5AG07
Nonlinear structural mechanics by finite element method.

Singularities in linear problems

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References

References on singularities in linear problem:

- Szabo and Babuška, Ch. 6

  https://classes.engineering.wustl.edu/mase5510/Chapter_6.pdf
- J.J.Marigo, Elasticité et Rupture, Ed Ecole Polytechnique Ch.4.2

  https://moodle.polytechnique.fr/pluginfile.php/30014/mod_resource/content/1/ElasticiteRupture.pdf

Singularities

The regularity of the solution depends on the regularity of the

- Boundary conditions
- Source terms
- Geometry
- Material

If any of the above is not regular, the solution will not be smooth. Knowning the more common type of singularity is fundamental to analyse the solution and choose a suitable discretisation.

We distinguish between:

- (A) Smooth problems, where no singularities are present
- (B) Problem with singularities.

Geometric singular points

Non-smooth boundaries (notches and angles) implies point singularities in 2D (white point)

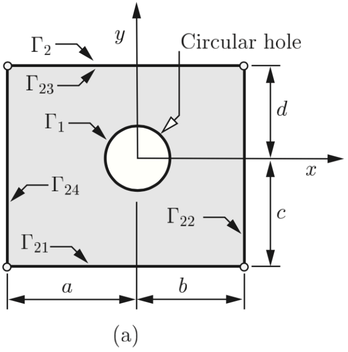


Figure 6.1 Typical geometric singular points associated with planar domains.

from Szabo and Babuška

Singularities: examples

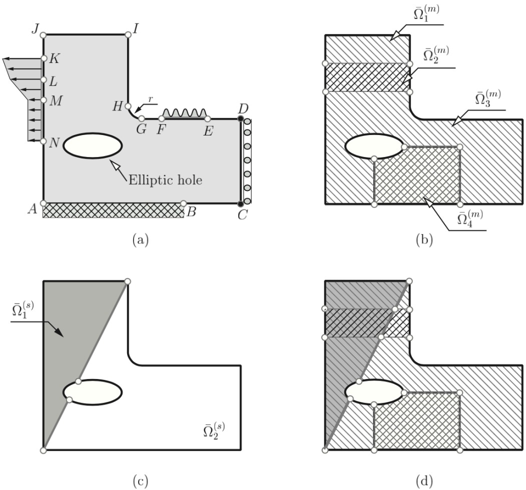
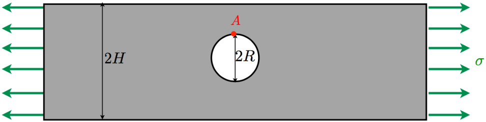


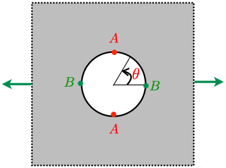
Figure 6.2 Typical singular points associated with (a) boundary conditions, (b) material interfaces, (c) source terms, (d) a combination of material interfaces and source terms.

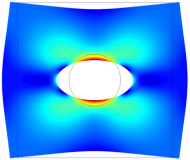
from Szabo and Babuška

Holed plate

- Calcul de la concentration des contraintes en A







- En milieu infini

au bord de la cavité $\boxed{\sigma_{\theta\theta}(R,\theta) = (1 - 2\cos 2\theta)\sigma}$

$$\Longrightarrow \begin{cases} \sigma_{\theta\theta}(A) = \boxed{3\sigma} \\ \sigma_{\theta\theta}(B) = \boxed{-\sigma} \end{cases}$$

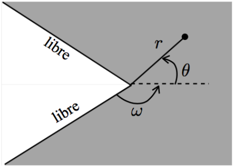
- Dans une bande de largeur H

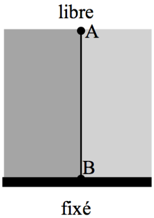
$$\boxed{\sigma_{\theta\theta}(A) = f\left(\frac{R}{H}\right)\sigma}$$

R/H	0	1/3	1
f	3	3.75	∞

From Marigo

Singularities





- **forme des déplacements**

$$\underline{\xi} = \sum_{i=1}^{N} K_i \, r^{\alpha_i} \, \underline{U}^i(\theta) + \cdots$$

$$\left\{ \begin{array}{lcl} N & : & \text{nombre de singularités} \\ \alpha_i & : & \text{puissance de la i-ème singularité} \\ \underline{U}^i & : & \text{fonction angulaire de la i-ème singularité} \\ K_i & : & \text{facteur d'intensité de la i-ème singularité} \end{array} \right.$$

- **forme des contraintes**

$$\underline{\underline{\sigma}} = \sum_{i=1}^{N} K_i \, r^{\alpha_i - 1} \, \underline{\underline{S}}^i(\theta) + \cdots$$

