Algebra

A1 For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

Answer. $b_n = \frac{N}{2}$.

Solution. First, we have LHS > 0, so RHS > 0 must hold, too. This immediately gives $b_N(x^2-2x+1)+x > 0$ for all $x \geq 0$, so $(2b_N-1)^2 - 4b_N^2 < 0$, or $(2-\frac{1}{b_N})$, so $b_N > \frac{1}{4}$. Conversely when $b_N > \frac{1}{4}$ this constraint is guaranteed.

Now with both sides positives, we can freely raise power on both sides. Raising both sides by power of N and subtracting x^N, we get the following inequality:

$$\frac{1}{2}(x-1)^2(x^{N-1}+\cdots+x+1)^2 = \frac{1}{2}(x^N-1)^2 \leq (b_N(x-1)^2+x)^N - x^N$$

$$= b_N(x-1)^2\left(\sum_{k=0}^{N-1}(b_N(x-1)^2+x)^k x^{N-1-k}\right)$$

So we need

$$(x^{N-1}+\cdots+x+1)^2 \leq 2b_N\left(\sum_{k=0}^{N-1}(b_N(x-1)^2+x)^k x^{N-1-k}\right)$$

whenever $x \neq 1$.

We first see that by taking $x = 1$,

$$(x^{N-1}+\cdots+x+1)^2 = N^2 \qquad \sum_{k=0}^{N-1}(b_N(x-1)^2+x)^k x^{N-1-k} = N$$

so if $b_N < \frac{N}{2}$ there exists some x in the neighbourhood of 1 such that

$$(x^{N-1}+\cdots+x+1)^2 > 2b_N\left(\sum_{k=0}^{N-1}(b_N(x-1)^2+x)^k x^{N-1-k}\right)$$

which isn't allowed. Therefore $b_n \geq \frac{N}{2}$ is necessary.

Now we show this b_N works. We'll proceed by induction: base case $N = 1$ just gives $(x^{N-1}+\cdots+x+1)^2 = 1$, and $\sum_{k=0}^{N-1}(b_N(x-1)^2+x)^k x^{N-1-k} = 1$, so equality holds on both sides. Now suppose that for some $N \geq 2$, $(b_k(x-1)^2+x)^k \geq \frac{1}{2}(x^{2k}+1)$ for all $k < N$. We have $b_N > b_k$, so

$$\sum_{k=0}^{N-1}(b_N(x-1)^2+x)^k x^{N-1-k} \geq \sum_{k=0}^{N-1} x^{N-1-k}\left(\frac{1}{2}(x^{2k}+1)\right) = \frac{1}{2}\sum_{k=0}^{N-1}\left(x^{N-k-1}+x^{N+k+1}\right)$$

$$= \frac{1}{2}(x^{N-1}+\cdots+x+1)(x^{N-1}+1)$$

To proceed further, we first consider the x nonnegative case: this way, $x^{N-1}+1$ majorizes $x^k + x^{N-1-k}$, so Muirhead's inequality means that

$$\frac{N}{2}(x^{N-1}+1) \geq x^{N-1}+\cdots+x+1$$

which establishes our conclusion. Now when $x < 0$:

- When $x \leq -\frac{1}{2}$, $(x-1)^2 \geq (-x-1)^2$ and therefore

$$\sqrt[N]{\frac{x^{2N}+1}{2}} = \sqrt[N]{\frac{(-x)^{2N}+1}{2}} \leq b_N(-x-1)^2 + x \leq b_N(x-1)^2 + x$$

- When $x < 0$ and $|x| < \frac{1}{2}$, the LHS $\sqrt[N]{\frac{x^{2N}+1}{2}} \leq 1$. On the other hand, $b_N \geq 1$ so $b_N(x-1)^2 + x \geq (x-1)^2 + x = (x-\frac{1}{2})^2 + \frac{3}{4} \geq 1$ whenever $x < 0$.

A2 Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i+j+k \geq n$.

Answer. $n = 4$.

Solution. We first identify the following members of $\mathcal{B}$. Notice also that $\mathcal{B}$ is closed under scalar multiplication so it's legal to subtract and add (in a sense).

- $x^i y^j z^k$: just take $R(x,y,z) = x^{i-1}y^{j-1}z^{k-1}$
- $x^a y^b$ with $a \geq 2, b \geq 2$ (with the symmetric sums): consider $Q(x,y,z) = x^{a-1}y^{b-1}$, giving
$$(xy+yz+zx)Q(x,y,z) = x^a y^b + x^{a-1}y^b z + x^a y^{b-1} z$$
But then we already had $x^{a-1}y^b z$, $x^a y^{b-1} z$ both in $\mathcal{B}$. (Similarly so for each component in the symmetric sum of $x^a y^b$)
- $x^a y$ with $a \geq 3$. Consider $P(x,y,z) = x^{a-1}y$. Then
$$(x+y+z)P(x,y,z) = x^a y + x^{a-1}y^2 + x^{a-1}yz$$
which, with $a \geq 3$, gives both $x^{a-1}y^2$ and $x^{a-1}yz$ in $\mathcal{B}$.
- x^a, $a \geq 4$. Now let $P = x^{a-1}$, giving
$$(x+y+z)P(x,y,z) = x^a + x^{a-1}y + x^{a-1}z$$
Since $a \geq 4$, both $x^{a-1}y$ and $x^{a-1}z$ are in $\mathcal{B}$.

Hence any $n \geq 4$ would work.

To show $n \geq 4$ is necessary, we show that we cannot have x^2y in $\mathcal{B}$. Observe that, when limiting in the case $x^i y^j z^k$ with $i+j+k = 3$, it suffices to consider P, Q, R homogeneous with degrees $2, 1, 0$, respectively. We also notice that

$$Q = ax+by+cz \Rightarrow (xy+yz+zx)Q(x,y,z) = a(x^2y+x^2z)+b(y^2z+y^2x)+c(z^2x+z^2y)+(a+b+c)xyz$$

Meanwhile, having x^2 as part of P introduces x^3 term that cannot be found anywhere, so the coefficient of x^2, y^2, z^2 of P must be 0. Thus let $P = pxy + qyz + rzx$, which gives

$$p(x^2y + xy^2) + q(y^2z + yz^2) + r(zx^2 + z^2x) + (p+q+r)xyz$$

Since R is constant (and cannot produce terms x^2y), in order to get only x^2y we need: $r = -a$ (eliminate x^2z), $b = -p$ (eliminate xy^2), $c = a$ (eliminate xz^2), $q = p$ (eliminate y^2z), giving us

$$(xy+yz+zx)Q(x,y,z) + (x+y+z)P(x,y,z) = (a+p)(x^2y + yz^2)$$

but we need $a+p = 1$, i.e. contradiction.

A3 Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Answer. 8.

