# Posterior Derivation

For each hurricane $i$ and $k_i$'s time points:

$Y_i(t+6)=\beta_{0i}+\beta_{ij}Y_i(t)+\beta_x\Delta_{i1}(t)+\beta_{ii}\Delta_{i2}(t)+\beta_{4i}\Delta_{i3}(t)+\varepsilon_i(t)$

$Y_i(t+6)=x_i(t)\cdot\beta_i+\varepsilon_i(t), where \beta_i=(\beta_{0i},\beta_{1i},\ldots,\beta_{4i})\sim N_5(\mu,\Sigma), \varepsilon_i(t)\sim N\left(0,\sigma^2\right)$

$Y_i\mid X_i\sim N_{k_i}\left(x_i\beta_i,\sigma^2 I_{k_i}\right)$

$f\left(y_i\mid\beta_i,\nabla^2\right)=\left[(2\pi)^{k_i}\cdot\det\left(\sigma^2 I_{k_i}\right)\right]^{-\frac{1}{2}}\cdot\exp\left\{-\frac{1}{2}\left(y_i-x_i\beta_i\right)^\top\cdot\left(\sigma^2 I_{k_i}\right)^{-1}\left(y_i-x_i\beta_i\right)\right\}$

Thus we can derive the Likelihood:

$$\begin{aligned}
f\left(y\mid B,\sigma^2\right)&=\prod_{i=1}^n f\left(y_i\mid\beta_i,\sigma^2\right)\\
&=\prod_{i=1}^n\left(\left[(2\pi)^{k_i}\cdot\det\left(\sigma^2 I_{ki}\right)\right]^{-\frac{1}{2}}\cdot\exp\left\{-\frac{1}{2}\left(y_i-x_i\beta_i\right)^\top\cdot\left(\sigma^2 I_{k_i}\right)^{-1}\left(y_i-x_i\beta_i\right)\right\}\right)
\end{aligned}$$

We have the Priors below:

$\beta_i=(\beta_{0i},\beta_{1i},\ldots,\beta_{4i})\sim N_5(\mu,\Sigma)$

$B=\left(\beta_1^\top,\beta_2^\top,\ldots,\beta_n^\top\right)^\top$, assume $\beta_1,\beta_2,\ldots,\beta_n$ are independent,

$\pi(B|\mu,\Sigma)=\prod_{i=1}^n f\left(\beta_i\right)=(2\pi)^{-5n/2}\cdot\det(\Sigma)^{-n/2}\cdot\exp\left\{-\frac{1}{2}\sum_i\left[\left(\beta_i-\mu\right)^\top\cdot(\Sigma)^{-1}\cdot\left(\beta_i-\mu\right)\right]\right\}$

$\pi\left(\sigma^2\right)\propto\frac{1}{\sigma^2},\pi(\mu)\propto 1,\pi\left(\Sigma^{-1}\right)\propto|\Sigma|^{-(d+1)}\cdot\exp\left(-\frac{1}{2}\Sigma^{-1}\right)$

We can then derive the Posterior:

$$g\left(B,\sigma^2\mid y\right)\propto g\left(y\mid B,\sigma^2\right)\cdot\pi(B\mid\mu,\Sigma)\cdot\pi\left(\sigma^2\right)\cdot\pi(\mu)\cdot\pi\left(\Sigma^{-1}\right)$$

$$\begin{aligned}
\pi\left(\sigma^2\mid\cdot\right)&\propto\prod_{i=1}^n\det\left(\sigma^2 I_{k_i}\right)^{-\frac{1}{2}}\cdot\exp\left\{-\frac{1}{2}\Sigma_i\left[\left(y_i-x_i\beta_i\right)^\top\left(\sigma^2 I_{ki}\right)^{-1}\left(y_i-x_i\beta_i\right)\right]\right\}\cdot\sigma^{-2}\\
&=\left(\sigma^2\right)^{-\frac{1}{2}\Sigma_i k_i}\cdot\exp\left\{-\frac{1}{2\sigma^2}\Sigma_i\left[\left(y_i-x_i\beta_i\right)^\top\cdot\left(y_i-x_i\beta_i\right)\right]\right\}\cdot\sigma^{-2}\\
&=\left(\sigma^2\right)^{-1-\frac{1}{2}\Sigma_i k_i}\cdot\exp\left\{-\frac{1}{2\sigma^2}\Sigma_i\Sigma_{ti}\left(y_{i,t}-x_{i,t}\cdot\beta_i\right)^2\right\},\text{ assume } w=\left(\sigma^2\right)^{-1}\\
&=w\left(1+\frac{1}{2}\Sigma_i k_i\right)\cdot\exp\left\{-w\cdot\frac{1}{2}\cdot\Sigma_i\Sigma_{ti}\left(y_{i,t}-x_{i,t}\cdot\beta_i\right)^2\right\}
\end{aligned}$$

Therefore, $w\sim\text{Gamma}\left(\frac{1}{2}\Sigma_i k_i+2,\frac{1}{2}\Sigma_i\Sigma_{t_i}\left(y_{i,t}-x_{i,t}\cdot\beta_i\right)^2\right)$.

