5AG07
Nonlinear structural mechanics by finite element method.

1 - Introduction to FEM

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What is FEM?

The finite element method is a framework and a recipe for discretization of differential equations

- Ordinary differential equations
- Partial differential equations
- Integral equations

- A recipe for discretization of PDE
- PDE $\rightarrow Ax = b$
- Different bases, stabilization, error control, adaptivity

The FEM cookbook

(i) $\mathcal{A}u = f$ — *Partial differential equation*

Multiply by v

(ii) $a(u, v) = L(v)$ — *Continuous variational problem*

Take $V_h \subset V$

(iii) $a(u_h, v) = L(v)$ — *Discrete variational problem*

Let $u_h = \sum_j U_j \phi_j$

(iv) $AU = b$ — *System of discrete equations*

The PDE (i)

Consider Poisson's equation, the Hello World of partial differential equations:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \partial\Omega \end{aligned}$$

Poisson's equation arises in numerous applications:

- heat conduction, electrostatics, diffusion of substances, twisting of elastic rods, inviscid fluid flow, water waves, magnetostatics, ...
- as part of numerical splitting strategies for more complicated systems of PDEs, in particular the Navier–Stokes equations

From PDE (i) to variational problem (ii)

The simple recipe is: multiply the PDE by a test function v and integrate over Ω:

$$-\int_\Omega (\Delta u) v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Then integrate by parts and set $v = 0$ on the Dirichlet boundary:

$$-\int_\Omega (\Delta u) v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s}_{=0}$$

We find that:

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

The variational problem (ii)

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

for all $v \in \hat{V}$

The trial space V and the test space $\hat{V}$ are (here) given by

$$V = \{v \in H^1(\Omega) : v = u_0 \text{ on } \partial\Omega\}$$
$$\hat{V} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$$

From continuous (ii) to discrete (iii) problem

We approximate the continuous variational problem with a discrete variational problem posed on finite dimensional subspaces of V and $\hat{V}$:

$$V_h \subset V$$
$$\hat{V}_h \subset \hat{V}$$

Find $u_h \in V_h \subset V$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

for all $v \in \hat{V}_h \subset \hat{V}$

From discrete variational problem (iii) to discrete system of equations (iv)

Choose a basis for the discrete function space:

$$V_h = \operatorname{span}\{\phi_j\}_{j=1}^N$$

Make an ansatz for the discrete solution:

$$u_h = \sum_{j=1}^N U_j \phi_j$$

Test against the basis functions:

$$\int_\Omega \nabla(\underbrace{\sum_{j=1}^N U_j \phi_j)}_{u_h} \cdot \nabla \phi_i \, \mathrm{d}x = \int_\Omega f \phi_i \, \mathrm{d}x$$

From discrete variational problem (iii) to discrete system of equations (iv), contd.

Rearrange to get:

$$\sum_{j=1}^{N} U_j \underbrace{\int_\Omega \nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x}_{A_{ij}} = \underbrace{\int_\Omega f\phi_i \,\mathrm{d}x}_{b_i}$$

A linear system of equations:

$$A\,U = b$$

where

$$A_{ij} = \int_\Omega \nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x \tag{1}$$

$$b_i = \int_\Omega f\phi_i \,\mathrm{d}x \tag{2}$$

The canonical abstract problem

(i) Partial differential equation:

$$\mathcal{A}u = f \quad \text{in } \Omega$$

(ii) Continuous variational problem: find $u \in V$ such that

$$a(u, v) = L(v) \quad \text{for all } v \in \hat{V}$$

(iii) Discrete variational problem: find $u_h \in V_h \subset V$ such that

$$a(u_h, v) = L(v) \quad \text{for all } v \in \hat{V}_h$$

(iv) Discrete system of equations for $u_h = \sum_{j=1}^{N} U_j \phi_j$:

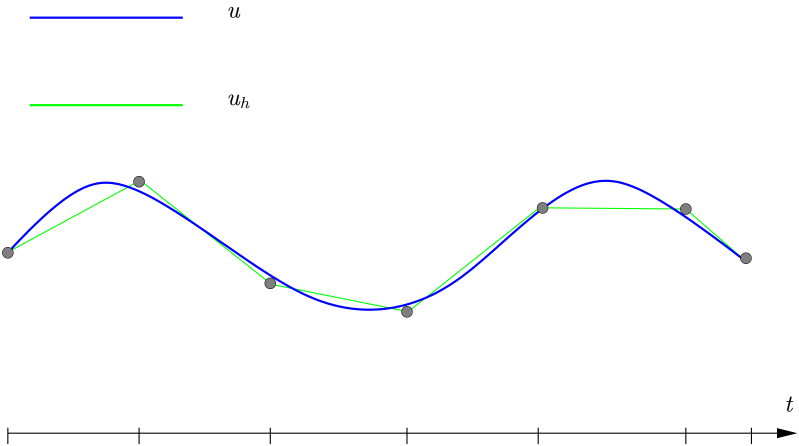
$$\begin{aligned} A\,U &= b \\ A_{ij} &= a(\phi_j, \phi_i) \\ b_i &= L(\phi_i) \end{aligned}$$

Important topics

- *How to choose V_h?*
- *How to compute A and b*
- *How to solve $AU = b$?*
- *How large is the error $e = u - u_h$?*
- Extensions to nonlinear problems

How to choose V_h

Finite element function spaces

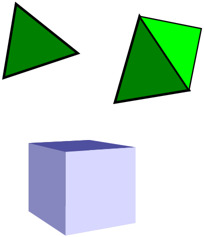
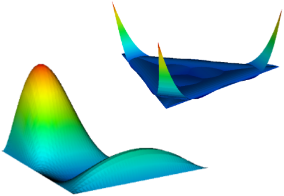


The finite element definition (Ciarlet 1975)

A finite element is a triple $(T, \mathcal{V}, \mathcal{L})$, where

- the domain T is a bounded, closed subset of $\mathcal{R}^d$ (for $d = 1, 2, 3, \dots$) with nonempty interior and piecewise smooth boundary
- the space $\mathcal{V} = \mathcal{V}(T)$ is a finite dimensional function space on T of dimension n
- the set of degrees of freedom (nodes) $\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_n\}$ is a basis for the dual space $\mathcal{V}'$; that is, the space of bounded linear functionals on $\mathcal{V}$

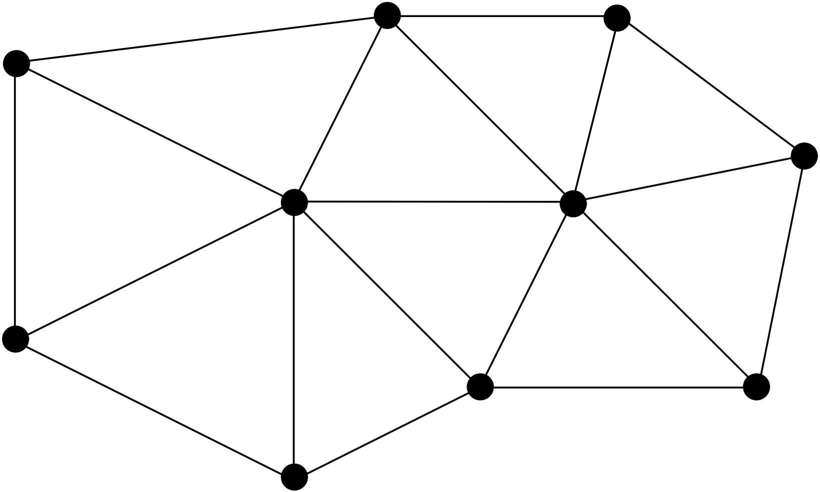
The finite element definition (Ciarlet 1975)

T	$\mathcal{V}$	$\mathcal{L}$
		$v(\bar{x})$ $v(\bar{x}) \cdot n$ $\int_T v(x) w(x) \, \mathrm{d}x$

