

Ravninske krivulje s pitagorejskim hodogramom

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Ravninske krivulje s pitagorejskim hodografom

Definicija

Hodograf parametrične krivulje $r(t)$ v \mathbb{R}^n je odvod krivulje same $r'(t)$ podan kot parametrična krivulja. Polinomska krivulja $r(t)$ v \mathbb{R}^n je krivulja s Pitagorejskim hodografom (PH), če vsota kvadratov vseh n polinomov na koordinatnih komponentah hodografa krivulje sovпада s kvadratom nekega polinoma $\sigma(t)$.

Torej za $r'(t) = (x'(t), y'(t))$ velja:

$$x'^2(t) + y'^2(t) = \sigma^2(t)$$

za nek polinom $\sigma(t)$.

$a^2(t) + b^2(t) = c^2(t)$ natanko tedaj, ko obstajajo polinomi $u(t)$, $v(t)$, $w(t)$, tako da

$$\begin{aligned}a(t) &= [u^2(t) - v^2(t)]w(t), \\b(t) &= 2u(t)v(t)w(t), \\c(t) &= [u^2(t) + v^2(t)]w(t),\end{aligned}\tag{1}$$

kjer imata $u(t)$ in $v(t)$ paroma različne ničle.

Ravninska krivulja s PH $r(t) = (x(t), y(t))$ definirana z zamenjavo treh polinomov $u(t)$, $v(t)$, $w(t)$ v izrazih

$$\begin{aligned}x'(t) &= [u^2(t) - v^2(t)]w(t) \\ y'(t) &= 2u(t)v(t)w(t)\end{aligned}\tag{2}$$

in z integriranjem.

Bézierjeve kontrolne točke krivulj s PH

$$\begin{aligned}x'(t) &= (u_0^2 - v_0^2)B_0^2(t) + \\&\quad (u_0u_1 - v_0v_1)B_1^2(t) + (u_1^2 - v_1^2)B_2^2(t), \\y'(t) &= 2u_0v_0B_0^2(t) + (u_0v_1 + u_1v_0)B_1^2(t) + 2u_1v_1B_2^2(t).\end{aligned}$$

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{3}(u_0^2 - v_0^2, 2u_0v_0),$$

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{3}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0),$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{1}{3}(u_1^2 - v_1^2, 2u_1v_1),$$

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{5}(u_0^2 - v_0^2, 2u_0v_0),$$

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{5}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0),$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{2}{15}(u_1^2 - v_1^2, 2u_1v_1) +$$

$$\frac{1}{15}(u_0u_2 - v_0v_2, u_0v_2 + u_2v_0),$$

$$\mathbf{p}_4 = \mathbf{p}_3 + \frac{1}{5}(u_1u_2 - v_1v_2, u_1v_2 + u_2v_1),$$

$$\mathbf{p}_5 = \mathbf{p}_4 + \frac{1}{5}(u_2^2 - v_2^2, 2u_2v_2),$$

Parametrična hitrost in dolžina loka

$$\sigma(t) = |r'(t)| = \sqrt{x'^2(t) + y'^2(t)} = u^2(t) + v^2(t)$$

$$\sigma(t) = \sum_{k=0}^{n-1} \sigma_k B_k^{n-1}(t),$$

kjer so

$$\sigma_k = \sum_{j=\max(0, k-m)}^{\min(m, k)} \frac{\binom{m}{j} \binom{m}{k-j}}{\binom{n-1}{k}} (u_j u_{k-j} + v_j v_{k-j}),$$

$$k = 0, \dots, n-1.$$

$$\sigma_0 = u_0^2 + v_0^2,$$

$$\sigma_1 = u_0 u_1 + v_0 v_1,$$

$$\sigma_2 = u_1^2 + v_1^2.$$

$$\sigma_0 = u_0^2 + v_0^2,$$

$$\sigma_1 = u_0 u_1 + v_0 v_1,$$

$$\sigma_2 = \frac{2}{3}(u_1^2 + v_1^2) + \frac{1}{3}(u_0 u_2 + v_0 v_2),$$

$$\sigma_3 = u_1 u_2 + v_1 v_2,$$

$$\sigma_4 = u_2^2 + v_2^2.$$

$$s(t) = \sum_{k=0}^n s_k B_k^n(t),$$

kjer je $s_0 = 0$ in $s_k = \frac{1}{n} \sum_{j=0}^{k-1} \sigma_j$, $k = 1, \dots, n$.

$$S = s(1) = \frac{\sigma_0 + \sigma_1 + \dots + \sigma_{n-1}}{n}.$$

$$\Delta s = S/N$$

Začetni približek:

$$t_k^{(0)} = t_{k-1} + \frac{\Delta s}{\sigma(t_{k-1})}$$

Newton-Raphson:

$$t_k^{(r)} = t_k^{(r-1)} - \frac{s(t_k^{(r-1)}) - k\Delta s}{\sigma(t_k^{(r-1)})}, r = 1, 2, \dots$$

Lastnosti odvoda krivulje

$$\mathbf{t} = \frac{(u^2 - v^2, 2uv)}{\sigma}$$

$$\mathbf{n} = \frac{(2uv, v^2 - u^2)}{\sigma}$$

$$\kappa = 2 \frac{uv' - u'v}{\sigma^2}.$$

Racionalni odmiki krivulj s PH

$$r_d(t) = r(t) + d\mathbf{n}(t),$$

kjer je $\mathbf{n}(t)$ enotska normala .

Racionalni odmiki krivulj s PH

$$\mathbf{P}_k = (W_k, X_k, Y_k) = (1, x_k, y_k), \quad k = 0, \dots, n.$$

$$\Delta \mathbf{P}_k = \mathbf{P}_{k+1} - \mathbf{P}_k = (0, \Delta x_k, \Delta y_k), \quad k = 0, \dots, n-1$$

$$\Delta \mathbf{P}_k^\perp = (0, \Delta y_k, -\Delta x_k).$$

Racionalni odmiki krivulj s PH

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Racionalni odmik

$$r_d(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)} \right),$$

kjer je $\mathbf{O}_k = (W_k, X_k, Y_k)$, $k = 0, \dots, 2n-1$, določeno z

$$\mathbf{O}_k = \sum_{j=\max(0, k-n)}^{\min(n-1, k)} \frac{\binom{n-1}{j} \binom{n}{k-j}}{\binom{2n-1}{k}} (\sigma_j \mathbf{P}_{k-j} + dn \Delta \mathbf{P}_j^\perp).$$

Jan ...