IMO 2016 Training Camp 3

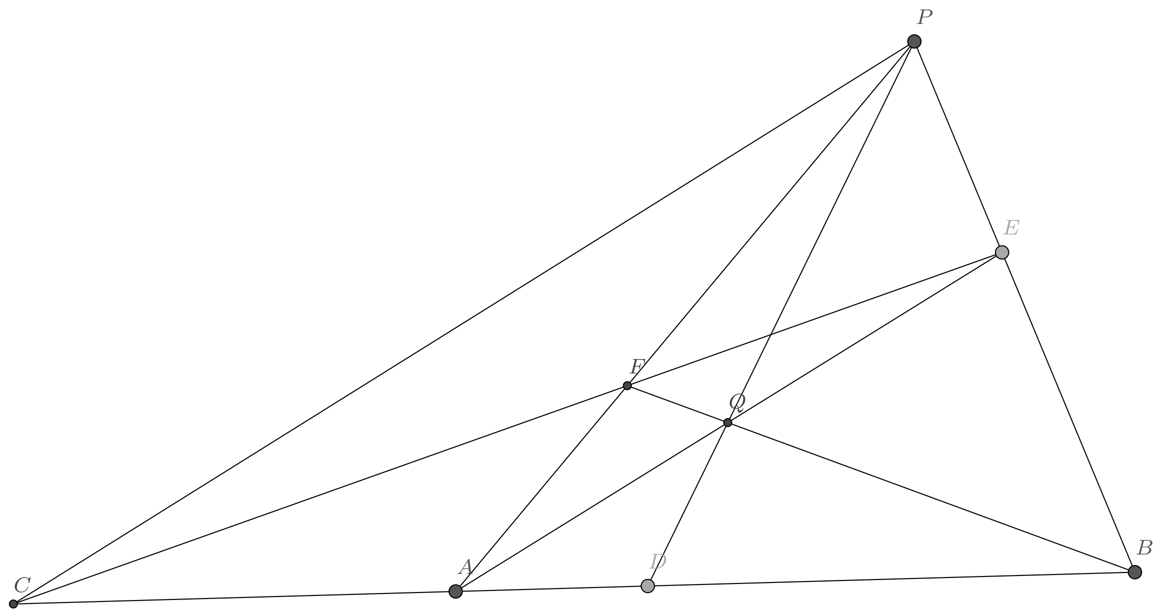
Big guns in geometry

5 March 2016

At this stage you should be familiar with ideas and tricks like angle chasing, similar triangles and cyclic quadrilaterals (with possibly some ratio hacks). But at the IMO level there is no guarantee that these techniques are sufficient to solve contest geometry problems. It is therefore timely to learn more identities and tricks to aid our missions.

1 Harmonics.

Ever thought of the complete quadrilateral below?



By Ceva's theorem we have $\frac{EB}{EP} \cdot \frac{FP}{FA} \cdot \frac{AD}{DB} = 1$ and by Menelaus' theorem, $\frac{EB}{EP} \cdot \frac{FP}{FA} \cdot \frac{AC}{CB} = -1$ (note: the negative ratio simply denotes that C is not on segment AB.) We therefore have:

$$\frac{AD}{DB} : \frac{AC}{CB} = -1.(*)$$

We call any family of four *collinear* points $(A, B; C, D)$ satisfying (*) a **harmonic bundle**.

A pencil $P(A, B, C, D)$ is the collection of lines PA, PB, PC, PD. We name it a **harmonic pencil** if $(A, B; C, D)$ is harmonic (so the line $P(A, B, C, D)$ above is indeed a harmonic pencil). As $\frac{AD}{DB} = \frac{PA}{PB} \cdot \frac{\sin\angle APD}{\sin\angle BPD}$ and $\frac{AC}{CB} = \frac{PA}{PB} \cdot \frac{\sin\angle APC}{\sin BPC}$ we know that

$|\dfrac{AD}{DB} : \dfrac{AC}{CB}| = \dfrac{\sin\angle APD}{\sin\angle BPD} : \dfrac{\sin\angle APC}{\sin\angle BPC}$ we know that $\dfrac{\sin\angle APD}{\sin\angle BPD} = \dfrac{\sin\angle APC}{\sin\angle BPC}$ iff $P(A,B,C,D)$ is harmonic pencil (assumming that PC and PD are different lines, of course).

These imply that for a harmonic bundle $(A,B;C,D)$ and a point P not on the line connecting A,B,C,D, if a line ℓ intersects PA,PB,PC,PD at A',B',C',D', then $(A',B';C',D')$ is harmonic.

Let's turn back to the diagram above. We can see that if PD,BF,AE are concurrent at Q and A,F,E are collinear then $(A,B;C,D)$ is harmonic. We can generalize above to the following:
Let triangle PAB with points D,E,F on sides AB,BP,PA. Let C be a point on line AB. Then any two of the three below imply the third:

1. $(A,B;C,D)$ is harmonic.
2. PD,BF,AE are concurrent.
3. C,F,E are collinear.

Notice that if EF is parallel to AB then C is actually 'point of infinity', so $\frac{CA}{CB} \to -1$. D is therefore the midpoint of AB if $(A,B;C,D)$ is harmonic.

