Solutions to APMO 2024

Problem 1. Let ABC be an acute triangle. Let D be a point on side AB and E be a point on side AC such that lines BC and DE are parallel. Let X be an interior point of $BCED$. Suppose rays DX and EX meet side BC at points P and Q, respectively, such that both P and Q lie between B and C. Suppose that the circumcircles of triangles BQX and CPX intersect at a point $Y \neq X$. Prove that the points A, X, and Y are collinear.

Solution. The task is equivalent to showing that A lies on the radical axis of circles BQX and CPX. Let circle BQX intersect AB at B and T, and CQX intersect CPX intersect AC at C and U. Note that our goal is to show that $AB \cdot AT = AC \cdot AU$. However, since $\frac{AD}{AB} = \frac{AE}{AC}$ (given DE is parallel to BC), it suffices to show that $AD \cdot AT = AE \cdot AU$. We may now angle chase to obtain

$$\angle EDX = \angle XPQ = \angle XUC = \angle EUX$$

So U lies on circle EDX and similarly T lies on circle EDX. We thus conclude that D, T, U, E, X are concyclic, and therefore $AD \cdot AT = AE \cdot AU$.

Problem 2. Consider a 100×100 table, and identify the cell in row a and column b, $1 \leq a, b \leq 100$, with the ordered pair (a, b). Let k be an integer such that $51 \leq k \leq 99$. A k-knight is a piece that moves one cell vertically or horizontally and k cells to the other direction; that is, it moves from (a, b) to (c, d) such that $(|a-c|, |b-d|)$ is either $(1, k)$ or $(k, 1)$. The k-knight starts at cell $(1, 1)$, and performs several moves. A sequence of moves is a sequence of cells $(x_0, y_0) = (1, 1)$, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ such that, for all $i = 1, 2, \ldots, n$, $1 \leq x_i, y_i \leq 100$ and the k-knight can move from (x_{i-1}, y_{i-1}) to (x_i, y_i). In this case, each cell (x_i, y_i) is said to be reachable. For each k, find $L(k)$, the number of reachable cells.

Answer.

$$L(k) = \begin{cases} 100^2 - (2k-100)^2 & k \text{ even} \\ \frac{100^2 - (2k-100)^2}{2} & k \text{ odd} \end{cases}$$

Solution. We first note the following: if k is odd, then each move flips both the parity of the coordinates (and therefore all reachable cells have coordinates that are of the same parity). In addition, since $k > \frac{100}{2}$, each reachable cell must have one coordinate that is either $\geq k+1$, or $\leq 100-k$.

It now remains to show that these are all the conditions we need. Thus, $L(k)$ can be counted in the following way: the fact that the coordinates cannot be both in the range $\{101-k, \cdots, k\}$ (thus excluding the middle $(2k-100^{)}2$ cells). Of those, exactly half has coordinates of matching parity and half do not, further reducing $L(k)$ into half when k is odd.

By symmetry, if (m, n) is reachable, so is (n, m). Thus we may consider just one side of these. We isolate each of the cases, one by one:

m odd, n odd, $m \not{i}n[101-k, k]$. We first do $(1,1) \to (2, k+1) \to (3, 1) \to (4, k+1) \to (5, 1) \to \cdots \to (m, 1)$, and one of the following two:

$$(m+k, 2) \to (m, 3) \to (m+k, 4) \to (m, 5) \to \cdots \to (m, n)$$

$$(m-k, 2) \to (m, 3) \to (m-k, 4) \to (m, 5) \to \cdots \to (m, n)$$

depending on whether $m \leq 100-k$ or $m \geq k+1$.

