# Zlepki nad triangulacijami

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**Predavanja iz RPGO**

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# Triangulacija

**Triangulacija**

Končna množica trikotnikov

$$\triangle = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_N\}$$

je (regularna) triangulacija območja $\Omega$ s poligonskim robom, če je

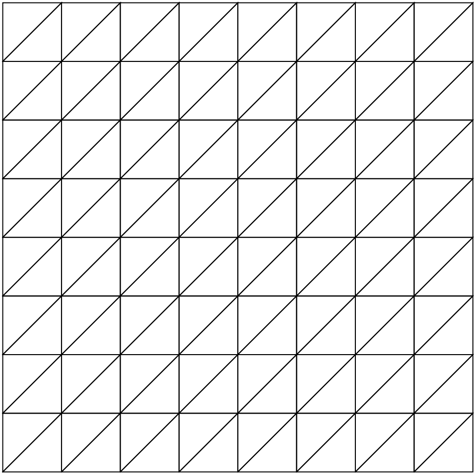
$$\Omega = \cup_{i=1}^{N} \mathcal{T}_i$$

in če velja, da se poljubna različna trikotnika iz množice $\triangle$ z nepraznim presekom sekata bodisi v skupnem oglišču bodisi vzdolž skupne stranice.

Označimo z $\mathcal{V}$ množico točk triangulacije in z $\mathcal{E}$ množico stranic triangulacije.

# Primeri triangulacij

Regularna triangulacija tipa 1:



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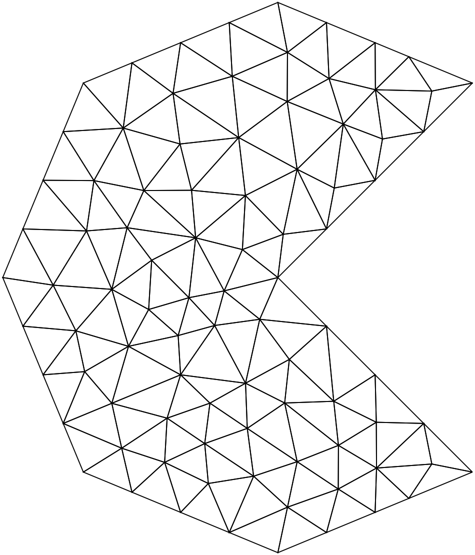

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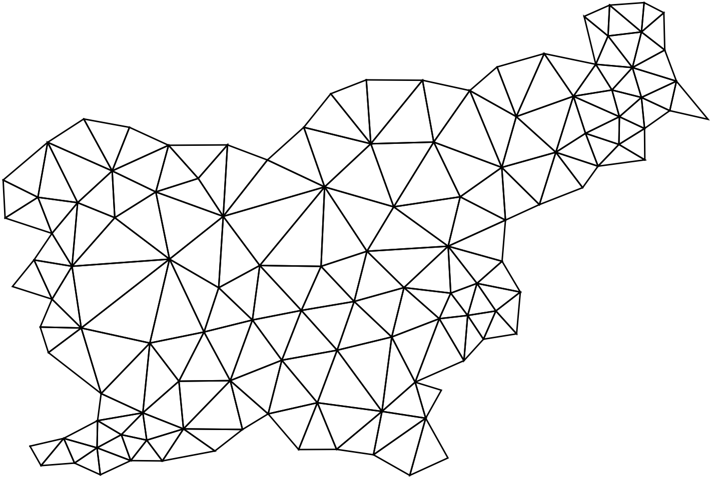

# Zlepki nad triangulacijami

**Prostor $\mathcal{C}^r$ zlepkov stopnje $n$**

$$S_n^r(\triangle) := \left\{ s \in \mathcal{C}^r(\Omega) : \ s\big|_{\mathcal{T}} \in \mathbb{P}_n^2, \quad \forall \mathcal{T} \in \triangle \right\}$$

Vprašanja:

- Dimenzija?

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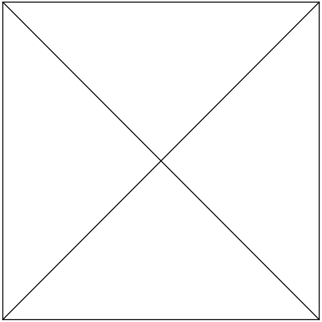
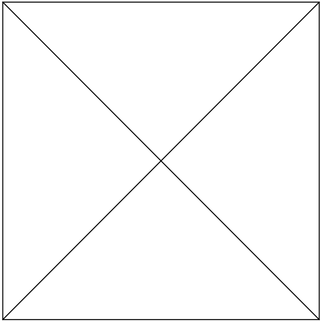
Vprašanja:

- Dimenzija?
- Konstrukcija zlepkov?
- Konstrukcija stabilnih baznih zlepkov?