Two other lemmas.
• If C,A,D,B are collinear in that order and M is th midpoint of AB, then $(A,B;C,D)$ is harmonic $\Leftrightarrow CA\cdot CB = CD\cdot CM \Leftrightarrow MA^2 = MC\cdot MD$.
• Points (C,A,B,D) lie on a line in this order. P is a point not on on this line. Then any two of the following conditions imply the third:

1. $(A,B;C,D)$ is harmonic.
2. PD is the angle bisector of $\angle CPD$.
3. $AP \perp PB$.

Harmonic quadrilateral. A cyclic quadrilateral $ABCD$ is named harmonic quadrilateral if $\dfrac{AB}{BC} = \dfrac{AD}{DC}$. Some facts about this quadrilateral:

1. Let ω be the circle circumsribing $ABCD$. Then point of intersection of tangents to ω at A and C lie on BD. Likewise, the point of intersection of tangents to ω at B and D lie on AC.
2. Let P be any point on ω. Then $1 = \dfrac{AB}{BC} : \dfrac{AD}{DC} = \dfrac{\sin APB}{\sin BPC} : \dfrac{\sin APD}{\sin DPC}$. Therefore, $P(A,C;B,D)$ is harmonic.
3. Let Q be the common point of line AC, tangent to ω at B and tangent to ω at D.Let $R = AC\cap BD$. Then $(Q,R;A,C)$ is harmonic.

2 Poles and polars.

Given a circle ω with center O and radius r and any point $A \neq O$. Let A' be the point on ray OA such that $OA\cdot OA' = r^2$. The line l through A' perpendicular to OA is called the polar of A with respect to ω. A is called the pole of l with respect to ω. Notice that if A lies outside ω and C and D are on ω such that AC and AD are tangent to ω, then $CD \equiv l$, the polar of A.

La Hire's Theorem: A point X lies on the polar of a point Y with respect to a circle ω. Then Y lies on the polar of X with respect to ω. From this identity, we know that if points A and B have polars l_A and l_B, then the pole of line AB is the point of intersection of l_A and l_B.

Brokard's Theorem: The points A, B, C, D lie in this order on a circle ω with center O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of $\triangle PQR$. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to ω.

3 Other heavy machineries.

• **Pascal's Theorem**: Given a hexagon $ABCDEF$ inscribed in a circle, let $P = AB \cap ED$, $Q = BC \cap EF$, $R = CD \cap AF$. Then P, Q, R are collinear. (An easy way to remember - the three points of intersection of pairs of opposite sides are collinear).

Note: Points A, B, C, D, E, F do not have to lie on the circle in this order.

textbf Note: It is sometimes useful to use degenerate versions of Pascal's Theorem. For example if $C \equiv D$ then line CD becomes the tangent to the circle at C.

• **Brianchon's Theorem:** Given a hexagon $ABCDEF$ circumscribed about a circle, the three diagonals joining pairs of opposite points are concurrent, i.e. AD, BE, CF are concurrent.

Note: It is sometimes useful to use degenerate versions of Brianchon's Theorem. For example if $ABCD$ is a quadrilateral circumscribed about a circle tangent to BC, AD at P, Q then PQ, AC, BD are concurrent.

• **Desargues' Theorem:** Given two triangles $A_1B_1C_1$ and $A_2B_2C_2$ we say that they are perspective with respect to a point when A_1A_2, B_1B_2, C_1C_2 are concurrent. We say that they are perspective with respect to a line when $A_1B_1 \cap A_2B_2$, $A_1C_1 \cap A_2C_2$, $C_1B_1 \cap C_2B_2$ are collinear. Then two triangles are perspective with respect to a point iff they are perspective with respect to a line. Proof? Use Menelaus' theorem for a few times.

• **Sawayama-Thebault's Theorem:** A point D is on side BC of $\triangle ABC$. A circle ω_1 with centre O_1 is tangent to AD, BD and Γ, the circumcircle of $\triangle ABC$. A circle ω_2 with centre O_2 is tangent to AD, DC and Γ. Let I be the incentre of $\triangle ABC$. Then O_1, I, O_2 are collinear.

4 Homothety: triangles and circles.

Refer to Desargues' theorem above: what happened if $A_1B_1 \parallel A_2B_2$, $A_1C_1 \parallel A_2C_2$, $C_1B_1 \parallel C_2B_2$? The fact is, as long as the triangles are not congruent, the lines A_1A_2, B_1B_2, C_1C_2 are still concurrent. (If congruent and are of similar orientation then the three lines are parallel!) The point P at which they concur is called the centre of homothety.

Definition. Homothety under a center P and ratio k is the mapping of any point A to A' in a plane such that $A'P = k \cdot AP$, and A, P, P' collinear. (Again, if $k < 0$ then A' and A are at the different side of P).

Fun fact 1: If $\triangle A'B'C'$ is the image of $\triangle ABC$ under some homothety, then all relevant points of $\triangle A'B'C'$ (e.g. centroid, orthocentre, and even circumcircle) is the image of of the corresponding point of $\triangle ABC$.

Fun fact 2: Denote $\triangle A'B'C'$ and $\triangle ABC$ as above. Then if the centre of homothety lies on one relevant line (e.g. Euler's line, $A-$ altitude) of $\triangle A'B'C'$, then this corresponding relevant line of $\triangle ABC$ coincides with the former; otherwise, two such lines are parallel to each other.

