Putnam 2013

A1 Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Solution. Each face corresponds to exactly 3 vertices, so on average each vertex corresponds to $20 \times 3 \div 12 = 5$ faces. Since this icosahedron is regular, each vertex corresponds to 5 faces. Suppose that for each vertex, the number written is different. Then the sum of the 5 faces joining a vertex is at least $0+1+2+3+4 = 10$. Since each vertex corresponds to 3 faces and there are 12 vertices, the total sum of 20 faces is at least $10 \times 12 \div 3 = 40$, contradiction.

A2 Let S be the set of all positive integers that are not perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Solution. Suppose that $f(k_1) = f(k_2)$ for some $k_1 < k_2$. Let $k_1 < a_1 < \cdots < a_r = f(k_1)$ and $k_2 < b_1 < \cdots < b_s = f(k_2)$ be such choices for k_1 and k_2. Given that $k_1 \cdot a_1 \cdots a_r$ and $k_2 \cdot b_1 \cdots b_s$ are both perfect square, their product $k_1 \cdot a_1 \cdots a_r \cdot k_2 \cdot b_1 \cdots b_s$ is also a perfect square. Suppose that some number g appears in both sequence $\{a_i\}$ and $\{b_i\}$, then removing g from the combined sequence $k_1 \cdot a_1 \cdots a_r$ and $k_2 \cdot b_1 \cdots b_s$ yields that $k_1 \cdot a_1 \cdots a_r \cdot k_2 \cdot b_1 \cdots b_s / g^2$ is still a perfect square. Now, we remove all such repeated elements and sort the numbers, we get $k_1 < c_1 < \cdots < c_t$, since k_1 only appears once ($k_1 < k_2$) and since $a_r = b_s$, this number is removed from both sides and $c_t < a_r = f(k_1)$, contraidcting the minimality of a_r.

A3 Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Solution. First, since $0 < x < 1$, each sequence $1 - x^n = 1 + x^n + x^{2n} + x^{3n} + \cdots$ converges absolutely. Hence we are free to permute the sequence and get the sum in the following sense:

$$0 = \frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = \sum_{i=1}^{n} a_i \left(\sum_{j=0}^{\infty} x^{ij} \right) = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{n} a_i (x^j)^i \right)$$

Let $b_j = \sum_{i=0}^{n} a_i (x^j)^i$ for all $j \geq 0$, then it follows that $\sum_{i=0}^{\infty} b_j$ also converges absolutely to 0. Now let k to be the minimal index such that $b_k \neq 0$. If $b_j = 0$ for some $j > k$ then we are done, since we can just pick $y = x^j$ and since $j > k \geq 0$, $y \in (0,1)$. Otherwise, since $\sum_{i=k}^{\infty} b_j = \sum_{i=0}^{\infty} b_j = 0$, there exists a $j \geq k$ such that $b_j < 0$ and $b_{j+1} > 0$, or vice versa. In either case, $a_0 + a_1 y + \cdots + a_n y^n = 0$ for some $y \in (x^{j+1}, x^j)$, which obviously lies in $(0,1)$.

A4 A finite collection of digits 0 and 1 is written around a circle. An arc of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that

$|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \text{ and } N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

Solution. Let n be the number of bits written around the circle, and m the number of 1's written. We first prove the following lemma: for each $0 \leq k \leq n$, consider $S_k = \{Z(w) : |w| = k\}$ where $|w|$ is the length of w. Then $S_k = \{\lfloor \frac{km}{n} \rfloor, \lceil \frac{km}{n} \rceil\}$. To prove this, we notice that from the problem statement, $\max(S_k) - \min(S_k) \leq 1$. We consider two possible cases:

- The quantity $\frac{km}{n}$ is not an integer. Then $\lceil \frac{km}{n} \rceil - \lfloor \frac{km}{n} \rfloor = 1$. Now consider the n arcs $W_1, W_2, \cdots, W_n$ with length k; each point belongs to exactly k of the arcs. Thus $\sum_{i=1}^{n} Z(W_i) = km$ since there are m one's written, and the average of $Z(w_i)$ is $\frac{km}{n}$. By piegonholw principle, there is at least one W_i with $Z(W_i) \geq \lceil \frac{km}{n} \rceil$ and one W_i with $Z(W_i) \leq \lceil \frac{km}{n} \rceil$. Since $\max(W_i) - \min(W_i) \leq 1$ and $\lceil \frac{km}{n} \rceil - \lfloor \frac{km}{n} \rfloor$, the conclusion follows.
- Now that the quantity $\frac{km}{n}$ is an integer, meaning that $\lceil \frac{km}{n} \rceil = \frac{km}{n} = \lfloor \frac{km}{n} \rfloor$. Denote the n arcs by $W_1, \cdots, W_n$, and by the logic above, the average of $Z(w_i)$ is $\frac{km}{n}$. If there is w_i with $Z(w_i) < \frac{km}{n}$, i.e. $Z(w_i) \leq \frac{km}{n} - 1$, then there must be w_i with $Z(w_i) > \frac{km}{n}$, i.e. $Z(w_i) \geq \frac{km}{n} + 1$. This is a contradiction that $\max(S_k) - \min(S_k) \leq 1$, so we have $Z(W_i) = \frac{km}{n}$ for each i.

