

IMO 2016 Training Camp 4

Three theorems in geometry

9 April 2016

1 Simson's theorem

Statement. Given a point D and a triangle ABC , the feet of perpendiculars from D to sides BC, CA, AB are collinear iff D lies on the circumcircle of triangle ABC . (Proof? Simple angle chasing, or better still, oriented angle.)

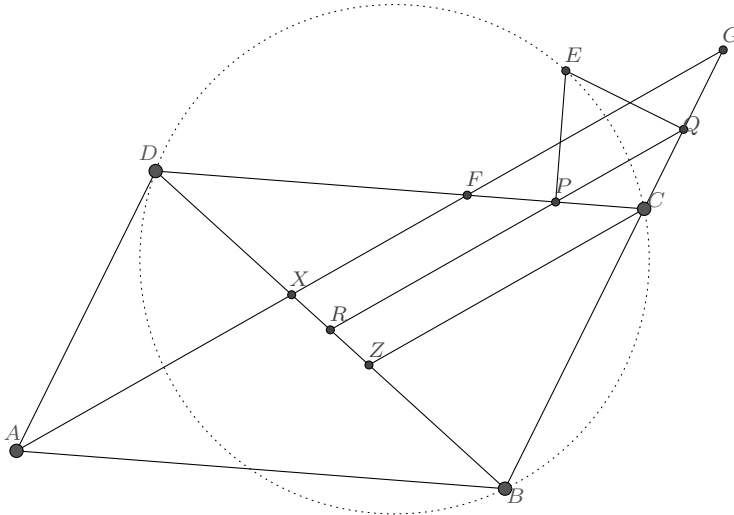
Looks simple? The application to this theorem can be very very profound.

1. *IMO 2003, #4.* Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC .

Solution. By Simson's theorem P, Q, R are collinear. Now $\frac{PQ}{QR} = \frac{PD}{DR} \cdot \frac{\sin \angle PDQ}{\sin \angle QDR} = \frac{PD}{DR} \cdot \frac{\sin \angle BAC}{\sin \angle BCA} = \frac{PD}{DR} \cdot \frac{BC}{BA}$. Now $\frac{PD}{DR} = \frac{\sin \angle PBD}{\sin \angle RBD} = \frac{\sin \angle ACD}{\sin \angle CAD} = \frac{AD}{CD}$. Therefore $\frac{PQ}{QR} = \frac{AD}{CD} \cdot \frac{BC}{BA}$, from which we know that the ratio is 1 iff $\frac{BA}{BC} = \frac{AD}{CD}$, which is equivalent to that the internal angle bisectors of $\angle B$ and $\angle D$ concur on AC . ■

2. *IMO 2007, #2.* Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A . Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G . Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB .

Solution.



Obviously E is the circumcenter of triangle CFG . If we denote P as the midpoint of CF and Q the midpoint of CG , then $EP \perp CD$ and $EQ \perp BC$. Notice also that $PQ \parallel \ell$. Let the perpendicular from E to BD intersect at R , then P, Q, R are collinear by Simson's theorem.

Let ℓ intersect BD at X , and let the line parallel to ℓ , passing through C intersect BD at Z . Now $PQ \parallel \ell$ implies R is the midpoint of XZ . But since $ABCD$ is a parallelogram, X and Z are the reflections of each other in the midpoint of BD . (Proof: triangles ABD and CDB are congruent, so the triangles in the following pairs are also congruent: $\{ADX, CBZ\}, \{ABX, CDZ\}$.) This implies that R is the midpoint of BD and since $ER \perp BD$, $EB = ED$ and from the fact that E lies on the circumcircle of CBD we know that EC is the external angle bisector of $\angle BCD$. This implies that $CF = CG$, so $\angle DAF = \angle FGC = \angle GFC = \angle GAB$. (elegant, no?) ■

3. *TOT Spring Fall 2008, Senior A-Level, #7.* Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that its circumcircle passes through the midpoint of the segment joining the centres of the circles.

