Putnam 2015

A1 Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Solution. Let the coordinates of A to be $(x_A, \frac{1}{x_A})$ and the coordinates of B to be $(x_B, \frac{1}{x_B})$. Same goes for $(x_P, \frac{1}{x_P})$. Thus the area is given by:

$$\frac{1}{2}\left|\frac{x_A}{x_B} + \frac{x_B}{x_P} + \frac{x_P}{x_A} - \frac{x_B}{x_A} - \frac{x_P}{x_B} - \frac{x_A}{x_P}\right|$$

Ignoring absolute value, differentiating with respect to x_P we get that any stationary point happens when $(x_B - x_A)\left(\frac{1}{x_P^2} - \frac{1}{x_A x_B}\right) = 0$. This happens when $x_P = \pm\sqrt{x_A x_B}$. W.l.o.g. we assume that both x_A and x_B are both positive, and thus x_P must also be positive. Thus $x_P = \sqrt{x_A x_B}$. Also w.l.o.g. we assume that $x_A < x_P < x_B$. Since the area is the lowest possible (i.e. 0) when $x_P = x_A$ or $x_P = x_B$, and positive at other times, and also since $x_P = \sqrt{x_A x_B}$ is the only stationary point, this area must be nondecreasing in the interval $x_P \in (x_A, \sqrt{x_A x_B})$ and nonincreasing in the interval $x_P \in (\sqrt{x_A x_B}, x_B)$, we know that $x_P = \sqrt{x_A x_B}$ is indeed the point where the area attains the maximum. Now the area of bounded by the hyperbola and the chord AP is given by the following:

$$\frac{1}{2}(x_P - x_A)\left(\frac{1}{x_P} + \frac{1}{x_A}\right) - \int_{x_A}^{x_P} \frac{1}{x}dx = \frac{1}{2}\left(\frac{x_P}{x_A} - \frac{x_A}{x_P}\right) - (\ln x_P - \ln x_A)$$

substituting $x_P = \sqrt{x_A x_B}$ we get

$$\frac{1}{2}\left(\frac{\sqrt{x_A x_B}}{x_A} - \frac{x_A}{\sqrt{x_A x_B}}\right) - (\ln\sqrt{x_A x_B} - \ln x_A) = \frac{1}{2}\left(\sqrt{\frac{x_B}{x_A}} - \sqrt{\frac{x_A}{x_B}}\right) - \frac{1}{2}(\ln x_B - \ln x_A)$$

Similarly the area bounded by PB and the hyperbola is given by

$$\frac{1}{2}(x_B - x_P)\left(\frac{1}{x_P} + \frac{1}{x_B}\right) - \int_{x_P}^{x_B} \frac{1}{x}dx = \frac{1}{2}\left(\frac{x_B}{x_P} - \frac{x_P}{x_B}\right) - (\ln x_B - \ln x_P)$$

and since $x_P = \sqrt{x_A x_B}$ we get

$$\frac{1}{2}\left(\frac{x_B}{\sqrt{x_A x_B}} - \frac{\sqrt{x_A x_B}}{x_B}\right) - (\ln x_B - \ln\sqrt{x_A x_B}) = \frac{1}{2}\left(\sqrt{\frac{x_B}{x_A}} - \sqrt{\frac{x_A}{x_B}}\right) - \frac{1}{2}(\ln x_B - \ln x_A)$$

hence showing that they have the same area.

A2 Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$.

Find an odd prime factor of a_{2015}.

Answer. 181.
Solution. The characteristic polynomial of this recurrence equation is $x^2 - 4x + 1 = 0$, which has roots $\frac{4\pm\sqrt{4^2-4}}{2} = 2 \pm \sqrt{3}$. Thus $a_n = a(2+\sqrt{3})^n + b(2-\sqrt{3})^n$, and since $a + b = 1$ and $a(2+\sqrt{3}) + b(2-\sqrt{3}) = 2$, we get $a = b = \frac{1}{2}$. Thus we have $a_n = \frac{1}{2}((2-\sqrt{3})^n + (2+\sqrt{3})^n)$. Now $\frac{a_{2015}}{a_5} = \sum_{i=0}^{402}(-1)^i(2-\sqrt{3})^{5i}(2-\sqrt{3})^{2010+5i}$, and since both a_{2015} and a_5 are both integers, this expression must also be a rational number. From the right hand side we can also deduce that this ratio is in the form of $x + y\sqrt{3}$ with x, y both integers and since $x+y\sqrt{3} \in \mathbb{Q}$, $y = 0$ hence $\frac{a_{2015}}{a_5}$ is actually an integer. So it suffices to find a prime factor of a_5. Finally, since $a_5 = 362 = 2 \times 181$ and 181 is a prime, this is a possible answer.

