Algebra

A1 (IMO 1) Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Answer. Such f is a constant (say c), where $c = 0$ or $1 \le c < 2$.

Solution. We first verify our solution, noting that LHS is the constant c, while the RHS is $c\lfloor c \rfloor = c$ since $c = 0$ or $\lfloor c \rfloor = 1$. Now plugging $x = y = 0$ gives

$$f(0) = f(0)\lfloor f(0) \rfloor \tag{1}$$

i.e. $f(0)(\lfloor f(0) \rfloor - 1) = 0$. This means either $f(0) = 0$ or $\lfloor f(0) \rfloor = 1$.

In the second case with $\lfloor f(0) \rfloor = 1$, plugging just $y = 0$ gives $f(0) = f(x)\lfloor f(0) \rfloor = f(x)$, so $f(x) = f(0)$ for all x and we have $f \equiv c$ for some constant $c \in [1, 2)$.

Now consider the case $f(0) = 0$. Let $0 \le x < 1$ be arbitrary, then $\lfloor x \rfloor = 0$ so in this case plugging $y = x$ gives $0 = f(0) = f(x)\lfloor f(x) \rfloor$, meaning that $f(x) = 0$ or $\lfloor f(x) \rfloor = 0$, though the first condition actually means $\lfloor f(x) \rfloor = 0$ too. Therefore $0 \le x < 1$ means $\lfloor f(x) \rfloor = 0$ for all x. Now for any real number z, choose an integer x such that $|x| > |z|$ and both x and z have the same sign. Then $0 < z/x < 1$. Let $y = z/x$ and we have $f(z) = f(xy) = f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor = f(x)(0) = 0$, so $f \equiv 0$.

A2 Let the real numbers a, b, c, d satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

$$36 \le 4\left(a^3 + b^3 + c^3 + d^3\right) - \left(a^4 + b^4 + c^4 + d^4\right) \le 48.$$

Solution. For the right inequality, all we need is the following AM-GM inequality:

$$\begin{aligned}
\frac{a^4 + b^4 + c^4 + d^4 + 48}{2} &= \frac{(a^4 + 4a^2) + (b^4 + 4b^2) + (c^4 + 4c^2) + (d^4 + 4d^2)}{2} \\
&\ge \sqrt{4a^6} + \sqrt{4b^6} + \sqrt{4c^6} + \sqrt{4d^6} \\
&\ge 2|a^3| + 2|b^3| + 2|c^3| + 2|d^3| \\
&\ge 2a^3 + 2b^3 + 2c^3 + 2d^3
\end{aligned} \tag{2}$$

rearranging gives $4\left(a^3 + b^3 + c^3 + d^3\right) - \left(a^4 + b^4 + c^4 + d^4\right) \le 48$.

For the left inequality, we notice the following: $(a-1)^4 = (a^4 - 4a^3) + 6a^2 - 6a + 1$ and if we denote $S_k = a^k + b^k + c^k + d^k$ we have

$$(a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4 = S_4 - 4S_3 + 6S_2 - 6S_1 + 4 = S_4 - 4S_3 + 52 \tag{3}$$

using $S_1 = 6$ and $S_2 = 12$. This means,

$$4S_3 - S_4 = 52 - \left[(a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4\right]$$

On the other hand, $(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 = S_2 - 2S_1 + 4 = 12 - 2(6) + 4 = 4$. Considering the following identity for all reals $x_1, \cdots, x_n$

$$\left(\sum_{i=1}^{n} x_i^2\right)^2 = \sum_{i=1}^{n} x_i^4 + 2\sum_{i<j} x_i^2 x_j^2 \ge \sum_{i=1}^{n} x_i^4$$

(since all squares are nonnegative), we have

$$(a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4 \le ((a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2)^2 = 4^2 = 16$$

so $4S_3 - S_4 \ge 52 - 16 = 36$, as desired.

A4 A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$ $x_1 + x_2 + \ldots + x_n \geq 0$.

Solution. Let $s_n = x_1 + x_2 + \ldots + x_n$. Suppose on the contrary, there exists n such that $s_n < 0$; consider the smallest such n. Note that all x_i's are ± 1, so it must be that $s_{n-1} = 0$ and $x_n = -1$. In addition, $s_n \equiv n \pmod 2$ since all terms are ± 1, so n must be odd.

We now follow up with the following observations:

- $x_{4k+1} = (-1)^{2k+1+1}x_{2k+1} = x_{2k+1}$ while $x_{4k+2} = -x_{2k+1}$, so $s_{4k} = s_{4k+2}$ for all $k \geq 0$.
- $x_{4k-1} = (-1)^{2k+1}x_{2k} = -x_{2k} = x_{4k}$, so we actually have:

$$s_{4k} = \sum_{i=1}^{k} x_{4i-3} + x_{4i-2} + x_{4i-1} + x_{4i} = \sum_{i=1}^{k} x_{2i-1} - x_{2i-1} - x_{2i} - x_{2i} = -2\sum_{i=1}^{k} x_{2i} = 2\sum_{i=1}^{k} x_i = 2s_k$$

In particular, if $s_{4k} = 0$, so is s_k and therefore k is even.

If $n-1 = 4k$ for some k, then $0 = s_{n-1} = s_{4k} = s_k$ and $s_n = x_n = x_{4k+1} = (-1)^{2k+2}x_{2k+1} = x_{2k+1} = (-1)^{k+2}x_{k+1} = x_{k+1}$ (since k is even), and if $n-1 = 4k+2$ for some k then $0 = s_{4k+2} = s_{4k} = s_k$ and $s_n = x_n = x_{4k+3} = (-1)^{2k+3}x_{2k+2} = -x_{2k+2} = x_{k+1}$. In both cases we have $s_n = x_{k+1}$ and since $s_k = 0$, $x_{k+1} = s_{k+1}$ too. Since $s_n = -1$, we must have $s_{k+1} = -1$, too. By the minimality of n, we have $4k+1 \leq n \leq k+1$, which can only happen when $k = 0$. However, when $k = 0$, we have $s_1 = x_1 = 1$, contradiction. This means s_n must be nonnegative for all n.

A5 Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2y\right) = x^3 f(xy).$$

Answer. $f(x) = \frac{1}{x}$, which is rational for rational x and both sides equal to $\frac{x^2}{y}$.

Solution. We split into the following steps:

Step 1: injectivity of f and $f(1) = 1$. If $f(x_0) = f(x_1)$, then plugging $x = x_0, x_1$ and $y = 1$ for both yields

$$x_0^3 f(x_0) = f\left(f(x_0)^2\right) = f\left(f(x_1)^2\right) = x_1^3 f(x_1)$$

so $x_0^3 = x_1^3$, and therefore $x_0 = x_1$. This means f is injective. With $x = 1$ and injectivity, we have $y = f(1)^2 y$, and so $f(1) = 1$.

Step 2: multiplicativity of f. Now, consider $y = \frac{1}{x}$, then $f\left(\frac{f(x)^2}{x}\right) = x^3$. Meanwhile, set $y = \frac{1}{f(x)^2}$ and we have $1 = f(1) = x^3\left(\frac{x}{f(x)^2}\right)$. Changing x to $\frac{1}{x}$ we have

$$f\left(\frac{f(x)^2}{x}\right) = x^3 = \left(\frac{1/x}{f(1/x)^2}\right)$$

and so, by injectivity again,

$$\frac{1/x}{f(1/x)^2} = \frac{f(x)^2}{x}$$

which means that $f(x)f(1/x) = 1$ for all $x \in \mathbb{Q}$. In particular, $x_1x_2y_1y_2 = 1$ then $f(x_1x_2)f(y_1y_2) = 1$. To use this to our advantage, we have

$$f\left(f(x_1)^2y_1\right) f\left(f(x_2)^2y_2\right) = x_1^3 f(x_1y_1)x_2^3 f(x_2y_2) = x_1^3x_2^3$$

or, after substituting $y_1 = \frac{f(x_2)^2}{x_1x_2}, y_2 = \frac{1}{f(x_2)^2}$, we have

$$f\left(\frac{f(x_1)^2 f(x_2)^2}{x_1x_2}\right) = x_1^3x_2^3$$

If we substitute x_1 with x_1x_2 and x_2 with 1, we also have

$$f\left(\frac{f(x_1x_2)^2f(1)^2}{x_1x_2}\right) = x_1^3x_2^3$$

and by injectivity again, $f(x_1)f(x_2) = f(x_1x_2)$, as claimed.

Step 3: establish $f(x) = \frac{1}{x}$. To do this, we focus on $f(f(x))^2 = f(f(x)^2) = x^3f(x)$, the left allowed given the multiplicity of f.

Denote (a, b) as good if $x^af(x)^b \in \mathbb{Q}$; this would hold if a and b are both integers. Next, given that $f(f(x)) = x^{3/2}f(x)^{1/2}$, if (a, b) is good then we have

$$f(f(x^af(x)^b)) = (x^af(x)^b)^{3/2}f(x^af(x)^b)^{1/2} = x^{3a/2}f(x)^{3b/2}f(x)^{a/2}f(f(x))^{b/2}$$
$$= x^{3a/2}f(x)^{3b/2}f(x)^{a/2}(x^{3/2}f(x)^{1/2})^{b/2} = x^{3a/2+3b/4}f(x)^{a/2+3b/2+b/4} = x^{3a/2+3b/4}f(x)^{a/2+7b/4}$$

is also in $\mathbb{Q}$. It follows that (a, b) is good implies that $\left(\frac{3a}{2} + \frac{3b}{4}, \frac{a}{2} + \frac{7b}{4}\right)$ is also good. This motivates us to consider the matrix

$$M = \begin{pmatrix} \frac{3}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{7}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 6 & 3 \\ 2 & 7 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{9}{4} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^{-1}$$

Now if v is a vector of good pair, then Mv is good. It then follows that M^kv is good for all $k \geq 0$. But we also have

$$M^k = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{9^k}{4^k} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & \frac{9^k}{4^k} \\ -2 & \frac{9^k}{4^k} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 + 2 \cdot \frac{9^k}{4^k} & -3 + 3 \cdot \frac{9^k}{4^k} \\ -2 + 2 \cdot \frac{9^k}{4^k} & 2 + 3 \cdot \frac{9^k}{4^k} \end{pmatrix}$$

In particular, take v to represent $(1, 0)$, then $\frac{1}{5}(3+2\cdot\frac{9^k}{4^k}.-2+2\cdot\frac{9^k}{4^k})$ is good. Since the good sequence is closed under scalar multiplication of integers and addition / subtraction of integer pairs, we have $(2 \cdot \frac{9^k}{4^k}.2 \cdot \frac{9^k}{4^k})$ is good. That is, $(xf(x))^{2\cdot\frac{9^k}{4^k}} \in \mathbb{Q}$. We conclude that $xf(x)$ has to be a perfect $\frac{4^k}{2}$-th power of rationals, which can only happen when $xf(x) = 1$.