Hint. Anything special about the Simson's line?

2 Casey's theorem (a.k.a. generalized Ptolemy's theorem)

Statement. Let $\omega_1, \omega_2, \omega_3$ and ω_4 be four non-intersecting and mutually exclusive circles, and let t_{ij} be the length of common exterior bitangent (i.e. segment connecting $A_i A_j$ s.t. A_i on ω_i , A_j on ω_j and $A_i A_j$ tangent to both circles externally.) Then there exists a circle tangent internally to all four circles (in the order of $\omega_1, \omega_2, \omega_3$ and ω_4) if and only if:

$$t_{12}t_{34} + t_{23}t_{14} = t_{13}t_{24}.$$

Proof. Denote T_i as the tangency point of ω_i and the big circle, ω . Denote the radius of ω as R and radius of ω_k as R_k ($k = 1, 2, 3, 4$). We prove that $t_{ij} = T_i T_j \cdot \sqrt{(1 - \frac{r_i}{r})(1 - \frac{r_j}{r})}$, thereby reducing the statement into ordinary Ptolemy's theorem (which is far easier to prove!) Note that this proof only works to prove necessity, not sufficiency (which is left as an exercise).

Now, by Monge's theorem the line $T_i T_j$ passes through point E , the exsimilicentre of ω_i and ω_j . Denote A_i and A_j the same way we define them in the theorem statement. Then EA_i^2 and EA_j^2 is the power of point of E w.r.t. ω_i and ω_j , respectively, and $\frac{r_i}{r_j} = \frac{EA_i^2}{EA_j^2}$. Denote U_i, U_j as the second intersection of line $T_i T_j$ and ω_i, ω_j respectively. By the definition of exsimilicentres we have ratio $r_1 : r_2 : r = T_i U_i : T_j U_j : T_i T_j$. By the definition of point E we can write ET_i, EU_i, EU_j, ET_j (assuming, w.l.o.g., that they lie on the line in that order) as $r_1 a, r_1 b, r_2 a, r_2 b$. Now $EA_i^2 = ET_i \cdot EU_i$ and $EA_j^2 = EU_j \cdot ET_j$ so $A_i A_j = (r_2 - r_1)\sqrt{ab}$, while

$$T_i T_j \cdot \sqrt{(1 - \frac{r_i}{r})(1 - \frac{r_j}{r})} = T_i T_j \cdot \sqrt{(1 - \frac{T_i U_i}{T_i T_j})(1 - \frac{T_j U_j}{T_i T_j})} = \sqrt{(T_i T_j - T_i U_i)(T_i T_j - T_j U_j)} = \sqrt{(T_i U_j)(T_j U_i)} = \sqrt{a(r_2 - r_1) \cdot b(r_2 - r_1)} = (r_2 - r_1)\sqrt{ab}. \blacksquare$$

Below is a generalization of Problem 4 in IMO 1978, which assumes that $AB = AC$.

Exercise: A circle is tangent internally to the circumcircle of $\triangle ABC$ and also the sides AB, AC at P, Q , respectively. Prove that the midpoint of segment PQ is the incentre of $\triangle ABC$.

This lemma may come handy in solving the following:

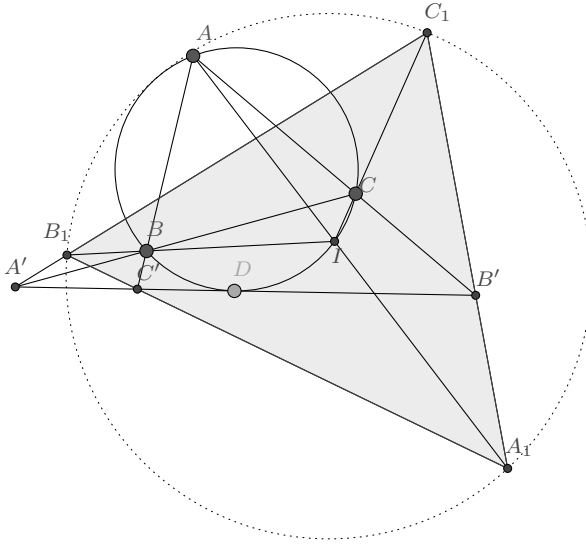
APMO 2006, #4. Let A, B be two distinct points on a given circle O and let P be the midpoint