Now going back to the problem. For each arc w, we have $|w| = Z(w) + N(w)$. Thus considering the $w_1, \cdots, w_k$ given in the problem we have

$$Z + N = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) + \frac{1}{k}\sum_{j=1}^{k} Z(w_j) = \frac{1}{k}\sum_{j=1}^{k}(Z(w_j) + N(w_j)) = \frac{1}{k}\sum_{j=1}^{k}|w_j|$$

And in addition, for each w_j we have $Z(w_j) \in \{\lfloor \frac{|w_j|m}{n} \rfloor, \lceil \frac{|w_j|m}{n} \rceil\}$ so $\frac{|w_j|m}{n} - 1 < Z(w_j) < \frac{|w_j|m}{n} + 1$. This means, $\frac{1}{k}\sum_{j=1}^{k}(\frac{|w_j|m}{n} - 1) < Z < \frac{1}{k}\sum_{j=1}^{k}(\frac{|w_j|m}{n} + 1)$, i.e. $(\frac{m}{kn}\sum_{j=1}^{k}|w_j|) - 1 < Z < (\frac{m}{kn}\sum_{j=1}^{k}|w_j|) + 1$, and since $\frac{1}{k}\sum_{j=1}^{k}|w_j| = Z + N$, we have

$$(\frac{m}{n}(Z + W)) - 1 < Z < (\frac{m}{n}(Z + N)) + 1$$

Since Z is an integer, this also implies that $Z \in \{\lfloor \frac{m}{n}(Z + N) \rfloor, \lceil \frac{m}{n}(Z + N) \rceil\}$. This means we can find an arc w of length $Z + N$ that has $Z(w) = Z$, and therefore $N(W) = N$.

B1 For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Answer. -1.

Solution.

$$\begin{aligned}
\sum_{n=1}^{2013} c(n)c(n+2) &= c(1)c(3) + \sum_{n=1}^{1006} c(2n)c(2n+2) + \sum_{n=1}^{1006} c(2n+1)c(2n+3) \\
&= c(1)c(3) + \sum_{n=1}^{1006} c(n)c(n+1) + \sum_{n=1}^{1006} (-1)^n c(n)(-1)^{n+1}c(n+1) \\
&= c(1)c(3) + \sum_{n=1}^{1006} c(n)c(n+1) + \sum_{n=1}^{1006} (-1)^{2n+1} c(n)c(n+1) \\
&= c(1)c(3) + \sum_{n=1}^{1006} c(n)c(n+1) - \sum_{n=1}^{1006} c(n)c(n+1) \\
&= c(1)c(3) \\
&= c(1)(-1)^1 c(1) \\
&= -1
\end{aligned}$$

B2 Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and
(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

Answer. 3.
Solution. Consider the following:

$$\begin{aligned}
f(x) &= 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}\cos(4\pi x) \\
&= 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}(2\cos^2(2\pi x) - 1) \\
&= \frac{1}{3}(1 + 4\cos(2\pi x) + 4\cos^2(2\pi x)) \\
&= \frac{1}{3}(1 + 2\cos(2\pi x))^2
\end{aligned}$$

which is clearly nonnegative all the time. We also have $f(0) = 1 + \frac{4}{3} + \frac{2}{3} = 3$, establishing the equality. To show that 3 is indeed the maximum, it suffices to show that $\sum_{n=1}^{N} a_n \leq 2$ at all times. But plugging $x = \frac{1}{3}$ gives $\cos(\frac{2}{3}n\pi) = -\frac{1}{2}$ if n is not divisible by 3, and 1 otherwise. Considering that $a_n = 0$ whenever n is a multiple of 3, we have $f(\frac{1}{3}) = 1 - \frac{1}{2}\sum_{n=1}^{N} a_n \geq 0$. Thus $\sum_{n=1}^{N} a_n \leq 2$ must hold. Finally note the motivation to get the example $f(x)$ as shown in the beginning: we simply find a suitable a such that $2ax^2 + (2-a)x + (1-a)$ is always nonnegative, which is essentially asking for the discriminant $(2-a)^2 - 4(2a)(1-a) \leq 0$, and we get $a = \frac{2}{3}$ as the sole answer.

B3 Let P be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in P$, then $S \cup S' \in P$ and $S \cap S' \in P$, and

(ii) if $S \in P$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in P$ and T contains exactly one fewer element than S.