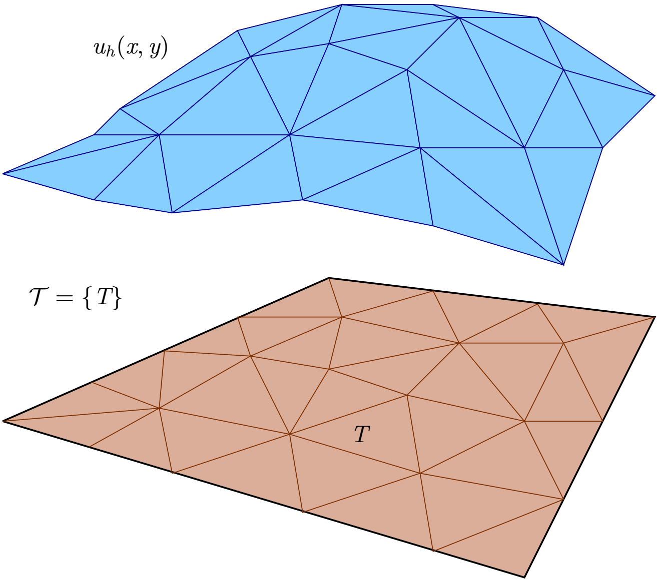
The linear Lagrange element: $(T, \mathcal{V}, \mathcal{L})$

- T is a line, triangle or tetrahedron
- $\mathcal{V}$ is the first-degree polynomials on T
- $\mathcal{L}$ is point evaluation at the vertices

The linear Lagrange element: $\mathcal{L}$



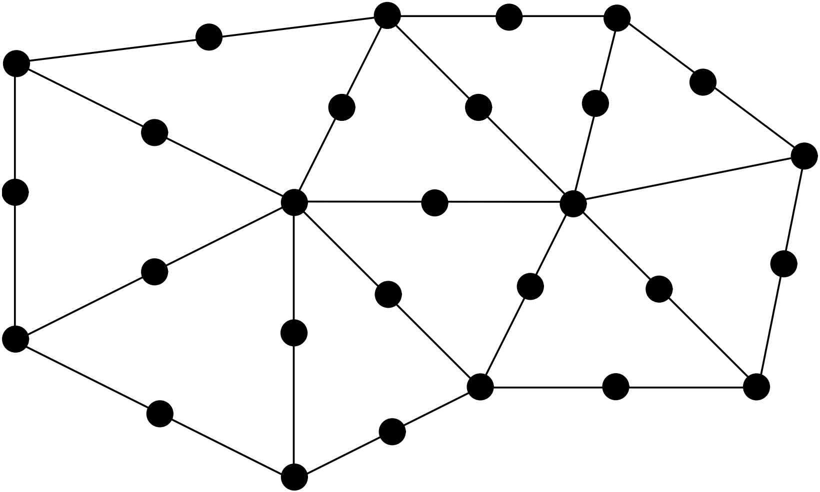
The linear Lagrange element: V_h



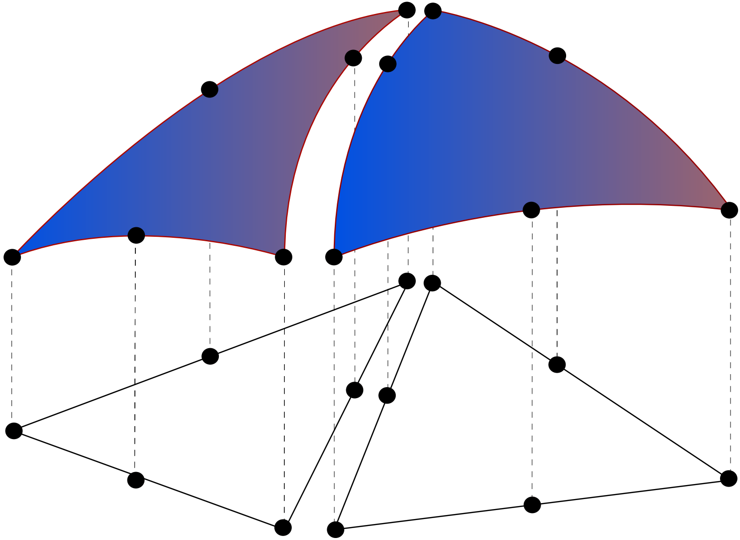
The quadratic Lagrange element: $(T, \mathcal{V}, \mathcal{L})$