A6 Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers.

Prove that $f(n) = g(n)$ for all positive integer n.

Solution. We first notice the following identity: for any integers m, n we have

$$f(m) = f(n) \to g(f(m) = g(f(n))) \to g(m) = g(n) \tag{4}$$

For each integer n, consider the following two sets:

$$S_n \triangleq \{b : f(b) = n\} \qquad T_n \triangleq \{b : f(b) = n\} \tag{5}$$

The problem is then equivalent to showing that $S_n = T_n$ for all n.

First, we consider the ranges of f and g. That is, the set $\mathsf{Range}(f) \triangleq \{n : S_n \neq \emptyset\}$. Let $m_f = \min_n\{S_n \neq \emptyset\}$ and $\mathsf{Range}(f) = \{n : n \geq m_f\}$. Then from the problem statement, $m \in S_n \to g(m) \in S_{n+1}$, so $S_n \neq \emptyset$ for all $n \geq m_f$. Considering the same for g we have

$$\mathsf{Range}(f) = \{n : n \geq m_f\} \qquad \mathsf{Range}(g) = \{n : n \geq m_g\} \tag{6}$$

We then show that for each n and m,

$$\text{either } S_n = T_m \text{ or } S_n \cap T_m = \emptyset \tag{7}$$

Consider S_n for any $n \geq m_f$, let $b \in S_n$ (so $f(b) = n$), and let $g(b) = m$ for some integer m. Then from 4, $c \in S_n \to f(b) = f(c) \to g(b) = g(c) \to c \in T_m$, and similarly $c \in T_m \to c \in S_n$. It then follows that $T_m = S_n$.

Next, let's show the following: for each set S_k:

- If $k = m_f$, then no element in f can be a value of g;
- If $k > m_f$, then exactly one element in S_k is a value of g.

To start with uniqueness, suppose $x, y \in S_k$ with $g(a) = x, g(b) = y$ (possibly, $x = y$). Then $f(a)+1 = f(g(a)) = f(x) = k = f(y) = f(g(b)) = f(b)+1$. This means $k \geq m_f + 1$, proving the first bullet point. Now, with $f(a) = f(b) = k-1$ we have $g(a) = g(b)$ too, and therefore $x = y$, showing that at most one element in S_k can be a value of g. To show existence for the second bullet point, for each $n \geq m_f$, we have $m \in S_n \to f(m) = n \to f(g(m)) = n+1 \to g(m) \in S_{n+1}$. so for each $n \geq m_f + 1$ there's one element in S_n that's a value in g.

We also notice that $x \in \mathsf{Range}(g)$ if and only if $x \geq m_g$. Therefore, for each $n \geq m_f$, denote the number

$$M_f(n) = \max_k \{k \in S_n\} \tag{8}$$

(we may define M_g similarly). Then we have $M_f(m_f) < m_g$, while $M_f(x) \geq m_g$ for all $x \geq m_f + 1$. But since $\cup_{i=m_f}^{\infty} S_i = \mathbb{N}$, each $k \geq m_g$ is in S_u for some u, and by our lemma we have $u \geq m_f + 1$, and $k = M_f(u)$. This means that

$$M_f(m_f) < m_g \qquad \{M_f(k) : k \geq m_f + 1\} = \{m_g, m_g + 1, \cdots\} \tag{9}$$

which, in a similar way, also implies

$$M_g(m_g) < m_f \qquad \{M_g(k) : k \geq m_g + 1\} = \{m_f, m_f + 1, \cdots\} \tag{10}$$

But since each pair of sets (S_m, T_n) are either equal or disjoint, and $\cup S_m = \cup T_n = \mathbb{N}$, $\{S_m\}$ and $\{T_n\}$ are just bijection of each other. This means the numbers $M_f(\cdot)$ and $M_g(\cdot)$ are also mappings (bijection) of each other. We therefore have

$$\{M_f(m_f)\} \cup \{m_g, m_g+1, \cdots\} = \cup_{k \geq m_f} M_k = \cup_{\ell \geq m_g} M_\ell = \{M_g(m_g)\} \cup \{m_f, m_f + 1, \cdots\} \tag{11}$$

and with $M_f(m_f) < m_g$ and $M_g(m_g) < m_f$ we have $M_f(m_f) = M_g(m_g)$, and consequently $m_f = m_g$, and also $S_{m_f} \cap T_{m_g} \neq \emptyset$ (and therefore equal). This means there's a number $b \in S_{m_f}$ such that $f(b) = m_f = m_g = g(b)$.

Finally, consider the following logical chain:

$$\begin{aligned} \forall b \geq m_f : S_b = T_b &\to f(M_f(b)) = f(M_g(b)) = g(M_g(b)) = g(M_f(b)) = b \\ &\to f(g(M_f(b))) = b+1 = g(f(M_g(b))) \to b \in S_{b+1} \cap T_{b+1} \to S_{b+1} = T_{b+1} \end{aligned} \tag{12}$$

and therefore with $S_{m_f} = T_{m_g}$ and $m_f = m_g$, we have $S_b = T_b$ for all $b \geq m_f$, as desired.

A7 (IMO 6) Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \leq s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

Solution. Let's first preprocess the sequence (which is unnecessary, but gives a clearer idea on how to solve this). Fix a real number M and consider $b_n = Mn - a_n$. Then for all $n > s$ we have the following recursion:

$$b_n = Mn - a_n = Mn - \max\{a_k + a_{n-k}\} = Mn - \max\{Mk - b_k + M(n_k)b_{n-k}\} = \min\{b_k + b_{n-k}\}$$

Thus our objective stays the same: showing there's an $\ell \leq s$ such thaat $b_n = b_\ell + b_{n-\ell}$. Next, consider:

$$M = \max\{\frac{a_k}{k} : 1 \leq k \leq s\} \quad \ell = \arg\max\{\frac{a_k}{k} : 1 \leq k \leq s\}$$

Then $b_\ell = 0$ and for all $1 \le k \le s$ we have $b_k \ge 0$. We now proceed to the following lemma:

Lemma. Denote S_n for all $n > s$ as the set containing sequences $(s_1, \cdots, s_m)$ with $1 \le s_i \le s$, $s_i + s_j > s$ for some i, j, and $s_1 + \cdots + s_m = n$. Then for all $n > s$:

$$b_n = \min\{\sum_{(s_1,\cdots,s_m)\in S_n} \sum_{i=1}^{m} b_{s_i}\}$$

Proof: we proceed by induction. For base case, $n = s+1$. Here we have

$$S_n = \{(1, s), (2, s-1), \cdots, (s, 1)\}$$

(the $s_i + s_j > s$ requirement means we cannot have more than two parts when $n = s+1$). By definition we have

$$b_n = \min\{b_1 + b_s, b_2 + b_{s-1}, b_s + b_1\}$$

which proves our base case.

Now suppose that this holds for $s+1, \cdots, n-1$. We have $b_n = b_u + b_v$ for some $1 \le u, v < n$. If $u \le s$ and $v \le s$ then $(u, v) \in S_n$. Otherwise, say $u > s$. By induction hypothesis we have

$$b_u = b_{s_1} + \cdots + b_{s_p}$$

for some p, where $\sum s_i = u$, and $s_i + s_j > s$ for some (i, j). We also have $b_v = b_{t_1} + \cdots + b_{t_q}$ for some q, and $\sum t_i = v$ (if $v \le s$ we just have $q = 1$ and $t_1 = v$). Thus

$$s_1 + \cdots + s_p + t_1 + \cdots + t_q = u + v = n \quad b_n = \sum b_{s_i} + \sum b_{t_j}$$

and $s_i + s_j > s$ so $(s_1, \cdots, s_p, t_1, \cdots, t_q) \in S_n$.

To show that this sequence $(s_1, \cdots, s_p, t_1, \cdots, t_q)$ does give the minimal sum (that is, $\sum b_{s_i} + \sum b_{t_j}$ is the minimal possible). Suppose otherwise, that

$$\sum_{i=1}^{w} b_{r_i} < b_n$$

for some $(r_1, \cdots, r_w) \in S_n$. If $w = 2$ then $r_1, r_2 < s$ but $r_1 + r_2 = n$ and so $b_{r_1} + b_{r_2} < b_n$. Otherwise, w.l.o.g. we have $r_1 + r_2 > s$. By induction hypothesis we have

$$b_{n-r_w} \le b_{r_1} + \cdots + b_{r_{w_1}}$$

and so

$$b_{n-r_w} + b_{r_w} \le b_{r_1} + \cdots + b_{r_{w_1}} + b_{r_w} < b_n$$

so in both cases there are u, v with $u+v = n$ and $b_u + b_v < n$, which contradicts $b_n = \min\{b_u + b_v : u+v = n\}$. Thus this induction step establishes the lemma. $\square$

Now (TODO)

A8 Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a+c+e = S$ and $b+d+f = T$. Prove that

$$2ST > \sqrt{3(S+T)\left(S(bd+bf+df) + T(ac+ae+ce)\right)}.$$

Solution. Denote $P = \sqrt{(a-c)^2 + (a-e)^2 + (c-e)^2}$, and $Q = \sqrt{(b-d)^2 + (b-f)^2 + (d-f)^2}$. It then follows that

$$3(bd+bf+df) = T^2 - \frac{1}{2}Q^2 \qquad 3(ac+ae+ce) = S^2 - \frac{1}{2}P^2 \tag{13}$$

Thus taking squares on both sides and taking the difference gives

$$4S^2T^2 - (S+T)(S(T^2 - \frac{1}{2}Q^2) + T(S^2 - \frac{1}{2}P^2)) = \frac{1}{2}(S+T)(SQ^2 + PT^2) - ST(T-S)^2 \tag{14}$$

I.e. our task is to show that

$$S^2Q^2 + T^2P^2 + ST(P^2 + Q^2) > 2ST(T - S)^2 \tag{15}$$

Let's now show that $P + Q > \sqrt{2}(T - S)$. Indeed, with $a < c < e$ we have

$$\begin{aligned} P^2 &= (a-c)^2 + ((c-a) + (e-c))^2 + (c-e)^2 \\ &= 2((c-a)^2 + (e-c)^2 + (c-a)(e-c)) \\ &= 2[(c-a) + \frac{1}{2}(e-c)]^2 + \frac{3}{2}(e-c)^2 \\ &> 2[(c-a) + \frac{1}{2}(e-c)]^2 \end{aligned} \tag{16}$$

and similarly $Q^2 > 2[(f-d) + \frac{1}{2}(d-b)]^2$. Thus combining them gives

$$\begin{aligned} P + Q &> \sqrt{2}((c-a) + \frac{1}{2}(e-c) + (f-d) + \frac{1}{2}(d-b)) \\ &> \sqrt{2}(f - a - (e + c - d - b)) = \sqrt{2}(T - S) \end{aligned} \tag{17}$$

The inequality is due to that $e > d$ and $c > b$. It then follows that

$$S^2Q^2+T^2P^2+ST(P^2+Q^2) = (SQ-TP)^2+ST(P+Q)^2 > (SQ-TP)^2+2ST(T-S)^2 \geq 2ST(T-S)^2 \tag{18}$$

as desired.