(a) $\dim S_2^1(\Delta) = 8$



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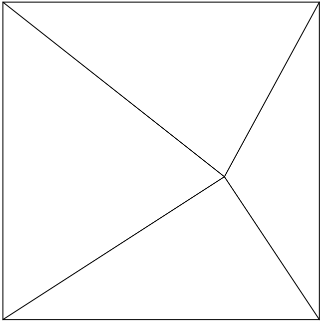
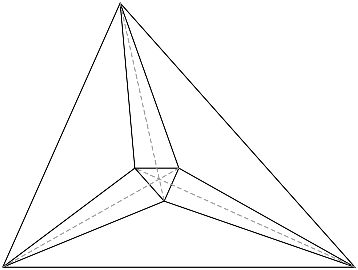


(b) $\dim S_2^1(\Delta) = 7$



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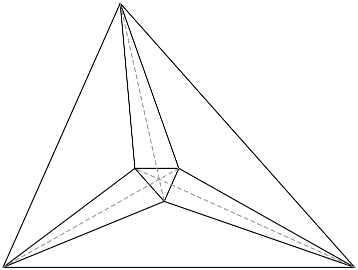

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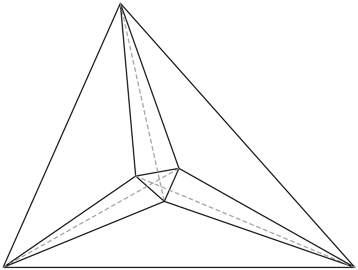

(b) $\dim S_2^1(\Delta) = 6$

# Super-zlepki in makro elementi nad triangulacijami

**Super-zlepki nad triangulacijami**

$$S_n^{r,\rho}(\triangle) := \{s \in S_n^r(\triangle) : \ s \in \mathcal{C}^\rho(\boldsymbol{p}), \ \forall \boldsymbol{p} \in \mathcal{V}\}$$

# Super-zlepki in makro elementi nad triangulacijami

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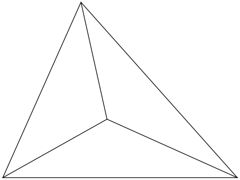
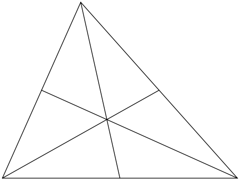
### Prostori makro elementov

Prostori zlepkov nad deljeno triangulacijo.

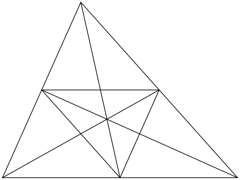
Primeri delitev:



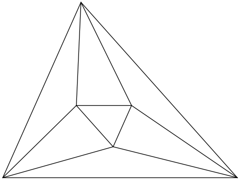
(a) Clough–Tocherjeva 3-delitev



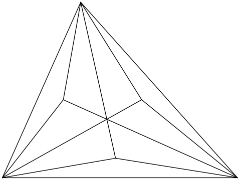
(b) Powell–Sabinova 6-delitev



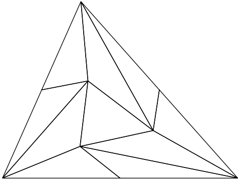
(c) Powell–Sabinova 12-delitev



(d) Morgan–Scott-ova 7-delitev



(e) dvojna C–T delitev

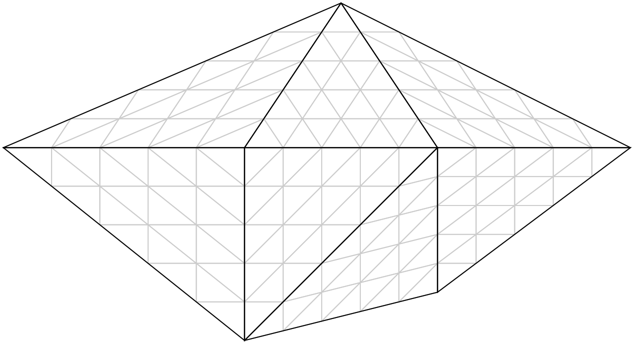


(f) 10-delitev

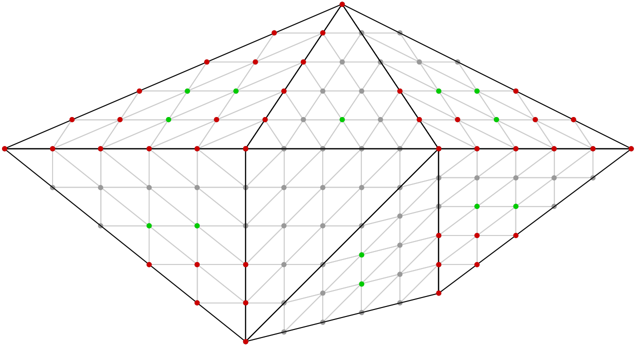
# Prostor $\mathcal{C}^1$ super zlepkov stopnje 5

$$\mathbb{A} := S_5^{1,2}(\triangle) = \left\{ s \in \mathcal{C}^1(\Omega) : \quad s\big|_{\mathcal{T}} \in \mathbb{P}_5^2,\ \forall \mathcal{T} \in \triangle, \quad s \in \mathcal{C}^2(\boldsymbol{p}),\ \forall \boldsymbol{p} \in \mathcal{V} \right\}$$

Zlepke iz tega prostora imenujemo tudi Argyrisovi elementi.

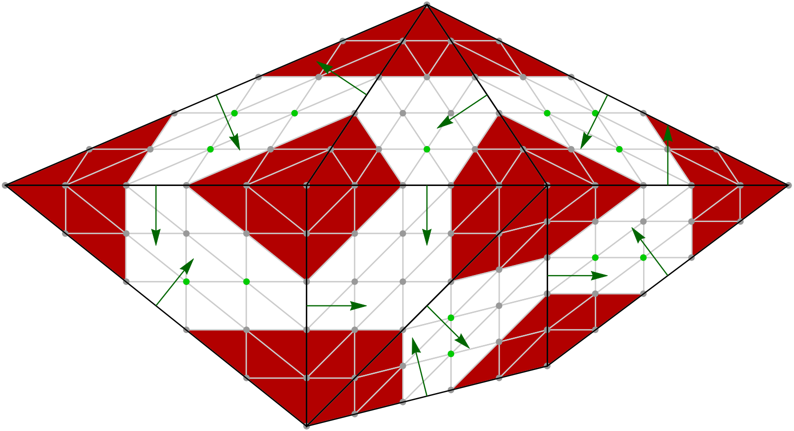


Minimalna določitvena množica:



Dimenzija: $\dim S_5^{1,2}(\triangle) = 6|\mathcal{V}| + |\mathcal{E}|$

# Interpolacijski problem



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Zlepek $s \in S_5^{1,2}(\triangle)$ je natančno določen z

- vrednostjo, prvimi in drugimi odvodi v točkah triangulacije:

$$D_x^\alpha D_y^\beta s(\boldsymbol{p}_i) = f_i^{(\alpha,\beta)}, \quad 0 \le \alpha + \beta \le 2, \quad \boldsymbol{p}_i \in \mathcal{V}$$

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- vrednostjo normalnega odvoda na vsaki stranici $\boldsymbol{e}_{ij} = \langle \boldsymbol{p}_i, \boldsymbol{p}_j \rangle$ triangulacije:

$$D_{\boldsymbol{n}_{ij}} s(\boldsymbol{r}_{i,j}) = g_{i,j}, \quad \boldsymbol{r}_{i,j} := \frac{1}{2} \left( \boldsymbol{p}_i + \boldsymbol{p}_j \right), \quad \boldsymbol{n}_{ij} \perp \boldsymbol{e}_{ij}$$

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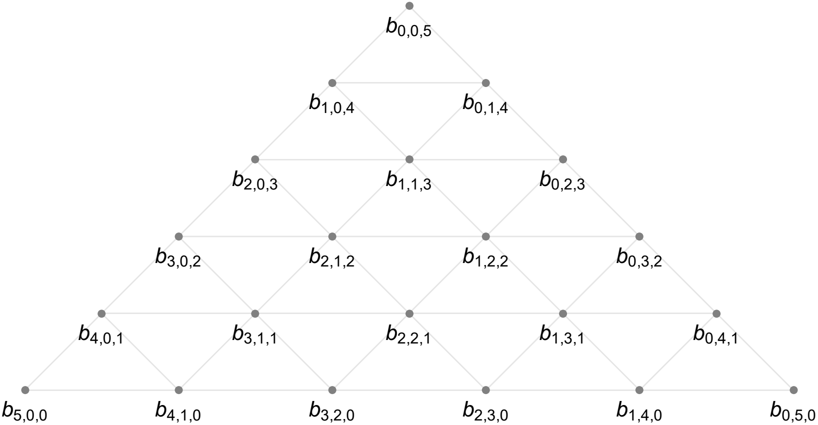
Konstrukcija je povsem lokalna.