- T is a line, triangle or tetrahedron
- $\mathcal{V}$ is the second-degree polynomials on T
- $\mathcal{L}$ is point evaluation at the vertices and edge midpoints

The quadratic Lagrange element: $\mathcal{L}$

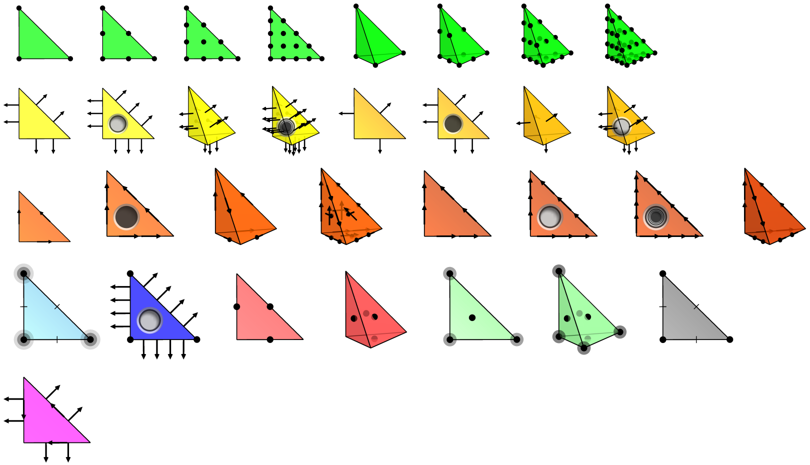


The quadratic Lagrange element: V_h



Families of elements

Families of elements



Computing the sparse matrix A

Naive assembly algorithm

$A = 0$

for $i = 1, \dots, N$

for $j = 1, \dots, N$

$A_{ij} = a(\phi_j, \phi_i)$

end for

end for

The element matrix

The global matrix A is defined by

$$A_{ij} = a(\phi_j, \phi_i)$$

The *element matrix* A_T is defined by

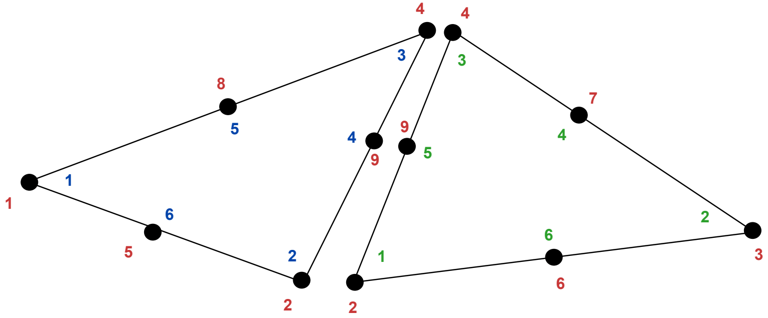
$$A_{T,ij} = a_T(\phi_j^T, \phi_i^T)$$

The local-to-global mapping

The global matrix ι_T is defined by

$$I = \iota_T(i)$$

where I is the *global index* corresponding to the *local index* i



The assembly algorithm

$A = 0$

for $T \in \mathcal{T}$

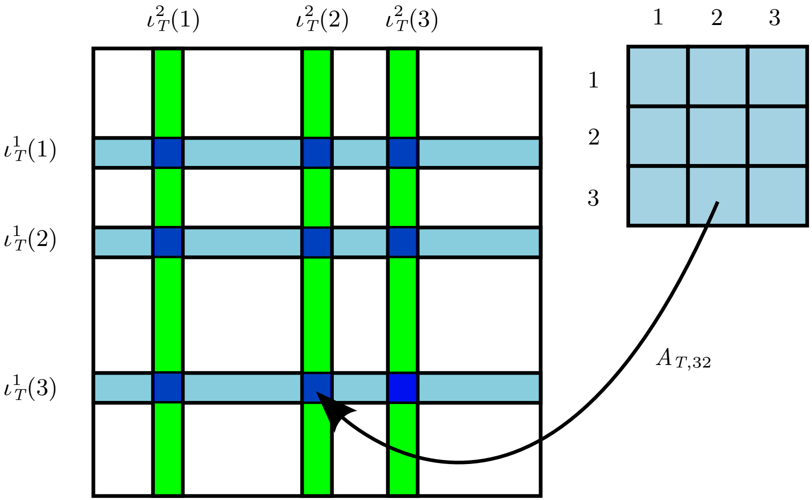
  Compute the element matrix A_T

  Compute the local-to-global mapping ι_T

  Add A_T to A according to ι_T

end for

Adding the element matrix A_T



Solving $AU = b$

Direct methods

- Gaussian elimination
  - Requires $\sim \frac{2}{3} N^3$ operations
- LU factorization: $A = LU$
  - Solve requires $\sim \frac{2}{3} N^3$ operations
  - Reuse L and U for repeated solves
- Cholesky factorization: $A = LL^\top$
  - Works if A is symmetric and positive definite
  - Solve requires $\sim \frac{1}{3} N^3$ operations
  - Reuse L for repeated solves

Iterative methods

Krylov subspace methods

- GMRES (Generalized Minimal RESidual method)
- CG (Conjugate Gradient method)
    - Works if A is symmetric and positive definite
- BiCGSTAB, MINRES, TFQMR, ...

Multigrid methods

- GMG (Geometric MultiGrid)
- AMG (Algebraic MultiGrid)

Preconditioners

