Binary quadratic form + n-th power residues.

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1. *JOM 2013, N5.* Let p be an odd prime such that $2^p + 1 | p^p + 1$. Prove that any prime divisor of $2^p + 1$ other than 3 is greater than $6p$.
   Generalization. For any prime number p we have: each prime divisor of $\gcd(2^p + 1, p^p + 1)$ is equal to 3 or is greater than $6p$.

2. *APMO 2014, # 3.* Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

3. *APMO 2012, # 3.* Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

4. *IMO 2012, N6.* Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

5. *IMO 2012, N8.* Prove that for every prime $p > 100$ and every integer r, there exist two integers a and b such that p divides $a^2 + b^5 - r$.

6. *IMO 2011, N8.* Let $k \in \mathbb{Z}^+$ and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, \ldots, g_{n-1}$, such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

7. *IMO 2010, N4.* Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n-good if $n | P(m) - P(k)$ implies $n | m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.
   (a) Find a pair (a, b) which is 51-good, but not very good.
   (b) Show that all 2010-good pairs are very good.

8. *IMO 2008, # 3.* Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.