# Bézierjeve ordinate nad poljubnim trikotnikom $\mathcal{T} = \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$



# Bézierjeve ordinate nad poljubnim trikotnikom

Iz interpolacije vrednosti v točkah dobimo

$$b_{5,0,0} = f_1^{0,0}, \quad b_{0,5,0} = f_2^{0,0}, \quad b_{0,0,5} = f_3^{0,0}.$$

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Interpolacija prvih odvodov:

- Izberimo $\boldsymbol{d} = \boldsymbol{p}_2 - \boldsymbol{p}_1$, $\mathrm{bar}(\boldsymbol{d}; \mathcal{T}) = (-1, 1, 0)$:

$$D_{\boldsymbol{d}} s(\boldsymbol{p}_1) = \mathrm{grad}(s)(\boldsymbol{p}_1) \cdot \boldsymbol{d} = 5 \underline{b}\left[(1,0,0)^{\langle 4 \rangle}, (-1,1,0); \mathcal{T}\right]$$

$$\Longrightarrow \quad b_{4,1,0} = f_1^{0,0} + \frac{1}{5}\left(f_1^{1,0}, f_1^{0,1}\right) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_1).$$

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Podobno dobimo:

$$b_{1,4,0} = f_2^{0,0} + \frac{1}{5}\left(f_2^{1,0}, f_2^{0,1}\right) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2),$$
$$b_{0,4,1} = f_2^{0,0} + \frac{1}{5}\left(f_2^{1,0}, f_2^{0,1}\right) \cdot (\boldsymbol{p}_3 - \boldsymbol{p}_2),$$
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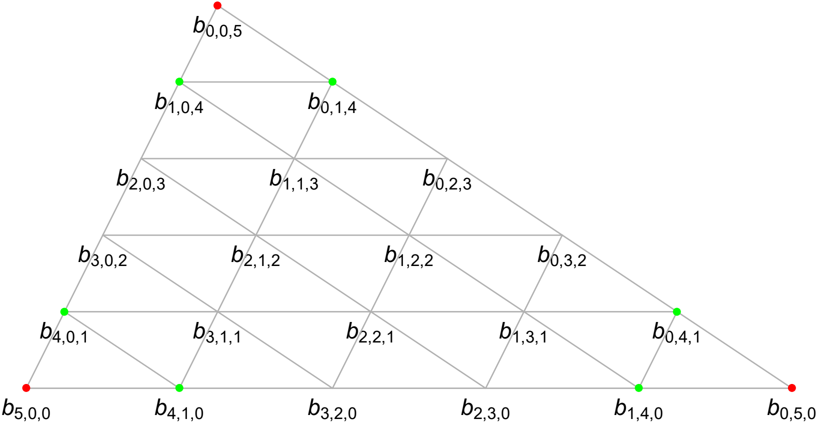
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Interpolacija drugih odvodov:

- Definiramo $H_i := \begin{pmatrix} f_i^{2,0} & f_i^{1,1} \\ f_i^{1,1} & f_i^{0,2} \end{pmatrix}$.
- Izberimo $\boldsymbol{d} = \boldsymbol{p}_2 - \boldsymbol{p}_1$, $\mathrm{bar}(\boldsymbol{d}; \mathcal{T}) = (-1, 1, 0)$:

$$D_{\boldsymbol{d}}^2 s(\boldsymbol{p}_1) = \boldsymbol{d}^T H_1 \boldsymbol{d} = 20\underline{b}\left[(1,0,0)^{\langle 3\rangle}, (-1,1,0)^{\langle 2\rangle}; \mathcal{T}\right]$$

$$\Longrightarrow \quad b_{3,2,0} = 2b_{4,1,0} - b_{5,0,0} + \frac{1}{20}(\boldsymbol{p}_2 - \boldsymbol{p}_1)^T H_1 (\boldsymbol{p}_2 - \boldsymbol{p}_1)$$