Solution. By a slew of AM-GM inequality (in the form of $x + y \geq 2\sqrt{xy}$) we have

$$\begin{aligned}
\left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right) &= \geq 2\sqrt{\frac{ac}{bd}} + 2\sqrt{\frac{bd}{ac}} \\
&= \frac{2(ac+bd)}{\sqrt{abcd}} \\
&= \frac{(a+c)(b+d)}{\sqrt{abcd}} \\
&= 2\left(\sqrt{\frac{ab}{cd}} + \sqrt{\frac{ad}{bc}} + \sqrt{\frac{bc}{ad}} + \sqrt{\frac{cd}{ab}}\right) \\
&\geq 2\left(2\sqrt[4]{\frac{abcd}{abcd}} + 2\sqrt[4]{\frac{abcd}{abcd}}\right) \\
&= 8
\end{aligned}$$

Equality case when $a = c = 2 + \sqrt{3}$ and $b = d = 1$, i.e. the solution to the equation $a^2 + b^2 = 4ab$.

A4. (IMO 2) The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a+b+c+d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

Solution. The weighted AM-GM inequality says that $\sqrt[a+b+c+d]{(a^a b^b c^c d^d)} \leq \frac{a^2+b^2+c^2+d^2}{a+b+c+d}$. Thus it suffices to prove that

$$(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2) < 1$$

We have this identity:

$$(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2)$$

$$= \left(\frac{5}{2} - \frac{1}{2}(3(a-d) + (b-c)\right)\left(\frac{1}{4} + \frac{1}{4}((a-b)^2 + (b-c)^2 + (c-d)^2 + (a-c)^2 + (b-d)^2 + (a-d)^2)\right)$$

Thus it suffices to show that

$$(5 - (3(a-d) + (b-c))\left(1 + ((a-b)^2 + (b-c)^2 + (c-d)^2 + (a-c)^2 + (b-d)^2 + (a-d)^2)\right) < 8$$

Using the identity $\sum_{i=1}^{n} a_i^2 \leq (\sum_{i=1}^{n} a_i)^2$ for all $a_i \geq 0$, we have $(a-b)^2 + (b-c)^2 + (c-d)^2 \leq (a-d)^2$. In addition, with $a \geq b \geq c \geq d$ we have $(a-c)^2 + (b-d)^2 \leq (a-d)^2 + (b-c)^2$. Therefore substituting $a - d$ with x and $b - c$ with y (with $0 \leq y \leq x$), we are left with proving

$$(5 - (3x + y))\left(1 + (2x^2 + x^2 + y^2)\right) < 8$$

We now show that $3x^2 + y^2 \leq \frac{(3x+y)^2}{3}$. Expanding this and subtracting like terms from both sides give this as equivalent to $\frac{2y^2}{3} \leq 2xy$ but since $y \geq 0$, this is the as $y = 0$ or $\frac{y}{3} \leq x$. The conclusion immediately follows give $0 \leq y \leq x$. Therefore all we need to show is $(5-(3x+y))\left(1+\frac{(3x+y)^2}{3}\right)<8$, or, after substituting $3x+y$ with z, we are left with $(5-z)(1+\frac{z^2}{3})<8$.

Finally $(5-z)(1+\frac{z^2}{3})<8$ iff $z<3$ there's a root at $z=3$, the rest is just quadratic equation with no root. In addition, $x=a-d$ and $y=b-c$, so $3x+y\le 3(a-d)+3(b-c)\le 3(a+b-c-d)<3(a+b+c+d)=3$. Thus $z<1$ and the desired inequality follows.

A5 A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1<\cdots<x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1),\ldots,P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience. Can the magician find a strategy to perform such a trick?

Answer. No.

Solution. We claim the following:

Lemma. There exists polynomial P and Q of degree n, and values $a_1,\cdots,a_{2n}$, not all equal, such that (indices taken modulo $2n$)

$$P(x_i)=Q(x_{i+1})=a_i,\forall i=1,\cdots,2n$$

Proof: let's treat a_i's as random variables. Then Lagrange interpolation formula says that

$$P(x)\equiv\sum_{i=1}^{2n}a_i\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}\quad Q(x)\equiv\sum_{i=1}^{2n}a_i\prod_{j\neq i+1}\frac{x-x_j}{x_{i+1}-x_j}$$

(again, indices taken modulo $2n$).

Now, this gives us a polynomial of degree at most $2n-1$, so we need the coefficients of x^k to be 0 for $k>n$, for both P and Q (as well as having the coefficient at x^n to be nonzero). Notice each product is actually a polynomial: for each i, there exist $b_{i,0},\cdots,b_{i,2n-1}$ such that

$$\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}\equiv\sum_{j=0}^{2n-1}b_{ij}x^j$$

so we essentially have the following:

$$\begin{pmatrix} b_{1,2n-1} & \cdots b_{2n,2n-1} \\ b_{1,2n-2} & \cdots b_{2n,2n-2} \\ \vdots & \\ b_{1,n+1} & \cdots b_{2n,n+1} \\ b_{2,2n-1} & \cdots b_{1,2n-1} \\ b_{2,2n-2} & \cdots b_{1,2n-2} \\ \vdots & \\ b_{2,n+1} & \cdots b_{1,n+1} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_{2n}\end{pmatrix}=0$$

This is an equation of $2n$ variables, with $2n-2$ equations to solve, so the null space has degree at least 2. One guaranteed solution would be when $a_1=\cdots=a_{2n}$ (i.e. the span of $(1,1,\cdots,1)$). Thus there's a solution that's not in the span of $(1,\cdots,1)$, i.e. a_i's not all equal.

(TODO: ensure that the coeffs at n is nonzero...check $b_{1,n},\ldots,b_{2n,n}$ is not a linear combination of these??)

Thus from our examples, P and Q are two possible polynomials to return, and a_i not all the same means P and Q are not equal.

Combinatorics

C1. Let n be a positive integer. Find the number of permutations $a_1, a_2, \ldots a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \le 2a_2 \le 3a_3 \le \cdots \le na_n$$

Answer. F_n, the n-th Fibonacci sequence defined as $F_1 = 1$, $F_2 = 2$ and $F_n = F_{n-1}+F_{n-2}$ for all $n \ge 3$.

Solution. We consider the set

$$S = \{k : 1 \le k \le n, \{a_1, \cdots, a_k\} \neq \{1, 2, \cdots, k\}\}$$

Suppose for some k, we have $k \in S$, and either $k = 1$ or $k - 1 \notin S$. This means $a_k \neq k$. Let $\ell > k$ such that $a_\ell = k$, which also means $a_{\ell-1} > k$. We have $(\ell - 1)a_{\ell-1} \le \ell a_\ell = k\ell$, and therefore

$$a_{\ell-1} \le k(\frac{\ell}{ell-1}) = k(1 + \frac{1}{\ell-1}) \le k(1 + \frac{1}{k}) = k + 1$$

and with $a_{\ell-1} > k$, all the inequality above are equality, so $a_{k+1} = k$ and $a_k = k+1$. This means $k + 1 \notin S$.

Thus S must contain numbers $\le n - 1$ such that no two members are consecutive. In addition, for each $k \in S$ we have $a_k = k + 1$ and $a_{k+1} = k$. This means the set S uniquely determines such permutations, and it therefore reduces to finding the number of such set S, which is the same as the number of subsets of $\{1, 2, \cdots, n - 1\}$ such that no two elements are consecutive.

