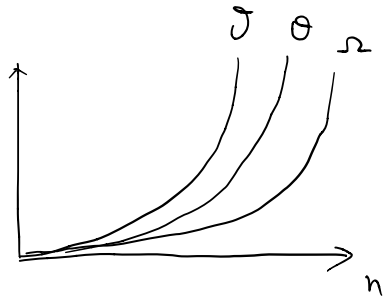
Measure Time / complicity.

- RAM model
- Bit model
- Oracle model 随机预言模型. { $x \to \square \to f(x)$; greater, less, equal; $X \in A$?
- other { communication complicity; cach

LANDAU Symble

O, Ω, Θ



Time complexity.

- Worse case $T_n = \max T(a_0 \cdots a_n)$
- Average case $T_n = \frac{1}{\#} \Sigma T(a_1 \cdots a_n)$

Fibonacci example.

if $x_n = 2^n$ $\quad 2^n = 2^{n-1} + 2^{n-2} \Rightarrow 2^2 = 2^1 + 1$

$$x_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$ 无理数不好算.

Code $X(n)$: case $n=0$: return 0 指数时长

$n=1$: return 1

当 n 很大，会不好用.　　$n>1$: return $X(n-1)+X(n-2)$　　① ① ①

设进① T_n = time to compute $X(n)$

$T_0 = T_1 = 1$

$T_n = T_{n-1} + T_{n-2} + 1$, $n \geq 2$

$(T_n+1) = (T_{n-1}+1) + (T_{n-2}+1)$

$T_0+1 = 2 = T_1 + 1$ ....

T_n+1 = 2 2 4 6 10 16 .... double.

设② Array ... pic　　$X[i] = X[i-1] + X[i-2]$

设③ Matrix　　pic $\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$

② 怎么算

Jan 9th

ex. Fast exponentiation　　x^n ?

exp(x, n)

Case　$n=0 \rightarrow 1$

$n=1 \rightarrow x$

$n>1$, even.　$(exp(x, n/2))^2$

$n>1$, odd　$(exp(x, n/2))^2 \cdot x$

time taken　$T_n = \begin{cases} ① & n=0,1 \\ T_{n/2} + ① & \text{recursion} \end{cases}$

Ex. 2. (4.6)　chip testing

Given:

chip ——— chip

Good: ever lie

bad: cannot be trusted

Reply: GG: both good or bad.
GB: ≥1 bad
BB: ≥1 bad

g: set of good chips
B: set of bad clips.

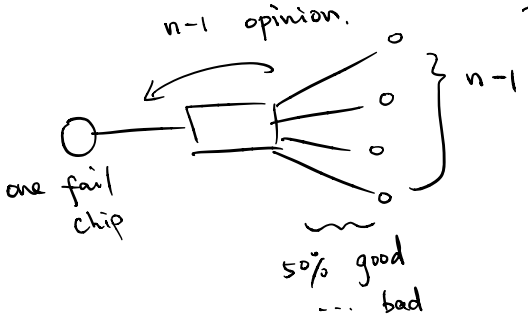
$|g| + |B| = n$

$|g| > |B|$ (assumption)

Question: identify g.

model: oracle model.

want: Time = $O(n)$



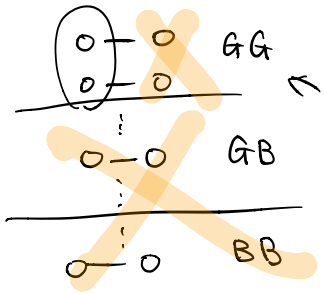
→ majority vote

$|g| = |B| + 1$

Problem: Find one good chip + n-1 test to determine g.

n even

$\frac{n}{2}$ pairs.



① $|g| = |B| + 1$ always hold.
$|g| > |B|$ after give up GB, BB.

not empty.

② Give up half of GG, even

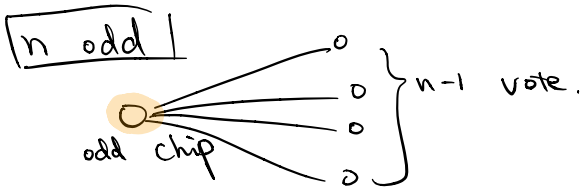
$T_n \le \frac{n}{2} + T_{n/2}$

n is power of 2.

$T_n \le \frac{n}{2} + \frac{n}{4} + \frac{n}{8} \cdots + T_1$ ← 0 only left must be good

$= n\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$

n odd !



if it is good → done

if it is bad → even case

$$T_n \le n-1 + \frac{n-1}{2} + T_{\frac{n-1}{2}}$$

$$\le 3n/2 + T_{\frac{n-1}{2}}$$

Claim $T_n \le 3n$

ex. 3. Multiplying two n-bit integer.