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- Izberimo:
  $\boldsymbol{d}_1 = \boldsymbol{p}_2 - \boldsymbol{p}_1$, $\mathrm{bar}(\boldsymbol{d}_1; \mathcal{T}) = (-1, 1, 0)$
  $\boldsymbol{d}_2 = \boldsymbol{p}_3 - \boldsymbol{p}_1$, $\mathrm{bar}(\boldsymbol{d}_2; \mathcal{T}) = (-1, 0, 1)$

  $$D_{\boldsymbol{d}_1} D_{\boldsymbol{d}_2} s(\boldsymbol{p}_1) = \boldsymbol{d}_1^T H_1 \boldsymbol{d}_2 = 20 \underline{b} \left[ (1,0,0)^{\langle 3 \rangle}, (-1,1,0), (-1,0,1); \mathcal{T} \right]$$

  $$\Longrightarrow \quad b_{3,1,1} = b_{4,0,1} + b_{4,1,0} - b_{5,0,0} + \frac{1}{20} (\boldsymbol{p}_3 - \boldsymbol{p}_1)^T H_1 (\boldsymbol{p}_2 - \boldsymbol{p}_1)$$

Podobno dobimo

$$
\begin{aligned}
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\end{aligned}
$$

in

$$
\begin{aligned}
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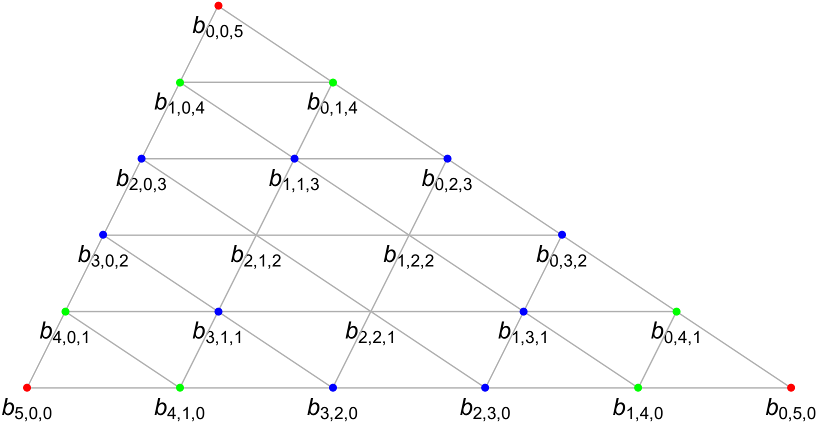
$$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$b_{0,4,1}$
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Interpolacija smernih odvodov na stranicah:

Naj bodo $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ baricentrične koordinate smeri $\boldsymbol{n}_{1,2}$ glede na trikotnik $\mathcal{T} = \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$.

Enačba:

$$D_{\boldsymbol{n}_{1,2}} s(\boldsymbol{r}_{1,2}) = g_{1,2} = 5\underline{b}\left[(1/2, 1/2, 0)^{\langle 4 \rangle}, \boldsymbol{\tau}; \mathcal{T}\right]$$

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$$\begin{aligned} \underline{b} \left[ (1/2, 1/2, 0)^{\langle 4 \rangle}, \boldsymbol{\tau}; \mathcal{T} \right] = & \frac{\tau_1}{16} \left( b_{5,0,0} + 4 b_{4,1,1} + 6 b_{3,2,0} + 4 b_{2,3,0} + b_{1,4,0} \right) + \\ & \frac{\tau_2}{16} \left( b_{0,5,0} + 4 b_{1,4,0} + 6 b_{2,3,0} + 4 b_{3,2,0} + b_{4,1,0} \right) + \\ & \frac{\tau_3}{16} \left( b_{4,0,1} + 4 b_{3,1,1} + 6 {\color{red} b_{2,2,1}} + 4 b_{1,3,1} + b_{0,4,1} \right) \end{aligned}$$