Remark. In the case of non-strict inequality, i.e. $a \leq b \leq c \leq d \leq e \leq f$, we see a necessary and sufficient condtion for equality is when $e = c$, $d = b$, and $SQ = TP$. Thus $b = c = d = e$, and given this, $(a + 2c)(f - c) = (2c + f)(c - a)$.

Let's describe how did we get to this – first, it's natural to expand the RHS while keeping the $bd + bf + df$ term intact, and subtracting S^2T^2 by $3S^2(bd + bf + df)$ does give $\frac{1}{2}S^2Q^2$ (and similarly to the $\frac{1}{2}T^2P^2$ out there). The remaining ones are $ST(2ST - 3(bd + bf + df + ac + ae + ce))$, which then rearranges into the terms above.

The next step is to compare $T - S$ and the magnitude of P, Q, where the intuition tells us that P, Q cannot be too small since a, b, c, d, e, f are in that order. (Another way to see it: how do we bound $\mu_y - \mu_x$ in terms of σ_y, σ_x where $\mu_x, \mu_y, \sigma_x, \sigma_y$ are the mean / standard deviation of $\{a, c, e\}$ and $\{b, d, f\}$ respectively). Also by keeping track the pairwise difference of $b-a, c-b, d-c, e-d, f-e$ (by writing as, say, x_1, y_1, x_2, y_2, x_3) we can (probably) assme that $c = b$ and $e = d$ (while making constant shifts to a and f), so that P, Q decreases while $T - S$ stays. The lemma on $P^2 > 2[(c-a) + \frac{1}{2}(e-c)]^2$ is a further manifest on assuming that x_2 can be made 0 too (albeit a bit more technicality).

An initial hurdle is, of course, that S^2, T^2, ST seem to have no relation with $P, Q, T - S$. Fortunately, this is sorted out rather beautifully in the end.

Combinatorics

C1 In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

Answer. Yes.

Solution. TODO

C2 On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Answer. $M = 2^{N-2} + 1$.
Solution. $M = 2^{N-2}$ is not enough: consider all the 2^{N-2} distinct flags where the top field is blue and the bottom field is yellow. No matter how we choose the N flags and arrange them, the top top-left corner of the diagonal will be blue and the bottom-right yellow.

To show that $M = 2^{N-2}+1$ is enough, consider an arbitrary set of M flags. Consider a bipartite graph with one side being the M flags and the other side being the k fields. In a blue matching, an edge exists between a flag and the i-th node if and only if the i-field of the flag is blue. Define yellow matching similarly. The goal is to find either a blue or yellow matching such that all the nodes on the right (the fields) are matched. By Hall's marriage theorem, this is possible if and only if for any k with $1 \le k \le N$ and any subset of fields of size k, at least k of the flags have an edge to at least one of the k fields.

Suppoe the condition fails for blue matching. That is, there exists k fields such that less than k flags have at least one of the k fields being blue. W.L.O.G. let the k fieids be the first k fields. Now, we know that all but at most $k-1$ of the flags have the first k fields being yellow, while there are 2^{N-k} distinct flags with first k fields being yellow. Thus we have $2^{N-2} + 1 \le 2^{N-k} + k - 1$. Since $2^{N-k} < 2^{N-k+1}$ for all real numbers k, for $k \le N$, both 2^{N-k} and 2^{N-k+1} are integers, we have $2^{N-k} \le 2^{N-k+1} - 1$ and therefore $2^{N-k} + k - 1 \le 2^{N-k+1} - 1 + k - 1 = 2^{N-(k-1)} + (k-1) - 1$, which means that the function $2^{N-k} + k - 1$ is decreasing. But when $k = 2$ we have $2^{N-k} + k - 1 = 2^{N-2} + 1$, so we have $k = 2$ here. We now split into cases:

- If $k = 1$, then all the M flags have top field yellow. Consider, now, the Hall condition on the yellow matching, and consider an arbitrary subset of ℓ fields. If this ℓ fields contain the first field then all (i.e. $2^{N-2} + 1 \ge N$) flags match to the first field and the Hall condition is easily satisfied. Otherwise, among the 2^{N-1} flags with first field yellow, there are only $2^{N-1-\ell}$ flags with all of the ℓ fields blue. In addition, $2^{N-2} + 1 - (2^{N-1-\ell} + \ell) \le 2^{N-2} + 1 - (2^{N-1-1} + 1) = 0$ (we use the fact from above that $2^{N-\ell} + \ell$ is decreasing), hence always nonnegative. This means, there will always be ℓ flags with at least one of the ℓ fields yellow.
- If $k = 2$, then all (exept possible one) flags have top two fields yellow, meaning we have at least 2^{N-2} flags with top two fields yellow. However, there are only 2^{N-2} distinct flags with the top two fields yellow, so we have all the distinct collection of flags with the first two fields yellow. Now, choose two of the 2^{N-2} flags arbitrarily, and for $3 \le i \le N$, the i-th flag is choosen as the one with 1st, 2nd, i-th field yellow and the rest blue. This gives the desired diagonalization.

C3 2500 chess kings have to be placed on a 100×100 chessboard so that

- no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
- each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Answer. There are (only) two such arrangements.
Solution. Now, parition our chessboard into 2500 2×2 squares and call each of them megasquare. We call the 2×100 squares containing the 50 megasquares a megarow, and define a megacolumn similarly. Within each megasquare, any two of the four squares share a vertex, so one megasquare cannot have more than a king. Since we need 2500 chess kings in total, each megasquare must have exactly a king.

Now, we have four types of megasquares: top-left (TL), top-right (TR), bottom-left (BL), bottom-right (BR), depending where we put our king. TL megasquares and TR megasquares are called T-megasquares, define B-, L-, R- megasquares similarly. We need the following lemmas:

Step 1. Each megarow consists of 25 T-megasquares and 25 B-megasquares. This is because each row contains exactly 25 kings. Similarly, each megacolumn consists of 25 L-megasquares and 25 R-megasquares. This means the number of T-, B-, L-, R-megsquares are each $25 \times 50 = 1250$.

Step 2. Consider any B-megasquare M_1, and consider either of the two bottom squares of the megasquare (the bottom row). If there is a megasquare M_2 located below this megasquare, each square of the top row of this megasquare will have a common vertex with the bottom row of the original megasquare. Hence, this megasquare must also be a B-megasquare.

Step 3. With steps 1 and steps 2 in mind, consider the top megarow, with 25 of them being B-megasquares. For each of the B-megasquares, by the lemma above, all the 49 megasquares below it (in the same row) must also be B-megasquares, so these 25 megarows will have all B-megasquares. These 25 megarows give rise of 25×50 megasquares, and therefore the remaining 25 megarows must all be T-megasquares. This means the notion of T- and B-megacolumns are well-defined: the megarow with either all T-megasquares or B-megasquares. Similarly the notion of L- and R-megarows are also well-defined.

Step 4. Obviously, the intersection of a T-megacolumn and a L-megarow is a TL-megasquare (and similarly for other combinations). Now that there are 25 T- and 25 B-megacolumns, we consider the adjacent megarows of different types: either T-megacolumn followed by B-megacolumn or vice versa. Consider also the adjacent megarows of different type: either L-megarow followed by R-megrow or vice versa. Their intersections together give rise of 4 megasquares. The bottom-right corner of the top-left megasquare shares a vertex with the top-left corner of the bottom-right corner, so the condition that top-left megasquare is BR and the bottom-right megasquare is TL cannot happen simultaneously: in other words, we cannot have B-megacolumn left to T-megacolumn and R-megarow above L-megarow simultaneously (call this a BT-switch and RL-switch). Similarly, by considering the top-right and bottom-left megasquares we cannot have TB-switch and LR-switch simultaneously.

Step 5. Now consider the 50 megacolumns from left to right. If the switch happens more than once, then the switches must alternate, which means both TB- and BT-switch must happen. Given that either LR- or RL-switch must also happen, this violates our finding in step 4. Hence only once switch can happen, which means that the leftmost 25 megacolumns are T-types and rightmost 25 megacolumns B-types, or vice versa. Similarly, only one switch (LR- or RL) can happen, which means the uppermost 25 megarows are L and bottommost R, or vice versa. In the case where the leftmost 25 megacolumns are T-types and rightmost 25 megacolumns B-types, TB-switch happens so LR-switch cannot happen: it must be an RL switch. In the other case, BT-switch happens so we must only have RL-switch. Each of these cases give rise of one configuration, so two configurations are possible.

To show that these two configurations work, each megasquare has only a king, so any two kings that could possibly attack each other must be put in two different megasquares. These two megasquares must also share a vertex, which gives two main cases to consider:

- One is above the other, hence in the same megacolumn. By our construction, each megacolumn is either T-type of B-type, so either both are on the top row of their respective megacolumns or on the bottom row, hence cannot attack each other. The case where one is to the left of the other is completely analogous.
- The two megasquares share exactly one vertex, and WLOG (the other case is similar) let's say one is on the top left of the other. If the kings were to attack, then we have BT-switch and RL-switch simultaneously. Both our configurations avoid that.

C5 $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i-th player. Prove that

$$\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0.$$

Solution. Consider a directed graph G where for each pair of nodes u and v, either $u \to v$ or $v \to u$ holds but not both. We consider $u \to v$ as symbolizing u beats v, then w_u, l_u are the outdegrees and indegrees of node u, respectively. For a disjoint subset A, B of G, denote $A \to B$ if for all $a \in A, b \in B$ we have $a \to b$.

Denote d_u as $w_u - l_u$, then we need to show that $\sum_u d_u^3 \geq 0$. In this directed graph the notion of bad company can be defined similarly. For convenience (probably not needed in this problem) we may define a good company of four nodes as the opposite: where a node has outgoing edge to all three other nodes, and the remaining three nodes form a directed cycle of length 3 on its own.