Long multiplication.

```
    1 1 0 1
  x   1 0 1
  -------
    1 1 0 1
    0 0 0 0      } n rows
  1 1 0 1
  ---------------
  1 0 0 0 0 0 1
  <= 2n.
```

2n x n matrix

cost = $\Theta(n^2)$

Do better

KARATUBA's Method (Fractal MUL)

$a = a_1 \times 2^{n/2} + a_2$ (a_1, a_2: n/2 bits)

$b = b_1 \times 2^{n/2} + b_2$ (b_1, b_2: n/2 bits)

$ab = a_1 b_1 2^n + (a_1 b_2 + a_2 b_1) 2^{n/2} + a_2 b_2$ → n-bit

$a_1 b_1$: 2n-bit, cost n, ×3 → 2n

$(a_1 b_2 + a_2 b_1)$: n/2 bits, cost n → n

$a_2 b_2$: n/2 bits, cost n/2 → 3n/2

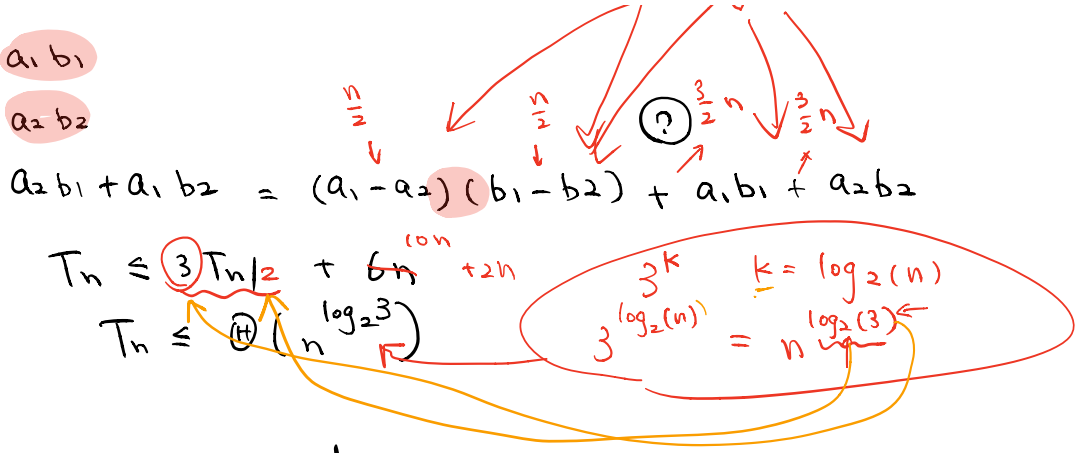
6n total

$$T_n \le 6n + 4T_{n/2} + 2n$$ (shifting)

$$\le O(n^2)$$

$a_1 b_1$

$a_2 b_2$



* Toom-Cook Method. $T_n \le 5T_{n/3} + 10n$?

$T_n = \Theta(n^{\log_3 5})$

Best known. $O(n \log n)$

Ex. 4 # Merge Sort.

Given : n number array $A[1 \cdots n]$

model : 1 time unit = 1 comparison.

Solution : MergeSort (A)

$A' \leftarrow A[1 \cdots n/2]$

$A'' \leftarrow A[n/2 + 1 \cdots n]$

$B' \leftarrow$ mergesort A'.

$B'' \leftarrow A''$

return Merge (B', B'')

$n-1$ on number comparison.

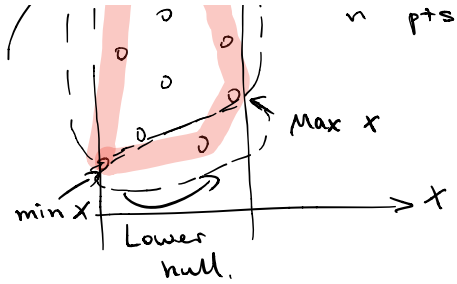
$T_n = T_{\lfloor n/2 \rfloor} + T_{\lceil n/2 \rceil} + n - 1$

$\approx 2T_{n/2} + n \Rightarrow \Theta(n \log n)$

Computational Geometry

Convex hull

n^2



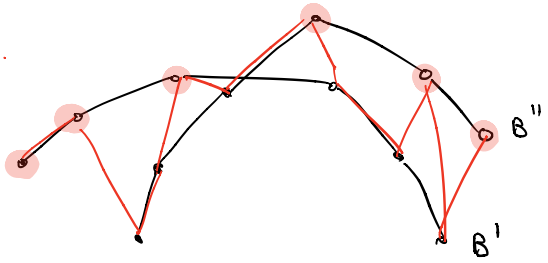
$A' \leftarrow A[1 \cdots n/2]$ ←

$A'' \leftarrow A[n/2+1 \cdots n]$ ←

$B' \leftarrow$ mergesort A'. ←

$B'' \leftarrow A''$ ←

Same.



In time(n) find upper hull