# Statistical Method

## Likelihood

For each hurricane $i$ and $k_i$'s time points, we have the following Bayesian model:

$$Y_i(t+6)=\beta_{0i}+\beta_{1i}Y_i(t)+\beta_{2i}\Delta_{i1}(t)+\beta_{3i}\Delta_{i2}(t)+\beta_{4i}\Delta_{i3}(t)+\varepsilon_i(t),$$

where $Y_i(t)$ is the wind speed at time $t$, $\Delta_{i1}$, $\Delta_{i2}$, and $\Delta_{i3}$ are the changes of latitude, longitude, and the wind speed between time point $t$ and $t-6$, respectively. $\varepsilon_i(t)$ follows a normal distributions with mean zero and variance $\sigma^2$. The above Bayesian model can be simplified as:

$$Y_i(t+6)=x_i(t)+\varepsilon_i(t),$$

where $\beta_i=(\beta_{0i},\beta_{1i},...,\beta_{4i})\sim N\left(0,\sigma^2\right)$. Based on the property of the multivariate linear regression model, for each hurricane $i$, we have:

$$Y_i \mid X_i \sim N_{k_i}\left(x_i\beta_i,\sigma^2 I_{k_i}\right),$$

where $I_{k_i}$ is an identity matrix with $k_i$ dimensions.

Thus, we can consider the following distribution of each hurricane $i$:

$$f\left(y_i \mid \beta_i,\sigma^2\right)=\left[(2\pi)^{k_i}\cdot\det\left(\sigma^2 I_{k_i}\right)\right]^{-\frac{1}{2}}\cdot\exp\left\{-\frac{1}{2}\left(y_i-x_i\beta_i\right)^\top\left(\sigma^2 I_{k_i}\right)^{-1}\left(y_i-x_i\beta_i\right)\right\}$$

.

From above, we derive the following likelihood function:

$$\begin{aligned}
f\left(y \mid B,\sigma^2\right)&=\prod_{i=1}^{n} f\left(y_i \mid \beta_i,\sigma^2\right)\\
&=\prod_{i=1}^{n}\left(\left[(2\pi)^{k_i}\cdot\det\left(\sigma^2 I_{ki}\right)\right]^{-\frac{1}{2}}\cdot\exp\left\{-\frac{1}{2}\left(y_i-x_i\beta_i\right)^\top\left(\sigma^2 I_{k_i}\right)^{-1}\left(y_i-x_i\beta_i\right)\right\}\right)
\end{aligned}$$

## Prior distributions

We assume the following non-informative prior distributions:

$$\beta_i=(\beta_{0i},\beta_{1i},\ldots,\beta_{4i})\sim N_5(\mu,\Sigma),$$

where $B=\left(\beta_1^\top,\beta_2^\top,\ldots,\beta_n^\top\right)^\top$ and $n$ is the number of hurricanes. So,

$$\pi(B|\mu,\Sigma^{-1})=\prod_{i=1}^{n} f\left(\beta_i\right)\propto\det(\Sigma)^{-n/2}\cdot\exp\left\{-\frac{1}{2}\sum_i\left[\left(\beta_i-\mu\right)^\top(\Sigma)^{-1}\left(\beta_i-\mu\right)\right]\right\}.$$

Also, $\pi\left(\sigma^2\right)\propto\frac{1}{\sigma^2}$; $\pi(\mu)\propto 1$; $\pi\left(\Sigma^{-1}\right)\propto|\Sigma|^{-(d+1)}\cdot\exp\left(-\frac{1}{2}\Sigma^{-1}\right)$.

## Conditional posteriors

The posterior distribution is the product of the likelihood and the prior:

$$g\left(B, \sigma^2, \mu, \Sigma^{-1} \mid y\right) \propto f\left(y \mid B, \sigma^2\right) \cdot \pi(B \mid \mu, \Sigma^{-1}) \cdot \pi\left(\sigma^2\right) \cdot \pi(\mu) \cdot \pi\left(\Sigma^{-1}\right),$$

so we have:

$$\begin{aligned}
\pi\left(\sigma^2 \mid \cdot\right) &\propto \prod_{i=1}^{n} \det\left(\sigma^2 I_{k_i}\right)^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\Sigma_i \left[\left(y_i - x_i\beta_i\right)^\top \left(\sigma^2 I_{ki}\right)^{-1}\left(y_i - x_i\beta_i\right)\right]\right\} \cdot \sigma^{-2} \\
&= \left(\sigma^2\right)^{-\frac{1}{2}\Sigma_i k_i} \cdot \exp\left\{-\frac{1}{2\sigma^2}\Sigma_i \left[\left(y_i - x_i\beta_i\right)^\top \left(y_i - x_i\beta_i\right)\right]\right\} \cdot \sigma^{-2} \\
&= \left(\sigma^2\right)^{-1-\frac{1}{2}\Sigma_i k_i} \cdot \exp\left\{-\frac{1}{2\sigma^2}\Sigma_i\Sigma_{ti}\left(y_{i,t} - x_{i,t}\beta_i\right)^2\right\}
\end{aligned}$$

Therefore, $\sigma^2 \sim$ Inverse Gamma $\left(\frac{1}{2}\Sigma_i k_i, \frac{1}{2}\Sigma_i\Sigma_{t_i}\left(y_{i,t} - x_{i,t}\beta_i\right)^2\right)$.