A3 Compute

$$\log_2 \left(\prod_{a=1}^{2015} \prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right) \right)$$

Here i is the imaginary unit (that is, $i^2 = -1$).

Answer.
Solution. Since $e^x = e^{2\pi i + x}$, we will consider everything in the cycle of $2\pi i$. In this context, if $ab \equiv k \pmod{2015}$, and let $ab - k = 2015c$ with c an integer, then $e^{2\pi i ab/2015} = e^{2\pi i(k+2015c)/2015} = e^{2\pi i k/2015} e^{2\pi i c} = e^{2\pi i k/2015}$. Thus we can consider everything modulo 2015.

Let $d = \gcd(a, 2015)$. Then $2015|ab$ if and only if $c = \frac{2015}{d}|b$. In addition, $\{a, 2a, \cdots ca\} = \{d, 2d, \cdots, cd = 2015\}$ in modulo 2015. Thus we have

$$\prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right) = \left(\prod_{b=1}^{c} \left(1 + e^{2\pi i bd/2015}\right) \right)^d = \left(\prod_{b=1}^{e} \left(1 + e^{2\pi i b/c}\right) \right)^d$$

Bearing in mind that c is odd, we now investigate this sum. Now, it is given that $\prod_{b=1}^{c} \left(x - e^{2\pi i b/c}\right) = x^c - 1$, since $e^{2\pi b/c}$ are all the roots of unity for $b = 1, 2, \cdots, c$. Substituting $c = -1$ we get $\prod_{b=1}^{e} \left(-1 - e^{2\pi i b/c}\right) = x^c - 1 = -1 - 1 = -2$ since c is odd. Reversing the sign we get $\prod_{b=1}^{e} \left(1 + e^{2\pi i b/c}\right) = (-2)(-1)^c = 2$. Therefore we have $\prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right) = 2^d$. Summing up we get

$$\log_2 \left(\prod_{a=1}^{2015} \prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right) \right) = \log_2 \left(\prod_{a=1}^{2015} 2^{\gcd(2015, a)} \right) = \sum_{a=1}^{2015} \gcd(2015, a)$$

By the Euler's totient function, there are $\phi(2015) = \phi(5 \cdot 13 \cdot 31) = 4 \cdot 12 \cdot 30 = 1440$ such a's with $\gcd(a, 2015) = 1$. The number of a's with $\gcd(a, 2015) = d$ is $\phi(\frac{2015}{d})$, so this gives the total as

$$\begin{aligned}
\sum_{a=1}^{2015} \gcd(2015, a) &= \sum_{d|2015} \phi(d) \frac{2015}{d} \\
&= 1440 + 4(12 \cdot 30) + 13(4)(30) + 5(13)(30) + 4(13)(31) + 5(12)(31) + 5(13)(30) + 5(13) \\
&= 13725
\end{aligned}$$

A5 Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

Solution. We first eliminate the case where $q = pr$ with $p > 1, r > 1$ and $\gcd(p, r) = 1$. First w.l.o.g. (to make our computations easier) that r is a prime power, say s^k. We first calculate the number M_p of integers a with $0 < a < q/4 = pr/4$ and $\gcd(a, p) = 1$. Notice that $a < pr/4$ if and only if $a/p < r/4$. Consider the numbers in the intervals $[1, p], [p + 1, 2p], \cdots, [(d - 1)p + 1, dp]$ where $d = \lfloor r/4 \rfloor$. Each number mentioned is less than $q/4$, and in each category, there are $\phi(p)$ numbers relatively prime to p. So these sets contributed $\phi(p) \cdot d$ to M_p, which is even since $\phi(p)$ is always even for $p > 2$. It remains to investigate contribution of the interval $[dp, (d+1)p]$ to M_q. Now $dp + k < q/4$ if and only if $k < (r/4 - \lfloor r/4 \rfloor)p$. If $r \equiv 1 \pmod 4$ then the bound is $p/4$, in which case the contribution is precisely N_p. Otherwise, the bound is $3p/4$, and the contribution is precisely $\phi(p) - N_p$. Thus $M_q \equiv N_p \pmod 2$, as always.

To investigate the relation between M_q and N_q, we note that if a number counts into N_q, then it counts into M_q. Conversely, an integer a counts into M_q but not N_q if and only if $a = st$ with $\gcd(t,p) = 1$ and $t < q/4s = ps^{k-1}/4$. To count the number of such t, we notice that the number of such t with $t \le p\lfloor s^{k-1}/4 \rfloor$ is $\lfloor s^{k-1}/4 \rfloor \phi(p)$, which is again even. As of the case above, it remains to consider the contribution of such t in the next set of p numbers. Similar to above, if $s^{k-1} \equiv 1 \pmod 4$ then this contribution is N_p, and if $s^{k-1} \equiv 3 \pmod 4$ then this contribution is $\phi(p) - N_p$. In either case it's congruent to $N_p \pmod 2$. Thus $N_q \equiv M_q - N_p \equiv N_p - N_p = 0 \pmod 2$.