of the line segment AB . Let O_1 be the circle tangent to the line AB at P and tangent to the circle O . Let l be the tangent line, different from the line AB , to O_1 passing through A . Let C be the intersection point, different from A , of l and O . Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC . Prove that the circle O_2 is tangent to the circle O .

Below we present the application to a very hard IMO problem, with only six complete solutions submitted.

1. *IMO 2011, # 6.* Let ABC be an acute triangle with circumcircle Γ . Let ℓ be a tangent line to Γ , and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB , respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ .

Solution.



Because we are using Casey's theorem, please expect some ratio bashing (including the notorious trigonometry!) Denote A_1, B_1, C_1 as $\ell_b \cap \ell_c, \ell_a \cap \ell_c, \ell_a \cap \ell_b$, respectively. We claim that AA_1, BB_1, CC_1 concur on Γ .

Now denote the intersection of BC, CA, AB with ℓ as A', B' and C' , respectively. let us consider triangle $B_1A'C'$. Since $A'B$ bisects $\angle B_1A'C'$ and $C'B$ bisects $\angle B_1C'A'$, we know that B is either the incenter or excenter of $B_1A'C'$ and it follows that B_1B bisects angle $A'B_1C'$, which then follows that (well known: verify it yourself!) BB_1 passes through the circumcentre of $BA'C'$. Using the notation of directed angles it follows that $\angle(BB_1, AB) = \angle(BB_1, BC') = 90^\circ - \angle(A'C', A'B) = 90^\circ - \angle(\ell, BC)$ and $\angle(BB_1, BC) = \angle(BB_1, BA') = 90^\circ - \angle(A'C', C'B) = 90^\circ - \angle(\ell, AB)$. In a similar way, $\angle(CC_1, AC) = 90^\circ - \angle(\ell, BC)$ and $\angle(CC_1, BC) = 90^\circ - \angle(\ell, AC)$. Now $\angle(BB_1, CC_1) = \angle(BB_1, BC) - \angle(CC_1, BC) = (90^\circ - \angle(\ell, AB)) - (90^\circ - \angle(\ell, AC)) = \angle(AB, AC)$, yielding that BB_1 and CC_1 indeed intersect on Γ . The concurrence of AA_1, BB_1 and CC_1 (namely I) follows from the fact that they intersect at either the incenter or excenter of triangle $A_1B_1C_1$, hence we are done.

Now we need the fact that I is the incenter (not excentre opposite to any of the sides), given that ABC is an acute triangle. Now let's investigate the relationship between $\angle A$,

$\angle B, \angle C$ (of $\triangle ABC$) and $\angle A_1, \angle B_1, \angle C_1$ (of $\triangle A_1B_1C_1$). Take B again. If $\angle A'BC' < 90^\circ$ then $\angle B = \angle A'BC'$, and it can be verified that B is now the excentre of $\triangle B_1A'C'$ and $\angle B_1A'C' = 180^\circ - 2\angle A'BC = 180^\circ - 2\angle B$. On the other hand, if $\angle A'BC$ is obtuse then $\angle B = 180^\circ - \angle A'BC$ and B is the incentre of $\triangle B_1A'C'$ so $\angle B_1A'C' = 2\angle A'BC - 360^\circ = 2(180^\circ - \angle B) - 360^\circ = 180^\circ - 2\angle B$. So $\angle B_1$ is either $2\angle B$ or $180^\circ - 2\angle B$, depending whether BB_1 is the external or internal angle bisector of angle B_1 , respectively. Same goes for line angles A_1 and C_1 . If I is the excentre, then exactly two of the lines AA_1, BB_1, CC_1 are the external angle bisectors of the respective angles. So let say AA_1 and BB_1 are the external angle bisectors of angles A_1 and B_1 then angles A_1, B_1, C_1 are $2\angle A, 2\angle B$ and $180^\circ - 2\angle C$ respectively. As the three angles sum up to 180° and $\angle A + \angle B + \angle C = 180^\circ$, comparing the two equations yields $\angle C = 90^\circ$ (contradiction).