We first show the following auxilliary observations.

Lemma 1. *A graph with n vertices has $\sum d_u^2 \leq \frac{n^3-n}{3}$.*

Proof. A quick calculation yields that this is the count in the case we have a directed acyclic graph (henceforth called DAG), where we may order the vertices in $1, 2, \cdots, n$ and $u \to v$ if $u < v$ (topological sort), giving the count d_u as $n-1, n-3, \cdots, -(n-3), -(n-1)$. We show that if a graph attains its maximum for $\sum d_i$ then it has to be acyclic.

Suppose we have a directed cycle of length ≥ 3, then considering the cycle itself there exist vertices $u \to v$ in the cycle and we have $d_u \leq d_v$. By changing the direction from $u \to v$ to $v \to u$, the count $\sum d_u^2$ has nett change $[(d_u-2)^2 - d_u^2] + [(d_v-2)^2 - d_v^2] = 8 - 4d_u + 4d_v > 0$, i.e. $\sum d_i^2$ is not maximized. $\square$

Lemma 2. *Let G be such a directed graph, partitioned into A and B. Suppose that:*

- *A has no directed cycle of size 3;*
- *There do not exist $a \in A$, and $b_1 \neq b_2 \in B$, such that $b_1 \to a, a \to b_2, b_1 \to b_2$.*

Then A is a DAG, and B is a block DAG in the following sense: consider the relation $u \sim v$ for $u, v \in B$ if for all $a \in A$, either we have both $u \to a$ and $v \to a$, or both $a \to u$ and $a \to v$. Then the blocks (not the individual nodes) also form a DAG in the following sense: $U \to V$ means that for each $u \in U$ and $v \in V$, we have $u \to v$.

Proof. We first show that the nodes in A is DAG. Otherwise, consider the smallest one $u_1 \to u_2 \to \cdots \to u_k \to u_1$ for some $k \geq 3$. If $k > 3$, then we either have $u_2 \to u_k$ or $u_k \to u_1$. In the first case we have a smaller cycle $u_1 \to u_2 \to u_k \to u_1$; in the second case $u_2 \to u_3 \to \cdots \to u_k \to u_2$, again another smaller cycle. These contradict the minimality, so the fact that there is no directed cycle of size 3 means that there is no directed cycle at all, and therefore we may use topological sort on A.

Next, we show that B is a block DAG. We first see that each direct cycle must all be in the same block: otherwise, choose $a \in A$ and a cycle $u_1 \to u_2 \cdots \to u_k \to u_1$ such that $a \to u_i$ and $u_j \to a$, then somewhere in the cycle we have $u_i \to a, a \to u_{i+1}$ (indices taken modulo k), which is a contradiction. To see why for two different blocks U and V we must have either $u \to v$ for all $u \in U$ and $v \in V$, or $v \to u$ for all $u \in U, v \in V$, w.l.o.g. there is an $a \in A$ such that $a \to U$, $V \to a$ ($\to$ is well-defined here due to equivalence relation). Then we automatically have $u \to v$ for all $u \in U$ and $v \in V$. These two observations together establish our claim.

$\square$

Lemma 3. *Consider A and B as per the previous lemma, and in addition suppose that G has no bad company; Denote $D = |A| - |B|$. Then $\sum_{i \in A} d_i^2 - \sum_{j \in B} d_j^2 \geq \frac{D^3 - D}{3}$.*

Proof. Notice that A is a DAG, so when $|B| = 0$, $\sum_{i \in A} d_i^2$ is maximized as per 1. For the case where $|B| > 0$, perform induction on the number of blocks of B. We use the previous case where there's 0 block as base case.

Now for induction hypothesis, we consider what happens as we add a block W of size w to B, and suppose that for all blocks $U \in B$ we have $U \to W$. We further subdivide A into A_1 and A_2, such that $A_1 \to W$ and $W \to A_2$. By the problem condition, for all U that's part of block B, with $U \to W$ we have $A_1 \to U$, too (which essentially means $A_1 \to B$). This gives a quick inequality on $\sum_{a_1 \in A_1} d_{a_1} \geq |A_1|(|B| - |A_2|)$ (i.e. the minimum / worst case being $A_2 \to A_1$). Since $D = |A_1| + |A_2| - |B|$ we consider the following notation on e_u:

- If $u \in A$ or $u \in B \backslash W$, e_u is the degree not considering W;
- If $u \in W$, e_u denotes the degree considering the vertices only in W (so $d_u = e_u + |A_2| - |A| - |B|$).

Denote $B' = |B\backslash W|$, and let $D' = |A| - |B'|$, so $D = D' - w$. Note that $\sum_{w \in W} e_w = 0$ and $\sum_{u \in A \cup B'} e_u = 0$. We'll also see later that $|A_2| - |A_1| - |B| = D' - 2|A_1|$ and $|B| - |A_2| = |A_1| - D'$. This gives

$$\begin{aligned}
\sum_{a \in A} d_a^2 - \sum_{b \in B} d_b^2 &= \sum_{a_1 \in A_1} (e_{a_1} + w)^2 + \sum_{a_1 \in A_1} (e_{a_2} - w)^2 - \sum_{b \in B \backslash W} (e_b + w)^2 - \sum_{w \in W} (e_w + |A_2| - |A_1| - |B|)^2 \\
&= \sum_{a \in A} (e_a^2 + w^2) - \sum_{b \in B'} (e_b^2 + w^2) - \sum_{w \in W} e_w^2 + (|A_2| - |A_1| - |B|)^2 \\
&+ 2w(\sum_{a_1 \in A_1} e_{a_1} - \sum_{a_2 \in A_2} e_{a_2} - \sum_{b \in B'} e_b) \\
&= w^2(|A| - |B'|) - w(|A_2| - |A_1| - |B|)^2 + 4w \sum_{a_1 \in A_1} e_{a_1} + \sum_{a \in A} e_a^2 - \sum_{b \in B'} e_b^2 - \sum_{w \in W} e_w^2 \\
&\geq D'w^2 - w(D' - 2|A_1|)^2 + 4w|A_1|(|A_1| - D') + \frac{D'^3 - D'}{3} - \frac{w^3 - w}{3} \\
&= \frac{D'^3 + 3D'w^2 - 3D'^2 w - w^3 - (D' - w)}{3} \\
&= \frac{D^3 - D}{3}
\end{aligned}$$

as desired.

□

Now we can complete the solution by doing induction on n. For the graph with 4 nodes, the score function $\sum_u d_u^3$ is -24 if the company is bad, $+24$ is the company is good, and 0 for all other cases. (We can check manually that this count is always 0 with ≤ 3 nodes). Suppose that our main problem statement holds for a graph G with n vertices, which we now consider adding the $n+1$-th node v_{n+1} to form G'. Denote A, B the partition of G such that $A \to v_{n+1}$ and $v_{n+1} \to B$, then $d_{v_{n+1}} = |B| - |A|$, and if e_i denotes the nett degree considering v_{n+1} then we have

$$\sum d_u^3 = \sum_{a \in A} (e_a + 1)^3 + \sum_{b \in B} (e_b - 1)^3 + (|B| - |A|)^3 = \sum_{a \in A} (e_a^3 + 3e_a^2 + 3e_a + 1) + \sum_{b \in B} (e_b^3 - 3e_b^2 + 3e_b - 1) + (|B| - |A|)^3$$

$$\geq 3 \sum_{a \in A} e_a^2 - 3 \sum_{b \in B} 3_b^2 + |A| - |B| + (|B| - |A|)^3 \geq 0$$

where we used $\sum_{a \in A} e_a^3 + \sum_{b \in B} e_b^3 \geq 0$ (by induction hypothesis), $\sum_{a \in A} e_a + \sum_{b \in B} e_b = 0$, and 3.

C6 Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

(ii) Next, one cuts each chosen string into two parts differing in length by at most one.

(For instance, if there are strings of $5, 4, 4, 2$ black pearls, strings of $8, 4, 3$ white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4), (3, 2), (2, 2)$ and $(2, 2)$ respectively.)

The process stops immediately after the step when a first isolated white pearl appears. Prove that at this stage, there will still exist a string of at least two black pearls.

Solution. Throughout the solution we use black string (repsectively, white string) to denote string of black (respectively, white) pearls. Let's consider the moment right before the occurence of the first isolated white pearl, which we partition into the following cases. Denote M as the number of steps to reach this.

Case 1. The number of strings never exceeds k in steps $1, \cdots, M-1$. This means, up to step M, all the strings are cut. At step i, the shortest white pearl has length $\lfloor \frac{w}{2^i} \rfloor$. Therefore if m steps have been taken, then the shortest white string after step $m-1$ has length $\lfloor \frac{w}{2^{m-1}} \rfloor \geq 2$ and therefore $b > w \geq 2^{m-1}$, i.e. the shortest black string has length ≥ 2 too (i.e. no isolated black pearl). The longest black string has length $\lceil \frac{b}{2^{m-1}} \rceil \geq \lceil \frac{w+1}{2^{m-1}} \rceil \geq \lceil \frac{2^m+1}{2^{m-1}} \rceil = 3$, so after the next step (where we see the first isolated white pearl), this longest black string is either uncut (length ≥ 3) or cut into two parts, one of which has length at least $\lceil \frac{3}{2} \rceil$.

Case 2. The number of strings exceeds k at some point before $M-1$; let $N < M$ be the number of steps for this to happen. Intuitively, there will be enough white strings to 'shield' some black strings of length 2 (i.e. at critical length) from being cut at each stage, which we will show it rigorously.

We now establish the following:

Lemma 4. *Denote N as the moment where the number of strings exceed k for the first time. Then between after each step $\ell = N+1, \cdots, M$, the number of white strings is at least k.*

Proof. Up to step M, there's no isolated white string, so each cut of a white string only increases the number of white strings of length ≥ 2. Thus it suffices to show that at step $N+1$ we have at least k white string.

As per case 1, we can show that up to step N, all white strings have length ≥ 2 (by assumption), so do all black strings. In addition, each step $i \leq N$ sees 2^i cuts, resulting in 2^i white and 2^i black strings. At step $N+1$, k of the strings are being cut, at least $k - 2^i$ of which are white. It then follows that there are at least $k - 2^i + 2^i = k$ white strings after the procedure. $\square$

Now, to show the problem conclusion, modifying proof from Case 1 yields a black string of length ≥ 3 at step N, so a black string of length ≥ 2 remains after step $N+1$. We show that at each step $i = N+2, \cdots, M, M+1$, if there's a black string of length ≥ 2 at the beginning of step i then this still holds after step i. Consider the shortest non-isolated black string (if tied, choose the one placed last). If its length is ≥ 3, this string will change into another string of length ≥ 2 regardless if it's cut. Otherwise if it's length is ≥ 2, this string will be placed last among all strings of length ≥ 2, and since by our lemma there are at least k white strings, all of which are length ≥ 2, this string will not be cut, hence doesn't get isolated.