Thus the case of q having more than two primes have been eliminated. If $q = 1$ then $N_1 = 0$, which serves as an edge case. If $q = p^k$ with $k \ge 1$, then N_q is the number of the integers between 1 and $\lfloor q/4 \rfloor$ minus the number of integers in this range and divisible by p. This gives the bound $\lfloor (p^k)/4 \rfloor - \lfloor (p^{k-1})/4 \rfloor$. Letting $p^{k-1} = 4\ell + a$ with $a \in \{1,3\}$ we get $\lfloor (p^{k-1})/4 \rfloor = \ell$ and $\lfloor (p^k)/4 \rfloor = \lfloor p(4\ell + a)/4 \rfloor = \ell p + \lfloor ap/4 \rfloor$. Since p is odd we have $\lfloor (p^k)/4 \rfloor - \lfloor (p^{k-1})/4 \rfloor = \ell(p-1) + \lfloor ap/4 \rfloor \equiv \lfloor ap/4 \rfloor \pmod 2$. If $a = 1$, then $\lfloor p/4 \rfloor$ is odd if and only if $p \equiv 5, 7 \pmod 8$. If $a = 3$, then again writing $p = 8c + d$ we get $\lfloor 3(8c+d)/4 \rfloor = 6c + \lfloor 3d/4 \rfloor \equiv \lfloor 3d/4 \rfloor \pmod 2$ we only need to consider the cases where $d \in \{1,3,5,7\}$, which gives the values $\lfloor 3/4 \rfloor, \lfloor 9/4 \rfloor, \lfloor 15/4 \rfloor, \lfloor 21/4 \rfloor = 0, 2, 3, 5$. Hence only $p \equiv 5, 7 \pmod 8$ satisfies this condition.

B3 Let S be the set of all 2×2 real matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

Answer. $M = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ and $M = \begin{pmatrix} -3a & -a \\ a & 3a \end{pmatrix}$ for any real number a.

Solution. If $M = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ then $M^2 = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}$ which is also in S. Hence now we only consider those M with nonzero common difference.

First, consider $M = \begin{pmatrix} a & a+d \\ a+2d & a+3d \end{pmatrix}$ with d as the common difference, then the characteristic polynomial is $x^2 - (2a+3d)x - 2d^2$, which has discriminant $(2a+3d)^2 + 8d^2 > 0$. Hence M has two real and distinct eigenvalues, which implies that M is diagonalizable. Write $M = PDP^{-1}$ where P is the matrix determined by M's eigenvectors, and D is the diagonal matrix symbolizing the eigenvalues. We proceed with the following claim:

Lemma: If $M^k \in S$ with $k \ge 1$, then $M^k = cM$ for some constant c.
Proof: First, notice that S is closed under matrix addition (that is, if M_1 and M_2 are both in S then $aM_1 + bM_2 \in S$ for all constants a and b). Next, we also have $M^k = (PDP^{-1})^k = PD^kP^{-1}$ with D^k remains diagonal. Suppose that there exist real constants a and b such that $aD + bD^k = I$ with I being the identity matrix. Then $aM + bM^k = P(aD)P^{-1} + P(bD^k)P^{-1} = PIP^{-1} = I$, which is not in S. So in this case, either M or M^k cannot be in S. This happens if the eigenvalues of M and M^k, when each treated as a 2-dimensional vector, is linearly independent. That is, if a, b are the eigenvalues of M, then a^k and b^k are the eigenvalues of M^k and thus $\begin{pmatrix} a & a^k \\ b & b^k \end{pmatrix}$ is linearly independent. To have M and M^k both in S, this matrix $\begin{pmatrix} a & a^k \\ b & b^k \end{pmatrix}$ must be linearly dependent, i.e. $ab^k - a^kb = 0$, or $ab(a^{k-1} - b^{k-1}) = 0$. If $a = 0$ or $b = 0$, then M has determinant 0, which implies that $-2d^2 = \det(M) = 0$, so the common difference is 0, contradiction (this case has been handled in the beginning of the proof). Hence we have $a^{k-1} = b^{k-1}$, which means $|a| = |b|$. The case where $a = b$ means $D = aI$ and so $M = aI \notin S$, so $a = -b$.

