Solution to IMO 2015 shortlisted problems.

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1 Algebra

1. **A2.**Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Solution. Plugging $y = f(x)$ yields $f(x - f(f(x))) = -1$, meaning that -1 is in the range of f. Take y such that $f(y) = -1$ we have $f(x + 1) = f(f(x))$. For the case where f is injective, we must have $f(x) = x + 1$, and this function obviously works with both sides equal $x - y$.

Now, let f not be injective, and let $f(x) = f(z)$ for some $x, z \in \mathbb{R}$. Then take y such that $f(y) = -1$ we have $f(x + 1) = f(f(x)) = f(f(z)) = f(z + 1)$. This means the function is periodic for sufficiently large domain (i.e. $\geq x_1$). Let M and m be real numbers $\geq x$ such that the numbers $f(M)$ and $f(m)$ are the maximum and minimum numbers in the period, respectively. Plug $x = M - 1$ yields R.H.S. as $f(f(M-1)) - f(y) - 1 = f(M) - f(y) - 1$, and with L.H.S. at most $f(M)$ we have $f(y) \geq -1$ for any real y (if $x - f(y) < x_1$ then we can change x to $M - 1 + kc$ where c is the length of functional period, so that $f(M) = f(M + kc)$ and k sufficiently large positive integer). A similar substitution yields R.H.S. as $f(m) - f(y) - 1$ and L.H.S. at least $f(m)$, so $f(y) \leq -1$. Combining both yields $f \equiv -1$, a constant, and this satisfies the functional equation too.

2. **A3.** Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n) x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, \cdots, 2n$.

Solution. Let $x = x_1$ and fix other indices, we know that this is a linear function in x (with constant term $\sum\limits_{2 \leq i < j \leq 2n} (j - i - n) x_i x_j$ and coefficient of x as $\sum\limits_{2 \leq i < 2n} (i - 1 - n) x_i$ and thereore the maximum value can be attained when $x = 1$ or -1. We can then safely assume that $\mid x_i \mid = 1, \forall i \in [1, 2n]$.

Now let p, q be the number of 1's and -1's in the sequence respectively, with $p + q = 2n$. W.L.O.G. let $p \leq q$ (replacing x_i by $-x_i$ for all i yields the same sum). We further name $a_1 < a_2 < \cdots < a_p$ and $b_1 < b_2 < \cdots < b_q$ such that $x_{a_i} = 1$ and $x_{b_i} = -1$. Now, the required sum becomes $\sum\limits_{1 \leq i < j \leq p} (a_j - a_i - n) + \sum\limits_{1 \leq i < j \leq q} (b_j - b_i - n) - \sum (|a_j - b_i| - n)$
$= \sum\limits_{1 \leq i < j \leq p} (a_j - a_i) + \sum\limits_{1 \leq i < j \leq q} (b_j - b_i) - \sum (|a_j - b_i|) - n(\binom{p}{2} + \binom{q}{2} - pq) = 2(\sum\limits_{1 \leq i < j \leq p} (a_j - a_i)$

$+\sum\limits_{1\le i<j\le q}(b_j-b_i))-\sum\limits_{1\le r<s\le 2n}(s-r)$ $-n(\binom{2n}{2}-2pq)$, since $\binom{2n}{2}=\binom{p+q}{2}=\binom{p}{2}+\binom{q}{2}+pq$ and $\sum\limits_{1\le r<s\le 2n}(s-r)=\sum\limits_{1\le i<j\le p}(a_j-a_i)+\sum\limits_{1\le i<j\le q}(b_j-b_i)+\sum(|a_j-b_i|)$. The aim is therefore to maximize $\sum(a_j-a_i)+\sum(b_j-b_i)+pqn$.