m even, n even, $m \notin [101-k, k]$. We first show that $(100, 100)$ is always reachable, and then we can use symmetry to act as if we started at $(1, 1)$. Indeed, for k odd we may do

$$(1,1) \to (2, k+1) \to (3,1) \to (4, k+1) \to \cdots \to (100, k+1)$$

and then

$$(100, k+1) \to (100-k, k+2) \to (100, k+3), \cdots, (100, 100)$$

while for k even, we may first do $(1,1) \to (2, k+1) \to (k+2, k+2)$ and then

$$(k+2, k+2) \to (k+3, 2) \to (k+4, k+2) \to \cdots \to (100, k+2)$$

and finally

$$(100, k+2) \to (100-k, k+3) \to (100, k+4) \to \cdots \to (100, 100)$$

m, n different parity, k even. Suppose m is even, n is odd. We first shift our piece to $(2, 1)$ via the following steps:

$$(1,1) \to (2, k+1) \to (k+2, k) \to (2, k-1) \to \cdots \to (2, 1)$$

If $m \notin [101-k, k]$, we continue with

$$(2,1) \to (3, k+1) \to (4,1) \to \cdots \to (m, 1) \to (m \pm k, 2) \to (m, 3) \cdots \to (m, n)$$

Again, $m \pm k$ is $m+k$ if $m \leq 100-k$ and $m-k$ if $m \geq k+1$. Otherwise if $n \notin [101-k, k]$, we continue with

$$(2,1) \to (k+2, 2) \to (2,3) \to \cdots \to (2, n) \to (3, n \pm k) \to (4, n) \to \cdots \to (m, n)$$

The sign of $n \pm k$ also depends whether $n \leq 100-k$ or $n \geq k+1$.

Problem 3. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive reals. Show that

$$\sum_{i=1}^{n} \frac{1}{2^i} \left(\frac{2}{1+a_i}\right)^{2^i} \geq \frac{2}{1+a_1 a_2 \ldots a_n} - \frac{1}{2^n}.$$

Solution. We first show that $(\frac{2}{1+x})^{2^k} + (\frac{2}{1+y})^{2^k} \geq 2(\frac{2}{1+xy})^{2^{k-1}}$ for all $k \geq 1$ and $x, y > 0$. To start with, consider $k=1$, then we're supposed to show that $\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy}$. Indeed, by cross-multiplying, the inequality becomes

$$(1+xy)((1+x)^2 + (1+y)^2) \geq (1+x)^2(1+y)^2$$

Subtracting the left with right, we're left with the following term:

$$(1+xy)^2 + (x-y)^2(1+xy) - (x+y)^2 \geq (1+xy)^2 + (x-y)^2 - (x+y)^2 = (1-xy)^2 \geq 0$$

which establishes the claim for $k=1$. Thus for $k \geq 2$ we may use induction hypothesis (together with $a^2 + b^2 \geq \frac{(a+b)^2}{2}$) to get

$$\left(\frac{2}{1+x}\right)^{2^k} + \left(\frac{2}{1+y}\right)^{2^k} \geq \frac{1}{2}\left(\left(\frac{2}{1+x}\right)^{2^{k-1}} + \left(\frac{2}{1+x}\right)^{2^{k-1}}\right)^2 \geq 2\left(\frac{2}{1+xy}\right)^{2^{k-2} \cdot 2} = 2\left(\frac{2}{1+xy}\right)^{2^{k-1}}$$

as claimed.

Therefore, to solve the original problem, we use the lemma above, together with $x^2 \geq 2x - 1$ to do the following conversion:

$$\frac{1}{2^i}\left(\frac{2}{1+a_i}\right)^{2^i} \geq \frac{1}{2^{i-1}}\left(\frac{2}{1+a_i \cdot \dots \cdot a_n}\right)^{2^{i-1}} - \frac{1}{2^i}\left(\frac{2}{1+a_{i+1} \cdot \dots \cdot a_n}\right)^{2^i}, \forall i = 1, \cdots, n-1$$

$$\frac{1}{2^n}\left(\frac{2}{1+a_n}\right)^{2^n} \geq \frac{1}{2^{n-1}}\left(\frac{2}{1+a_n}\right)^{2^n} - \frac{1}{2^n}$$

to yield the following: we may just do telescoping sum, together with $x^2 \geq 2x - 1$ to yield