Geometry

G1 Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Solution. We let P_1 to be the P closer to B and P_2 the further one, and define Q_1 and Q_2 correspondingly. Now,

$$\angle Q_1PA = \pi - \angle APB = \angle ACB = \angle BFD = \angle AFQ_1$$

so quadrilateral Q_1P_1FA is cyclic. Moreover $\angle ACB = \angle AFE$ since $BCFE$ is cyclic. With respect to the circumcircle of Q_1P_1FA, AP_1 subtends $\angle AFP_1 = \pi - \angle AFE = \pi - \angle ACB$ and AQ_1 subtends $\angle AFQ_1 = \angle ACB$, hence supplementary and therefore $AP_1 = AQ_1$.

Regarding P_2 and Q_2, we also have $\angle AP_2B = \angle AP_2Q_2 = \angle ACB = \angle AFD = \angle AFQ_2$ and therefore Q_2P_2FA is cyclic. Using the well-known fact that AF is the external angle bisector of $\angle DFE$ which is identical to $\angle Q_2EP_2$ due to collinearity, and that this external angle bisector intersects the circumcircle of Q_2FP_2 at A, we must have $AP_2 = AQ_2$. (Essentially, the idea of solving these two are similar, but worded rather differently).

G2 (IMO 4) Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that $SC = SP$ if and only if $MK = ML$.

Solution. Now we have the following equivalence:

$$MK = ML \leftrightarrow \angle MLK = \angle MKL \leftrightarrow \angle MLB + \angle KLB = \angle LKA + \angle AKM$$

$$\leftrightarrow \angle MCB + \angle KAB = \angle LBA + \angle ACM \leftrightarrow \angle PCB + \angle PAB = \angle PBA + \angle ACP$$

which becomes

$$MK = ML \leftrightarrow \angle PCA - \angle PCB = \angle PAB - \angle PBA \tag{19}$$

Let CP intersect AB at Q. W.l.o.g. let $CA < CB$. Since SC is tangent to Γ, by angle chasing we have

$$\angle CSA = \angle CSB = \angle CAB - \angle CBA \tag{20}$$

If $SC = SP$, then SP is also tangent to Γ, and

$$\angle PSA = \angle PSB = \angle PAB - \angle PBA \tag{21}$$

In the meantime,

$$\begin{aligned}\angle PCA - \angle PCB &= (\angle PCS - \angle SCA) - (\angle SCB - \angle PCS) \\ &= 2\angle PCS - \angle SCA - \angle SCB \\ &= 2\angle PCS - \angle SCA - \angle SAC \\ &= (180^\circ - \angle CSP) - (180^\circ - SCA) \\ &= \angle PSA \\ &= \angle PAB - \angle PBA\end{aligned}$$

which proves the claim (note that $2\angle PCS = 180^\circ - \angle CSP$ because $SC = SP$).

which proves $SC = SP \to MK = ML$.

To prove the other side, fix line SP and vary P along those. If P is moved closer to A, then $\angle PCA - \angle PCB$ decreases (since $\angle PCA$ decreases but $\angle PCB$ increases), but then $\angle PAB - \angle PBA$ increases ($\angle PAB$ increases, $\angle PBA$ decreases). We can also get an opposite analysis if we move P closer to B. This means that there's at most one P that could fulfill $\angle PCA - \angle PCB = \angle PAB - \angle PBA$, which we showed that it will always happen when $SC = SP$. Thus now $SC = SP$ becomes a necessary condition for this.

G3 Let $A_1A_2 \ldots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \ldots, P_n$ onto lines $A_1A_2, \ldots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \ldots, X_n$ on sides $A_1A_2, \ldots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \ldots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

Solution. We have $\displaystyle\sum_{i=1}^{n} \angle X_iPX_{i+1} = \sum_{i=1}^{n} \angle P_iPP_{i+1} = 2\pi$ with indices taken modulo n. This implies an index j (again taken modulo n) with $\angle X_iPX_{i+1} \geq \angle P_iPP_{i+1}$. We show that this follows that $XX_{i+1} \geq PP_{i+1}$.

Consider the circle $A_{i+1}P_iP_{i+1}$; its diameter is the segment $A_{i+1}P$. Moreover, from the sine rule extended to the circumcircle of A_iPP_{i+1} we have

$$A_{i+1}P = \frac{A_{i+1}P}{\sin \angle A_{i+1}P_iP} = \frac{P_iP_{i+1}}{\sin \angle P_iA_{i+1}P_{i+1}} = \frac{P_iP_{i+1}}{\sin \angle A_iA_{i+1}P_{i+2}}$$

and similarly if $A_{i+1}X$ were to be the diameter of the circle $A_{i+1}X_iX_{i+1}$ then

$$A_{i+1}X = \frac{A_{i+1}X}{\sin \angle A_{i+1}X_iX} = \frac{X_iX_{i+1}}{\sin \angle X_iA_{i+1}X_{i+1}} = \frac{X_iX_{i+1}}{\sin \angle A_iA_{i+1}A_{i+2}}$$

Now, $\angle X_iPX_{i+1} \geq \angle P_iPP_{i+1} = \angle X_iXX_{i+1}$, the last equality are both the same as $\pi - \angle A_iA_{i+1}A_{i+2}$. This means, with respect to the circle $A_{i+1}X_iXX_{i+1}$, P is either on or inside the circle. From the fact that $A_{i+1}X$ is the dimater of the circle we have that $A_{i+1}P \leq A_{i+1}X$, and therefore $P_iP_{i+1} \leq X_iX_{i+1}$.

G4 (IMO 2) Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Solution. Let EI intersect Γ again at H (this H is unique once A, B, C, E are fixed). All we need to show is that D, G, H are collinear.

Now, we have $\angle BAF = \angle CAE$ and since DI bisects $\angle BAC$, we also have $\angle FAI = \angle EAI = \angle DHI$; the last equality follows from that A, H, D, E are concyclic. This means if J is the intersection of HD and AF then $\angle JHI = \angle JAI$ and therefore J, H, A, I are concyclic.

Now, extend AJ to intersect Γ again at K, we get $\angle AKE = \angle AHE = \angle AHI = \angle AJI$, showing that $JI \parallel KE$. With $\angle KBC = \angle KAC = \angle BAE = \angle BCE$ we have $KE \parallel BC$ too and therefore $JI \parallel BC$. This means, if AD and BC intersect at L we have $\frac{FJ}{JA} = \frac{LI}{IA}$. Finally, using the well-known fact $BI = DI = IC$, and that $\angle LBD = \angle CAD = \angle LAB$, so DBL and DAB are, in fact, similar. This means that $DI^2 = BI^2 = DL \cdot DA$, i.e. $\frac{DI}{DL} = \frac{DA}{DI}$. Therefore,

$$\frac{FJ}{JA} = \frac{LI}{IA} = \frac{DI - DL}{DA - DI} = \frac{DI(1 - DL/DI)}{DA(1 - DI/DA)} = \frac{DI}{DA}$$

and by Menelaus theorem on $\triangle FIA$, the line DJ will intersect FI in its midpoint, i.e. G.

G5 Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

Solution. Denote the reflection of D in M as F. Since M is the common midpoint of lines CE and DF, $CDEF$ is a parallelogram. Moreover, from $\angle DMO = 90°$ we get MO as the perpendicular bisector of DF, hence $DO = FO$. This means F lies on the cirumcenter of triangle BCD.

The other observation is that, if G on segment AB is such that $BC = BG$, then from $AB = BC + AE$ we have $GA = AE$. But since $BC \parallel AE$, we have $\angle CBG + \angle GAE = 180°$ and therefore

$$\angle CGE = 180° - \angle CGB - \angle AGE = 180° - (90° - \angle CBG/2) - (90° - \angle GAE/2) = 90°$$

which means CE is the diameter of the circumcircle of CGE. With M as the midpoint of CE we have $CM = MG = ME$. From $BG = BC$, too, MB is the perpendicular bisector of CG; in particular, $MB \perp CG$ and similarly $MA \perp EG$ and with $CG \perp EG$ we have $MA \perp MB$. That is, $\angle AMB = 90°$. We also have MB bisects $\angle ABC$ for this reason.

Now with the first two points set up, define H as the point on ray BD and satisfying $BH = BF$. Let L be the intersection of AM and FH. We claim that M is the midpoint of AL. Let N be the midpoint of FH, and by $BF = BH$ we have $BN \perp FH$, or $\angle BNL = 90° = \angle BML$, which means B, N, M, L are concyclic. Since M is the midpoint of FD, we also have $MN \parallel HD$, and therefore $\angle BFH = \angle BHF = \angle KHD = \angle HNM = \angle LBM$, the last equality following from that B, N, M, L concyclic. In addition, from the fact that MB bisects $\angle ABC$ we have

$$\angle MBC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE = \frac{1}{2}(180° - \angle FCD) = \frac{1}{2}(180° - \angle FBD) = \angle BFH = \angle MBL$$

(we subtly used the fact that $FCDE$ is a parallelogram and that F, B, C, D are concyclic). This means that B, L, C are actyally collinear! I.e. BM bisects $\angle ABL$ too, so with A, M, L collinear by definition and that $BM \perp AL$ we have $AM = ML$ as required.

Finally, with M as the common midpoint of AL and DF, $AFLD$ is a parallelogram. Thus

$$\angle BDA = \angle BDF + \angle FDA = \angle HDF + \angle LFD = \angle BHF = \frac{1}{2}CDE$$

(the last step is due to the previous equation), as desired.

G6 The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Prove that if the incenter of triangle ABC lies outside triangle XYZ, then one of the angles of triangle ABC is greater than 120°.

Solution. Suppose all angles in triangle ABC are at most 120°, and that the incenter I of ABC lies insode triangle AYZ. Let A_1 be the second intersection of line AI with the circumcenter of triangle AYZ; define B_1 and C_1 similarly. Denote also B_2 as the midpoint of XZ and C_2 the midpoint of XY. Then we have $A_1Y = A_1Z$ and $\angle YA_1Z \geq 60^\circ = \angle YXZ$ means that A_1 lies in triangle XYZ, and similarly so do points B_1, C_1.