We now proceed by induction, if $n = 1$ then we only have a set $\emptyset$; if $n = 2$ we have $\emptyset, \{1\}$. For $n \ge 3$, depending on whether $n - 1 \in S$, it reduces to the number of sets among $\{1, 2, \cdots, n - 2\}$ or $\{1, 2, \cdots, n - 3\}$, which by induction hypothesis gives us F_{n-1} and F_{n-2} such sets, respectively. Therefore the number of such sets is $F_{n-2} + F_{n-1} = F_n$.

C2 In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and every quadrilateral Q_i has three corners of one color and one corner of the other color.

Solution. We'll use the following claim:

Lemma. Consider a $4k$-gon with b black and w white vertices ($b + w = 4k$), such that $k < b < 3k$ and $b \neq 2k$. Then there exists four consecutive corners with odd number of each colours.

Proof: Let $A_1, \cdots, A_{4k}$ be the vertices in that order, and a_i be the number of black vertices among $A_i, A_{i+1}, A_{i+2}, A_{i+3}$. Then $|a_{i+1} - a_i| \le 1$, and the average among a_i is $\frac{b}{k}$, which is a non-integer lying strictly between 1 and 3. It therefore follows that we must be able to choose i and j with $a_i = \lfloor \frac{b}{k} \rfloor$, and $a_j = \lceil \frac{b}{k} \rceil$. This means either $a_i = 1, a_j = 2$, or $a_i = 2, a_j = 3$. Thus we can always choose an i with a_i odd in both cases. □

Now in addition, if $b > w$ we take the 4 vertices with 3 being black, and if $b < w$ we take 4 vertices with 1 being black, we see that we have, in each cases:

$$2k - 2 \le b - 3 < 3k - 3 \quad k - 1 < b - 1 \le 2k - 2$$

so with b' black vertices left we still have $(k - 1) < b' < 3(k - 1)$.

Now let's do the algorithm as per the lemma. That is, at each step, if we have b, w black, white vertices with $b+w = 4k$ with $k < b < 3k$, $b \neq 2k$, then if $b > w$ choose 4 consecutive vertices with 3 blacks, and if $b < w$ choose 4 consecutive vertices with 1 black. If we get to a point where the condition no longer holds then $b = w = 2k$, which we do the following:

- If $k = 0$, stop.
- If there exists 4 consecutive vertices with odd number of black, choose that and continue as usual.
- Otherwise, each 4 consecutive vertices have 2 blacks and 2 whites.

The final case is of our interest, so we'll tackle just on that. Notice that by the $|a_{i+1} - a_i| = 1$ in the lemma we have: either the whites and blacks alternate, or they come in pairs (i.e. 2 blacks, 2 whites, 2 blacks, 2 whites, etc in that order). Moreover, up to this point, by choosing along the consecutive vertices, we ensure that the quadrilaterals formed are disjoint, and will be disjoint with the ones we form later on the remaining vertices. Thus we only need to worry about those remaining ones.

Let our vertices be $A_1, \cdots, A_{4k}$. In the first case (alternate) we choose $k-1$ quadrilaterals according to the following (indices modulo $4k$)

$$(i-1, 3(i+1), 3(i+1)+1, 3(i+1)+2), i = 1, \cdots, k-1$$

where we can see that each selection is congruent to $(i-1, i+1, i, i+1) \bmod 2$ so odd number of them is black. In the second case, assume that A_1, A_2 black and A_3, A_4 white. Then we do the following:

$$(4n-1, 1, 2, 3), (4n-2i, 4n-1-2i, 2i+2, 2i+3), i = 1, 2, \cdots, k-2$$

which we can see that $2i+2$ and $2i+3$ are different colours, and also $4n-2i, 4n-1-2i$ are of the same colour. Also the construction above guarantees that the $k-1$ vertices are distjoint so we're done.

Finally, the conclusion follows since $k = 25$ and $n = 41$ has $25 < 41 < 75$.

C3. (IMO 4) There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Answer. $k = n^2 - n + 1$.

Solution. We first show why $k = n^2 - n$ won't work. Consider the following construction:

- $A : i \to i+1, \forall i \in \{an + b : 0 \le a \le n-1, 1 \le b \le n-1\}$. Pictorially, this gives a segmentation of $[1, n], [n+1, 2n], \cdots, [n^2 n - n + 1, n^2]$ where each segment is a chain of $1 \to 2 \to \cdots n$.
- $B : i \to i + n, \forall i : 1 \le i \le n^2 - n$.

Then in A, two stations i and j are linked if and only if they are in the same "segment" (i.e. $\lceil \frac{i}{n} \rceil = \lceil \frac{j}{n} \rceil$), while in B, two stations i and j are linked if and only if $i \equiv j \pmod{n}$. Thus no two stations are linked by both companies.

Notice that the proof above also means any $k < n^2 - n$ won't work: we simply remove a subset of the cable car links from both companies.

Now we give an example why $n^2 - n + 1$ works. We define a chain of cable cars $a_0 \to a_1 \to \cdots a_m$ as follows:

- There's a cable car from a_i to a_{i+1}.

- There's no cable car ending at a_0, and no cable car starting at a_m (i.e. this chain is "maximal").

We see that two stations are linked if and only if they belong to the same chain (also, notice that these chains are disjoint union of the stations since all the starting points are different, and all the ending points are different).

The case where station i is not part of any cable car (starting or ending) is considered a degenerate chain of length 1 on its own. Given that both A and B has $k = n^2 - n + 1$ cable car services, there are $n-1$ chains in total ($n-1$ is the number of stations that's not a starting point of any cable car, and also the number of stations that's not a ending point of any cable car).

The longest chain in A now has length at least

$$\frac{n^2}{n-1} > n+1 > n-1$$

and since there are $n-1$ chains in B, two of the stations in this longest chain must also in the same chain in B, thereby being linked by both companies.

Comment. To generalize to the case where there are n stations (i.e. not square), the same idea applies: we need to find the largest integer m such that $\frac{n}{m} > m$. In this case $m = \lfloor\sqrt{n-1}\rfloor$ which gives the bound $k = n - \lfloor\sqrt{n-1}\rfloor$. The construction of counterexample for $k-1$ can also be done like above by splitting into segments of either length $\lfloor\sqrt{n}\rfloor$ or $\lceil\sqrt{n}\rceil$.

Geometry

G1. Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

Solution. We now have $QPCF$ an isoceles trapezoid, and with angle chasing we can conclude that triangles FQP and FAB are similar. Triangles FAQ and FBP are also similar. Therefore,

$$\frac{CQ}{CP} = \frac{FP}{FQ} = \frac{FB}{FA} = \frac{PB}{QA} = \frac{DP}{DQ}$$

Let $\angle QCD = a$ and $\angle PCD = b$. We have $\angle CQD = \angle CPD = 90°$, so C, Q, P, D lie on a circle. If the circle has diameter d, we get

$$\frac{CQ}{CP} = \frac{d\cos a}{d\cos b} = \frac{\cos a}{\cos b} \qquad \frac{DP}{DQ} = \frac{d\sin b}{d\sin a} = \frac{\sin b}{\sin a}$$

so we essentially have $\cos a \sin a = \cos b \sin b$, or $\sin 2a = \sin 2b$. This means, $a = b$ or $a + b = 90°$. The former is only possible if $CP = CQ$, which entails $AD = DB$ which is impossible here. Therefore $\angle ACB = a + b = 90°$.