Consider two circles ω_1, ω_2 with centres O_1, O_2. There are two unique points P, Q, such that a homothety with centre P and positive coeffcient carries ω_1 to ω_2, and a a homothety with centre Q and negative coeffcient carries ω_1 to ω_2. P is called the exsimilicentre, and Q is called the insimilicentre ofω_1, ω_2.We assumed that the two circles are not concentric (otherwise the only centre of simlarity is the centre!) and not congruent (otherwise the exsimilicenter is the point of infinity).

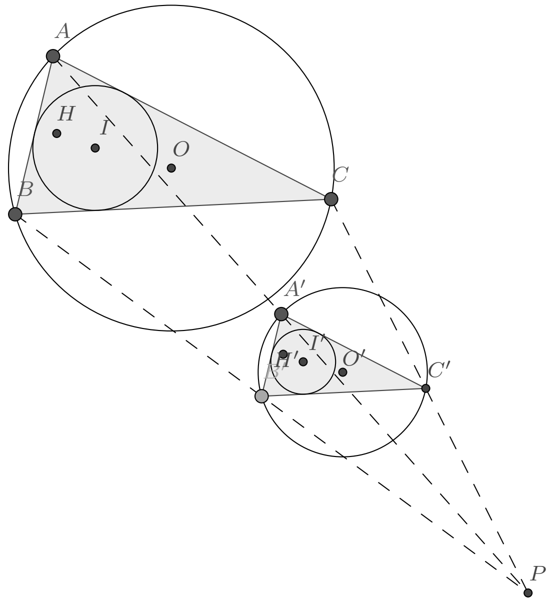


Figure showing how triangles, relevant points and circles can be homothetic.

Some useful facts:

1. P is the intersection of external tangents to ω_1, ω_2. Q is the intersection of internal tangents to ω_1, ω_2.
2. Let ω_1 to ω_2 intersect at $S;, R$, and PA_1, PA_2 are tangents toω_1 to ω_2 so that A_1, A_2 are on the same side of O_1O_2 as S. Then PR is tangent to $\triangle A_1RA_2$.
3. $(P, Q; O_1, O_2)$ is harmonic. (Why?)
4. **Monge's Theorem:** Given three circles $\omega_1, \omega_2, \omega_3$. Then the exsimilicentres of ω_1, ω_2, of ω_1, ω_3, and of ω_2, ω_3 are collinear.
   Proof: Let O_1, O_2, O_3 be the centres of the circles. Let K_1 be the intersection of the common tangents of ω_1, ω_2 and ω_1, ω_3. Define K_2, K_3 similarly. Then K_iA_i is the angle bisector of $\angle K_i$ in $\triangle K_1K_2K_3$. Hence K_1A_1, K_2A_2, K_3A_3 are concurrent. The result follows by Desargues' theorem.
   A proof without using Deargues' theorem: use Menelaus' theorem to the triangle formed by the centres of the three circles.
5. **Monge-d' Alembert's theorem:** Given three circles $\omega_1, \omega_2, \omega_3$. Then the insimilicentres of ω_1, ω_2, of ω_1, ω_3, and the exsimilicentre of ω_2, ω_3 are collinear.
6. If the exsimilicentre of two circles ω_1 and ω_2 lies on the circumference of ω_1 then ω_1 is internally tangent to ω_2, and the exsimicenter is the point of tangency of the circles.
7. If the insimilicentre of two circles ω_1 and ω_2 lies on the circumference of ω_1 then ω_1 is externally tangent to ω_2, and the insimicenter is the point of tangency of the circles.

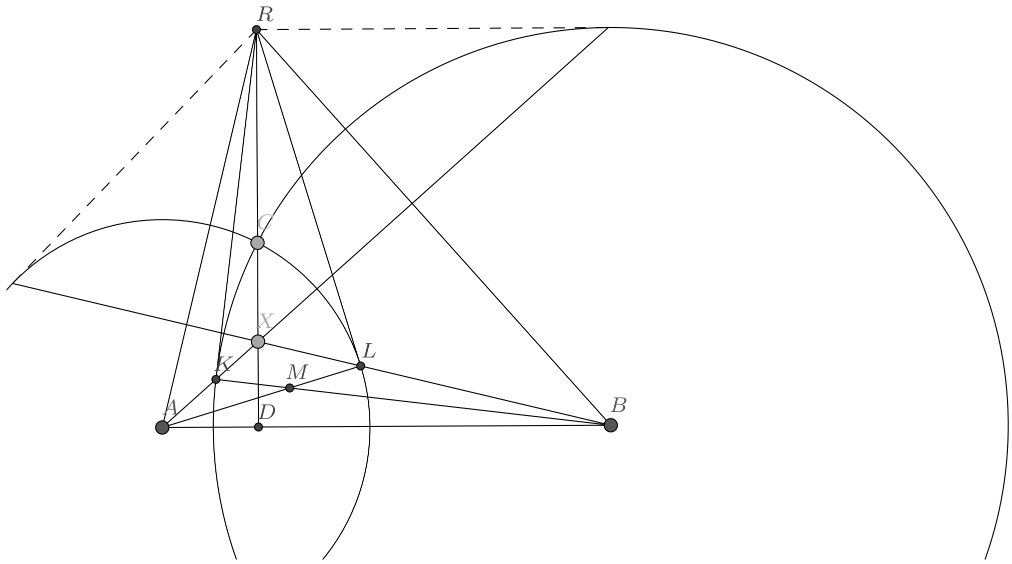
5 Examples.

1. *IMO 2012, #5.* Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

   Show that $MK = ML$.
   Solution.