Now let's proceed to the solving of the main problem. Denote T as the tangent point of ℓ and Γ , and a, b, c as $\angle(TA, \ell), \angle(TB, \ell), \angle(TC, \ell)$. Speaking in modulo 180° , $\angle A$ is congruent to $b - c$, so $\sin \angle A = |\sin(b - c)|$. Similarly $\sin \angle B = |\sin(c - a)|$ and $\sin \angle C = |\sin(a - b)|$. (we can change the \pm sign to modulus, though). By applying Casey's theorem to degenerate circles A_1, B_1, C_1 and circle Γ we need one of $A_1B_1 \cdot t(C_1), C_1A_1 \cdot t(B_1), B_1C_1 \cdot t(A_1)$ to be the sum of the other two, which is evident, if, by changing some of the terms into its negative, the resulting three terms add up to 0. ($t(K)$ is the length of tangent from point K to Γ .) Notice that it is the ratio $A_1B_1 \cdot t(C_1) : C_1A_1 \cdot t(B_1) : B_1C_1 \cdot t(A_1)$ that matters, so we can divide each of them by a constant whenever necessary.

By power of point theorem we have $t(A_1) = \sqrt{A_1I \cdot A_1A}$. Define r as the inradius of triangle $A_1B_1C_1$ and r_A the distance from A to lines A_1B', A_1C' and $B'C'$ (or ℓ) (the distance to the three lines are the same because A is the incentre or excentre of $A_1B'C'$.)

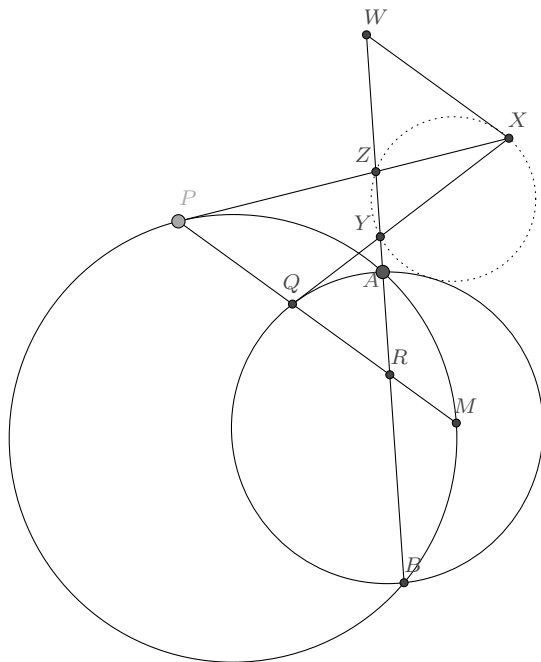
Now $A_1A = \frac{r_A}{\sin \angle A_1B'} = \frac{r_A}{\sin(90^\circ - \angle A)} = \frac{r_A}{\cos \angle A}$ as $\angle A_1 = 180^\circ - 2\angle A$. Similarly $\angle A_1I = \frac{r}{\cos \angle A}$ so $t(A_1) = \sqrt{A_1I \cdot A_1A} = \frac{\sqrt{r \cdot r_A}}{\cos \angle A}$. Now $B_1C_1 = D \sin \angle A_1 = D \sin \angle(180^\circ - 2\angle A) = D \sin 2\angle A$, where D is the diameter of circumcircle $A_1B_1C_1$ so $B_1C_1 \cdot t(A_1) = D \sin 2\angle A \cdot \frac{\sqrt{r \cdot r_A}}{\cos \angle A} = 2D \sin \angle A \cdot \sqrt{r \cdot r_A}$. The original ratio now becomes $\sin \angle A \cdot \sqrt{r_A} : \sin \angle B \cdot \sqrt{r_B} : \sin \angle C \cdot \sqrt{r_C}$ by eliminating constants D and r . Now if d is the diameter of Γ then it is not hard to notice that $r_A = \text{distance from } A \text{ to } \ell = TA \sin \angle(TA, \ell) = (d \sin a) \cdot \sin a = d \sin^2 a$. Meanwhile $\sin \angle A = |\sin(b - c)|$. So our original ratio becomes $\sin a \cdot |\sin(b - c)| : \sin a \cdot |\sin(c - a)| : \sin c \cdot |\sin(a - b)|$. But $\sin a \cdot \sin(b - c) + \sin a \cdot \sin(c - a) + \sin c \cdot \sin(a - b) = 0$, which confirms the identity we want to prove. ■