- **restrictions sur la puissance de la singularité**
  - contraintes non bornées : $\alpha_i < 1$
  - énergie élastique finie : $\alpha_i > 0$

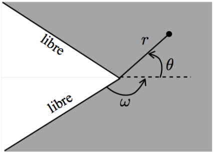
$$\frac{1}{2} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV \sim r^{\alpha_i - 1} \, r^{\alpha_i - 1} \, r \, dr \, d\theta \sim r^{2\alpha_i - 1} \, dr \, d\theta$$

$$\boxed{0 < \alpha_i < 1}$$

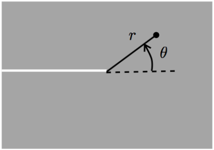
From Marigo

Notches (antiplane elasticity: Laplacian)

singularités en fond d'entaille en élasticité anti-plane



$$\underline{\xi}(\underline{x}) = \begin{pmatrix} 0 \\ 0 \\ \xi_z(r,\theta) \end{pmatrix}$$



- **Coin avec bords libres**
  - élasticité linéaire isotrope
  - forces volumiques régulières

$$\boxed{\frac{\pi}{2} < \omega \leq \pi}$$

$$\xi_z = K\, r^{\frac{\pi}{2\omega}} \sin \tfrac{\pi}{2\omega}\theta + \cdots$$

- **Fissure avec bords libres**
  - élasticité linéaire isotrope
  - forces volumiques régulières

$$\boxed{\omega = \pi}$$

$$\xi_z = K \sqrt{r} \sin \frac{\theta}{2} + \cdots$$

<u>Remarque</u> : la fissure correspond à la puissance la plus faible (donc à la singularité la plus forte)

Remarques :

- la constante multiplicative K (le facteur d'intensité de la singularité) reste indéterminée à ce stade
- le facteur d'intensité est une <u>quantité globale</u> qui dépend de l'ensemble des données (géométrie, élasticité, chargement)
- les forces volumiques ne jouent pas de rôle dans la mesure où elles ne sont pas (trop) singulières. (Elles interviennent dans les termes réguliers et dans la valeur de K).
- on obtient le même résultat si les bords de l'entaille sont soumis à des forces surfaciques pas (trop) singulières

From Marigo

Point load (Flamant problem)

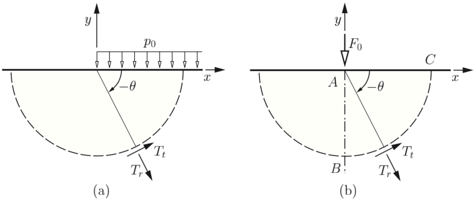


Figure 6.8 (a) Loading by a step function. (b) Loading by a concentrated force.

Solution using Airy stress function, polar coordinates and separation of variables:

$$\mathrm{Airy}(r,\theta) = -\frac{F_0}{\pi} r\theta \sin(\theta) \qquad \Rightarrow \qquad \boxed{\sigma_r = \frac{2F_0}{r\pi}, \quad u(r \to 0) \to \infty} \qquad (1)$$

- Stress are singular, displacement are singular, the energy is infinite
- In the finite element solution the energy of the solution depends on the mesh, as well as the maximum strain and displacement
- However, the solution far from the singular point does not depend on the mesh. You can use it but ignore the local solution.

from Szabo and Babuška. See also `https://en.wikipedia.org/wiki/Flamant_solution`

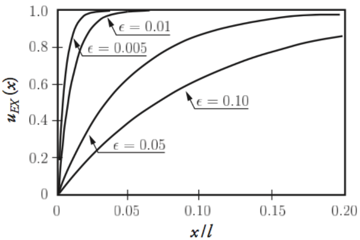
Boundary layers

Consider the problem of a bar on a elastic fundation of stiffness $k = \epsilon^2$

$$-u'' + \epsilon^2 u = 0, \quad u'(0) = u'(L) = 0 \tag{2}$$

The solution is in the form

$$u(x) = A\cosh(\sqrt{k}x) + B\sinh(\sqrt{k}x) \tag{3}$$



The mesh size should be adapted to $\sqrt{k}$ at the ends of the bar to resolve the boundary layer.

How to deal with singularities in FE computation

- Preprocessing
  - List and analyze the possible type of singularities you can have from geometry, BC, loadings, materials, etc.
  - Suitably refine the mesh around the singularity
- Postprocessing
  - Understand the limitation of the numerical results due to singularities
  - Look only for meaningful results: for example do not look for maximal stress around a singular point, where stress are going to infinity