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$$D_{\boldsymbol{n}_{1,2}} s(\boldsymbol{r}_{1,2}) = g_{1,2} = 5\underline{b}\left[(1/2, 1/2, 0)^{\langle 4 \rangle}, \boldsymbol{\tau}; \mathcal{T}\right]$$

$$\begin{aligned}
\underline{b}\left[(1/2, 1/2, 0)^{\langle 4 \rangle}, \boldsymbol{\tau}; \mathcal{T}\right] = & \frac{\tau_1}{16}\left(b_{5,0,0} + 4b_{4,1,1} + 6b_{3,2,0} + 4b_{2,3,0} + b_{1,4,0}\right) + \\
& \frac{\tau_2}{16}\left(b_{0,5,0} + 4b_{1,4,0} + 6b_{2,3,0} + 4b_{3,2,0} + b_{4,1,0}\right) + \\
& \frac{\tau_3}{16}\left(b_{4,0,1} + 4b_{3,1,1} + 6b_{2,2,1} + 4b_{1,3,1} + b_{0,4,1}\right)
\end{aligned}$$

$$\begin{aligned}
\implies \quad b_{2,2,1} = & \frac{8}{15\tau_3} g_{1,2} - \frac{1}{6}\left(b_{4,0,1} + 4b_{3,1,1} + 4b_{1,3,1} + b_{0,4,1}\right) + \\
& - \frac{1}{6}\frac{\tau_1}{\tau_3}\left(b_{5,0,0} + 4b_{4,1,1} + 6b_{3,2,0} + 4b_{2,3,0} + b_{1,4,0}\right) + \\
& - \frac{1}{6}\frac{\tau_1}{\tau_3}\left(b_{0,5,0} + 4b_{1,4,0} + 6b_{2,3,0} + 4b_{3,2,0} + b_{4,1,0}\right)
\end{aligned}$$

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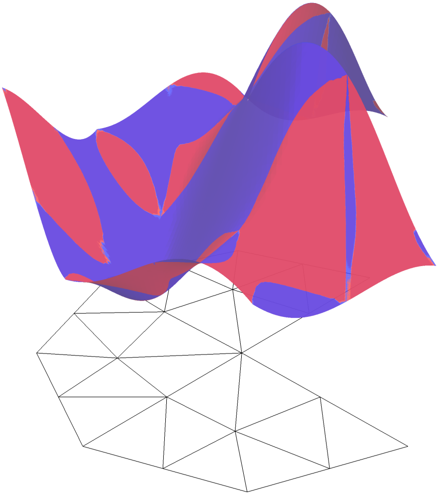
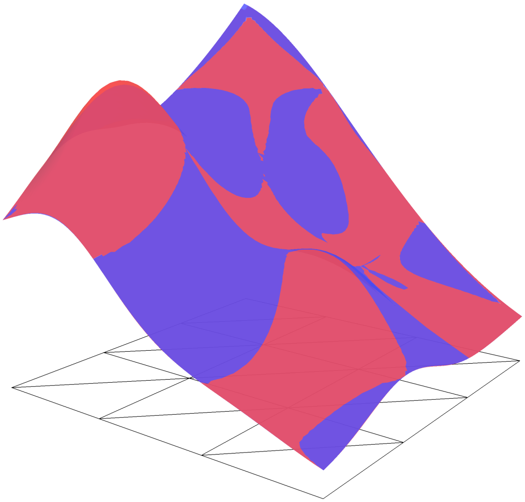
$$
\begin{aligned}
b_{1,2,2} =& \frac{8}{15\tau_1} g_{2,3} - \frac{1}{6}\left(b_{1,0,4} + 4b_{1,1,3} + 4b_{1,3,1} + b_{1,4,0}\right) + \\
& - \frac{1}{6}\frac{\tau_3}{\tau_1}\left(b_{0,0,5} + 4b_{0,1,4} + 6b_{0,2,3} + 4b_{0,3,2} + b_{0,4,1}\right) + \\
& - \frac{1}{6}\frac{\tau_2}{\tau_1}\left(b_{0,1,4} + 4b_{0,2,3} + 6b_{0,3,2} + 4b_{0,4,1} + b_{0,5,0}\right) \\
b_{2,1,2} =& \frac{8}{15\tau_2} g_{1,3} - \frac{1}{6}\left(b_{0,1,4} + 4b_{1,1,3} + 4b_{3,1,1} + b_{4,1,0}\right) + \\
& - \frac{1}{6}\frac{\tau_1}{\tau_2}\left(b_{1,0,4} + 4b_{2,0,3} + 6b_{3,0,2} + 4b_{4,0,1} + b_{5,0,0}\right) + \\
& - \frac{1}{6}\frac{\tau_3}{\tau_2}\left(b_{0,0,5} + 4b_{1,0,4} + 6b_{2,0,3} + 4b_{3,0,2} + b_{4,0,1}\right)
\end{aligned}
$$

# Primer



# Primer



# Primer