$$\begin{aligned}
\pi(\Sigma^{-1} \mid \cdot) &\propto \det(\Sigma)^{-n/2} \cdot \exp\left\{-\frac{1}{2}\Sigma_i\left(\beta_i - \mu\right)^\top \Sigma^{-1}\left(\beta_i - \mu\right)\right\} \cdot \det(\Sigma)^{-(d+1)} \cdot \exp\left\{-\frac{1}{2}\Sigma^{-1}\right\} \\
&= \det(\Sigma)^{-(n/2+d+1)} \cdot \exp\left\{-\frac{1}{2}\left[\Sigma^{-1} + \Sigma_i\left(\beta_i - \mu\right)^\top \Sigma^{-1}\left(\beta_i - \mu\right)\right]\right\} \\
&\propto \det(\Sigma^{-1})^{(n+2d+2)/2} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1} \cdot \left(I + \Sigma_i\left(\beta_i - \mu\right)\left(\beta_i - \mu\right)^\top\right]\right\} \\
&\propto \det(\Sigma^{-1})^{(n+3d+3-d-1)/2} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1} \cdot \left(I + \Sigma_i\left(\beta_i - \mu\right)\left(\beta_i - \mu\right)^\top\right]\right\}
\end{aligned}$$

Thus $\Sigma^{-1} \sim$ Wishart $\left(n + 3d + 3, \left[I + \Sigma_i\left(\beta_i - \mu\right)\left(\beta_i - \mu\right)^\top\right]^{-1}\right)$, that is:

$$\Sigma \sim \text{Inverse Wishart}\left(n + 3d + 3, I + \Sigma_i\left(\beta_i - \mu\right)\left(\beta_i - \mu\right)^\top\right)$$

$$\begin{aligned}
\pi(\mu \mid \cdot) &\propto \exp\left\{-\frac{1}{2}\Sigma_i\left[\left(\beta_i - \mu\right)^\top \Sigma^{-1}\left(\beta_i - \mu\right)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\Sigma_i\left(\beta_i^\top \Sigma^{-1}\beta_i + \mu^\top \Sigma^{-1}\mu - 2\beta_i^\top \Sigma^{-1}\mu\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\Sigma_i\beta_i^\top \Sigma^{-1}\beta_i + \mu^\top n\Sigma^{-1}\mu - 2\Sigma_i\beta_i^\top \Sigma^{-1}\mu\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\mu^\top n\Sigma^{-1}\mu - 2\Sigma_i\beta_i^\top \Sigma^{-1}\mu + \Sigma_i - \beta_i^\top \Sigma^{-1}\beta_i\right)\right\} \\
&= \exp\left\{-\frac{1}{2}(\mu^\top \underbrace{n\Sigma^{-1}}_{M}\mu - 2\mu^\top \underbrace{\Sigma_i\Sigma^{-1}\beta_i}_{N} + \Sigma_i\beta_i^\top \Sigma^{-1}\beta_i)\right\} \\
&= \exp\left\{-\frac{1}{2}\left[\left(\mu - M^{-1}N\right)^\top M\left(\mu - M^{-1}N\right)\right]\right.
\end{aligned}$$

Therefore, $\mu \sim$ MVN().

$$
\begin{aligned}
&\pi(B) \propto \exp\left\{-\frac{1}{2}\Sigma_i\left[\left(y_i - x_i\beta_i\right)^\top\left(\sigma^2 I_{ki}\right)^{-1}\left(y_i - x_i\beta_i\right)\right]\right\} \cdot \exp\left\{-\frac{1}{2}\Sigma_i\left[\left(\beta_i - \mu\right)^\top \cdot (\Sigma)^{-1} \cdot \left(\beta_i - \mu\right)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\Sigma_i\left[\left(y_i - x_i\beta_i\right)^\top\left(\sigma^2 I_{ki}\right)^{-1}\left(y_i - x_i\beta_i\right) + \left(\beta_i - \mu\right)^\top \cdot \Sigma^{-1}\left(\beta_i - \mu\right)\right]\right\} \\
&= \exp\left\{y_i^\top \sigma^{-2} I_{ki} y_i^\top + \beta_i^\top x_i^\top \sigma^{-2} I_{k_i} x_i \beta_i - 2 y_i^\top \sigma^{-2} I_{ki} x_i \beta_i + \beta_i^\top \Sigma^{-1}\beta_i + \mu^\top \Sigma^{-1}\mu - 2\mu^\top \Sigma^{-1}\beta_i\right\} \\
&= \exp\left\{y_i^\top \sigma^{-2} I_{ki} y_i + \mu^\top \Sigma^{-1}\mu + \beta_i^\top\left(\Sigma^{-1} + x_i^\top \sigma^{-2} I_{ki} x_i\right)\beta_i - 2\left(y_i^\top \sigma^{-2} I_{ki} x_i + \mu^\top \Sigma^{-1}\right)\beta_i\right\}
\end{aligned}
$$

We can define the following terms:

$$
\begin{aligned}
R &= y_i^\top \sigma^{-2} I_{k_i} y_i + \mu^\top \Sigma^{-1}\mu \\
V &= \Sigma^{-1} + x_i^\top \sigma^{-2} I_{ki} x_i \\
M &= \sigma^{-2} x_i^\top y_i + \Sigma^{-1}\mu
\end{aligned}
$$

Thus, $\pi\left(B \mid \cdots\right) \propto \left(\beta_i - V^{-1}M\right)^\top \cdot V \cdot \left(\beta_i - V^{-1}M\right) \sim N\left(V^{-1}M, V^{-1}\right)$