Methods

Posterior Distribution and Conditional Distributions

Likelihood

For each hurricane i and k_i 's time points, we have the following Bayesian model:

$$Y_i(t+6) = \beta_{0i} + \beta_{1i}Y_i(t) + \beta_{2i}\Delta_{i1}(t) + \beta_{3i}\Delta_{i2}(t) + \beta_{4i}\Delta_{i3}(t) + \varepsilon_i(t),$$

where $Y_i(t)$ is the wind speed at time t , Δ_{i1} , Δ_{i2} , and Δ_{i3} are the changes of latitude, longitude, and the wind speed between time point t and $t-6$, respectively. $\varepsilon_i(t)$ follows a normal distributions with mean zero and variance σ^2 . The above Bayesian model can be simplified as:

$$Y_i(t+6) = x_i(t) + \varepsilon_i(t),$$

where $\beta_i = (\beta_{0i}, \beta_{1i}, \dots, \beta_{4i}) \sim N(0, \sigma^2)$. Based on the property of the multivariate linear regression model, for each hurricane i , we have:

$$Y_i | X_i \sim N_{k_i}(x_i\beta_i, \sigma^2 I_{k_i}),$$

where I_{k_i} is an identity matrix with k_i dimensions.

Thus, we can consider the following distribution of each hurricane i :

$$f(y_i | \beta_i, \sigma^2) = [(2\pi)^{k_i} \cdot \det(\sigma^2 I_{k_i})]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}(y_i - x_i\beta_i)^\top (\sigma^2 I_{k_i})^{-1} (y_i - x_i\beta_i)\right\}$$

.

From above, we derive the following likelihood function:

$$\begin{aligned} f(y | B, \sigma^2) &= \prod_{i=1}^n f(y_i | \beta_i, \sigma^2) \\ &= \prod_{i=1}^n \left([(2\pi)^{k_i} \cdot \det(\sigma^2 I_{k_i})]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}(y_i - x_i\beta_i)^\top (\sigma^2 I_{k_i})^{-1} (y_i - x_i\beta_i)\right\} \right) \end{aligned}$$

Prior distributions

We assume the following non-informative prior distributions:

$$\beta_i = (\beta_{0i}, \beta_{1i}, \dots, \beta_{4i}) \sim N_5(\mu, \Sigma),$$

where $B = (\beta_1^\top, \beta_2^\top, \dots, \beta_n^\top)^\top$ and n is the number of hurricanes. So,

$$\pi(B|\mu, \Sigma^{-1}) = \prod_{i=1}^n f(\beta_i) \propto \det(\Sigma)^{-n/2} \cdot \exp \left\{ -\frac{1}{2} \sum_i \left[(\beta_i - \mu)^\top (\Sigma)^{-1} (\beta_i - \mu) \right] \right\}.$$

Also, $\pi(\sigma^2) \propto \frac{1}{\sigma^2}$; $\pi(\mu) \propto 1$; $\pi(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \cdot \exp(-\frac{1}{2}\Sigma^{-1})$.

Conditional posteriors

The posterior distribution is the product of the likelihood and the prior:

$$g(B, \sigma^2, \mu, \Sigma^{-1} | y) \propto f(y | B, \sigma^2) \cdot \pi(B | \mu, \Sigma^{-1}) \cdot \pi(\sigma^2) \cdot \pi(\mu) \cdot \pi(\Sigma^{-1}),$$

so we have:

$$\begin{aligned} \pi(\sigma^2 | \cdot) &\propto \prod_{i=1}^n \det(\sigma^2 I_{k_i})^{-\frac{1}{2}} \cdot \exp \left\{ -\frac{1}{2} \sum_i \left[(y_i - x_i \beta_i)^\top (\sigma^2 I_{k_i})^{-1} (y_i - x_i \beta_i) \right] \right\} \cdot \sigma^{-2} \\ &= (\sigma^2)^{-\frac{1}{2} \sum_i k_i} \cdot \exp \left\{ -\frac{1}{2\sigma^2} \sum_i \left[(y_i - x_i \beta_i)^\top (y_i - x_i \beta_i) \right] \right\} \cdot \sigma^{-2} \\ &= (\sigma^2)^{-1 - \frac{1}{2} \sum_i k_i} \cdot \exp \left\{ -\frac{1}{2\sigma^2} \sum_i \sum_{t_i} (y_{i,t} - x_{i,t} \beta_i)^2 \right\} \end{aligned}$$

Therefore, $\sigma^2 \sim \text{Inverse Gamma} \left(\frac{1}{2} \sum_i k_i, \frac{1}{2} \sum_i \sum_{t_i} (y_{i,t} - x_{i,t} \beta_i)^2 \right)$.