Telescoping the terms we know that $\sum(a_j-a_i)=(a_2-a_1)+(a_3-a_2)+\cdots+(a_p-a_{p-1})+(a_3-a_1)+(a_4-a_2)+\cdots+(a_p-a_{p-2})+\cdots+(a_p-a_1)=\sum_{i=1}^{p}((i-1)-(p-i))a_i=\sum\limits_{i=1}^{p}(2i-p-1)a_i$. Similarly, $\sum(b_j-b_i)=\sum\limits_{i=1}^{q}(2i-q-1)b_i$. Since $\{a_i\}\cup\{b_j\}=[1,2n]$, by rearranging inequality, for fixed p,q the maximum is attained when the terms $1,2,\cdots,2n$ have coefficient $-(q-1),-(q-3),\cdots-(p+1),-(p-1),-(p-1),\cdots+(p-1),+(p-1),+(p+1),\cdots+(q-1)$ (basically, the weightage k appears two times if $|k|<p$ and $k\neq 0$, and one time otherwise.) (Remember that $p\equiv q \pmod 2$).

Now we will prove that the maximum of $\sum(a_j-a_i)+\sum(b_j-b_i)+pqn$ occurs iff $\{p,q\}=\{n,n\}$ or $\{n-1,n+1\}$. For $p=q=n$ we have pqn as n^3 and the coefficient (or weightage) of $1,2,\cdots 2n$ as $-(n-1),-(n-1),-(n-3),-(n-3),\cdots+(n-1),+(n-1)$ while for the second case we have pqn as $n(n-1)(n+1)$ (i.e. n less than the first case) and he coefficients as $-n,-(n-2),-(n-2),-(n-4),-(n-4),\cdots(n-2),(n-2),n$. The difference of coefficient between this and the first case will therefore be $-1,+1,-1,+1,\cdots-1,+1$, from which we know that the resulting difference between the second and the first case is $-1+2-3+4\cdots-(2n-1)+2n=n$. (i.e. n more than the first case). Hence, the sum in these two cases are equal. For other $p<n-1$, we split into two cases. For $p\equiv n \pmod 2$, subtracting the coefficients $-(q-1),-(q-3),\cdots-(p+1),-(p-1),-(p-1),\cdots+(p-1),+(p-1),+(p+1),\cdots+(q-1)$ by $-(n-1),-(n-1),-(n-3),-(n-3),\cdots+(n-1),+(n-1)$ yields $-(q-n),-(q-n-2),-(q-n-2),\cdots 0,0,0,\cdots+(q-n-2),+(q-n-2),+(q-n)$ so after multplying by $1,2,\cdots 2n$ and considering the difference $n(pq-n^2)=-n(q-n)^2$ (since $p+q=2n$) and the difference is now $(q-n)(2n-1)+(q-n-2)(2n-3)+(q-n-2)(2n-5)+\cdots-n(q-n)^2<2n((q-n)+2(q-n-2)+2(q-n-4)+\cdots+2(2))-n(q-n)^2=2n(q-n+4\cdot\frac{q-n-1}{2}\cdot\frac{q-n}{2})-n(q-n)^2=0$. If $p\equiv n-1 \mod 2$ then use the same weightage to subtract $-n,-(n-2),-(n-2),-(n-4),-(n-4),\cdots(n-2),(n-2),n$ we get $-(q-1-n),-(q-1-n),-(q-3-n),-(q-3-n),\cdots,0,0,0,\cdots+(q-1-n),+(q-1-n)$. The difference is $(q-1-n)(2n-1)+(q-1-n)(2n-3)+\cdots-n((q-n)^2-1)<2(2n-1)(2+4+\cdots+(q-1-n))-n((q-n)^2-1)=2(2n-1)\frac{q-n-1}{2}\cdot\frac{q-n+1}{2}-n((q-n)^2-1)<n(q-n-1)(q+n-1)-n((q-n)^2-1)=0$. Summing up, we know that for other p, the resulting sum is smaller so we can safely assume that $p=q=n$. If we let $x_1=x_3=\cdots=x_{2n-1}=1$ and $x_2=x_4\cdots=x_{2n}=-1$ then obviously, $a_i=2i-1$ and $b_i=2i$, so they have the same weightage $2i-n-1$. This means the equality case is attained here.