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{2^i}\left(\frac{2}{1+a_i}\right)^{2^i} &\geq \sum_{i=1}^{n-1}\left(\frac{1}{2^{i-1}}\left(\frac{2}{1+a_i \cdot \dots \cdot a_n}\right)^{2^{i-1}} - \frac{1}{2^i}\left(\frac{2}{1+a_{i+1} \cdot \dots \cdot a_n}\right)^{2^i}\right) \\ &+ \frac{1}{2^{n-1}}\left(\frac{2}{1+a_n}\right)^{2^n} - \frac{1}{2^n} \\ &= \frac{2}{1+a_1 \cdot \dots \cdot a_n} - \frac{1}{2^n}\end{aligned}$$

as desired.

Problem 4. Prove that for every positive integer t there is a unique permutation $a_0, a_1, \dots, a_{t-1}$ of $0, 1, \dots, t-1$ such that, for every $0 \leq i \leq t-1$, the binomial coefficient $\binom{t+i}{2a_i}$ is odd and $2a_i \neq t+i$.

Solution. We consider strong induction in t: for $t = 1$ the only permutation $a_0 = 0$ satisfies the property.

Now fix $t > 1$ and suppose that the assertion works for all smaller t's. Denote the remainder of m when divided by n as $\text{rem}(m, n)$ that saitsfies $m \equiv \text{rem}(m, n) \pmod n$ and $0 \leq \text{rem}(m, n) < n$. We note the following consequence of Lukas' theorem (modulo 2) : for each m, n,

$$\binom{m}{n} \text{ is odd if and only if for all } N \geq 1,\ \text{rem}(m, 2^N) \geq \text{rem}(n, 2^N)$$

Indeed, $\binom{m}{n}$ is odd if and only if $\lfloor\frac{m}{2^N}\rfloor = \lfloor\frac{n}{2^N}\rfloor + \lfloor\frac{m-n}{2^N}\rfloor$ for all N, which will require the fractional part to also satisfy $\{\frac{m}{2^N}\} = \{\frac{n}{2^N}\} + \{\frac{m-n}{2^N}\}$. One other key observation is also the following: if $m = n + 2^N$ for some $N \geq 0$ and the the digit coresponding to 2^N of n is 0, then $\binom{m}{n}$ is odd (the only place where the digits differ is at 2^N, which is 1 for m and 0 for n).

The task can now be viewed as a bijection f from $S_t = \{t, t+1, \cdots, 2t-1\}$ to $T_t = \{0, 1, \cdots, t-1\}$ such that for each $k \in S_t$, $\binom{k}{2f(k)}$ is odd, and also $k \neq 2f(k)$. We now proceed via the following three steps.

Step 1. Pick N and t' such that $t = 2^N + t'$ and $0 \leq t' < 2^N$. Note that S_t are integers in the range $[2^N + t', 2^{N+1} + 2t' - 1]$ inclusive. We first show that there is a unique way to assign f to $2^{N+1}, 2^{N+1} + 1, \cdots, 2^{N+1} + 2t' + 1$ (if $t' = 0$ this is vacuously true). When considered modulo 2^{N+1}, we have

$$2^{N+1}, 2^{N+1} + 1, \cdots, 2^{N+1} + 2t' - 1 \equiv 0, 1, \cdots, 2t' + 1$$

On the other hand, we have

$$0, 2, \cdots, 2(t-1) = 0, 2, \cdots, 2^{N+1} + 2t' - 2 \equiv 0, 2, \cdots, 2^{N+1} - 2, 0, 2, \cdots, 2t' - 2$$

i.e. each of $0, 2, \cdots, 2(t'-1)$ appears exactly two times. Thus for each $m = 1, 2, \cdots, t'$, we have exactly $2m$ numbers among $\{2^{N+1}, 2^{N+1} + 1, \cdots, 2^{N+1} + 2t' - 1\}$ and also $\{0, 2, \cdots, 2(t-1)\}$ with remainder at most $2m - 1 \pmod{2^{N+1}}$, which follows that

$$\{f(2^{N+1}), \cdots, f(2^{N+1} + 2m - 1)\} = \{0, 1, m-1, 2^N, \cdots, 2^N + m - 1\}$$