Suppose that $f : P \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in P.$$

Must there exist real numbers $f_1, \dots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in P$?

Answer. Yes.
Solution. Let S_0 be the subset such that $|S_0| \geq |S|$ for all $S \in P$. We first show that $S \subseteq S_0$ for all $S \in P$. Indeed, for an arbitrary set $S \in P$, we have $S \cup S_0 \in P$ and $|S \cup S_0| \geq |S_0|$. By the maximality of S_0 we must have $|S \cup S_0| = |S_0|$, which can only happen when $S \subseteq S_0$, as desired.

Now, w.l.o.g. let $S_0 = \{1, 2, \cdots, k\}$. By (ii) there exists $S_1 \subseteq S_0$ with one element fewer than S_0; w.l.o.g. let it be $\{1, 2, \cdots, k-1\}$. Continuing this trend we can assume that for all $0 \leq i \leq k$, $S_{k-i} = \{1, 2, \cdots, i\} \in P$. Consider, now, $f_1, f_2, \cdots, f_n$ such that $f_i = f(S_{k-i}) - f(S_{k-i+1})$ for $i = 1, 2, \cdots, k$, and arbitrary for $i = k+1, \cdots, n$. The identity $f(S) = \sum_{i \in S} f_i$ holds when $S = S_0, S_1, \cdots, S_k$ with S_k being the empty set (because $f(\emptyset) = 0$).

To show that this identity holds for all $S \in P$, we first notice that $S \subseteq S_0 = \{1, 2, \cdots, k\}$, so only $f_1, \cdots, f_k$ matter. We will proceed using the following premise with parameter $p = 0, 1, \cdots, k$: the $f(S) = \sum_{i \in S} f_i$ identity holds for all p-element subsets S. We are to prove this statement for all $p = 0, 1, \cdots, k$, and we will proceed by induction.

Base case: when $p = 0$ we have emptyset (proven above), and when $p = 1$ we have $S = \{j\}$ for some $1 \leq j \leq k$. We have $S_{k-j} = \{1, 2, \cdots, j\}$ and $S_{k-j+1} = \{1, 2, \cdots, j-1\}$. Moreover, $f(S_{k-j+1}) = \displaystyle\sum_{i=1}^{j-1} f_i$ and $f(S_{k-j}) = \displaystyle\sum_{i=1}^{j}$ by how f_i's are defined. Now by the definition of f,

$$f(S_{k-j}) = f(S \cup S_{k-j+1}) = f(S) + f(S_{k-j+1}) - f(S \cup S_{k-j+1})$$

since $S = \{j\}$. Considering that $S \cup S_{k-j+1} = \emptyset$, we have $f(S \cup S_{k-j+1}) = 0$ and therefore

$$f(S) = f(S_{k-j}) - f(S_{k-j+1}) = \sum_{i=1}^{j} f_i - \sum_{i=1}^{j-1} f_i = f_j$$

as desired.

Now let $2 \leq q \leq p$ be such that the preimise is true for all $p = 1, 2, \cdots, q-1$. Consider, now, any q-element subset $S = \{a_1, a_2, \cdots, a_q\}$. By condition (ii), there exists a subset of S in P with one fewer element; w.l.o.g. let it be $\{a_1, \cdots, a_{q-1}\}$. Consider, now, the set $S_{k-a_q} = \{1, 2, \cdots, a_q\} \in P$. Consider now the two equations:

$$f(S_{k-a_q} \cup \{a_1, \cdots, a_{q-1}\}) = f(S_{k-a_q}) + f(\{a_1, \cdots, a_{q-1}\}) - f(S_{k-a_q} \cap \{a_1, \cdots, a_{q-1}\})$$

$$f(S_{k-a_q} \cup \{a_1, \cdots, a_q\}) = f(S_{k-a_q}) + f(\{a_1, \cdots, a_q\}) - f(S_{k-a_q} \cap \{a_1, \cdots, a_q\})$$

First, notice that $\{a_1, \cdots, a_{q-1}\}$ and $\{a_1, \cdots, a_q\}$ differ only by an element a_q, and since $a_q \in S_{k-a_q}$, we have $\{1, 2, \cdots, a_q\} \cup \{a_1, \cdots, a_{q-1}\} = \{1, 2, \cdots, a_q\} \cup \{a_1, \cdots, a_q\}$. Comparing the two equations now give

$$f(\{a_1, \cdots, a_q\}) - f(\{a_1, \cdots, a_{q-1}\}) = f(S_{k-a_q} \cap \{a_1, \cdots, a_q\}) - f(S_{k-a_q} \cap \{a_1, \cdots, a_{q-1}\})$$