To compute the sum, notice that for each $k\in[1,2n-1]$ there are exactly $2n-k$ ordered pairs (r,s) with $s-r=k$ and $1\le r,s\le 2n$. In addition, $r+s\equiv k \pmod 2$ for those pairs satifsying this property, hence $x_rx_s=(-1)^{r+s}=-1^k$. This means our desired maximum sum now becomes $\sum\limits_{k=1}^{2n-1}(-1)^k(k-n)(2n-k)$. Ignoring the case where $k=n$ (which gives $(-1)^k(k-n)(2n-k)=0$ anyway) and pairing k with $2n-k$ $(1\le k\le 2n-1)$ gives $(-1)^k(k-n)(2n-k)+(-1)^{2n-k}(n-k)k=(-1)^k(n-k)(2k-2n)=2(-1)^{k+1}(n-k)^2$. Thus our sum now becomes $2((n-1)^2-(n-2)^2+\cdots+(-1)^{n-1}(1^2))=2((n-1)+(n-2)+\cdots+1)$

$= n(n-1)$.

3. **A4/IMO 5.** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Solution. Plugging $x = y = 0$ into the original equation gives $f(f(0)) = 0$, so if $f(0) = c$ then $f(c) = 0$. Plugging $x = c, y = 0$ yields $f(f(c)) + f(0) = f(c) + cf(0)$, or $c + c = c^2$, yielding $c = 0, 2$. When $y = 1$, $f(x + f(x + 1)) = x + f(x + 1)$, so $x + f(x + 1)$ **is a fixed point of f at all times** (1). Let $x = 0$ and y be a fixed point, then $y + f(0) = f(f(y)) + f(0) = f(y) + yf(0) = y + yf(0)$. In other words, $(y - 1)f(0) = 0$ so if $f(0) = 2$, then $y = 1$, i.e. 1 is the only fixed point of f. This means $x + f(x + 1) \equiv 1 = x - 1 + f(x)$, yielding $f(x) = 2 - x$. This function satisfies the equation as $f(x + f(x + y)) + f(xy) = 2 - (x + (2 - x - y)) + 2 - xy = 2 + y - xy = x + 2 - x - y + 2y - xy = x + f(x + y) + yf(x)$.

If $f(0) = 0$, we show that $f(x) \equiv x$. when $x = -1$, $-1 = -1 + 0 = -1 + f(-1 + 1)$ is a fixed point, too. When $x = 1, y = -1$, $f(1) + f(-1) = 1 + f(0) - f(1)$, or $2f(1) = 2$ so $f(1) = 1$. Now plugging $y = 0$ into the original equation yields $x + f(x)$ is a fixed point, because $f(xy) = yf(x) = 0$ (2). We show that $x + n + f(x)$ is a fixed point of f, $\forall n \in \mathbb{Z}_{\geq 1}$ (3). Indeed, if z and $z + 1$ are both fixed point of f, then plugging $x = 1$ and $y = z$ we have LHS$=$ $f(1 + f(y + 1)) + f(y) = f(1 + y + 1) + y = f(y + 2) + y$ while RHS is $1 + f(1 + y) + yf(1) = 1 + 1 + y + 1 + y$. With RHS=LHS, $f(y + 2) = y + 2$. Doing this repeatedly yields $y + 3, y + 4, \cdots$ fixed points of f. Now plug $z = x - 1 + f(x)$, any x. By (1), z is a fixed point and $z + 1 = x + f(x)$ is a fixed point by (2), too. We conclude that we established (3). In particular, $n = 1$ yields $x + 1 + f(x)$, or $x + f(x - 1)$ fixed points ($\forall x \in \mathbb{R}$), so plugging $y = -1$ yields $f(x + f(x - 1)) + f(-x) = x + f(x - 1) - f(x)$, or $f(x) = -f(x)$. (4)

Finally, replacing x by $-x$ and y by $-y$ in the problem yields $-x + f(-x - y){=}{-x} - f(x + y) = -(x + f(x + y)$ by (4), so by (4) again we have $f(-x + f(-x - y)) = -f(x + f(x + y))$. $-yf(-x) = yf(x)$, so $-f(x + f(x + y)) + f(xy) = -(x + f(x + y)) + yf(x)$. Adding this LHS to the original equation, and the RHS to the original equation, and equating them, we have $2f(xy) = 2yf(x)$ or $f(xy) = yf(x)$. In particular, $f(y) = yf(1) = y$ when $x = 1$, an this function obiously satisfies the problem condition. Q.E.D.

