Posterior Derivation

For each hurricane i and k_i's time points:

$Y_i(t+6) = \beta_{0i} + \beta_{ij} Y_i(t) + \beta_x \Delta_{i1}(t) + \beta_{ii} \Delta_{i2}(t) + \beta_{4i} \Delta_{i3}(t) + \varepsilon_i(t)$

$Y_i(t+b) = x_i(t) \cdot \beta_i + \varepsilon_i(t), where \beta_i = (\beta_{0i}, \beta_{1i}, \ldots, \beta_{4i}) \sim N_5(\mu, \Sigma), \varepsilon_i(t) \sim N\left(0, \sigma^2\right),$

$Y_i \mid X_i \sim N_{k_i}\left(x_i\beta_i, \nabla^2 I_{k_i}\right), f\left(y_i \mid \beta_i, \nabla^2\right) = \left[(2\pi)^{k_i} \cdot \det\left(\sigma^2 I_{k_i}\right)\right]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\left(y_i - x_i\beta_i\right)^\top \cdot \left(\sigma^2 I_{k_i}\right)^{-1}\left(y_i - x_i\beta_i\right)\right\}$

Likelihood:

$$\begin{aligned} g\left(y \mid B, \sigma^2\right) &= \prod_{i=1}^{n} f\left(y_i \mid \beta_i, \sigma^2\right) \\ &= \prod_{i=1}^{n}\left(\left[(2\pi)^{k_i} \cdot \det\left(\sigma^2 I_{ki}\right)\right]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\left(y_i - x_i\beta_i\right)^\top \cdot \left(\sigma^2 I_{k_i}\right)^{-1}\left(y_i - x_i\beta_i\right)\right\}\right) \end{aligned}$$

Prior:

$\beta_i = (\beta_{0i}, \beta_{1i}, \ldots, \beta_{4i}) \sim N_5(\mu, \Sigma), \mu : (5 \times 1), \Sigma : (5 \times 5)$

$B = \left(\beta_1^\top, \beta_2^\top, \ldots, \beta_n^\top\right)^\top$, assume $\beta_1, \beta_2, \ldots, \beta_n$ are independent, $B \cdot (n \times 5)$

$\pi(B|\mu, \Sigma) = \prod_{i=1}^{n} f\left(\beta_i\right) = (2\pi)^{-5n/2} \cdot \det(\Sigma)^{-n/2} \cdot \exp\left\{-\frac{1}{2}\sum_i\left[\left(\beta_i - \mu\right)^\top \cdot (\Sigma)^{-1} \cdot \left(\beta_i - \mu\right)\right]\right\}$

$\pi\left(\sigma^2\right) \propto \frac{1}{\sigma^2}, \pi(\mu) \propto 1, \pi\left(\Sigma^{-1}\right) \propto |\Sigma|^{-(d+1)} \cdot \exp\left(-\frac{1}{2}\Sigma^{-1}\right)$

Posterior:

$$g\left(B, \sigma^2 \mid y\right) \propto g\left(y \mid B, \sigma^2\right) \cdot \pi(B \mid \mu, \Sigma) \cdot \pi\left(\sigma^2\right) \cdot \pi(\mu) \cdot \pi\left(\Sigma^{-1}\right)$$