Since f is a bijection, considering this for each m separately we have

$$\{f(2^{N+1}+2m-2), f(2^{N+1}+2m-1)\} = \{m-1, 2^N+m-1\}$$

Finally, given that $2f(k) \neq k$, this forces

$$f(2^{N+1}+2m-2) = m-1 \qquad f(2^{N+1}+2m-1) = 2^N+m-1$$

as the only possible mapping. To show that this works, we have $2^{N+1}+2m-2-2f(2^{N+1}+2m-2) = 2^{N+1}$ and $2^{N+1}+2m-1-2(2^{N+1}+2m-1) = 1$, and the 2^{N+1}-digit of $2(m-1)$ is 0 ($m \leq t' < 2^N$) and $2(2^N+m-1)$ is odd.

Step 2. Now, we need to create f for the following domain / range:

$$f : \{t, t+1, \cdots, 2^{N+1}-1\} \to \{t', t'+1, \cdots, 2^N-1\}$$

Recall also that $t' < 2^N$, and note that we have $2^N - t'$ numbers on each side. If $t' = 2^N-1$, then f can only be $(2^{N+1}-1) = 2^N-1$, and note that and $\binom{2^{N+1}-1}{2(2^N)-1} = 2^{N+1}-1$ is odd, so this works. Otherwise, let $M \geq 1$ such that $2^N - 2^M \leq t' < 2^N - 2^{M-1}$, and for convenience denote also $t'' = t' - (2^N - 2^M)$; Note that $0 \leq t'' < 2^{M-1}$. When considered modulo 2^M, we have

$$t, t+1, \cdots, 2^{N+1}-1 \equiv t'', t''+1, \cdots, 2^M-1$$

(because the size of each set is $\leq 2^M$ but $> 2^{M-1}$), and

$$2t', 2t'+1, \cdots, 2(2^N-1) \equiv 2t'', \cdots, 2(2^{M-1}-1), 0, \cdots, 2(2^{M-1}-1)$$

Therefore, for each $m = t'', \cdots, 2^{M-1}-1$, the number of elements among each of $\{t, t+1, \cdots, 2^{N+1}-1\}$ and $\{2t', 2t'+1, \cdots, 2(2^N-1)\}$ with remainder at least $2m$ modulo 2^M is exactly $2(2^{M-1}-m)$. Therefore we have

$$\{f(2m+2^{N+1}-2^M), \cdots, f(2^{N+1}-1)\} = \{(2^N-2^M)+m, \cdots, 2^N-2^{M-1}-1, (2^N-2^{M-1})+m, \cdots, 2^N-1\}$$

By considering each such m individually, we have

$$\{f(2m+2^{N+1}-2^M), f(2m+1+2^{N+1}-2^M)\} = \{(2^N-2^M)+m, (2^N-2^{M-1})+m\}$$

and again, since $k \neq 2f(k)$, this forces $f(2m+2^{N+1}-2^M) = (2^N-2^M)+m$ and $f(2m+1+2^{N+1}-2^M) = (2^N-2^{M-1})+m$. Now $\binom{2m+1+2^{N+1}-2^M}{2(2^N-2^{M-1})} = 2m+1+2^{N+1}-2^M$ is odd, and between $2m+2^{N+1}-2^M$ and $2((2^N-2^M)+m)$, the difference is 2^M. Given that $m \leq t'' < 2^{M-1}$, the remainder of each number modulo 2^{M+1}) is $2m$ and $2m+2^M$, respectively, so the binomial coefficient is also odd.