Now that X, Y, Z are on the sides of triangle ABC, the quadrilaterals $BZYX$ and $CXZY$ are convex. Since B, B_1, I are collinear and both B, B_1 are in quadrilateral $BZYX$, I cannot be on segment BB_1. Thus B, B_1, I are on the line in that order, and similarly C, C_1, I are on the line in that order (and $B_1 \neq I, C_1 \neq I$). It then follows that the quadrilateral IB_1XC_1 is also convex.

We now have $\angle BIC = \angle B_1IC_1 = 90^\circ + \frac{\angle BAC}{2}$, so $\angle BIC$ is obtuse. In fact, with B, B_1, I in that order and C, C_1, I in that order we also have $\angle BAC > 60^\circ$, so $\angle B_1IC_1 > 120^\circ$. Since I is in triangle AYZ, while B_1, C_1 are in XYZ, segments IB_1 and IC_1 each intersect segment YZ. Thus we may choose a point I_1, lying on YZ and the interior of triangle $I_1B_1C_1$, and thus $\angle B_1I_1C_1 > \angle B_1IC_1 > 120^\circ$.

On the other hand, since $B_1X = B_1Z$, we have $B_2B_1 \perp XZ$ and similarly $C_2C_1 \perp XY$. Given also that I_1 is on side YZ, $IZ \leq IX$ and $IY \leq IX$, and therefore both B_2, C_2 are outside angle domain $\angle B_1I_1C_1$. Thus $\angle B_2I_1C_2 \geq \angle B_1I_1C_1$. However, both B_2, C_2 are on the incircle of XYZ which subtends an angle of 60° on the circumference, while I_1 can either be on or outside this incircle. It then follows that $\angle B_2I_1C_2 \leq 60^\circ$, giving us

$$120^\circ < \angle B_1IC_1 < \angle B_1I_1C_1 \leq \angle B_2I_1C_2 \leq 60^\circ \tag{22}$$

which is a contradiction.

G7 Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}, \widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}, \widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

Solution. The key to solving this problem is the following.

Lemma 5. *Cconsider O_1 as the set of all circles tangent to γ_1 externally and to γ_2 internally, and O_2 as the set of all circles tangent to γ_2 externally and to γ_3 internally. If the exsimilicenter (that is, the intersection of common external tangent) of ω_1 and ω_2 lies on AC for* some *$\omega_1 \in O_1$ and $\omega_2 \in O_2$, then the exsimilicenter of ω_1 and ω_2 lies on AC for* all *$\omega_1 \in O_1$ and $\omega_2 \in O_2$.*

Proof. Consider ω_A and ω_B both in O_1. Let A_1 and A_2 be the tangency points of ω_A with γ_1 and γ_2, respectively. Define B_1 and B_2 similarly. By Monge Alembert's theorem, A_1 and A_2 passes through the insimilicenter of the circles determined by arcs γ_1 and γ_2; we name them as Γ_1 and Γ_2.

Let the intersection of A_1A_2 and B_1B_2 as X, the insimilicenter of Γ_1 and Γ_2. By this definition, if A_2X intersects Γ_2 at A_2' again and B_2X intersects Γ_2 again at B_2' then $\dfrac{A_1X}{A_2X'} = \dfrac{B_1X}{B_2X'} = \dfrac{r_1}{r_2}$ with r_1, r_2 denoting the radii of Γ_1 and Γ_2 respectively. We now have:

$$XA_1 \cdot XA_2 = \frac{A_1X}{A_2X'}(XA_2' \cdot XA_2) = \frac{A_1X}{A_2X'}(XB_2' \cdot XB_2) = \frac{B_1X}{B_2X'}(XB_2' \cdot XB_2) = XB_1 \cdot XB_2$$

where the equality $XA_2' \cdot XA_2 = XB_2' \cdot XB_2$ follows from the fact that $A_2, B_2, A_2'B_2'$ all on Γ_2 and that A_2A_2' and B_2B_2' intersect at X. Therefore, A_1, A_2, B_1, B_2 are concyclic. Now, consider this circle containing the four points as Γ_0, we know that Γ_0 and Γ_1 intersect at A_1B_1, Γ_0 and Γ_2 at A_2B_2, Γ_1 and Γ_2 at AC. This means A_1B_1 and A_2B_2 concur on AC by the radical axis theorem. Finally, A_1, B_1 are the exsimilicenters of Γ_1 and ω_A, ω_B, respectively, so by Monge-Alembert theorem again the exsimilicenter of ω_A and ω_B must be on A_1B_1. Similarly, the same exsimilicenter must be on A_2B_2 (except A_2, B_2 are actually the insimilicenters of Γ_2 and the ωs). This means A_1B_1 and A_2B_2 intersect at the the exsimilicenter of ω_1 and ω_2: in other words, by the previous points, ω_1 and ω_2 have exsimilicenter on AC. By a similar logic, any two difference circles in O_2 also have their exsimilicenter on AC.

Now, suppose that ω_1 and ω_2 are such that the exsimilicenter lies on AC, say B. Consider all $\omega_2' \in O_2$ varying from A to C. From above, the exsimilicenter of ω_2 and ω_2' will also lie on AC (except when $\omega_2 = \omega_2'$). At most one of the circle ω_2' will have their exsimilicenter coincide with B. In other words, for all ω_2' except this circle and the circle coincide with ω_2, the exsimilicenter of ω_2 and ω_2' lies on AC other than B. By Monge-Alembert theorem again, the exsimilicenter of ω_1 and ω_2' will also be on AC for all those ω_2'. As ω_2' varies continuously, the exsimilicenter of ω_1 and ω_2' will also vary continuously. This means that this exsimilicenter of the two circles will also be on AC even for the two edge cases. Finally, if ω_1' is any other circle in O_1, then a similar logic yields that ω_1' and ω_2' have exsimilicenters on AC, which solves the lemma. $\square$

Now go back to the problem. Despite the fancy formulation of the problem, really all we need is this: let the circle inscribed in $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$ as ω_{11} and $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$ as ω_{12}; their exsimilicenters intersect at B. Let the circle inscribed in $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ be ω_{21}, and let ω_{22} to be the unique circle tangent to h_2, γ_2 and γ_3. By our lemma above, the exsimilicenter must be on AC. Since h_2 is an external common tangent of ω_{21} and ω_{22} and h_2 intersect AC at B, B is indeed the exsimilicenter. Thus, the other common tangent must also pass through B, and so must be h_3. This means h_3 is also tangent to ω_{22}. In other words, $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed. Q.E.D.

Number Theory

N1 Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right)\cdots\left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}.$$

Answer. 39.

Solution. We need $s_1 > 1$ since $s_1 = 0$ will make the product equal 0. If we sort $s_1, s_2, \cdots$ in increasing order then we have $s_i \geq i + 1$ for all i and therefore

$$\prod_{i=1}^{n}\left(1 - \frac{1}{s_i}\right) \geq \prod_{i=1}^{n}\left(1 - \frac{1}{i+1}\right) = \prod_{i=1}^{n}\left(\frac{i}{i+1}\right) = \frac{1}{n+1}$$

and therefore $\dfrac{51}{2010} \geq \dfrac{1}{n+1}$, i.e. $n + 1 \geq \dfrac{2010}{51} = 39 + \dfrac{21}{51}$. Thus $n \geq 39$.

Now let's show that $n = 39$ works. They are the numbers $2, 3, ..., 33,\ 35, 36, 37, 38, 39, 40, 67$, which telescopes into this somewhat nice form:

$$\frac{1}{33} \cdot \frac{34}{40} \cdot \frac{66}{67} = \frac{17}{670} = \frac{51}{2010}$$

Note: The original proposal has $\frac{42}{2010}$ instead of $\frac{51}{2010}$. The answer is 48 for the following reason: one of the numbers s_k has the be a multiple of 67 given that 2010 is. Therefore,

$$\frac{42}{2010} = \prod_{i=1}^{n}\left(1-\frac{1}{s_i}\right) \geq \frac{66}{67}\prod_{i=1}^{n-1}\left(1-\frac{1}{s_i}\right) \Rightarrow \prod_{i=1}^{n-1}\left(1-\frac{1}{s_i}\right) \leq \frac{42}{2010}\cdot\frac{67}{66} = \frac{7}{330}$$

so $n \geq \frac{330}{7} = 47 + \frac{1}{7}$ and so $n \geq 48$. The construction is given by

$$2, 3, \cdots, 33, 36, \cdots, 50, 67$$

N2 Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2\cdot 3^n = m\left(2^{n+1} - 1\right).$$

Answer. $(6,3), (9,3), (9,5), (54,5)$

Solution. Solving the quadratic equation in m gives $m = \dfrac{2^{n+1} - 1 \pm \sqrt{(2^{n+1}-1)^2 - 8\cdot 3^n}}{2}$, so the term $(2^{n+1}-1)^2 - 8\cdot 3^n$ needs to be a perfect square. Now, if $(2^{n+1}-1)^2 - 8\cdot 3^n = a^2$ then $(2^{n+1}-1-a)(2^{n+1}-1+a) = 8\cdot 3^n$. We now have the following observation:

- Given that $2^{n+1} - 1 - a \equiv 2^{n+1} - 1 + a \pmod 2$, both must be even for their product to be divisible by 8. This means one of the numbers $2^{n+1}-1-a$ and $2^{n+1}-1+a$ must be $2\cdot 3^{n_1}$ and the other $4\cdot 3^{n_2}$, with $n_1 + n_2 = n$.
- Adding $2^{n+1}-1-a$ and $2^{n+1}-1+a$ together gives $2(2^{n+1}-1) = 2\cdot 3^{n_1} + 4\cdot 3^{n_2}$, so $2^{n+1}-1 = 3^{n_1} + 2\cdot 3^{n_2}$, and therefore $3^{\min(n_1,n_2)} \mid 2^{n+1} - 1$.

We now consider the following.

Lemma 6. *For each $k \geq 1$, the order of 2 modulo 3^k is $2\cdot 3^{k-1}$.*

Proof. We will prove by induction that when $\ell = 2\cdot 3^{k-1}$,

$$3^k \mid 2^\ell - 1 \qquad 3^{k+1} \nmid 2^\ell - 1 \tag{23}$$

Base case: $k = 1$ we have $2^1 - 1$ not divisible by 3, and $2^2 - 1$ divisible by 3 not 9.