- ILU, ICC, SOR, AMG, Jacobi, block-Jacobi, additive Schwarz, ...

Which method should I use?

Rules of thumb

- Direct methods for small systems
- Iterative methods for large systems
- Break-even at ca 100–1000 degrees of freedom
- Use a symmetric method for a symmetric system
  - Cholesky factorization (direct)
  - CG (iterative)
- Use a multigrid preconditioner for Poisson-like systems
- GMRES with ILU preconditioning is a good default choice

Norms and apriori error estimations

Measuring functions: norms

Commonly used (semi-)norms

Let $u : x \in \Omega \subset \mathcal{R}^n \to u(x)\mathcal{R}^m$ be a generic function:

- sup-norm:

$$\|u\|_{\infty,\Omega} = \sup_{\Omega} |u(x)|$$

- L^2-norm:

$$\|u\|_{0,\Omega} = \sqrt{\int_{\Omega} u(x) \cdot u(x) \mathrm{d}x}$$

- H^1-norm:

$$\|u\|_{1,\Omega} = \sqrt{\int_{\Omega} u(x) \cdot u(x) + \int_{\Omega} \nabla u(x) \cdot \nabla u(x)} \equiv \sqrt{\int_{\Omega} \nabla u(x) \cdot \nabla u(x) \mathrm{d}x}$$

- Energy norm for the variational problem $\min_u \frac{1}{2} a(u, u) - l(u)$:

$$\|u\|_E = \sqrt{\frac{a(u, u)}{2}}$$

Measuring functions

Exemple in FEniCS

Let $u : x \in \Omega \subset \mathcal{R}^n \to u(x)\mathcal{R}^m$ be a generic function:

- L^2-norm:

$$\|u\|_{0,\Omega} = \sqrt{\int_\Omega u(x) \cdot u(x) \mathrm{d}x}$$

```
integral = assemble(inner(u,u)*dx)
norm_L2 = sqrt(integral)
```

Theory can help you to validate your implementation!

A priori estimates for the Poisson problem (or Linear elasticity)

If

- $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$
- $V_h = \{v_h \in C(\Omega) : v_h \in P^k(T)\ \forall\ T \in \mathcal{T}\}$

then

$$E_1(h) := \|u - u_h\|_{1,\Omega} \leq Ch^k \|u\|_{k+1,\Omega}$$

$$E_0(h) := \|u - u_h\|_{0,\Omega} \leq Ch^{k+1} \|u\|_{k+1,\Omega}$$

where $\|\cdot\|_{l,\Omega} = \|\cdot\|_{H^l(\Omega)}$ for $l = 0, 1, k+1$.