Step 3. We are now left with defining f for the following:

$$\{2^{N+1}-2^M+t'', \cdots, 2^{N+1}-2^M+2t''-1\} \to \{2^N-2^{M-1}, \cdots, (2^N-2^{M-1})+t''-1\}$$

Notice that all the elements in the LHS set has binary digit 1 at positons $M, \cdots, N$ and RHS has binary digit 1 at positions $M-1, \cdots, N-1$, so this is the same as defining f for:

$$\{t'', \cdots, 2t''-1\} \to \{0, \cdots, t''-1\}$$

i.e. solving this problem for t''. Therefore, this part of arrangement for t is valid if and only if it's also valid for t'', which by induction hypothesis there's exactly one such construction. Thus this completes the induction step.

Problem 5. Line ℓ intersects sides BC and AD of cyclic quadrilateral $ABCD$ in its interior points R and S, respectively, and intersects ray DC beyond point C at Q, and ray BA beyond point A at P. Circumcircles of the triangles QCR and QDS intersect at $N \neq Q$, while circumcircles of the triangles PAS and PBR intersect at $M \neq P$. Let lines MP and NQ meet at point X, lines AB and CD meet at point K and lines BC and AD meet at point L. Prove that point X lies on line KL.

Solution. Denote G as the intersection of BD and AC; we know that $(KB, KC; KL; KG)$ are harmonic pencil. By Ceva's theorem (trigo version) on triangle KBC we have

$$\frac{\sin\angle ABD}{\sin\angle DBC} \cdot \frac{\sin\angle BCA}{\sin\angle ACD} \cdot \frac{\sin\angle CKG}{\sin\angle GKA} = 1$$

Since $ABCD$ is cyclic, we may change some of the sines above to chord subtended on the circumcircle, giving (together with harmonics)

$$\frac{\sin\angle LKA}{\sin\angle CKL} = \frac{\sin\angle GKA}{\sin\angle CKG} = \frac{\sin\angle ABD}{\sin\angle DBC} \cdot \frac{\sin\angle BCA}{\sin\angle ACD} = \frac{AD}{CD} \cdot \frac{AB}{AD} = \frac{AB}{CD}$$

Next, we consider the same for tiangle KPQ via lines PM, NQ and KX which are concurrent. By spiral similarity, we have triangles $MAS \sim MBR$, and $MSR \sim MAB$, and therefore,

$$\frac{\sin\angle MPQ}{\sin\angle MPK} = \frac{\sin\angle MPS}{\sin\angle MPA} = \frac{MS}{MA} = \frac{RS}{AB}$$

where the middle equality is by considering the length of arcs subtended on circle $MPAS$ while the last is due to the triangle similarity. In a similar spirit we have

$$\frac{\sin\angle NQK}{\sin\angle NQP} = \frac{\sin\angle NQC}{\sin\angle NQR} = \frac{NC}{NR} = \frac{CD}{RS}$$

Therefore, by Ceva's theorem, we have

$$\frac{\sin\angle XKP}{\sin\angle QKX} = \frac{\sin\angle MPK}{\sin\angle MPQ} \cdot \frac{\sin\angle NQP}{\sin\angle NQK} = \frac{AB}{RS} \cdot \frac{RS}{CD} = \frac{AB}{CD}$$

so $\frac{\sin\angle LKA}{\sin\angle CKL} = \frac{\sin\angle XKP}{\sin\angle QKX}$.

Finally, though our computation of the ratio of sines are unsigned, we see that both the lines KL and KX are outside the angle domain of $\angle KPQ$: the former is because both K and L are outside the quadrilateral $ABCD$; the latter is because either PM is in angle domain $\angle KPQ$ or NQ is in angle domain QPK but not both. Thus KL and KX are the same line, as desired.