   Denote ω_A as the circle with centre A and radius AC, and ω_B as the circle with centre B and radius BC. Now considering ω_A and letting the pole of BX be R, we know that the pole of CD is B, so by La Hire's theorem R lies on CD. By La Hire's theorem the polar of X contains both B (as $X \in CD$) and R (polar of R is BX) and by definition this polar is perpendiular to AX. We therefore have $RX \perp AB$ and $RB \perp AX$, so R is the orthocentre of triangle ABX, and RL is tangent to BX (by definition again).

   By similar reasoning, R is the pole of AX with respect to ω_B, and therefore RK is tangent to ω_B. Now that R is on CD, the radical axis of ω_A and ω_B. Thus the lengths of tangent (i.e. the square root of power of point from R to the two circles) are equal, hence $RK = RL$. Now $MK^2 = RM^2 - RK^2 = RM^2 - RL^2 = ML^2$, or $MK = ML$.

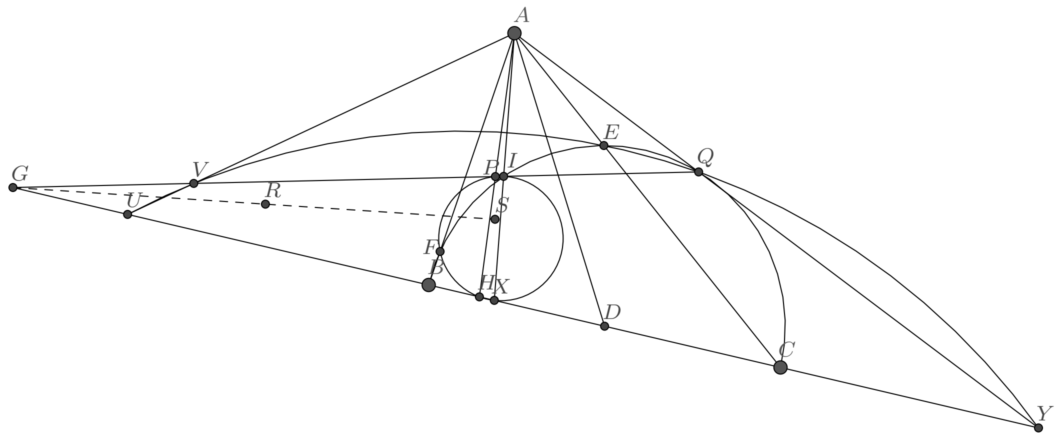


2. *IMO 2004, G8.* Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.
   Solution. Let P be the second point of intersection between CD and the circle (ABM), and let $G = AB \cap CD$. A very simple computation, based on the fact that $GD \cdot GC = GA \cdot GB = GM \cdot GP$ and M is the midpoint of CD will show that P is, in fact, the harmonic conjugate of C, D art G, so it belongs to EF.

   $ANBM$ is a harmonic quadrilateral, so $(PA, PB; PN, PM)$, or $P(A, B; N, G)$ is harmonic pencil. Now let $EF \cap AB = H$, we have $(G, H; A, B)$ harmonic, so $P(G, H; A, B)$, or $P(G, E; A, B)$ is a harmonic pencil too. This means lines PH and PN coincide, and we are done. ■

3. *JOM 2013, G7.* Given a triangle ABC, let l be the median corresponding to the vertex A. Let E, F be the feet of the perpendiculars from B, C to AC, AB. Reflect the points E, F across l to the points P, Q. Let AP, AQ intersect BC at X, Y. Let Γ_1, Γ_2 be the circumcircles of $\triangle EQY, \triangle FPX$.

   Prove that A lies on the line connecting the centers of Γ_1, Γ_2.
   Solution. Here $a \cap b$ refers to the intersection between objects a and b. Let $\Gamma_1 \cap BC = U, Y$, $\Gamma_1 \cap PQ = V, Q$. Let $\Gamma_2 \cap BC = H, X$, $\Gamma_2 \cap PQ = I, P$. Let O_1, O_2 be the centers of the circles Γ_1, Γ_2. Let D be the midpoint of BC. Let $PQ \cap BC = G$.



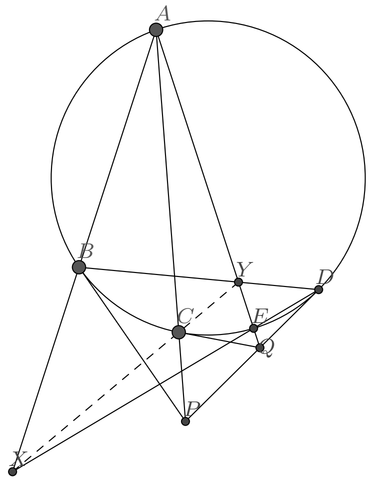
   Firstly, notice that $EP \perp AD$ and $AE \perp BE$, thus $\angle EAD = 90° - \angle AEP = \angle PEB = \angle AFP = \angle AQE = \angle EUY = \angle EUD$, from which it follows that A, E, D, U are concyclic. It follows immediately that

$$\angle AUY = \angle AUD = \angle DEC = \angle DCE = \angle AFE = \angle AQP = \angle VUY$$

   which proves that A, U, V are collinear. Now note that if $UQ \cap VY = R$, then since the complete quadrilateral formed by U, V, Q, Y, G, A has V, U, Q, Y concyclic on Γ_1, then by Brokard's Theorem, GR is the polar of A with respect to Γ_1. Analogously, if $IX \cap PH = S$, then GS is the polar of A with respect to Γ_2. But since $(GA, GP; GS, GX)$ and $(GA, GQ; GR, GY)$ are both harmonic pencils, G, R, S are collinear.