Taking log on each side

$$\log(E_1(h)) \leq \log(Ch^k \|u\|_{k+1,\Omega}) = k \log(h) + \log(C\|u\|_{k+1,\Omega})$$

Method of manufactured solutions

Recipe

1. Take a suitable function u
2. Compute $-\Delta u$ to obtain f
3. Compute boundary values (trivial if only Dirichlet boundary conditions are used)
4. Solve the corresponding variational problem

$$a(u_h, v) = L(v)$$

   for a sequence of meshes Ω_h and compute the error $E_i(h) = \|u - u_h\|_{i,\Omega_i}$ for $i = 0, 1$
5. Plot $\log(E_i(h))$ against $\log(h)$ and determine k

Exercice

See TP 1.2

Types of error estimators

- *A priori* estimates

  (from mathematical analysis before calculating the solution):

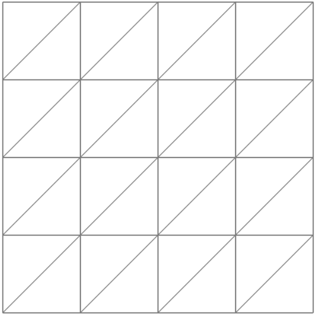
$$\|u - u_h\| \leqslant Ch^k \|u\|_{\Omega,k+1}$$

- *A posteriori* estimates (based on the postprocessing of the solution):
  - Explicit residual-based error estimators
  - Implicit error estimator based on local problems
  - Gradient recovery estimators
  - Hierarchic error estimators
  - Goal-oriented error estimators

A priori estimates

If $u \in H^{k+1}(\Omega)$ and $V_h = P^k(\mathcal{T}_h)$ then

$$\|u - u_h\| \leqslant Ch^k \|u\|_{\Omega,k+1}$$
$$\|u - u_h\|_{L^2(\Omega)} \leqslant Ch^{k+1} \|u\|_{\Omega,k+1}$$



These estimates are based on the regularity of the solution: it can be wrost than this in the presence of singularities!!

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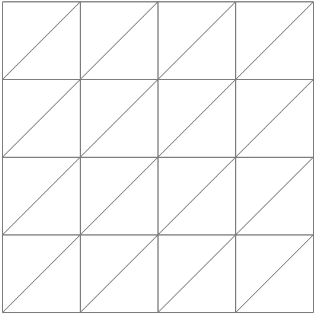

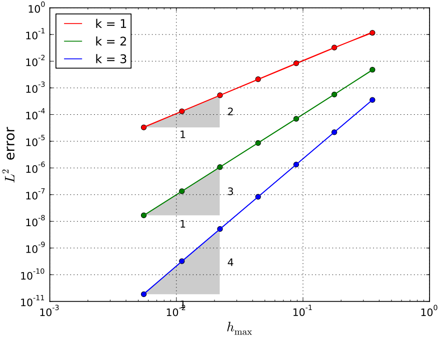

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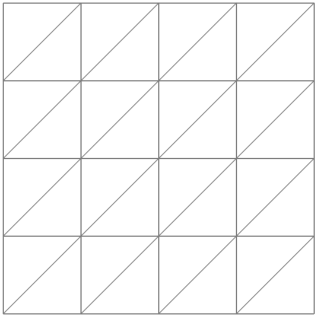

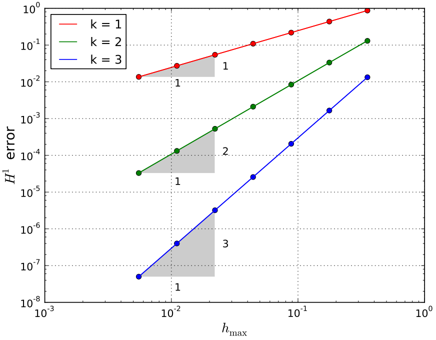

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