2 Combinatorics

1. **C1.** In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Solution. We induct on the n, the number of towns. Let T_i, L_i, R_i be town i (counted from the left, so T_1 is leftmost and T_n is rightmost), left bulldozer of town i and right bulldozer of town i, respectively, $\forall i \in [1, n]$. For $n = 1$ there is nothing to prove; for $n = 2$, T_2 is swept $\Leftrightarrow R_1 > L_2 \Leftrightarrow T_1$ isn't swept.

By induction hypothesis we can assume that there is a unique town T_s that is not swept when there are n towns, and add a town T_{n+1} on the right. Obviously, there are no way for R_i and L_j to reach T_s, $\forall i \in [1, s-1], \forall j \in [s+1, n]$. We show that exactly one of T_s and T_{n+1} will be swept. Indeed, denote M such that $R_M = \max\{R_i \mid i \in [s, n]\}$, meaning that R_M can sweep $T_k, \forall k \in [M+1, n]$. If $R_M > L_{n+1}$, then after sweeping T_n, R_M can sweep T_{n+1} but L_{n+1} can't sweep T_M, so it can't sweep T_s (since $s \leq M$). By our hypothesis no other bulldozer can sweep T_s so here, T_{n+1} is swept but not T_s. Conversely, if $L_{n+1} > R_M$, then L_{n+1} can sweep $T_n, T_{n-1}, \cdots, T_M, \cdots, T_s$. No single town $T_i, \forall i \in [s, n]$ can sweep T_{n+1} and by the second sentence of this paragraph, no town $T_i, \forall i \in [1, s-1]$ can sweep T_{n+1}. Hene T_s can be swept, but not T_{n+1}.

2. **C2/IMO 1.** We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

   (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

   (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

   Solution. For n odd, taking $A_1A_2\cdots A_n$ a regular n-gon yields that the perpendicular bisector of A_iA_j passes through $A_{\frac{i+j}{2}}$ for $i+j$ even, or $A_{\frac{i+j+n}{2}}$ for $i+j$ odd (indices taken modulo n). This configuration is therefore balanced, and since P with $PA_i = PA_j = PA_k$ implies P is the centre of the polygon (i.e. $P \neq A_i$, $\forall \in [1, n]$), this configuration is also centre-free.

   Now, for even n, consider a circle with centre O and vertices A, B, C such that A, B, C lie on this circle in that order and that triangles AOB and BOC are both equlateral. Obviously, the perpendicular bisector of any point on the circle pass through O, and the perpendicular bisectors of OA, OB, OC pass through B, C, B, respectively. Hence $OABC$ is balanced. Now we add two points X, Y on the circumfence at a time, such that X, Y do not overlap the previous points and XOY equilateral. O is equidistant from any point on the circle, so we only need to consider lines XO, YO. However, their perpendicular bisectors pass through Y, X respectively (so the configuration is balanced too).

   Finally, if a configuration of n points $(A_1, A_2, \cdots, A_n)$ is centre-free, then we denote $f(i, j)$ $(i \neq j)$ such that $A_{f(i,j)}$ is equidistant from A_i and A_j (if there are more than one such point we take this f arbitrarily). Now $f(i, j) \notin \{i, j\}$ and $f(i, j) \neq f(i, k)$ if $j \neq k$ (otherwise, $A_{f(i,j)}$ is equidistant from A_i, A_j, A_k, contradiction. Therefore, for fixed i, $\{f(i, j) | j \in [1, n] \backslash \{i\}\} = \{j | j \in [1, n] \backslash \{i\}\}$. Summing this for all i entails that we counted each $f(i, j)$ twice, and for each number k, k is in $n-1$ of the sets $\{j | j \in [1, n] \backslash \{i\}\}$, $i = 1, 2, \cdots n$. Therefore, for each $k \in [1, n]$ there are $\frac{n-1}{2}$ unordered paris of i, j for which $f(i, j) = A_k$. Therefore n is odd. (Answer for (b): all odd n).