Going back to the proof, we now know that the eigenvalues of M are in the form of $e, -e$ for some e. This forces the characteristic polynomial of M to be $x^2 + e^2$, which also implies that $2a+3d=0$ in the beginning. Therefore we have $M = \begin{pmatrix} -3a & -a \\ a & 3a \end{pmatrix}$ for some real number a. To show that this is a valid example, $M^3 = 8a^3 \begin{pmatrix} -3 & -1 \\ 1 & 3 \end{pmatrix}$ which is indeed in S.

B5 Let P_n be the number of permutations π of $\{1,2,\ldots,n\}$ such that

$$|i-j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

for all i,j in $\{1,2,\ldots,n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n, and find its value.

Answer. This value is always 4.

Solution. For each n we denote Q_n as the number of permutations satisfying the conditions $i-j| = 1$ implies $|\pi(i)-\pi(j)| \leq 2$ and $\pi(n) = n$. Fix n, and we consider the number of such permutations when $\pi(n) = k$ for each $k = 1,2,\cdots,n$. When $k = n$ this number is Q_n as defined, and by symmetry this holds true when $k = 1$. Hence we proceed to consider the cases when $\pi(n) = 2,\cdots,n-1$. Now, denote $\pi(n) = k$ and we have we consider any j satisfying $\pi(j) < k$ and $\pi(j+1) > k$. Since $\pi(j+1) - \pi(j) \leq 2$, we must have $\pi(j) = k-1$ and $\pi(j+1) = k+1$. Similarly, if $\pi(j) > k$ and $\pi(j+1) < k$ then we must have $\pi(j) = k+1$ and $\pi(j+1) = k-1$. Since permutation is a bijection, exactly one of the above happens and exactly one j satisfies this condition. Thus the numbers $\pi(1),\cdots,\pi(n-1)$ are partitioned into two consecutive regions, one with values $< k$ and the other $> k$. In other words exactly one of the following holds: $\pi(j) < k$ for all $1 \leq j \leq k-1$ and $\pi(j) > k$ for all $k \leq j \leq n$, or $\pi(j) > k$ for all $1 \leq j \leq n-k$, and $\pi(j) < k$ for $n-k+1 \leq j \leq n-1$. In the first case, $\pi(k-1)$ must be equal to $k-1$ and $\pi(k) = k+1$, so this gives Q_{k-1} ways to arrange $\pi(1),\cdots\pi(k-1)$. We now claim that there's only one way to arrange $\pi(k),\cdots,\pi(n-1)$ given the constraint. To begin with, since $\pi(n) = k$ and $\pi(k) = k+1$ and everything in between has $\pi(j) > k$, we have $\pi(n-1) = k+2, \pi(k+1) = k+3$. Repeating the process gives a unique arrangement given by $\pi(k),\cdots,\pi(n-1) = k+1, k+3, \cdots n-1, n, n-2, \cdots k+2$ for $k \equiv n \pmod 2$, and $\pi(k),\cdots,\pi(n-1) = k+1, k+3, \cdots, n-2, n, n-1, \cdots, k+2$ otherwise. This gives a total of Q_{k-1}. For the second case, similarly, we have Q_{n-k} ways of arranging the first $n-k$ permutation numbers, and exactly 1 way for the next $k-1$ numbers. Thus summing above and considering all k we get, for all $n \geq 2$,

$$P_n = 2Q_n + \sum_{i=2}^{n-1} Q_{k-1} + Q_{n-k} = 2(Q_n + \sum_{i=1}^{n-2} Q_i)$$

Now the desired value becomes $2(Q_{n+5} + \sum_{i=1}^{n+3} Q_i) - 2(Q_{n+4} + \sum_{i=1}^{n+2} Q_i) - 2(Q_{n+3} + \sum_{i=1}^{n+1} Q_i) + 2(Q_n + \sum_{i=1}^{n-2} Q_i) = 2(Q_{n+5} - Q_{n+4} - Q_{n+1} - Q_{n-1})$ To calculate the above, we find an iterative formula for Q_n for all $n \geq 4$. Since $\pi(n) = n$, we have $\pi(n-1) = n-1$ or $n-2$. For $\pi(n-1) = n-1$, Q_{n-1} permutation arises as claimed above. For $\pi(n-1) = n-2$, we can use the argument above to establish that $Q_{n-3} + Q_1$ permutations arise. Thus $Q_n = Q_{n-1} + Q_{n-3} + Q_1$. This means that for all $n \geq 2$ we have $2(Q_{n+5} - Q_{n+4} - Q_{n+1} - Q_{n-1}) = 2(Q_{n+4} + Q_{n+2} + Q_1 - Q_{n+4} - Q_{n+1} - Q_{n-1}) = 2(Q_{n+1} + Q_{n-1} + Q_1 + Q_1 - Q_{n+1} - Q_{n-1}) = 4Q_1$. It's not hard to see that $Q_1 = 1$ so our desired answer must be 4.