$$\begin{aligned} \pi\left(\sigma^2 \mid \cdot\right) &\propto \prod_{i=1}^{n} \det\left(\sigma^2 I_{k_i}\right)^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\Sigma_i\left[\left(y_i - x_i\beta_i\right)^\top\left(\sigma^2 I_{ki}\right)^{-1}\left(y_i - x_i\beta_i\right)\right]\right\} \cdot \sigma^{-2} \\ &= \left(\sigma^2\right)^{-\frac{1}{2}\Sigma_i k_i} \cdot \exp\left\{-\frac{1}{2\sigma^2}\Sigma_i\left[\left(y_i - x_i\beta_i\right)^\top \cdot \left(y_i - x_i\beta_i\right)\right]\right\} \cdot \sigma^{-2} \\ &= \left(\sigma^2\right)^{-1-\frac{1}{2}\Sigma_i k_i} \cdot \exp\left\{-\frac{1}{2\sigma^2}\Sigma_i\Sigma_{ti}\left(y_{i,t} - x_{i,t} \cdot \beta_i\right)^2\right\}, \text{ assume } w = \left(\sigma^2\right)^{-1} \\ &= w^{\left(1+\frac{1}{2}\Sigma_i k_i\right)} \cdot \exp\left\{-w \cdot \frac{1}{2} \cdot \Sigma_i\Sigma_{ti}\left(y_{i,t} - x_{i,t} \cdot \beta_i\right)^2\right\} \\ \therefore w &\sim \text{ Gamma}\left(\frac{1}{2}\Sigma_i k_i + 2, \frac{1}{2}\Sigma_i\Sigma_{t_i}\left(y_{i,t} - x_{i,t} \cdot \beta_i\right)^2\right) \end{aligned}$$

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\begin{aligned}
\pi\left(\Sigma^{-1} \mid .\right) &\propto \det(\Sigma)^{-n/2} \cdot \exp\left\{-\frac{1}{2}\Sigma_i\left(\beta_i-\mu\right)^\top \Sigma^{-1}\left(\beta_i-\mu\right)\right\} \cdot \det(\Sigma)^{-(d+1)} \cdot \exp\left\{-\frac{1}{2}\Sigma^{-1}\right\} \\
&= \det(\Sigma)^{-(n|2+d+1)} \cdot \exp\left\{-\frac{1}{2}\left[\Sigma^{-1}+\Sigma_i\left(\beta_i-\mu\right)^\top \Sigma^{-1}\left(\beta_i-\mu\right)\right]\right\} \\
&\propto \det(\Sigma)^{-(n+2d+2)/2} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1}\cdot\left(I+\Sigma_i\left(\beta_i-\mu\right)\left(\beta_i-\mu\right)^\top\right]\right\} \\
&\propto \det(\Sigma)^{-(n+d+1+d+1)/2} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1}\left(I+\Sigma_i\left(\beta_i-\mu\right)\left(\beta_i-\mu\right)^\top\right]\right\} \\
&\therefore \Sigma^{-1} \sim \text{ inverse Wishart }\left(n+d+1, I+\Sigma_i\left(\beta_i-\mu\right)^\top\left(\beta_i-\mu\right)\right)
\end{aligned}
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\begin{aligned}
\pi(\mu) &\propto \exp\left\{-\frac{1}{2}\Sigma_i\left[\left(\beta_i-\mu\right)^\top \Sigma^{-1}\left(\beta_i-\mu\right)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\Sigma_i\left(\beta_i^\top\Sigma^{-1}\beta_i+\mu^\top\Sigma^{-1}\mu-2\beta_i^\top\Sigma^{-1}\mu\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\Sigma_i\beta_i^\top\Sigma^{-1}\beta_i+\mu^\top n\Sigma^{-1}\mu-2\Sigma_i\beta_i^\top\Sigma^{-1}\mu\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\mu^\top n\Sigma^{-1}\mu-2\Sigma_i\beta_i^\top\Sigma^{-1}\mu+\Sigma_i-\beta_i^\top\Sigma^{-1}\beta_i\right)\right\} \\
&= \exp\left\{-\frac{1}{2}(\mu^\top \underbrace{n}_{M}\Sigma^{-1}\mu-2\mu^\top\underbrace{\Sigma_i\Sigma^{-1}\beta_i}_{N}+\Sigma_i\beta_i^\top\Sigma^{-1}\beta_i)\right\} \\
&= \exp\left\{-\frac{1}{2}\left[\left(\mu-M^{-1}N\right)^\top M\left(\mu-M^{-1}N\right)\right]\right.
\end{aligned}
$$