   Since by definition $AO_1 \perp GR$, $AO_2 \perp GS$, it follows immediately that AO_1O_2 must be collinear, and we're done. ■

4. *APMO 2013, #5.* Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.
   Solution. The problem is equivalent to proving BE, AD, CQ are collinear. If we denote $X = DE \cap AB$, $Y = AE \cap BD$ then by Brokard's theorem Y is the polar of the point $BE \cap AD$ so the problem statement of proving that CQ contains this point is equivalent of proving that XY contains the pole of CQ, which is C (all pole and polar relations are w.r.t. ω.)



Recall from the trigonometry notes that (don't do this is contests–the trigonometric identity isn't well known!) it suffices to prove that $\frac{\sin\angle AXY}{\sin\angle DXY} = \frac{\sin\angle AXC}{\sin\angle DXC}$. Now from $\angle XAY = \angle XDY$ we have $\frac{\sin\angle AXY}{\sin\angle DXY} = \frac{AY\cdot\sin\angle XAY}{DY\sin\angle XDY} = \frac{\sin\angle ADY}{\sin\angle DAY} = \frac{AB}{DE}$. $\frac{\sin\angle AXC}{\sin\angle DXC} = \frac{AC\cdot\sin\angle XAC}{CD\cdot\sin\angle XDC} = \frac{AC\cdot BC}{CD\cdot CE}$. Therefore it is clear that we what we need is $AC\cdot BC\cdot DE = CD\cdot CE\cdot AB$. Now since $ABCD$ and $ACED$ are harmonic quadilaterals, $CD\cdot AB{=}BC\cdot AD$ and $AC\cdot ED{=}AD\cdot EC$. Therefore $AC\cdot BC\cdot DE{=}AD\cdot BC\cdot EC{=}AB\cdot CD\cdot EC$. ■

5. *IMO 2007, G8.* Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

   Solution. Let J be the centre of the circle Γ tangent to sides AB, BC, AD (or the intersection of internal angle bisectors of $\angle BAD$ and $\angle ABC$). Then we prove that E and F are the exsimilicentres and insimilicentre of ω and Γ, so they have to lie on line IJ!

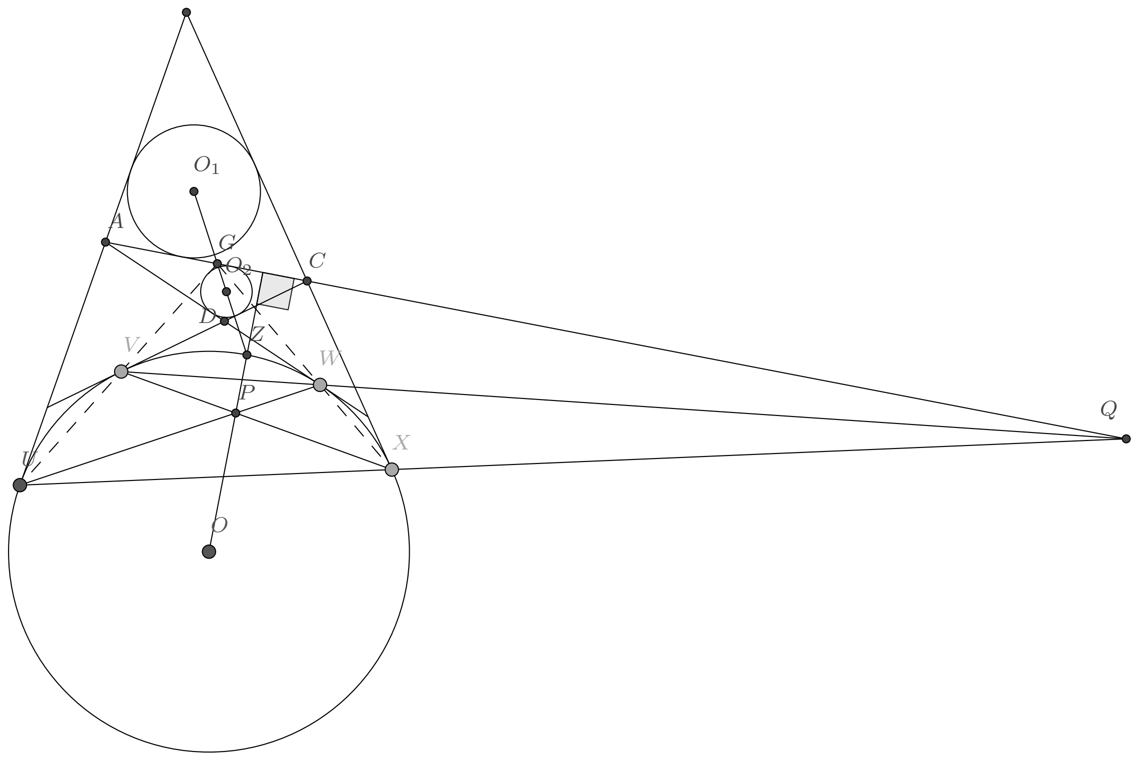
   Denote incircles of APD and BPC as ω_1 and ω_2, respectively. As incircles of triangles APD and CPD are tangent to each other, there exists a circle tangent to sides AP, PC, CD, DA (the reverse of this identity holds true, and is well known. However, as an exercise try to prove it!) Name the circle Γ_1. Similarly there is a circle Γ_2 inscribed in quadrilateral $PBCD$.