Induction step: now suppose that this is true for some $k \geq 1$, such that $\ell = 2\cdot 3^{k-1}$ we have $3^k \mid 2^\ell - 1$ but $3^{k+1} \nmid 2^\ell - 1$. Let $2^\ell = 3^k x + 1$ with $3 \nmid x$. Let ℓ_1 be the smallest number with $3^{k+1} \mid 2^{\ell_1} - 1$; we have $\ell \mid \ell_1$ so let $\ell_1 = y\ell$. This means:

$$2^{\ell_1} - 1 = 2^{y\ell} - 1 = (3^k x + 1)^y - 1 = \sum_{i=1}^{y}\binom{y}{i}(3^k x)^i$$

and for the purpose of modulo 3^{k+2}, for all $k \geq 1$ we have $3k \geq k+2$ so we have

$$\sum_{i=1}^{y}\binom{y}{i}(3^k x)^i \equiv \binom{y}{2}(3^k x)^2 + \binom{y}{1}(3^k x) \pmod{3^{k+2}}$$

now if this is divisible by 3^{k+1}, since $2k \geq k+1$, we have $\binom{y}{2}(3^k x)^2 + \binom{y}{1}(3^k x) \equiv 3^k xy \pmod{3^{k+1}}$. This means $3 \mid xy$ and with $3 \nmid x$ we have $3 \mid y$. Thus the smallest such y is $y = 3$. Now, for this $y = 3$ we have

$$\sum_{i=1}^{y}\binom{y}{i}(3^k x)^i \equiv \binom{3}{2}(3^k x)^2 + \binom{3}{1}(3^k x) = 3(3^{2k})x^2 + 3^{k+1}x = 3^{k+1}(3^k x^2 + x) \pmod{3^{k+2}}$$

and since $3^k x^2 + x \equiv x \not\equiv 0 \pmod 3$, we have $3^{k+2} \nmid 2^{3\ell} - 1$, as desired. $\square$

Now going back to the problem, where we left off at $3^{\min(n_1,n_2)} \mid 2^{n+1}-1$. If $k=\min(n_1,n_2)$ then we have $2\cdot 3^{k-1} \mid n+1$. In particular, $n+1 \geq 2\cdot 3^{k-1}$. Moreover, we have

$$2^{n+1}-1 = 3^{n_1}+2\cdot 3^{n_2} \leq 3^{n-k}+2\cdot 3^k > 3^{n-k}$$

In other words $8^{(n+1)/3} > 2^{n+1}-1 > 3^{n-k} = 9^{(n-k)/2}$ so $\dfrac{n+1}{3} > \dfrac{n-k}{2}$, i.e. $2n+2 > 3n-3k$ so $n < 3k+2$. On the other hand we have $n+1 \geq 2\cdot 3^{k-1}$ so $2\cdot 3^{k-1}-1 \leq n \leq 3k+1$. It's not hard to see that this only works when $k \geq 2$. We would now have the $2^{n+1}-1 > 3^{n-k} \geq 3^{n-2}$ but by above, $n < 3k+2 \geq 3(2)+2 = 8$ so we only need to test all $n \leq 7$. Nevertheless, for $n = 6, 7$ we have $n+1 = 7, 8$ which is not divisible by 6, so for them $k \geq 1$ and $3(1)+2 = 5 < 6$ so these cases can be disregarded and we have $n \leq 5$ to test.

Recall that the discriminant is $(2^{n+1}-1)^2 - 8\cdot 3^n$. For $n = 0,1,2,3,4,5$ these discriminants are

$$-7, -15, -23, 9, 313, 2025$$

so only $n = 3, 5$ work as perfect squares. Plugging these into the quadratic formula we have:

$$n=3: \frac{2^{3+1}-1 \pm \sqrt{(2^{3+1}-1)^2 - 8\cdot 3^n}}{2} = \frac{15\pm 3}{2} = 6, 9;$$

$$n=5: \frac{2^{5+1}-1 \pm \sqrt{(2^{5+1}-1)^2 - 8\cdot 3^n}}{2} = \frac{63\pm 45}{2} = 9, 54.$$

Since these are obtained from the quadratic equations, they work as fine, giving us our desired pairs.

N3 Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2+7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

Answer. $n = 5$.

Solution. The example $n=5$ is realized by taking f_i as $x, 2, 1, 1, 1$ for $i = 1, \cdots, 5$.

To show $n \leq 4$ doesn't work, if f_1, f_2, f_3, f_4 (some of them possibly 0) work, then there exist integers a_i, b_i $(i = 1,2,3,4)$ and k such that

$$\begin{aligned} a_1^2+a_2^2+a_3^2+a_4^2 &= k^2 \\ b_1^2+b_2^2+b_3^2+b_4^2 &= 7k^2 \\ a_1b_1+a_2b_2+a_3b_3+a_4b_4 &= 0 \end{aligned} \tag{24}$$

This also gives $(a_1\pm b_1)^2+\cdots(a_4\pm b_4)^2 = 8k^2$. In addition, we can choose k such that k is minimal; in particular, one of k, a_i, b_i (for some $i = 1,2,3,4$) is odd.

Since x^2 is $\equiv 1 \pmod 8$ if x is odd and divisible by 4 if x is even, we have must have all four numbers $a_1+b_1, \cdots a_4+b_4$ to be even for their sum of squares to be divisible by 8. If k is odd, then $k^2 \equiv 1 \pmod 8$ and $7k^2 \equiv 7 \pmod 8$, which means only exactly one of a_i's is odd (by considering a_i mod 4), while exactly three of b_i's is odd. This is a contradiction.

Hence k is even and both $\sum a_i^2, \sum b_i^2$ are divisible by 4. Since $a_i \equiv b_i \pmod 2$, and one of $\bigcup_{i=1}^4 \{a_i, b_i\}$'s is odd, by considering a_i^2 mod 4 we have all a_i, b_i odd. Again, we reuse the previous logic that for x even, x^2 is either divisible by 32, or $\equiv 4 \pmod{32}$ (depending on whether x is divisible by 4). Since k is even, $\sum(a_i+b_i)^2 = 8k^2$ is divisible by 32, and so is $\sum(a_i-b_i)^2$. Hence each of a_i+b_i and a_i-b_i must be divisible by 4, implying that a_i, b_i are all even. This is again a contradiction.

N4 Let a, b be integers, and let $P(x) = ax^3+bx$. For any positive integer n we say that the pair (a,b) is n-good if $n|P(m)-P(k)$ implies $n|m-k$ for all integers m, k. We say that (a,b) is *very good* if (a,b) is n-good for infinitely many positive integers n.

(a) Find a pair (a,b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

Solution. (a) (Courtesy of the official solution). The desired pair is $x^3 - 51^2x = x(x^2 - 51)$. Since $P(0) = P(51)$, it is only m-good for m that are divisors of 51, i.e. finite. On ther other hand, to show it's 51-good, notice that $x^3 - 51x^2 \equiv x^3 \pmod{17}$.

- $x^3 \equiv 0, 1, 2$ for $x = 0, 1, 2 \pmod 3$ so it's 3-good.
- If $x^3 \equiv y^3 \pmod{17}$ (here assume $x, y \neq 0$; the 0 case can be isolated), then rasing both sides to the power of 11 we have $x^{33} \equiv y^{33} \pmod{17}$. But by Fermat's little theorem we have $x^{32} \equiv y^{32} = 1$ so $x \equiv x^{33} \equiv y^{33} \equiv y$, so this is also 17-good.

Finally, as $lcm(3, 17) = 51$ and (a, b) is both 3- and 17-good, it's 51-good.

(b) Let (a, b) be 2010-good. We split this into the following steps:

Step 1. It's also 67-good.
Proof: Now, $\{P(0), \cdots, P(2009) \pmod{2010}\} = \{0, 1, \cdots, 2009\}$. In particular, since $67 \mid 2010$, the numbers $0, 1, \cdots, 66$ appears the same number of times (i.e. 30) in the sequence $i \pmod{67}$ with $i = 0, 1, \cdots, 2009$. Bearing in mind that $n - m \mid P(n) - P(m)$ for all $n \neq m$ and all integer polynomials P, if we consider the sequence $P(0), \cdots, P(2009)$ modulo 67, it will simply be $P(0), \cdots, P(66)$ repeated 30 times. But since $P(0), \cdots, P(2009)$ modulo 67 is also $0, 1, \cdots, 66$ repeated 30 times, we have $\{P(0), \cdots, P(66) \pmod{67}\} = \{0, \cdots, 66\}$. Hence (a, b) is 67-good.

Step 2. $67 \mid a$ and $67 \nmid b$.
Proof: now that (a, b) is 67-good, for each m, n with $67 \nmid m - n$ we have $67 \nmid P(m) - P(n) = a(m^3 - n^3) + b(m - n) = (m - n)(a(m^2 - mn + n^2) + b)$, i.e. $67 \nmid a(m^2 + mn + n^2) + b$. We now have two cases:

- If $67 \mid b$ then $P(m) - P(n) \equiv a(m^3 - n^3) \pmod{67}$. Choose $n = 1$ and $m = g^{22}$ where g is a primitive root modulo 67, then $m^3 \equiv g^{66} \equiv 1 \equiv n^3 \pmod{67}$, so $67 \mid P(m) - P(n)$. But then $67 \nmid m - n$ since $m \not\equiv 1 \pmod{67}$. Thus this case gives rise of a contradiction.
- Now assume $67 \nmid b$. We recall the identity $67 \nmid a(m^2 + mn + n^2) + b$ for all $m \not\equiv n \pmod{67}$. If, there exist x such that $67 \mid a(3x^2) + b$, from $b \not\equiv 0$ we have $3ax^2 \not\equiv 0$ and therefore $x \not\equiv 0$. Now, choose $m = 2x$ and $n = -x$ we have $a((2x)^2 - 2x^2 + x^2) + b = 3ax^2 + b \equiv 0 \pmod{67}$. But $m - n = 3x \not\equiv 0 \pmod{67}$, contradicting that (a, b) is good. Thus we can further assume that $67 \nmid a(m^2 + mn + n^2) + b$ for any integer pairs (m, n).

  We now claim that $\{m^2 + mn + n^2 : m, n \in \mathbb{Z}\}$ attains all integers in $\mathbb{Z}_{67}$. For 0 it's easy: take $m = n = 0$. Choosing $m = 0$ we have n^2, which gives all the quadratic residues modulo 67. Choosing $m = n$ we have $3m^2$, which gives $3\times$ all the quadratic residues modulo 67. Now, $8^2 = 64 \equiv -3 \pmod{67}$ is a quadratic residue modulo 67, and since $67 \equiv 3 \pmod 4$, -1 is a quadratic non-residue modulo 67, and therefore $3 = (-1)(-3)$ is also a quadratic non-residue modulo 67. Thus for all n with $67 \nmid 67$, $3n^2$ is a quadratic non-residue mod 67, so $\{3n^2 : n = 1, \cdots, 66\}$ covers all quadratic non-residues modulo 67. Thus, $(m, n) = (0, 0), (0, n), (n, n)$ together covers all integers in $\mathbb{Z}_{67}$.