Now another elegant key to the following APMO problem, also among the hardest in that paper.

2. *APMO 2014, #5.* Circles ω and Ω meet at points A and B . Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P , and let ℓ_Q be the tangent line to Ω at Q . Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω .

Solution. Denote X, Y, Z as $\ell_P \cap \ell_Q, \ell_Q \cap AB$ and $AB \cap \ell_P$, respectively. A simple conjecture yields that if T is the tangent point we need, then X, Z, Y, T are on a circle in that order. Again, from Casey's theorem we need $XY \cdot t(Z) = XZ \cdot t(Y) + ZY \cdot t(X)$, where $t(K)$ is the length of tangent from point K to Ω . Now $t(X) = XQ = XY + YQ$ and $t(Y) = YQ$. Since X lies on AB , the radical axis of ω and Ω , the tangent from X to

both circles are the same, and $t(X) = XP$. Now the original desired equation becomes $XY \cdot ZP = XZ \cdot YQ + YZ \cdot (XY + YQ)$, or $XY \cdot (ZP - YZ) = YQ \cdot (XZ + YZ)$.

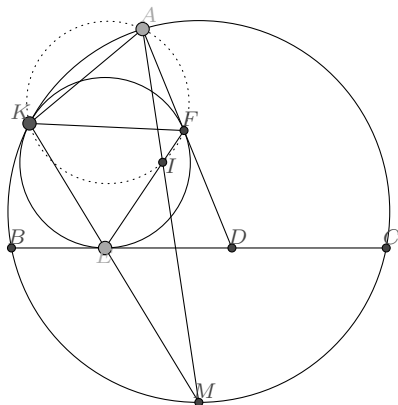


A crucial identity in this problem is that, if $R = MP \cap AB$ then $ZP = ZR$. (actually this is well-known and the proof is left as an exercise: think of the tangent to ω at M which is parallel to AB !) Now let W on AB s.t. $WX \parallel PR$, we have $\frac{WZ}{XZ} = \frac{ZR}{ZP} = 1$, so $XZ + YZ = YZ + WZ = WY$. Furthermore $ZP - YZ = ZR - YZ = YR$. So now we need $XY \cdot YR = YQ \cdot WY$, or $\frac{XY}{WY} = \frac{YQ}{YR}$. But this follows from the fact $WX \parallel PM$ (so we are done!) ■

3 Sawayama Thebault's theorem.

Statement. Let I be the incentre of $\triangle ABC$, D a point on line BC . If a circle is tangent to the circumcircle of triangle ABC , to segment DC at E and segment DA at F . Then E, I, F are collinear.

Proof (found in Awesome Math notes).