G2. (IMO 1) Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Solution. The claim is that the lines will meet at the circumcenter O of ABP. Here's how:

- As $OA = OB = OP$, it's already on the perpendicular bisector of AB.
- Now $\angle PAD + \angle DPA < 180^\circ$ (they are part of interior angles of the triangle PAD), we have $\angle PBA = \frac{\angle PAD + \angle DPA}{2} < 90^\circ$ based on the angle condition. This gives the following angle condition:

  $$\angle POA = 2\angle PBA = \angle PAD + \angle DPA = 180^\circ - \angle PDA$$

  and therefore $DAOP$ is concyclic. As $OP = OA$, it then follows that DO bisects $\angle ADP$.
- Similarly, CO bisects $\angle PCB$.

G3 Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90$, $CDA > 90$ and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

Solution. We first claim that the circumcircles of triangles ALD and AKB are tangent to each other. This is the same as showing that $\angle AKB + \angle ALD = \angle BAD$. Indeed, given that E is the reflection of A in BC, $AE \perp BC$ and therefore $\angle AKB = 90^\circ - \angle CDB$ and similarly $\angle ALD = 90^\circ - \angle CBD$. Therefore,

$$\angle AKB + \angle ALD = 180^\circ - \angle CDB - \angle CBD = \angle BCD = \angle BAD$$

as claimed.

Now, reflect A in BD to get G. To avoid cases on whether K lies between E and A we use directed angles here. From E being the reflection of A in BC we get $\angle(AE, AB) = \angle(EB, AE)$ and similarly by definition of G we get $\angle(AE, AB) = \angle(AK, AB) = \angle(BG, GK)$. Therefore, $\angle(BG, GK) = \angle(BE, EA) = \angle(BE, EK)$ so B, G, K, E are concyclic. Similarly, L, D, F, G are concyclic. This means that G is an intersection of circumcircles of BDK and DFL. Finally, since circles BAK and ALD are tangent to each other, same goes to circles BDK and DFL as they are simply the reflections of BAK and ALD in line BD.

G5 Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of $\triangle ANK, \triangle BKL, \triangle CLM, \triangle DMN$.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

Solution. Let O_A, O_B, O_C, O_D be the centers of $\omega_A, \omega_B, \omega_C, \omega_D$. Denote the common internal tangency point of incircles BAD and BCD as P, and the common internal tangency point of incircles ABC and ADC as Q. Denote, also, the inradius of triangle XYZ as $r(XYZ)$. We see that P lies on BD and Q lies on AC.

Let's first show the following:

$$\frac{BP}{PD} = \frac{r(BKL)}{r(DMN)}$$

Indeed, we see that triangles BKL and BAC are similar, with similitude, say, β. Similarly, AKN and ABD are similar, with similitude, say, α. Then $\alpha + \beta = \frac{AK+KB}{KB} = 1$. It then follows that the similitude of DMN and DCA is also β, and the similitude of CLM and CBD is α. Thus $\frac{r(BKL)}{r(DMN)} = \frac{r(BAC)}{r(DCA)}$, so it suffices to show that $\frac{BP}{PD} = \frac{r(BAC)}{r(DCA)}$.

Now P is the intersection of BD and I_AI_C, where I_A is the incenter of ABD and I_C the incenter of CBD. Then we have (using $|\triangle|$ as the area of a triangle)

$$\frac{BP}{PD} = \frac{|BI_AI_C|}{|DI_AI_C|} = \frac{BI_A \cdot BI_C \cdot \sin\angle I_ABI_C}{DI_A \cdot DI_C \cdot \sin\angle I_ADI_C} = \frac{\sin\angle I_ADB \cdot \sin\angle I_CDB \cdot \sin\angle I_ABI_C}{\sin\angle I_ABD \cdot \sin\angle I_CBD \cdot \sin\angle I_ADI_C}$$

(the last equality by sine rule). Meanwhile, denoting I_B the incenter of ABC and I_D the incenter of ADC we have

$$\frac{r(BAC)}{r(DCA)} = \frac{|AI_BC|}{|AI_DC|} = \frac{AI_B \cdot I_BC \cdot \sin\angle AI_BC}{AI_D \cdot I_DC \cdot \sin\angle AI_DC} = \frac{(AC\frac{\sin\angle I_BCA}{\sin\angle AI_BC})\cdot(AC\frac{\sin\angle I_BAC}{\sin\angle AI_BC})\cdot\sin\angle AI_BC}{(AC\frac{\sin\angle I_DCA}{\sin\angle AI_DC})\cdot(AC\frac{\sin\angle I_DAC}{\sin\angle AI_DC})\cdot\sin\angle AI_DC}$$

$$= \frac{\sin\angle I_BCA\sin\angle I_BAC\sin\angle AI_DC}{\sin\angle I_DCA\sin\angle I_DAC\sin\angle AI_BC}$$

We also have

$$\angle I_ADB = \frac{1}{2}\angle ADB = \frac{1}{2}\angle ACB = \frac{1}{2}\angle I_BCA \qquad \angle I_CDB = \frac{1}{2}\angle CDB = \frac{1}{2}\angle CAB = \frac{1}{2}\angle I_BAC$$

$$\angle I_ABD = \frac{1}{2}\angle ABD = \frac{1}{2}\angle ACD = \frac{1}{2}\angle I_DCA \qquad \angle I_CBD = \frac{1}{2}\angle CBD = \frac{1}{2}\angle CAD = \frac{1}{2}\angle I_DAC$$

and finally,

$$\angle AI_DC = 90^\circ + \frac{\angle ADC}{2} = 180^\circ - \frac{\angle ABC}{2} = 180^\circ - \angle I_ABI_C$$

$$\angle AI_BC = 90^\circ + \frac{\angle ABC}{2} = 180^\circ - \frac{\angle ADC}{2} = 180^\circ - \angle I_ADI_C$$

so the two ratios are indeed equal, as desired. In a similar way we also have

$$\frac{AQ}{QC} = \frac{r(AKN)}{r(CLM)}$$

Next, let R be on PQ such that $\frac{PR}{RQ} = \frac{\beta}{\alpha}$, i.e. $R = \alpha P + \beta Q$. We show that R is the desired point of concurrency. To see why, let $\frac{AQ}{QC} = \frac{r(AKN)}{r(CLM)} = \frac{r(ABD)}{r(CBD)} = \frac{\gamma}{\mu}$ for some $\gamma, \mu > 0$ and $\gamma + \mu = 1$. Then we have $Q = \mu C + \gamma A$, $P = \mu I_C + \gamma I_A$. We also have A, O_A, I_A collinear with $O_A = \beta A + \alpha I_A$ and C, O_C, I_C collinear with $O_C = \beta C + \alpha I_C$. Thus the common internal tangents of ω_A, ω_C intersect at the following point:

$$\mu O_C + \gamma O_A = \mu(\beta C + \alpha I_C) + \gamma(\beta A + \alpha I_A) = \beta(\mu C + \gamma A) + \alpha(\mu I_C + \gamma I_A) = \beta Q + \alpha P$$

so this point is indeed R. In a similar way we can also show that the common internal tangents of ω_B, ω_D will intersect at R.