   Now, A is the exsimilicentre of Γ_1 and Γ; C is the exsimilicentre of Γ_1 and ω. Then by Monge's theorem the exsimilicentre of ω and Γ lies on AC. Similarly this exsimilicentre lies on BD (by considering exsimilicentres of ω, Γ and Γ_2). The exsimilicentre must then be $AC \cap BD = E$). Next, A is the exsimilicentre of ω_1 and Γ, K is the insimilicentre of ω_1 and ω. By Monge d'Almebert's theorem the insimilicentre of ω and Γ lies on AK. Similarly this insimilicentre lies on BL. Therefore, $AK \cap BL$ is the insimillicentre of ω and Γ. ■

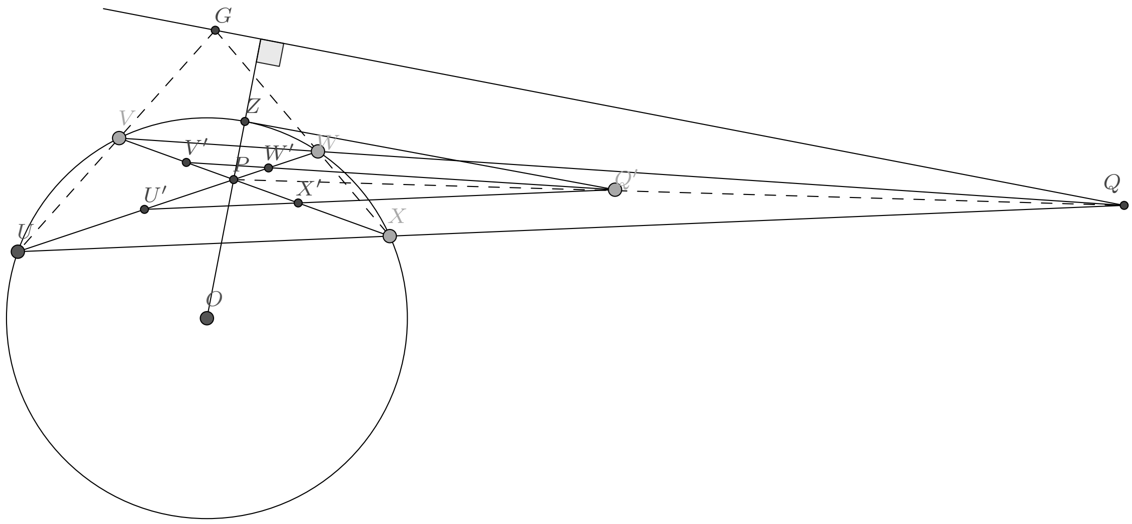
6. *IMO 2008, #6.* Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect

on ω.

Solution. Throughout the problem the pole-polar relation is with respect to ω. Denote $G = AC \cap BD$. Also denote the point of tangency of lines AB, CD, AD, BC as U, V, W, X respectively. We prove that G is th intersection of UV and WX. Indeed, the polar of A is UW and the polar of C is VX, so the pole of AC is $UW \cap VX$, namely P. In a similar way we deduce that the pole of BD is $UX \cap VW = Q$. So the polar of $G = AC \cap BD$ is PQ. By Brokard's theorem we know that the UV and WX intersect at the polar of PQ, which is G. (note the profuse usage of La Hire's theorem!) Also that pole of AC (i.e. P) lies on the pole of Q (i.e. PG), we know that Q lies on AC (La Hire's theorem again).



By Monge D' Alembert's theorem we get that exsimilicenter of ω and ω_1 (i.e. B), insimilicenter of ω and ω_2 (i.e. D), and insimilicenter of ω_1 and ω_2 is collinear. But AC is a common inner tangent of ω_1 and ω_2, we infer that G is itself the insimilienter of ω_1 and ω_2. Also that, if we denote O_1 and O_2 as centers of ω_1 and ω_2 and Z as the exsimilicenter we desire, then $(Z, G; O_1, O_2)$ is harmonic bundle and $(OZ, OG; OO_1, OO_2)$ is a harmonic pencil. Since $OO_1 \perp UX$, $OO_2 \perp VW$ $OG \perp PQ$ (pole and polar relation) and $(QG, QP; QU, QV)$ (or $(AC, QP; UX, VW)$) is harmonic pencil, we have $OZ \perp AC$. and since $OP \perp AC$ (P= pole of AC), P, O, Z collinear.