$$\begin{aligned} \pi(\Sigma^{-1} | \cdot) &\propto \det(\Sigma)^{-n/2} \cdot \exp \left\{ -\frac{1}{2} \sum_i (\beta_i - \mu)^\top \Sigma^{-1} (\beta_i - \mu) \right\} \cdot \det(\Sigma)^{-(d+1)} \cdot \exp \left\{ -\frac{1}{2} \Sigma^{-1} \right\} \\ &= \det(\Sigma)^{-(n/2+d+1)} \cdot \exp \left\{ -\frac{1}{2} \left[\Sigma^{-1} + \sum_i (\beta_i - \mu)^\top \Sigma^{-1} (\beta_i - \mu) \right] \right\} \\ &\propto \det(\Sigma^{-1})^{(n+2d+2)/2} \cdot \exp \left\{ -\frac{1}{2} \text{tr} \left[\Sigma^{-1} \cdot \left(I + \sum_i (\beta_i - \mu) (\beta_i - \mu)^\top \right) \right] \right\} \\ &\propto \det(\Sigma^{-1})^{(n+3d+3-d-1)/2} \cdot \exp \left\{ -\frac{1}{2} \text{tr} \left[\Sigma^{-1} \cdot \left(I + \sum_i (\beta_i - \mu) (\beta_i - \mu)^\top \right) \right] \right\} \end{aligned}$$

Thus $\Sigma^{-1} \sim \text{Wishart} \left(n + 3d + 3, [I + \sum_i (\beta_i - \mu) (\beta_i - \mu)^\top]^{-1} \right)$, that is:

$$\Sigma \sim \text{Inverse Wishart} \left(n + 3d + 3, I + \sum_i (\beta_i - \mu) (\beta_i - \mu)^\top \right)$$

$$\begin{aligned} \pi(\mu | \cdot) &\propto \exp \left\{ -\frac{1}{2} \sum_i \left[(\beta_i - \mu)^\top \Sigma^{-1} (\beta_i - \mu) \right] \right\} \\ &= \exp \left\{ -\frac{1}{2} \sum_i \left(\beta_i^\top \Sigma^{-1} \beta_i + \mu^\top \Sigma^{-1} \mu - 2\beta_i^\top \Sigma^{-1} \mu \right) \right\} \\ &= \exp \left\{ -\frac{1}{2} \left(\sum_i \beta_i^\top \Sigma^{-1} \beta_i + \mu^\top n \Sigma^{-1} \mu - 2 \sum_i \beta_i^\top \Sigma^{-1} \mu \right) \right\} \\ &= \exp \left\{ -\frac{1}{2} \left(\mu^\top n \Sigma^{-1} \mu - 2 \sum_i \beta_i^\top \Sigma^{-1} \mu + \sum_i \beta_i^\top \Sigma^{-1} \beta_i \right) \right\} \\ &= \exp \left\{ -\frac{1}{2} \left(\underbrace{\mu^\top n \Sigma^{-1} \mu}_M - 2 \underbrace{\mu^\top \sum_i \Sigma^{-1} \beta_i}_N + \sum_i \beta_i^\top \Sigma^{-1} \beta_i \right) \right\} \\ &= \exp \left\{ -\frac{1}{2} \left[(\mu - M^{-1} N)^\top M (\mu - M^{-1} N) \right] \right\} \end{aligned}$$

Therefore, $\mu \sim MVN(M^{-1}N, M^{-1})$.

$$\begin{aligned}
\pi(B \mid \cdot) &\propto \exp \left\{ -\frac{1}{2} \Sigma_i \left[(y_i - x_i \beta_i)^\top (\sigma^2 I_{k_i})^{-1} (y_i - x_i \beta_i) \right] \right\} \cdot \exp \left\{ -\frac{1}{2} \Sigma_i \left[(\beta_i - \mu)^\top (\Sigma)^{-1} (\beta_i - \mu) \right] \right\} \\
&= \exp \left\{ -\frac{1}{2} \Sigma_i \left[(y_i - x_i \beta_i)^\top (\sigma^2 I_{k_i})^{-1} (y_i - x_i \beta_i) + (\beta_i - \mu)^\top \Sigma^{-1} (\beta_i - \mu) \right] \right\} \\
&= \exp \left\{ -\frac{1}{2} \Sigma_i [y_i^\top \sigma^{-2} I_{k_i} y_i + \beta_i^\top x_i^\top \sigma^{-2} I_{k_i} x_i \beta_i - 2 y_i^\top \sigma^{-2} I_{k_i} x_i \beta_i + \beta_i^\top \Sigma^{-1} \beta_i + \mu^\top \Sigma^{-1} \mu - 2 \mu^\top \Sigma^{-1} \beta_i] \right\} \\
&= \exp \left\{ -\frac{1}{2} \Sigma_i [y_i^\top \sigma^{-2} I_{k_i} y_i + \mu^\top \Sigma^{-1} \mu + \beta_i^\top (\Sigma^{-1} + x_i^\top \sigma^{-2} I_{k_i} x_i) \beta_i - 2 (y_i^\top \sigma^{-2} I_{k_i} x_i + \mu^\top \Sigma^{-1}) \beta_i] \right\}
\end{aligned}$$

We can define the following terms:

$$\begin{aligned}
R &= y_i^\top \sigma^{-2} I_{k_i} y_i + \mu^\top \Sigma^{-1} \mu \\
V &= \Sigma^{-1} + x_i^\top \sigma^{-2} I_{k_i} x_i \\
M &= \sigma^{-2} x_i^\top y_i + \Sigma^{-1} \mu
\end{aligned}$$

Thus, $\pi(B \mid \cdot) \propto (\beta_i - V^{-1}M)^\top V (\beta_i - V^{-1}M) \sim MVN(V^{-1}M, V^{-1})$