G6 Let ABC be a triangle with $AB < AC$, incenter I, and A excenter I_A. The incircle meets BC at D. Define $E = AD \cap BI_A$, $F = AD \cap CI_A$. Show that the circumcircle of $\triangle AID$ and $\triangle I_AEF$ are tangent to each other.

Solution. We first claim that the circles AID and I_AEF have an intersection on the circle IBC.

Now, II_A is the diameter of circle IBC. Let G be another intersection of circles IBC and I_AEF. Then triangles GEF and GBC are similar, also GBE and GCF are similar. This means we have

$$\frac{GB}{GC} = \frac{BE}{CF}$$

and moreover by Menelaus' theorem on line DEF and triangle BI_AC,

$$\frac{EB}{EI_A} \cdot \frac{FI_A}{FC} \cdot \frac{DC}{DB} = -1$$

(notice that F lies outside segment I_AC so $\frac{FI_A}{FC}$ is assumed negative here), and also considering the triangle BCG and line GD,

$$\frac{BD}{DC} = \frac{BG}{GC} \cdot \frac{\sin\angle BGD}{\sin\angle CGD} = \frac{BE}{CF}\frac{\sin\angle BGD}{\sin\angle CGD}$$

so we ended up with

$$-1 = \frac{EB}{EI_A} \cdot \frac{FI_A}{FC} \cdot \frac{DC}{DB} = \frac{EB}{EI_A} \cdot \frac{FI_A}{FC} \cdot \frac{CF}{BE} \frac{\sin \angle CGD}{\sin \angle BGD} = \frac{FI_A}{EI_A} \frac{\sin \angle CGD}{\sin \angle BGD}$$

so ignoring sign and recognizing that E is on segment BI_A but F outside segment CI_A, we have

$$\frac{\sin \angle I_A EF}{\sin \angle I_A FE} = \frac{FI_A}{EI_A} = \frac{\sin \angle BGD}{\sin \angle CGD}$$

This means $\angle I_A EF = \angle BGD$ and $\angle I_A FE = \angle CGD$.

Now we claim that G also lies on the circumcircle of triangle AID. By angle chasing,

$$\angle DAI = \angle FAI_A = \angle II_A C - \angle AFI_A = \angle IGC - \angle DGC = \angle IGD$$

as desired.

It remains to show that G is in fact the tangency point of the circles AID and $I_A EF$. Given $AIDG$ on one circle and $GEFI_A$ on one circle, what we need to show now is

$$\angle IGE = \angle GAI + \angle GI_A E$$

The last quantity $\angle GI_A E$ is the same as $\angle GCB$; we also have

$$\angle GAI = \angle GAD + \angle IAD = \angle GID + \angle IGD$$

and finally

$$\angle IGE = \angle IGD + \angle DGE = \angle IGD + \angle BGE - \angle BGD$$

so now we need to prove that

$$\angle BGE = \angle GID + \angle GCB + \angle BGD$$

We notice also that $\angle BGD = \angle I_A EF = \angle BED$ so $BEDG$ is concyclic. This means $\angle BGE = \angle CDF$. Also, $\angle CDF - \angle BED = \angle DBE = \angle CBI_A$ so it suffices to show that $\angle CBI_A = \angle GID + \angle GCB$.

Let ID intersect the circle $I_A BC$ again at H, then $\angle GID = \angle GIH = \angle GI_A H$ and $\angle GCB = \angle GI_A B$ so$\angle GID + \angle GCB = HI_A B = HCB$. We also have $BC_I AH$ an isoceles trapezoid so $\angle HCB = \angle I_A BC$, as desired.

G7 Let P be a point on the circumcircle of acute triangle ABC. Let D, E, F be the reflections of P in the A-midline, B-midline, and C-midline. Let ω be the circumcircle of the triangle formed by the perpendicular bisectors of AD, BE, CF.

Show that the circumcircles of $\triangle ADP, \triangle BEP, \triangle CFP$, and ω share a common point.

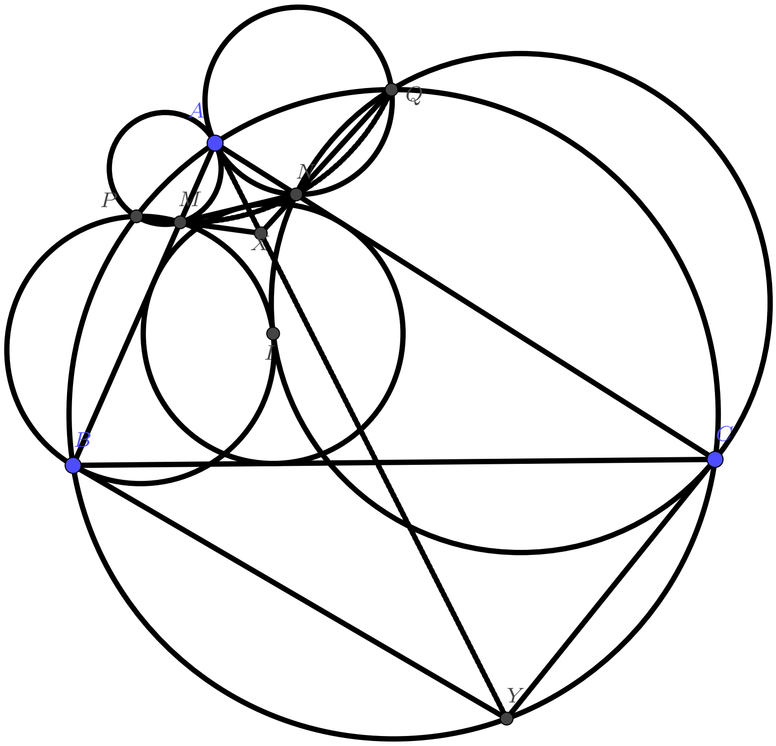
Solution. We first show that the first three circles are coaxial. If K, L and M are the perpendiculars from A, B, C to lines BC, CA, AB, respectively, then K is the reflection of A in the corresponding midline. Then $AKPD$ is an isoceles trapezoid, and therefore cyclic. Similarly, L lies on circumcicle of BEP and M lies on CFP.

(TODO...next time)

G8 Let ABC be a triangle with incenter I and circumcircle Γ. Circles ω_B passing through B and ω_C passing through C are tangent at I. Let ω_B meet minor arc AB of Γ at P and AB at $M \neq B$, and let ω_C meet minor arc AC of Γ at Q and AC at $N \neq C$. Rays PM and QN meet at X. Let Y be a point such that YB is tangent to ω_B and YC is tangent to ω_C.

Show that A, X, Y are collinear.

Solution. We break things down into several steps.



Step 1. Y lies on Γ. Indeed, from the tangency of ω_B and ω_C at I we got $\angle BIC = \angle BMI + \angle CNI$, and by definition of Y (tangent to circles) we have $\angle IBY = \angle BMI, \angle ICY = \angle INC$. Therefore,

$$\angle ABY + \angle ACY = \angle ABI + \angle ACI + \angle BMI + \angle CNI = \angle ABI + \angle ACI + \angle BIC = 180^\circ$$

Step 2. I is the excenter opposite A or triangle AMN.