  If $67 \nmid a$, then we can choose a number x such that $67 \mid ax + b$. From the previous point, there exists m, n such that $m^2 + mn + n^2 \equiv x \pmod{67}$, and thus $67 \mid a(m^2 + mn + n^2) + b$, which is a contradiction. Therefore, we need $67 \mid a$.

Combining the two cases yields the results of this lemma.

Step 3. (a, b) is 67^k good for all $k \geq 1$, which finishes the solution.
Proof: Let's say, $67^k \mid P(m) - P(n) = a(m^3 - n^3) + b(m - n) = (m - n)(a(m^2 + mn + n^2) + b)$. In modulo 67, we have $67 \mid a$ but $67 \nmid b$, so $a(m^2 + mn + n^2) + b \equiv b \not\equiv 0 \pmod{67}$. This means, $a(m^2 + mn + n^2) + b$ is relatively prime to 67, and therefore same to 67^k. This means, $67^k \mid m - n$, as desired.

N5 (IMO 3) Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Answer. $f(n) = n + c$ for some integer $c \geq 0$.

Solution. As per the official solution we'll use the lemma $p \mid g(m_1) - g(m_2) \to p \mid m_1 - m_2$ for any prime p. Indeed, let $kp = g(m_1) - g(m_2)$. If $p \nmid k$ then we can choose n such that $g(m_1) + n \equiv p^3 \pmod{p^4}$. It follows that $g(m_2) + n \equiv p^3 - kp \pmod{p^4}$. Otherwise, $p \mid k$ and we can choose n such that $g(m_1) + n \equiv g(m_2) + n \equiv p \pmod{p^2}$. In both cases, $v_p(g(m_1) + n), v_p(g(m_2) + n)$ are either 1 or 3, hence is odd. It then follows that there exists an n such that $p \mid g(n) + m_1$ and $p \mid g(n) + m_2$, so $p \mid m_1 - m_2$.

Now by this lemma, $g(n+1) - g(n) = \pm 1$. We first show that the direction has to be monotonic: otherwise, let n be the place it changes direction, then $g(n+1) - g(n) = -(g(n) - g(n-1))$, and $g(n-1) = g(n+1)$, which violates the lemma for all primes $p \neq 2$. If $g(n+1) - g(n) = -1$ for all n then $g(g(1)+1) = g(1) - g(1) = 0 \notin \mathbb{N}$, which isn't allowed either. Hence we have $g(n+1) - g(n) = 1$ for all n and therefore $g(n) = n + c$ for some integer $c \geq 0$. This function would work since the given equation will be $(m+n+c)^2$.

N6 The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been coloured with the following property being satisfied: for each $0 \leq i, j \leq 2^n - 1$, the j-th cell in the i-th row and the $(i+j)$-th cell in the j-th row have the same colour. (The indices of the cells in a row are considered modulo 2^n.) Prove that the maximal possible number of colours is 2^n.

Solution. Consider the sequence $S_{(i,j)} \in (\mathbb{Z}/2^n)^2$ modulo 2^n:

$$S_{i,j}(0) = (i, j) \quad S_{i,j}(k) = (a, b) \to S_{i,j}(k+1) = (b, a+b)$$

Recall that the Fibonacci sequence is defined as

$$F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}$$

and one can show that

$$S_{i,j}(k) = (F_{k-1}i + F_k j, F_k i + F_{k+1} j)$$

(which by convention we let $F_{-1} = 1$). From the problem statement, $S_{i,j}(k)$ and $S_{i,j}(k+1)$ must have the same colour for all k, so all members of the same sequence $S_{i,j}$ have the same colour.

We first notice that any Fibonacci sequence is periodic modulo k for any positive integer k. Indeed, the pairs $\{(F_i, F_{i+1})\}_{i \geq 1}$ repeats itself via Pigeonhole principle, and if $(F_i, F_{i+1}) = (F_j, F_{j+1})$ then $(F_{i+1}, F_{i+2}) = (F_{j+1}, F_{j+2})$.

With this, we proceed to the following.

Lemma 7. *The sequence $\{F_k\}_{k \geq 0}$ modulo 2^n has minimal period $3 \cdot 2^{n-1}$.*

Proof.

$$F_n = \frac{1}{\sqrt{5}}(\beta^n - \alpha^n)$$

where $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1-\sqrt{5}}{2}$. This follows from that $F_{k+2} = F_{k+1} + F_k$ has characteristic polynomial $x^2 - x - 1 = 0$, which has roots α, β. Define also the sequences

$$G_k = \frac{F_{2k}}{F_k} = \beta^k + \alpha^k \quad H_k = G_k - F_k \quad \forall k > 0$$

we see that G_k also follows the update rule $G_{k+2} = G_{k+1} + G_k$ (since it's in the form of $p\alpha^k + q\beta^k$), and analogously, $H_{k+2} = H_{k+1} + H_k$ (as it's the difference between two sequences that also follow the same form).

Given $F_0 = 0$ and $F_1 = 1$, the problem now becomes finding the minimal k such that

$$2^n \mid F_k \quad F_{k+1} \equiv 1 \pmod{2^n}$$

Let's claim the following:

- $2^{n+1} \mid F_{3 \cdot 2^{n-1}}, \forall n \geq 2$

- $F_{3\cdot 2^{n-1}+1} \equiv 2^n + 1 \pmod{2^{n+1}}, \forall n \geq 2.$

To see why these claims would imply our lemma, let $f(n)$ be the minimal period mod 2^n. The facts above do show that for all $n \geq 3$, $3 \cdot 2^{n-1}$ is a period of F_k mod 2^n, but $3 \cdot 2^{n-2}$ is not. Thus $f(n) \mid 3 \cdot 2^{n-1}$ but $f(n) \nmid 3 \cdot 2^{n-2}$. In other words $f(n)$ can either be $3 \cdot 2^{n-1}$ or 2^{n-1}. But the latter cannot be, because $F_{2^{n-1}}$ is odd (F_k is even iff $3 \mid k$). The smaller cases $n \leq 2$ can be verified manually.

For the first claim, we first notice that $G_1 = \frac{F_2}{F_1} = 1$ and $G_2 = \frac{F_4}{F_2} = \frac{3}{1} = 3$, so continuing this (via induction) we see that G_k is even iff $3 \mid k$. For base case $n = 2$ we have $F_6 = 8 \equiv 0 \pmod{2^3}$; for inducting step all we need is that $F_{3\cdot 2^n} = F_{3\cdot 2^{n-1}} \cdot G_{3\cdot 2^{n-1}}$ and $G_{3\cdot 2^{n-1}}$ is even.

For the second claim, againwe use induction to establish $F_7 = 13 \equiv 5 \pmod 8$. Let $n \geq 3$ now. We note that the first claim implies $F_{3\cdot 2^{n-1}+1} \equiv F_{3\cdot 2^{n-1}+2} \pmod{2^{n+1}}$. Also, investigating the sequence $\{H_k\}_{k\geq 1}$ gives $H_1 = 0, H_1 = 2, \cdots$, so we can deduces $H_k = 2F_{k-1}$. This means we got

$$\begin{aligned} F_{3\cdot 2^{n-1}+1} &\equiv F_{3\cdot 2^{n-1}+2} \pmod{2^{n+1}} \\ &= F_{3\cdot 2^{n-2}+1} \cdot G_{3\cdot 2^{n-2}+1} \\ &= F_{3\cdot 2^{n-2}+1}(F_{3\cdot 2^{n-2}+1} + H_{3\cdot 2^{n-2}+1}) \\ &= F_{3\cdot 2^{n-2}+1}(F_{3\cdot 2^{n-2}+1} + 2F_{3\cdot 2^{n-2}}) \\ &\equiv F^2_{3\cdot 2^{n-2}+1} \pmod{2^{n+1}} \end{aligned}$$

Assuming induction hypothesis for $n - 1$, writing $F_{3\cdot 2^{n-2}+1} = a2^{n-1} + 1$ for some odd a gives

$$F^2_{3\cdot 2^{n-2}+1} = a^2 2^{2n-2} + 1 + a2^n \equiv 2^n + 1 \pmod{2^{n+1}}$$

where $2n - 2 \geq n + 1$ since $n \geq 3$. $\square$

Now we shall investigate the minimal period of $S_{i,j}$. This is the same as finding the minimal positive k with

$$F_{k-1}i + F_k j \equiv i \quad F_k i + F_{k+1} j \equiv j$$

If k is a period of F_n then it's a period of $S_{i,j}$. Conversely, let's show that the minimal period $3 \cdot 2^{n-1}$ is also minimal period whenever i, j are not both even. By shifting (i, j) to $(j, i + j)$ and/or $(i + j, i + 2j)$, we may assume that i is odd and j is even. This means F_{k-1} has to be odd and F_k has to be even when $n \geq 1$ (by substituting to the equation above). Now, let $F_{k-1} = a$ and $F_k = b \cdot 2^m$, a odd. If $2^n \nmid F_k$ then we can choose b such that b is odd, and $m < n$. Thus in modulo 2^{m+1}, we have

$$i \equiv F_{k-1}i + F_k j = ai + b \cdot 2^m j \equiv ai \mod 2^{m+1}$$

(recall that j is even), so $a \equiv 1 \pmod{2^{m+1}}$. On the other hand,

$$j \equiv F_k i + F_{k+1} j \equiv (b \cdot 2^m)i + (b \cdot 2^m + a)j \equiv 2^m i + j = 2^m + j \not\equiv j \pmod{2^{m+1}}$$

since bi is odd, and bj is even. This is a contradiction.

Hence $m \geq n$ (that is, $2^n \mid F_k$), which we can also show that $F_{k-1} \equiv 1 \pmod{2^n}$. This would mean that this k must be a period of S_{ij}, and so the minimal such k is $3 \cdot 2^{n-1}$.

Now we can complete the proof. For $n = 0$ we have a single cell $(0, 0)$ so the answer is 1. If our conclusion holds for $n - 1$, then for n, those grids with (i, j) both even form 2^{n-1} cycles by induction hypothesis. For those (i, j) not both even, there are $2^{2n} - 2^{2n-2} = 3 \cdot^{2n-2}$ of those cells, each being a cycle of length $3 \cdot 2^{n-1}$. Hence the number of cycles is 2^{n-1}, making a total of 2^n cycles.