Since $a_q \in S_{k-a_q}$, we have $S_{k-a_q} \cap \{a_1, \cdots, a_{q-1}\} \subset S_{k-a_q} \cap \{a_1, \cdots, a_q\}$, differing only by an element a_q. If $\{a_1, \cdots, a_q\} = S_{k-a_q}$ then the condition $f(S) = \sum_{f_i \in S} f_i$ holds for this $S = \{a_1, \cdots, a_q\}$. Otherwise, $S_{k-a_q} \cap \{a_1, \cdots, a_q\}$ will have less than q elements. By the induction hypothesis, $f(S_{k-a_q} \cap \{a_1, \cdots, a_q\}) - f(S_{k-a_q} \cap \{a_1, \cdots, a_{q-1}\}) = f_{a_q}$, and therefore $f(\{a_1, \cdots, a_q\}) = f(\{a_1, \cdots, a_{q-1}\}) + f_{a_q}$. But by induction hypothesis again $f(\{a_1, \cdots, a_{q-1}\}) = f_{a_1} + \cdots + f_{a_{q-1}}$, and from here the conclusion follows.

B5 Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \geq 0$ such that $f^{(j)}(x) \leq k$.

[Here $f^{(j)}$ denotes the jth iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f\left(f^{(j)}(x)\right)$.]

Solution. We perform induction on n and $n-k$. When $k = n$ (when $n - k = 0$) then any function $f : X \to X$ works, so there are n^n such functions; when $k = n - 1$, the only requirement is that $f(n) \neq n$ so there are $(n-1)n^{n-1}$ such functions.

Thus now consider any $k \leq n - 2$. Observe that since $f^0(x) = x$, there is no restriction on $f(1), f(2), \cdots, f(k)$, giving n^k choices to each of them. We first make a following detour to a lemma: there exists $x > k$ with $f(x) \leq k$. Suppose not, then we have $f : \{k+1, \cdots, n\} \to \{k+1, \cdots, n\}$ and for each $x > k$ we have $f^{(j)}(x) > k$ for any $j \geq 0$, contradiction.

Now fix $m \in [1, n-k]$ such that exactly m of the numbers $k < x \leq n$ have $f(x) \leq k$. This gives rise of $\binom{n-k}{m}$ ways to choose those x, and each of them takes values $\{1, 2, \cdots, k\}$, giving rise to k^m of them. Now w.l.o.g. assume that those m elements are $k+1, \cdots, k+m$. To see how would $f(x)$ looks like for the other $x > k + m$'s, consider the problem of $g : \{1, 2, \cdots, n-k\} \to \{1, 2, \cdots, n-k\}$ where for each x, $g^{(j)}(x) \leq m$ for some $j \geq 0$. Here, $g(1), \cdots g(m)$ can be arbitrary, (i.e. $(n-k)^m$ choices), and by the induction hypothesis $n - k < n$ since $k \geq 1$, the number of such g's is $m \cdot (n-k)^{n-k-1}$, meaning that there are $(n-k)^{n-k-m-1}$ choices for $g(m+1), \cdots, g(n-k)$. Thus going back to our original problem here, we consider $f|_{k+m+1,\cdots,n}$ such that $f^{(j)}(x) \leq k + m$ for some j (this is because, given that $f(x) > k$ for all $x > k + m$, if j_0 is the minimum j with $f^{(j_0)}(x) \leq k$ then $f^{(j_0-1)}(x) \in \{k+1, \cdots, k+m\}$). This gives $m \cdot (n-k)^{(n-k-m-1)}$ choices on $f|_{k+m+1,\cdots,n}$, giving rise of $\binom{n-k}{m} \cdot k^m \cdot m \cdot (n-k)^{(n-k-m-1)}$ of them in total.

Hence considering all such $m \in [1, n-k]$ gives

$$
\begin{aligned}
\sum_{m=1}^{n-k} \binom{n-k}{m} \cdot k^m \cdot m \cdot (n-k)^{(n-k-m-1)} &= \sum_{m=1}^{n-k} \frac{(n-k)!}{m!(n-k-m)!} \cdot k^m \cdot m \cdot (n-k)^{(n-k-m-1)} \\
&= \sum_{m=1}^{n-k} \frac{(n-k)(n-k-1)!}{(m-1)!(n-k-m)!} \cdot k^{m-1} \cdot k \cdot (n-k)^{(n-k-m-1)} \\
&= k \sum_{m=1}^{n-k} \binom{n-k-1}{m-1} \cdot k^{m-1} (n-k)^{(n-k-m)} \\
&= k(k+n-k)^{m-k-1} \\
&= kn^{n-k-1}
\end{aligned}
$$

so combining with the arbitrary choice of $f|_{1,2,\cdots,k}$ we have $kn^{n-k-1}n^k = kn^{n-1}$, as desired.