Ravninske krivulje s pitagorejskim hodogramom

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Ravninske krivulje s pitagorejskim hodografom

Definicija

Hodograf parametrične krivulje $r(t)$ v $\mathbb{R}^n$ je odvod krivulje same $r\prime(t)$ podan kot parametrična krivulja. Polinomska krivulja $r(t)$ v $\mathbb{R}^n$ je krivulja s Pitagorejskim hodografom (PH), če vsota kvadratov vseh n polinomov na koordinatnih komponentah hodografa krivulje sovpada s kvadratom nekega polinoma $\sigma(t)$.

Torej za $r\prime(t) = (x\prime(t), y\prime(t))$ velja:

$$x\prime^2(t) + y\prime^2(t) = \sigma^2(t)$$

za nek polinom $\sigma(t)$.

$a^2(t)+b^2(t)=c^2(t)$ natanko tedaj, ko obstajajo polinomi $u(t), v(t), w(t)$, tako da

$$\begin{aligned} a(t) &= [u^2(t)-v^2(t)]w(t), \\ b(t) &= 2u(t)v(t)w(t), \\ c(t) &= [u^2(t)+v^2(t)]w(t), \end{aligned} \tag{1}$$

kjer imata $u(t)$ in $v(t)$ paroma različne ničle.

Ravninska krivulja s PH $r(t) = (x(t), y(t))$ definirana z zamenjavo treh polinomov $u(t), v(t), w(t)$ v izrazih

$$\begin{aligned} x\prime(t) &= [u^2(t) - v^2(t)]w(t) \\ y\prime(t) &= 2u(t)v(t)w(t) \end{aligned} \tag{2}$$

in z integriranjem.

Bézierjeve kontrolne točke krivulj s PH

$$
\begin{aligned}
x\prime(t) &= (u_0^2 - v_0^2)B_0^2(t) + \\
&\quad (u_0u_1 - v_0v_1)B_1^2(t) + (u_1^2 - v_1^2)B_2^2(t), \\
y\prime(t) &= 2u_0v_0B_0^2(t) + (u_0v_1 + u_1v_0)B_1^2(t) + 2u_1v_1B_2^2(t).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{3}(u_0^2 - v_0^2, 2u_0v_0), \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{3}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0), \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{1}{3}(u_1^2 - v_1^2, 2u_1v_1),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{5}(u_0^2 - v_0^2, 2u_0 v_0), \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{5}(u_0 u_1 - v_0 v_1, u_0 v_1 + u_1 v_0), \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{2}{15}(u_1^2 - v_1^2, 2u_1 v_1) + \\
&\quad \frac{1}{15}(u_0 u_2 - v_0 v_2, u_0 v_2 + u_2 v_0), \\
\mathbf{p}_4 &= \mathbf{p}_3 + \frac{1}{5}(u_1 u_2 - v_1 v_2, u_1 v_2 + u_2 v_1), \\
\mathbf{p}_5 &= \mathbf{p}_4 + \frac{1}{5}(u_2^2 - v_2^2, 2u_2 v_2),
\end{aligned}
$$

Parametrična hitrost in dolžina loka

$$\sigma(t) = |r\prime(t)| = \sqrt{x\prime^2(t) + y\prime^2(t)} = u^2(t) + v^2(t)$$

$$\sigma(t) = \sum_{k=0}^{n-1} \sigma_k B_k^{n-1}(t),$$

kjer so

$$\sigma_k = \sum_{j=max(0,k-m)}^{min(m,k)} \frac{\binom{m}{j}\binom{m}{k-j}}{\binom{n-1}{k}} (u_j u_{k-j} + v_j v_{k-j}),$$

$$k = 0, \ldots, n-1.$$

$$
\begin{aligned}
\sigma_0 &= u_0^2 + v_0^2, \\
\sigma_1 &= u_0 u_1 + v_0 v_1, \\
\sigma_2 &= u_1^2 + v_1^2.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_0 &= u_0^2 + v_0^2, \\
\sigma_1 &= u_0 u_1 + v_0 v_1, \\
\sigma_2 &= \frac{2}{3}(u_1^2 + v_1^2) + \frac{1}{3}(u_0 u_2 + v_0 v_2), \\
\sigma_3 &= u_1 u_2 + v_1 v_2, \\
\sigma_4 &= u_2^2 + v_2^2.
\end{aligned}
$$

$$s(t) = \sum_{k=0}^{n} s_k B_k^n(t),$$

kjer je $s_0 = 0$ in $s_k = \frac{1}{n} \sum_{j=0}^{k-1} \sigma_j, k = 1, \ldots, n.$

$S = s(1) = \frac{\sigma_0 + \sigma_1 + \ldots + \sigma_{n-1}}{n}.$

$\Delta s = S/N$
Začetni približek:

$$t_k^{(0)} = t_{k-1} + \frac{\Delta s}{\sigma(t_{k-1})}$$

Newton-Raphson:

$$t_k^{(r)} = t_k^{(r-1)} - \frac{s(t_k^{(r-1)}) - k\Delta s}{\sigma(t_k^{(r-1)})}, r = 1, 2, \ldots.$$

Lastnosti odvoda krivulje

$$
\begin{aligned}
\mathbf{t} &= \frac{(u^2 - v^2, 2uv)}{\sigma} \\
\mathbf{n} &= \frac{(2uv, v^2 - u^2)}{\sigma} \\
\kappa &= 2\frac{uv\prime - u\prime v}{\sigma^2}.
\end{aligned}
$$

Racionalni odmiki krivulj s PH

$$r_d(t) = r(t) + d\mathbf{n}(t),$$

kjer je $\mathbf{n}(t)$ enotska normala .

Racionalni odmiki krivulj s PH

$$\mathbf{P}_k = (W_k, X_k, Y_k) = (1, x_k, y_k), \qquad k = 0, \ldots, n.$$

$$\Delta\mathbf{P}_k = \mathbf{P}_{k+1} - \mathbf{P}_k = (0, \Delta x_k, \Delta y_k), \qquad k = 0, \ldots, n-1$$

$$\Delta\mathbf{P}_k^{\perp} = (0, \Delta y_k, -\Delta x_k).$$

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Racionalni odmik

$$r_d(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right),$$

kjer je $\mathbf{O}_k = (W_k, X_k, Y_k), \quad k = 0, \ldots, 2n-1$, določeno z

$$\mathbf{O}_k = \sum_{j=max(0,k-n)}^{min(n-1,k)} \frac{\binom{n-1}{j}\binom{n}{k-j}}{\binom{2n-1}{k}} (\sigma_j \mathbf{P}_{k-j} + dn\Delta\mathbf{P}_j^{\perp}).$$

Jan . . .