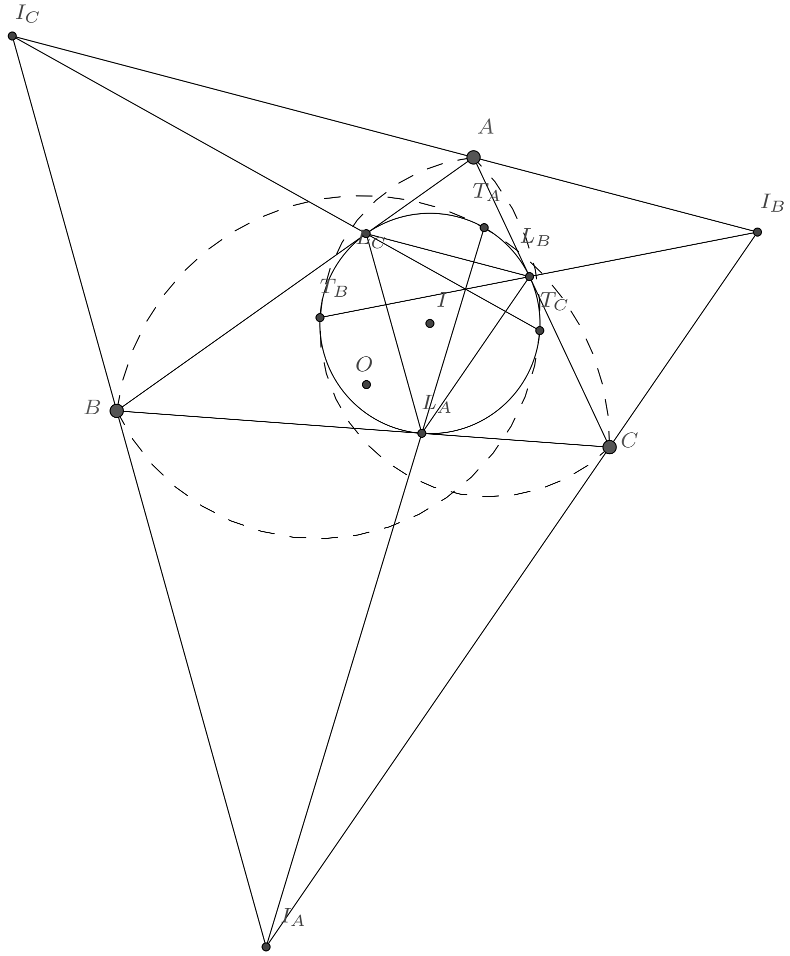
Now we try to locate the polars of O_1 and O_2. As B, O, O_1 are collinear, the polars of O_1 and B (i.e. UX) are collinear. Moreover, the pole of AO_1, U', lies on polar of A which is UW, and with AO_1 bisects $\angle BAC$, $AB \perp OU$, $AO_1 \perp OU'$ and $AC \perp OP$ we have OU' bisects $\angle UOP$. So the polar of O_1 is the line passing through U' and parallel to VX. Let this line intersect VX at X' and we have $\dfrac{UU'}{U'P} = \dfrac{XX'}{X'P} = \dfrac{r}{d}$ where r is the radius of ω and d is the distance OP (so AC is of distance $\dfrac{r^2}{d}$ from O.) Similarly if the polar of O_2 intersect VX and UW at V' and W' respectively then $V'W' \parallel VW$ and $\dfrac{VV'}{V'P} = \dfrac{WW'}{W'P} = \dfrac{r}{d}$. So $Q' = U'X' \cap V'W'$ is the pole of O_1O_2. Also notice that $U'V'W'X'$ is the image of $UVWX$ under the homothety at center P and ratio $\dfrac{d}{r+d}$, if we draw the line parallel to AC from P (namely l_1) and Q' (namely l_2) we see that $l_1, l_2 and AC$ are in that order, with l_1 at distance d from O (remember that $AC \perp OP$ and $l_1 \perp OP$) and AC at distance $\dfrac{r^2}{d}$ from O. So l_2 at distance $d + (\dfrac{r^2}{d} - d)(\dfrac{d}{r+d}) = d + (r-d) = r$ from O, i.e. l_2 is tangent to ω. ■

Finally, as OP and l_2 meet at the point of tangency of l_2 and W' this point is also on the polar of Q' (recall that l_2 passes through Q'). But with $Z \in O_1O_2$ and O_1O_2 has pole Q' we have the polar of Q' contain Z. So Z is indeed the tangency point, or in other words, $Z \in \omega$.

7. *RMM 2012, #6.* Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.
Solution. Lines AA', BB', CC' all pass through the radical centre of ω_A, ω_B and ω_C; the fact that the three lines of concurrent will be assumed below. Let T_A, T_B, T_C be the points of tangency of the incircle of triangle ABC (namely ω) and circles $\omega_A, \omega_B, \omega_C$, respectively. Clearly the tangent to ω at T_A is also tangent to ω_A, and is therefore the radical axis of these two circles. Similarly the tangent to ω to T_B is the radical axis of

ω and ω_B. Denote Z_C as the intersection of the two tangents which is also the radical centre of ω, ω_A and ω_B. Since C lies on both ω_B and ω_A, it is itself on the radical axis of these two circles. So C, C' and Z_C all lie on the radical axis of ω_A and ω_B.

Now let L_A, L_B and L_C be points of tangency of ω to BC, CA and AB respectively. We need to following lemma: T_AL_A, T_BL_B and CZ_C are concurrent. Why? By Pascal's theorem we have: $Z_C = T_AT_A \cap T_BT_B$, $T_AL_A \cap T_BL_B$ and $T_AL_B \cap T_BL_A$ are concurrent by considering the ordered pairs (T_A, T_B, L_B) and (T_B, T_A, L_A). By considering ordered pairs (L_A, T_B, L_B) and (L_B, T_A, L_A) we have $C = L_AL_A \cap L_BL_B$, $T_AL_A \cap T_BL_B$ and $T_AL_B \cap T_BL_A$. Now get it? C and Z_C both lie on the line pssing through $T_AL_A \cap T_BL_B$ and $T_AL_B \cap T_BL_A$, hence are collinear with $T_AL_A \cap T_BL_B$. With lines AA', BB', CC' concurrent we know that the required point is indeed $L_AT_A \cap L_BT_B \cap L_CT_C$ (these three lines all concur at the radical canter of ω_A, ω_B and ω_C.