Proof: by the tangency of ω's we have $\angle MCN = \angle ABI + \angle ACI = 90^\circ - \frac{BAC}{2}$. Now let the tangent to incircle of ABC from M to be line AB and MN' with N' on AC, then by angle chasing we can get

$$\angle MCN' = 180^\circ - \frac{1}{2}(\angle BMN' + \angle CN'M) = 180^\circ - \frac{1}{2}(180^\circ + \angle BAC) = 90^\circ - \frac{BAC}{2} = \angle MCN'$$

so indeed, $N = N'$ and therefore the incircle is tangent to MN, establishing this step.

Step 3. P, M, N, Q concyclic. Here we'll just angle chase:

$$\angle PMN = \angle PMA + \angle AMN = 180^\circ - \angle PMB + (180^\circ - 2\angle BMI)$$

while

$$\angle PQN = \angle AQN - \angle AQP = \angle AQC - \angle NQC - \angle ABP = (180^\circ - \angle ABC) - (180^\circ - \angle NIC) - \angle ABP$$

Finally, $\angle MIN + \angle BIC = 90^\circ + \frac{\angle BAC}{2} + 90^\circ - \frac{\angle BAC}{2} = 180^\circ$ so $\angle BIM + \angle NIC = 180^\circ$. Thus we're left with showing

$$\angle PMB + 2\angle BMI + \angle ABC + \angle MIB + \angle ABP = 360^\circ$$

Indeed,

$$\begin{aligned}
&\angle PMB + 2\angle BMI + \angle ABC + \angle MIB + \angle ABP \\
&= \angle PMB + \angle BPI + \angle ABC + (180^\circ - \frac{\angle ABC}{2}) + \angle ABP \\
&= (180^\circ - \angle MPI) + \angle ABC + (180^\circ - \frac{\angle ABC}{2}) \\
&= 360^\circ + \angle ABC - \frac{\angle ABC}{2} - \frac{\angle ABC}{2} \\
&= 360^\circ
\end{aligned}$$

as claimed.

Step 4. APM and AQN have circumcircles tangent at A, and common tangent line AY.

Proof: We have

$$\angle APM = \angle APB - \angle BPM = (180^\circ - \angle ACB) - (180^\circ - \angle MIB) = \angle MIB - \angle ACB$$

and also

$$\angle BAY = 180^\circ - \angle ABY - \angle AYB = \angle MIB - \angle ACB = \angle APM$$

so AY is tangent to the circle APM at A. Similarly AY is tangent to circle AQN at A, thus finishing the proof.

Now we can complete the solution. AY is the radical axis of APM and AQN, and since P, M, N, Q cyclic, their intersection X has $XM \cdot XP = XN \cdot XQ$. Therefore X is on the radical axis of these two circles (i.e. done).

G9 (IMO 6) Prove that there exists a positive constant c such that the following statement is true: Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ separates a set of points $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

Note. Weaker results with $cn^{-1/3}$ replaced by $cn^{-\alpha}$ may be awarded points depending on the value of the constant $\alpha > 1/3$.

(Mini-)Solution. I hereby attach my proof for $\alpha = \frac{1}{2}$ (which is worth 1 point).

Let D be the diameter of $\mathcal{S}$. That is, the maximum distance between any two points in $\mathcal{S}$. W.l.o.g. let the diameter to be $(0,0)$ and $(0,D)$. Then any point in $\mathcal{S}$ must have x-coordinate in $[-D,D]$ and y-coordinate in $[0,D]$. Consider, now, drawing $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$ boxes in the space $[-D,D] \times [0,D]$ space. Given that the distance between any two points in the box is at most 1, each point in $\mathcal{S}$ must be in different boxes. Thus there are at most $\lceil \sqrt{2}D \rceil \times \lceil 2\sqrt{2}D \rceil$ boxes. Omitting the ceiling functions for now (as they will be insignificant as D and n grow), we have at most $4D^2$ boxes. With n points in S, we have at least n boxes. Thus $D \geq \frac{\sqrt{n}}{2}$.

Now, as above we assumed that the two points have y-coordinates 0 and D, respectively. Putting a line parallel to the x-axis and with y-intercept between 0 and D will separate the two points (hence separating $\mathcal{S}$). Sorting the points by y-axis, it then follows that some consecutive points have gap at least $\frac{D}{n}$. Thus placing the line passing through the midpoint of these two points (or rather, perpendicular bisector) will ensure that any point has distance at least $\frac{D}{2n}$ from this line. With $D \geq \frac{\sqrt{n}}{2}$, we have distance from any points of $\mathcal{S}$ to ℓ at least $\frac{1}{4}n^{-\frac{1}{2}}$ (well maybe a bit lower to account for the ceiling function above).

Number Theory

N1 Given a positive integer k show that there exists a prime p such that one can choose distinct integers $a_1, a_2 \cdots, a_{k+3} \in \{1, 2, \cdots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \cdots, k$.

Solution. Let p be a random variable (for now), and allowing multiplicative inverse, we get the following requirement:

$$\frac{a_{i+4}}{a_i} = \frac{a_{i+1}a_{i+2}a_{i+3}a_{i+4}}{a_i a_{i+1} a_{i+2} a_{i+3}} = \frac{i+1}{i}$$

Now consider $q, r, s > k$ prime numbers, with $a_1 = q, a_2 = r, a_3 = \frac{1}{q}, a_4 = \frac{1}{r}$. Then $a_1 a_2 a_3 a_4 = 1$ and by obeying $\frac{a_{i+4}}{a_i} = \frac{i+1}{i}$, we get $a_i a_{i+1} a_{i+2} a_{i+3} = i$.

Next we see that $a_i \neq a_j$ for all i, j: if $i \not\equiv j \pmod 4$, then a_{4x+1} and a_{4x+2} have factors q and r on their numerators respectively (but not both); while a_{4x+3} and a_{4x+4} have those on their denominators respectively (but not both). This easily rules out the case where $i \not\equiv j \pmod 4$. For $i \equiv j \pmod 4$ we have $i < j$ implying

$$\frac{a_j}{a_i} = \prod_{x=0}^{(j-i)/4-1} \frac{i+4x+1}{i+4x}$$

which is a rational number that's strictly > 1. So $a_i \neq a_j$ in this case.

Finally, having established above, all we need to do is to assemble all the rational numbers $\{\frac{a_i}{a_j}\}_{i \neq j}$, and choose p such that:

- p is relatively prime to the numerators and denominators of all $\frac{a_i}{a_j}$.
- $\frac{a_i}{a_j} \not\equiv 1 \pmod p$. That is, choose p such that p does not divide the numerator of $a_i - a_j$ for all (i, j).

N2 For each prime p, construct a graph G_p on $\{1, 2, \ldots p\}$, where $m \neq n$ are adjacent if and only if p divides $(m^2 + 1 - n)(n^2 + 1 - m)$. Prove that G_p is disconnected for infinitely many p.

Solution. We claim that G_p is disconnected whenever $p \equiv 1 \pmod 3$. Notice that all the edges are given by $(m, m^2 + 1)$ for $m = 0, 1, \cdots, p-1$ since (m, n) are adjacent if and only if $p \mid m^2 + 1 - n$ or $p \mid n^2 + 1 - m$ (thus (m, n) can be written in the form $(m, m^2 + 1)$ or $(n^2 + 1, n)$). This means that there are at most p edges in G_p.

