Olympiad functions.

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1 IMO

1. *2015/5.* Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation
$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$
for all real numbers x and y.

2. *2013/5.* Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:
(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x + y) \geq f(x) + f(y)$;
(iii) there exists a rational number $a > 1$ such that $f(a) = a$.
Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$

3. *2012/4.* Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:
$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$
(Here $\mathbb{Z}$ denotes the set of integers.)

4. *2011/3.* Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies
$$f(x + y) \leq yf(x) + f(f(x))$$
for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

5. *2011/5.* Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

6. *2010/1.* Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$
where $\lfloor a \rfloor$ is greatest integer not greater than a.

7. *2010/3.* Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that
$$(g(m) + n)(g(n) + m)$$
is a perfect square for all $m, n \in \mathbb{N}$.

8. *2009/5.* Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths
$$a, f(b) \text{ and } f(b + f(a) - 1).$$
(A triangle is non-degenerate if its vertices are not collinear.)

9. *2008/4.* Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that
$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$
for all positive real numbers w, x, y, z, satisfying $wx = yz$.

10. *2002/5.* Find all functions f from the reals to the reals such that
$$(f(x) + f(z)) (f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$
for all real x, y, z, t.

2 APMO

1. *2016/5.* Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x),$$
for all positive real numbers x, y, z.

2. *2015/2.* Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that
$$f(a)f(b) = f(a^2 b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

3. *2011/5.* Determine all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:
1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.
2) For every pair of real numbers x and y,
$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$
is satisfied.

4. *2010/5.* Find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity $f(f(x)+f(y)+f(z)) = f(f(x)-f(y))+f(2xy+f(z))+2f(xz-yz)$

5. *2008/4.* Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :
(i) $f(0) = 0$;
(ii) $f(2n) = 2f(n)$ and
(iii) $f(2n + 1) = n + 2f(n)$ for all $n \geq 0$.
(a) Determine the three sets $L = \{n | f(n) < f(n + 1)\}$, $E = \{n | f(n) = f(n + 1)\}$, and $G = \{n | f(n) > f(n + 1)\}$.
(b) For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

3 TOT

1. Do there exists two functions f an g (integer to integer) such that for every integers x,
   (i) $f(f(x)) = x$, $g(g(x)) = x$, $f(g(x)) > x$, $g(f(x)) > x$,
   (ii) $f(f(x)) < x$, $g(g(x)) < x$, $f(g(x)) > x$, $g(f(x)) > x$.

4 IMO Shortlist

(ISL 2015)

1. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that
$$f(x - f(y)) = f(f(x)) - f(y) - 1$$
holds for all $x, y \in \mathbb{Z}$.

2. Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z} + 1$ satisfying
$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$
for every $x, y \in \mathbb{Z}$.

3. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:
   (i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$;
   (ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

   Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

4. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called k-good if $\gcd(f(m)+n, f(n)+m) \le k$ for all $m \neq n$. Find all k such that there exists a k-good function.

5. For every positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define
$$\mho(n) = \sum_{i:\, p_i > 10^{100}} \alpha_i.$$
That is, $\mho(n)$ is the number of prime factors of n greater than 10^{100}, counted with multiplicity.

   Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$\mho(f(a) - f(b)) \le \mho(a - b) \quad \text{for all integers } a \text{ and } b \text{ with } a > b.$$

(ISL 2014)

6. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$
for all integers m and n.

7. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

(ISL 2013)

8. Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

9. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

10. Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(ISL 2012)

11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1+xy) - f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and $f(-1) \neq 0$.

12. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

(ISL 2011)

13. Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

14. Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f)}_{k}(n)\ldots))$.

15. Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself, such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

(ISL 2010)

16. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

17. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n)+1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

    (ISL 2009)

18. Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x - f(y)\right) > yf(x) + x$$

19. Find all functions f from the set of real numbers into the set of real numbers which satisfy for all x, y the identity

$$f\left(xf(x+y)\right) = f\left(yf(x)\right) + x^2$$

20. Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

21. Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.