$$
\begin{aligned}
\pi(\Sigma^{-1}) &\propto \det(\Sigma)^{-n/2} \cdot \exp\left\{-\frac{1}{2}\Sigma_i \left(\beta_i - \mu\right)^\top \Sigma^{-1} \left(\beta_i - \mu\right)\right\} \cdot \det(\Sigma)^{-(d+1)} \cdot \exp\left\{-\frac{1}{2}\Sigma^{-1}\right\} \\
&= \det(\Sigma)^{-(n|2+d+1)} \cdot \exp\left\{-\frac{1}{2}\left[\Sigma^{-1} + \Sigma_i \left(\beta_i - \mu\right)^\top \Sigma^{-1} \left(\beta_i - \mu\right)\right]\right\} \\
&\propto \det(\Sigma)^{-(n+2d+2)/2} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1} \cdot \left(I + \Sigma_i \left(\beta_i - \mu\right)\left(\beta_i - \mu\right)^\top\right]\right\} \\
&\propto \det(\Sigma)^{-(n+2d+2)/2} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1} \cdot \left(I + \Sigma_i \left(\beta_i - \mu\right)\left(\beta_i - \mu\right)^\top\right]\right\}
\end{aligned}
$$

Thus $\Sigma^{-1} \sim$ Wishart $\left(n + 3d + 3, I + \Sigma_i \left(\beta_i - \mu\right)^\top \left(\beta_i - \mu\right)\right)$

$$
\begin{aligned}
\pi(\mu) &\propto \exp\left\{-\frac{1}{2}\Sigma_i \left[\left(\beta_i - \mu\right)^\top \Sigma^{-1} \left(\beta_i - \mu\right)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\Sigma_i \left(\beta_i^\top \Sigma^{-1}\beta_i + \mu^\top \Sigma^{-1}\mu - 2\beta_i^\top \Sigma^{-1}\mu\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\Sigma_i\beta_i^\top \Sigma^{-1}\beta_i + \mu^\top n\Sigma^{-1}\mu - 2\Sigma_i\beta_i^\top \Sigma^{-1}\mu\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\mu^\top n\Sigma^{-1}\mu - 2\Sigma_i\beta_i^\top \Sigma^{-1}\mu + \Sigma_i - \beta_i^\top \Sigma^{-1}\beta_i\right)\right\} \\
&= \exp\left\{-\frac{1}{2}(\mu^\top \underbrace{n\Sigma^{-1}}_{M}\mu - 2\mu^\top \underbrace{\Sigma_i\Sigma^{-1}\beta_i}_{N} + \Sigma_i\beta_i^\top \Sigma^{-1}\beta_i)\right\} \\
&= \exp\left\{-\frac{1}{2}\left[\left(\mu - M^{-1}N\right)^\top M \left(\mu - M^{-1}N\right)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\pi(B) \propto \exp\left\{-\frac{1}{2}\Sigma_i \left[\left(y_i - x_i\beta_i\right)^\top \left(\sigma^2 I_{ki}\right)^{-1}\left(y_i - x_i\beta_i\right)\right]\right\} \cdot \exp\left\{-\frac{1}{2}\Sigma_i \left[\left(\beta_i - \mu\right)^\top \cdot (\Sigma)^{-1} \cdot \left(\beta_i - \mu\right)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\Sigma_i \left[\left(y_i - x_i\beta_i\right)^\top \left(\sigma^2 I_{ki}\right)^{-1}\left(y_i - x_i\beta_i\right) + \left(\beta_i - \mu\right)^\top \cdot \Sigma^{-1}\left(\beta_i - \mu\right)\right]\right\} \\
&= \exp\left\{y_i^\top \sigma^{-2} I_{ki} y_i^\top + \beta_i^\top x_i^\top \sigma^{-2} I_{k_i} x_i \beta_i - 2y_i^\top \sigma^{-2} I_{ki} x_i \beta_i + \beta_i^\top \Sigma^{-1}\beta_i + \mu^\top \Sigma^{-1}\mu - 2\mu^\top \Sigma^{-1}\beta_i\right\} \\
&= \exp\left\{y_i^\top \sigma^{-2} I_{ki} y_i + \mu^\top \Sigma^{-1}\mu + \beta_i^\top \left(\Sigma^{-1} + x_i^\top \sigma^{-2} I_{ki} x_i\right)\beta_i - 2\left(y_i^\top \sigma^{-2} I_{ki} x_i + \mu^\top \Sigma^{-1}\right)\beta_i\right\}
\end{aligned}
$$

We can define the following terms:

$$
\begin{aligned}
R &= y_i^\top \sigma^{-2} I_{k_i} y_i + \mu^\top \Sigma^{-1}\mu \\
V &= \Sigma^{-1} + x_i^\top \sigma^{-2} I_{ki} x_i \\
M &= \sigma^{-2} x_i^\top y_i + \Sigma^{-1}\mu
\end{aligned}
$$

Thus, $\pi\left(B \mid \cdots\right) \propto \left(\beta_i - V^{-1}M\right)^\top \cdot V \cdot \left(\beta_i - V^{-1}M\right) \sim N\left(V^{-1}M, V^{-1}\right)$