Now, if $p \equiv 1 \pmod 3$, then $x^3 \equiv 1 \pmod 3$ has 3 distinct solutions (one of them being 1). Therefore we have, for some $x \not\equiv 1$,

$$p \mid x^3 - 1 = (x-1)(x^2 + x + 1)$$

so $p \mid x^2 + x + 1$ here. If $m = x + 1$, then $m^2 + 1 = (x+1)^2 + 2 = (x^2 + x + 1) + x + 1 \equiv x + 1 \pmod p$ so $(m, m^2 + 1)$ is actually (m, m) (i.e. a self-loop). In addition, if $p \mid m^2 + 1 - m$, then $(p - m + 1)^2 + 1 - (p - m + 1) \equiv (m-1)^2 + 1 + (m-1) = m^2 + 1 - m \equiv 0 \pmod p$. So both (m, m) and $(p - m + 1, p - m + 1)$ are self-loops.

Finally, we claim that $m \neq p - m + 1$, otherwise $2m + 1 = p$ and we have

$$p \mid (\frac{p+1}{2})^2 + 1 - (\frac{p+1}{2}) \equiv \frac{3}{4}$$

and therefore $p = 3$ here, which is a contradiction. This means that among the p edges just now, 2 of them are self-loops so there are at most $p - 2$ actual edges. This means that our graph can no longer be connected for such p. Finally, by Dirichlet's theorem, there are infinitely many such p with $p \equiv 1 \pmod 3$.

N3. (IMO 5) A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

Answer. All $n > 1$.

Solution. Call a set S of positive integers "good" if the arithmetic mean of every pair of integers is also the geometric pairs of some subset of integers in S. Consider $aS = \{ak : k \in S\}$ for some rational a such that $ak \in \mathbb{N}$ for all $k \in S$. The arithmetic and geometric mean of corresponding numbers are scaled by a, so S is good if and only if aS is good. Therefore, given an original collection S, we can divide by its gcd and assume that $\gcd(S) = \gcd\{k : k \in S\} = 1$.

In particular, some number $k \in S$ has to be odd. If $m \in S$ is even then $\frac{k+m}{2}$ is a half-integer. But $(\frac{k+m}{2})^d$ is never an integer for any integer $d > 0$. Thus $\frac{k+m}{2}$ cannot be geometric mean of any subset of S. We thus have all numbers odd.

Suppose also that the numbers are $a_1 \geq \cdots a_n$. If $a_1 > 1$, then there exists an odd prime p dividing a_1 but since $\gcd(a_1, \cdots, a_n) = 1$, there's some i with $p \nmid a_i$. Thus we can choose m that is the minimal index with $p \nmid a_m$ (i.e. a_m is the biggest number among them not divisible by p. Then $\frac{a_1+a_m}{2}$ must be an integer not divisible by p, and since $a_1 \neq a_m$, $a_1 > a_m$ and so $\frac{a_1+a_m}{2} > a_m$. But since $a_1, \cdots, a_{m-1}$ are divisible by p, the numbers that form geometric mean of $\frac{a_1+a_m}{2}$ must be taken from $\{a_m, \cdots, a_n\}$ which are at most a_m. This gives a contradiction. Thus $a_1 = 1$ and all numbers are equal.

N4 For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \ldots)$ is a p-sequence, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$.

(a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n?

(b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

Answer. Yes to both.

Solution. We see that $a_{n+1} = a_n + d_p(a_n) \equiv 2a_n$ so similarly $d_p(a_{n+1}) \equiv 2d_p(a_n)$.

Let c be the order of 2 modulo p. Then

$$(d_p(a_0), \cdots, d_p(a_{c-1})) = (d_p(a_0), 2d_p(a_0), \cdots, 2^{c-1}d_p(a_0)) \pmod p$$

and similarly,

$$(d_p(a_n), \cdots, d_p(a_{n+c-1})) = (d_p(a_0), 2d_p(a_n), \cdots, 2^{c-1}d_p(a_n)) \pmod p$$

for every n. Moreover, $d_p(a_c) = d_p(a_0)$. Therefore for each n,

$$a_{n+c} - a_n = \sum_{i=0}^{c-1} d_p(a_{n+i}) = \sum_{i=0}^{c-1} d_p(a_i)$$

is the same for all n.

Let's also investigate the properties of $\{d_p(a_0), \cdots, a_p(a_{c-1})\}$. Treating the nonzero residues modulo p as a group G, the subgroup $H = \{2^k : k = 0, \cdots, c-1\}$ paritions H into cosets, where $\{d_p(a_0), \cdots, a_p(a_{c-1})\}$ is one of them. Thus $\sum_{i=0}^{c-1} d_p(a_i)$ can also be viewed as a coset sum for the sequence (a_n). Now we can proceed with the following.

(a) We just need to make sure that:

- (a_n) and (b_n) have the same coset sum.
- There exists i and j with $a_i < b_i$ and $a_j > b_j$, which, with the condition before, $a_{i+cn} < b_{i+cn}$ and $a_{j+cn} > b_{j+cn}$ for all n.

We will in fact choose a_0 and b_0 such that they come from the same coset, and must have the same coset sum.

Consider any $p \equiv 1 \pmod 4$, and let $a_0 = 2^k$ for some k such that $a_0 < p < 2a_0$. With $p \geq 5$, $a_n \geq 4$ so $2a_0 \geq p+3$ (since $4 \mid p+3$ and $p \equiv 1 \pmod 4$). Let $b_0 = p+1$, and then $b_1 = p+2$. We now have

$$a_0 < p < p+1 = b_0 \qquad b_1 = p+2 < p+3 \leq 2a_0 = a_0 + d_p(a_0) = a_1$$

and notice that both a_0 and b_0 belong to the coset defined as

$$\{2^k : 0 \leq k \leq c-1\}$$

completing the construction. Finally, we note that there are infinitely many such primes p.

(b) We claim that all we need is having a_0 and b_0 such that the coset sum of a_0 is greater than that of b_0. Suppose that this condition is fulfilled. By adding p to b_0 as many times as we want, we may assume $a_0 < b_0$.

Consider the number

$$g = \min\{a_n - b_n : n = 0, 1, \cdots, c-1\}$$

and let coset sum of (a_n) and (b_n) to be x and y respectively with $x > y$. Then for all k and i,

$$a_{i+ck} - b_{i+ck} = (a_i + kx) - (b_i + ky) = k(x-y) + (a_i - b_i) \geq g + k(x-y)$$

so with $x - y > 0$, for all k sufficiently large and $i \geq 0$ we have $a_{i+ck} - b_{i+ck}$. This also means that for all n sufficiently large we have $a_n > b_n$. This means that there's a maximal index m such that $a_m < b_m$ (but $a_n > b_n$ for all $n > m$). All we need to do now is to shift both sequences by m spots to the left and replace a_0 with a_m and b_0 with b_m, which gives $a_0 < b_0$ but $a_n > b_n$ for all $n > 0$.

