Tricks on radical axis, power of point

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1 P.O.P and radical axis

Some background knowledge:

1. The power of a point A w.r.t. a circle with center O and radius r is given by $OA^2 - r^2$. Denote this number as $P(A)$.
2. Let BC be a chord on the circle, and A a point on line BC. Then $P(A) =$
   •$AB \cdot AC$ for A outside the chord (i.e. outside the circle).
   •$-AB \cdot AC$ for A on *segment* BC (i.e. inside the circle).
   •0 for A on the circle (i.e. coinciding with B or C).
3. $P(A)$ is independent of chords passing through it. In other words if chords BC and DE intersect at A then $AB \cdot AC = AD \cdot AE = |P(A)|$. (Property of a cyclic quadrilateral).
4. The *radical axis* of two circles is the line describing the locus of all points whose power of point are same to the two circles. (I.e. a point has equal P.O.P. w.r.t. the two circles iff it is on the radical axis).
5. In particular, if the circles intersect at two points, then the line joining the two points is their radical axis.
6. Given three circles ω_1, ω_2 and ω_3 with non collinear centres. Let the radical axis of ω_i and ω_{i+1} be ℓ_{i+2}. (Indices taken modulo 3). Then ℓ_1, ℓ_2 and ℓ_3 are concurrent at the radical centre of the three circles (the radical centre is the unique point which has equal P.O.P w.r.t. the three circles).
7. In case the centre of the three circles are collinear, then the three radical axes are parallel or coincide (in the latter case we say that the three circles are coaxial).

Problems.

1. *IMO 2006, G2.* Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

   Prove that the points P, Q, B and C are concyclic.

2. *IMO 2008, #1.* Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

   Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

3. *IMO 2008, G2.* Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

4. *IMO 2008, G3.* Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

5. *IMO 2009. #2.* Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

6. *IMO 2009, G3.* Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

7. *IMO 2011, G2.* Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

8. *IMO 2011, G4.* Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

9. *IMO 2011, G5.* Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

10. *IMO 2012, G4.* Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

11. *IMO 2012, #5.* Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

    Show that $MK = ML$.

2 Extra problems on spiral similarity.

1. *IMO 2006, G8.* Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

   $$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

   Prove that $AB + CD \geq BC + AD$.

2. *IMO 2013, G5.* Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

3. *IMO 2013, #3.* Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

3 Location of solutions.

Note: by the meaning of "official solution" of IMO shortlist it means the solution in the IMO shortlist, found here: http://imo-official.org/problems.aspx

P.O.P.

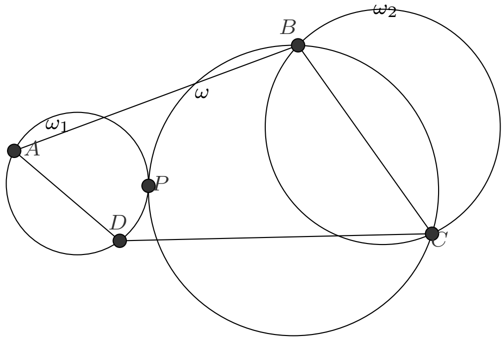
1. Official solution 1 (page 36.)
2. Official solution 2 (page 30.)
3. Official solution (page 31.)
4. Official solution (both) (pages 32, 33).
5. Official solution 1 (page 50).
6. Official solution (page 52).
7. Official aolution (page 46).
8. Official solution (page 50).
9. Official solution (page 52).
10. Official solution (page 32).
11. Official solution (page 34).

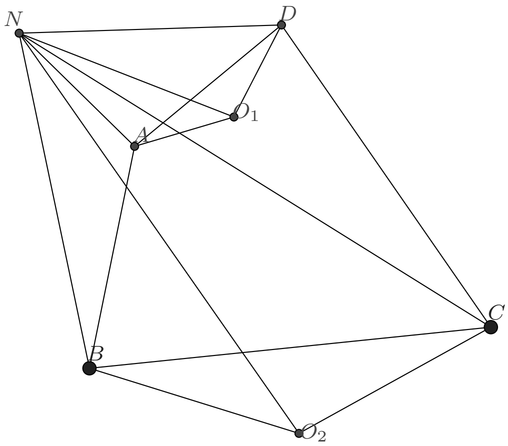
Spirals.

1. As in handout.
2. As in handout.
3. Official solution 1 (page 49).

4 Appendix: solution to IMO 2006, G8.

For any circle passing through A and D, define the *angle of circle* by $\angle AQD$ for any point Q inside the quadrilateral $ABCD$ and on the arc AD. Define similarly for circle passing through B and C. From the problem we know that $\angle APD + \angle BPC \leq 180°$. Define the circle pasing through A and D as ω_1., and the circle through B and C as ω. Let another circle ω_2 to pass through B and C such that sum of angles of ω_2 and ω_1 is $180°$. Since angle of ω_2, noted as $\angle\omega_2$ is at least ω, the region determined by line BC and arc BC of ω_2 is inside the quadrilateral lies inside the region of line BC and arc BC of ω. We infer that ω_1 and ω_2 are either externall tangent to each other, or are mutually exclusive.





Denote by O_1 and O_2 the centres of ω_1 and ω_2, respectively, and their radii as r_1, r_2. From above we have $O_1O_2 \geq r_1 + r_2$. To prove $O_1O_2 \geq r_1 + r_2 \Rightarrow AB + CD \geq AD + BC$, it suffices to proe that $\dfrac{AB+CD}{O_1O_2} \geq \dfrac{AD+BC}{r_1+r_2}$.

Assuming $ABCD$ is not a parallelogram, let N be the centre of spiral similarity that brings A to B and D to C. Since triangles AO_1D and BO_2C are similar (including the degenerate case where O_1 and O_2 are mispoints of AD and BC, respectively), this spiral similarity also brings O_1 to O_2. We therefore know that triangles NO_1O_2, NAB and NDC are similar. Therefore, $\dfrac{AB}{NA} = \dfrac{CD}{ND} = \dfrac{O_1O_2}{NO_1}$ and we have $\dfrac{AB+CD}{O_1O_2} = \dfrac{NA+ND}{NO_1}$. On the other hand, from the similarities of triangles BO_2C and AO_1D, and the fact that $r_1 = AO_1 = O_1D$, $r_2 = BO_2 = CO_2$ we have $\dfrac{AD}{r_1}$=$\dfrac{BC}{r_2}$=$AD + BCr_1 + r_2$. The whole inequality now becomes $\dfrac{NA+ND}{NO_1} \geq \dfrac{AD}{r_1}$, or $NA \cdot DO_1 + ND \cdot AO_1 \geq AD\dot{N}O_1$. This is obvious by Ptolemy's inequality, with equality holds iff N, A, O_1, D concyclic or collinear in this order.

Finally, for case where $ABCD$ is a parallelogram, it is not difficult to verify that $O_1O_2 =$

$AB = CD$, while $r_1 = r_2 \geq \frac{AD}{2} = \frac{BC}{2}$. Therefore $AB = O_1O_2 \geq r_1 + r_2 \geq \frac{AD}{2} + \frac{BC}{2}$ means $AB + CD \geq BC + AD$. Q.E.D.