Denote by Ω the circumcircle of ABC and by Γ the circle tangent to Γ , DC and DA . Let the two circles touch at K . Let M be the midpoint of arc BC on Ω not containing K . Then The homothety centred K that maps Γ to Ω maps point E to M , so K, E, M are collinear. Notice that A, I, M are collinear too, and $MI = MB = MC$.

Now let EI meet Γ at F' . To prove $F = F'$ it suffices to have AF' tangent to Γ . Note that $\angle KFE'$ is subtended by the arc KE in Γ and $\angle KAM$ is subtended by KM in Ω . Since KE and KM are however homothetic with center K , the two angles aforementioned are equal, implying that A, K, I, F' are concyclic.

We have $\angle BCM = \angle CBM = \angle CKM$. So triangles MCE and MKC are similar. Hence $MC^2 = ME \cdot MK$. Since $MC = MI$, we have $MI^2 = ME \cdot MK$, implying that MIE and MKI are in fact similar. Therefore, AF' is tangent to Ω . ■

Corollary 1. (Thebault) Denote A, B, C, D, I as above, and ω_1, ω_2 be two circles, each tangent to line AD , to line BC and to the circumcircle of $\triangle ABC$. (One tangent to segment BD and the other one tangent to segment CD . Then I lies on the line connecting centres of ω_1 and ω_2 .

Finally, we present the 6th (also the last) problem from some national Olympiad in 2010.

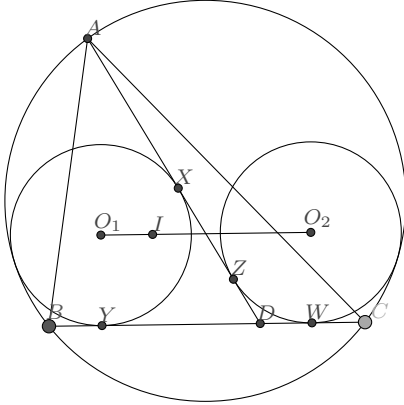
Corollary 2. Using the same notation with corollary 1, prove that B, C and the centres of ω_1 and ω_2 lie on a circle if and only if D is the tangency point of A -excircle with BC . *The actual contest only contains the "only if" part, but once this is established, the other direction can be settled quickly.

To solve the "only if" part, there are two ways: either using Thebault's theorem, or using Casey's theorem. But both require us to prove the following lemma:

If O_1 and O_2 are the centres of ω_1 and ω_2 , then B, C, O_1, O_2 are concyclic if and only if ω_1 and ω_2 are congruent.

Using the assumption above, we are left with Ehrmann-Pohoata's lemma. We offer a proof using Casey's theorem, bearing in mind that the two circles are congruent.

Proof. Let ω_1 tangent to AD at X , and segment BD at Y . Let ω_2 tangent to AD at Z and segment CD at W . Because of the congruence of the circles we can easily know that $BY = CW$, and $CY = BW$.



By Casey's theorem, $AB \cdot CY = BC \cdot AX + AC \cdot BY$ and $AC \cdot BW = BC \cdot AZ + AB \cdot CW$. Subtracting the first equation by the second equation yields $CY \cdot (AB - AC) = BC \cdot (AX - AZ) + BY \cdot (AC - AB)$, or $BC \cdot (AB - AC) = (CY + BY) \cdot (AB - AC) = BC \cdot (AX - AZ) = BC \cdot (DZ - DX) = BC \cdot (DW - DY) = BC \cdot (DC - DB)$, since $DC = DW + CW$, $DB = DY + BY$ and $CW = BY$. The equation now becomes $AB + BD = AC + CD$, and from here we conclude that D is the tangency point of the A -excircle with BC (well-known fact, again!) ■

4 References.

1. Cosmin Pohoata, 2012, AMY 2011-2012, *Homothety and Inversion*. AwesomeMath LLC.
2. <https://en.wikipedia.org/wiki/Casey>

4.1 Problem credit.

1. IMO 2003, 2007, 2011.
2. APMO 2006, 2014.
3. IMTOT Fall 2008.

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