It remains to show that we can indeed find infinitely many p with at least two cosets of different sum. We claim that all $p \equiv 7 \pmod 8$ fulfills this property. First, we quote a well-known identity that if $1 \equiv 1, 7 \pmod 8$, then 2 is a quadratic residue modulo 8. This means the number of cosets is even.

On the other hand, the sum of all cosets is

$$1 = 2 + \cdots + (p-1) = \frac{p(p-1)}{2}$$

and since $p \equiv 3 \pmod 4$ here, $\frac{p-1}{2}$ is odd and so is $\frac{p(p-1)}{2}$. Therefore with even number of cosets, the cosets cannot all have the same sum.

N5 Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

i. $f(n) \neq 0$ for at least one n;
ii. $f(xy) = f(x) + f(y)$ for every positive integers x and y;
iii. there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

Answer. $f(n) = c \cdot v_p(n)$ where p is a prime number, c any nonnegative constant and $v_p(n)$ the greatest power of p dividing n.

Solution. We first notice that if we prime factorize

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

then

$$f(n) = \sum_{i=1}^{k} \alpha_k f(p_k)$$

i.e. the function f is determined solely by the values of the primes.

We need at least one prime p to have $f(p) > 0$. On the other hand, we shall also prove that such p is unique.

Consider now, any such n such that $f(k) = f(n-k)$ for all $k < n$. Then for each k we have:

$$f(1) + f(2) + \cdots f(k) = f(k!) \qquad f(n-1) + \cdots + f(n-k) = f((n-1)\cdots(n-k))$$

and with $k!\binom{n-1}{k} = (n-1)\cdots(n-k)$, we have $f(\binom{n-1}{k}) = 0$. In other words, if p is one such prime with $f(p) \neq 0$, then $p \not| \binom{n-1}{k}$ for all $k \leq n-1$. By Lucas' theorem, this happens only when for all $k \leq n-1$, the p-ary representation of k has all digits at most the corresponding digits of $n-1$ when written in base p. On the other hand, if a non-leading digit (say p^u) of $n-1$ is not maximal (say, $v < p-1$) then pick $k = (v+1) \cdot p^u$ and the p^u digit of k is greater than that of $n-1$ but $k < n-1$ since p^u is not the leading digit of $n-1$.

Therefore $n-1$ must have $p-1$ in all its digits base p except for the leading digit, which means there exists $1 \leq m \leq p-1$ and a such that $n = m \cdot p^a$.

Now suppose that there are two primes p and q with $f(p)$ and $f(q)$ both nonzero. Then there exists m, r with $1 \leq m \leq p-1$ and $1 \leq r \leq q-1$, and $a, b \geq 0$ with $n = m \cdot p^a = r \cdot q^b$. Suppose also that $p \leq q$. Then $m < q$ and it would follow that $q \not| m\dot{p}^a$ so $b = 0$, or $n = r < q$. This means that the condition (iii) would only hold for finitely many n since they are all less than q, which is a contradiction.

Therefore only one prime p can have $f(p) \neq 0$. For this p, by using prime factorization of an integer n would yield $f(n) = f(p)v_p(n)$. To show that (iii) can be fulfilled, consider $n = p^\alpha$ for any $\alpha \geq 0$. Then for each $k < n$, if $b < \alpha$ is the highest power of p dividing k then $k = p^b \ell$ for some ℓ with ℓ not divisible by p. Then $n - k = p^b(p^{\alpha-b} - \ell)$ so $v_p(n-k) = b$, too, as claimed.

N6 For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime to n. Does there exist a constant C such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leq C$$

for all $n \geq 1$

Answer. No.

Solution. We first prove the following claim:

Lemma. Let k be an arbitrary positive integer. Then there exists an N such that if we take the first N primes $p_1, \cdots, p_N$, at least k of those p_i's is such that for all $j = 1, \cdots, N$, $p_i \nmid p_j - 1$.

Proof: In this collection, if p_i is odd and $p_i \geq \frac{p_N}{2}$ then $p_j - 1$ is either 1 or has prime divisors that are 2 and others at most $\frac{p_j - 1}{2} > p_i$. Thus it suffices to show that

N can be chosen such that at least k of those are at least $\frac{p_N}{2}$. By prime number theorem there exists function $f(N) \in o(1)$ such that the number of primes up to N is $(1+f(N))\frac{N}{\log N}$. Thus, when considering N and $2N$, big enough with $|f(N)| < 0.01$ and $|f(2N)| < 0.01$ for each of those we have the number of primes between N and $2N$ as

$$(1+f(2N))\frac{2N}{\log(2N)} - (1+f(N))\frac{N}{\log N} \geq 0.99\frac{2N}{\log 2 + \log(N)} - 1.01\frac{N}{\log N}$$
$$\geq \frac{0.95N}{\log N}$$

for N sufficiently big (to cover the addition $\log 2$ in denominator). Thus we can choose N that satisfies the aforementioned constraint, and such that $\frac{0.95N}{\log N} \geq k$ (here log is natural log).

Turning back to the problem, consider the collection $p_1, \cdots, p_\ell, q_1, \cdots, q_k$ as the first $\ell + k$ primes (in some order) such that $q_i - 1$ and $p_j - 1$ are not divisible by $q_x - 1$ for all i, j, x. Consider sequences $a_1, \cdots, a_\ell$ and the number

$$n = \prod_{i=1}^{\ell} p_i^{2^{a_i}-1} \prod_{j=1}^{k} q_j$$

Then

$$d(n) = 2^k \prod_{i=1}^{\ell} 2^{a_i} \quad \varphi(n) = \prod_{i=1}^{\ell} (p_i - 1) p_i^{2^{a_i}-2} \prod_{j=1}^{k} (q_j - 1)$$

By our construction, $p_i - 1$ and $q_j - 1$ are divisible by none of q_x's, and since we're taking the first $n + k$ primes, $p_i - 1$ and $q_j - 1$ must have prime divisors that are only in the p_x's. Thus let's name

$$\prod_{i=1}^{\ell}(p_i - 1) \prod_{j=1}^{k}(q_j - 1) = \prod_{i=1}^{\ell} p_i^{b_i}$$

and we have

$$\prod_{i=1}^{\ell}(p_i - 1)p_i^{2^{a_i}-2} \prod_{j=1}^{k}(q_j - 1) = \prod_{i=1}^{\ell} p_i^{2^{a_i}+b_i-2}$$

Next, since $d(n)$ is power of 2, $\varphi(d(n)) = \frac{1}{2}d(n)$ and so

$$\varphi(d(n)) = \varphi(2^k \prod_{i=1}^{\ell} 2^{a_i}) = 2^{k-1+\sum_{i=1}^{k} a_i}$$

$$d(\varphi(n)) = d\left(\prod_{i=1}^{\ell}(p_i - 1)p_i^{2^{a_i}-2} \prod_{j=1}^{k}(q_j - 1)\right) = \prod_{i=1}^{\ell}(2^{a_i} + b_i - 1)$$

so our ratio is now $2^{k-1}\prod_{i=1}^{\ell} \frac{2^{a_i}}{2^{a_i}+b_i-1}$. By choosing a_i large, each $\frac{2^{a_i}}{2^{a_i}+b_i-1}$ can stay close to 1 (while ℓ is finite), so we can make the ratio to be arbitrarily close to 2^{k-1}.