We proceed with the following:

Lemma 2: Line L_AT_A passes through the excentre of $\triangle ABC$ opposite A (namely I_A; define I_B and I_C similarly); similar condition for L_BT_B and L_CT_C.
Proof: Now let the perpendicular from I_A to BC be L'_A, we know that L_A and L'_A are reflections of each other in BC (why? We have done this in the trigonometry lecture in the previous camp; what you need to see is that I, B, C and II_A all on the circle with diameter II_A, namely Γ!) This implies that the midpoint M of $L_AL'_A$ lie on the perpendicular bisector of BC. Now, let K be another intersection of $L_AL'_A$ and Γ and L another intersection of $L_AL'_A$ and ω then $\angle IKI_A = 90°$. We then obtain K as the

midpoint of LL_A (IK the perpendicular bisector of LL_A.) Moreover, by the power of point theorem $BL_A \cdot CL_A = KL_A \cdot L_AI_A = 2KL_A \cdot \frac{1}{2}L_AI_A = LL_A \cdot LL_AM$, implying that $LBMC$ is cyclic and with $MB = MC$ LL_A bisects $\angle BLC$. Finally, let LB and LC intersect ω again at X and Y respectively; from th angle bisector condition we know that $L_AB = L_AC$ and $XY \parallel BC$, the tangent to ω at L_A. Now triangles LXY and LBC are similar, so they are homothetic at centre L. Same goes to their circumcircles and we conclude that ω and circle $LBMC$ is tangent to another. ω_A must therefore be the circumcircle of $LBMC$ and we conclude that $T_A = L$. and T_A, L_A, I_A collinear. (Theoretically, there are two circles passing through both B and C, and is tangent to ω. However, one such circle has been degenerated to line BC.

We are (finally) ready to present our proof, that is I_AL_A, I_BL_B and I_CL_C concur at IO. Now L_AL_B is the polar of C w.r.t. ω, hence perpendicular to IC. But I_AC and I_BC are both perpendicular to IC, so $I_AI_B \parallel L_AL_B$. In a similar manner for the other sides we have triangles $I_AI_BI_C$ and $L_AL_BL_C$ homothetic, so the whole problem statement becomes proving that the center of homothety lies on IO. Now for triangle $I_AI_BI_C$, I is its orthocenter (lies on Euler line) and O is the nine-point circle (also on Euler's line), and the Euler's lines of $L_AL_BL_C$ is parallel or will coincide with that of $I_AI_BI_C$, which is IO. But they have a common point I since I is the circumcenter of $L_AL_BL_C$, the two Euler lines coincide. This means the center of homothety lies on the Euler's line, which is IO. Q.E.D. (Phew!)

6 Practice problems.

1. Let $ABCD$ be a quadrilateral circumscribed around a circle ω. Let E, F, G, H be the point of tangency of lines AB, BC, CD, DA with ω. Prove that lines (diagonals) AC, BD, EG, FH are concurrent.

2. Starting from the previous practice problem, prove Brokard's theorem.

3. Two circles ω_1 and ω_2 are internally tangent at P (with ω_2 lying inside ω_1). Let l be a line tangent to ω_2 at Q and intersect ω_1 at points A and B. Let PQ intersect ω_1 again at M. Prove that $MA = MB$.

4. *JOM 2013, G2.* Let ω_1 and ω_2 be two circles, with centres O_1 and O_2 respectively, intersecting at X and Y. Let a line tangent to both ω_1 and ω_2 at A and B, respectively. Let E, F be points of O_1O_2 such that XE tangent to ω_1 and YF is tangent to ω_2. Let $AE \cap \omega_1 = A, C$ and $BF \cap \omega_2 = B, D$. Show that line BO_2 is tangent to the circumcircle of $\triangle ACD$ and line AO_2 is tangent to circumcircle of $\triangle BCD$.

5. *IMO 2005, G6.* Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

6. *RMM 2013, #3.* Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another.

7. *IMO 2014, G7 (lemma).* Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V , respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y . Prove that if the points I, X, and Y are collinear, then $VA = VC$.
   Note: The original problem asks that, if W and Z be the midpoints of AX and BC, respectively, then the points I, W, and Z are also collinear. Try it if you dare (after proving the lemma above)!

8. *RMM 2010, #3.* Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_1 to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i, A_iA_{i+1}$ and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_0 = A_4, A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

9. *RMM 2011, #3.* A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

7 References.

1. Alexander Remorov: 2010, *Projective Geometry-Part 2.* Canadian IMO Training.

2. Lindsey Shorer: 2015, *Summary of Some Concepts in Projective Geometry.* Canadian IMO Training.

3. Cosmin Phohata: 2012, AMY 2011-2012, *Segment 5, Homothety and Inversion.* AwesomeMath LLC.

7.1 Problem credit.

1. IMO 2004 (shortlist; solution's credit to an AoPS user), 2005 (shortlist), 2007 (shortlist and solution), 2008, 2012, 2014 (shortlist).

2. APMO 2013.

3. Romanian Masters in Mathematics (2010, 2011, 2012, 2013).

4. Junior Olympiad of Mathematics 2013 (solution of G7 is creditted to Justin, the problem proposer).