3. **C3.** For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to

the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

Solution. The largest element in A_1 cannot be greater than the smallest element in A_2, so if we sort the numbers in $a_1 < a_2 < \cdots < a_n$ then A_1=$\{a_1, a_2, \cdots, a_i\}$ for some $i \in [1, n]$ and $A_2 = \{a_{i+1}, a_{i+2}, \cdots, a_n\}$. Also, any element in A_1 must divide any element in A_2.

Denote k-partition as the partition of the set in A_1 and A_2 s.t. $A_1 = \{a_1, a_2, \cdots, a_k\}$ and $A_2 = \{a_{k+1}, a_{k+2}, \cdots, a_n\}$. Also denote the LCM and GCD of this k-partition as the LCM of A_1 and GCD of A_2, respectively. Suppose also that $k-1$-partition and k-partition are both good. We show that $3 \le k \le n-2$, and that $k+1$-partition and $k-2$-partition cannot be good. Indeed, every element in $\{a_1, a_2, \cdots, a_{k-1}\}$ divides $\{a_k, a_{k+1}, \cdots, a_n\}$ and every element in $\{a_1, a_2, \cdots, a_k\}$ divides $\{a_{k+1}, a_{k+2}, \cdots, a_n\}$. This entails the fact that a_k is a divisor of $a_{k+1}, a_{k+2}, \cdots, a_n$ and multiple of $a_1, a_2, \cdots, a_{k-1}$. This means the GCD of $k-1$-partition is a_k, (so same goes for the LCM of this partition) and the LCM of k-partition is also a_k (so same goes for the GCD of this partition). If $k = 2$ then the LCM of 1-partition is a_2. However, $A_1 = \{a_1\}$, contradiction. Similarly, k cannot be $n-1$ as well. Consider $k+1$-partition. Now in A_1, a_{k+1} is obviously a multiple of every other element (by hypothesis above) so the LCM is a_{k+1}. However, since the GCD of k-partition is a_k, there exists element a_i $(i > k+1)$ where $a_{k+1} \nmid a_i$, so the GCD of this $k+1$-partition cannot be a_{k+1}, and $k+1$-partition is not good. Similarly, $k-2$-partition is not good.

Finally, denote $1 \le x_1, x_2, \cdots x_m \le n-1$ be all indices such that x_i-partition is not good. From above, $x_1 \le 2$ and $x_m \ge n-2$, while $x_{i+1} - x_i \le 3$, $\forall i \in [1, m]$. Notice, also, that 2015=$n-1-m$. This means $n-2 \le x_m \le x_1 + 3(m-1) \le 2 + 3(m-1)$, so $n \le 3m+1 = 3(n-1-2015)+1 = 3n-6047$. We have $2n \ge 6047$, or $n \ge 3024$ since $n \in \mathbb{N}$. This can be achieved by taking $a_{3i+1} = 2^{i+1} \cdot 3^i$, $a_{3i+2} = 2^i \cdot 3^{i+1}$, $a_{3i+3} = 2^{i+1} \cdot 3^{i+1}$, $\forall i \in [0, 1007]$. where a $3i+2$-partition will give both LCM and GCD of $2^{i+1} \cdot 3^{i+1}$ $(\forall i \in [0, 1007])$ and a $3i+3$-partition will give both LCM and GCD of $2^{i+1} \cdot 3^{i+1}$.

4. **C5/IMO 6.** The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

   (i) $1 \le a_j \le 2015$ for all $j \ge 1$, (ii) $k + a_k \ne \ell + a_\ell$ for all $1 \le k < \ell$.

   Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \le 1007^2$$

   for all integers m and n such that $n > m \ge N$.

   Solution. Denote b_n as $a_n + n$, and we know that b_i is distinct for every b. Let $x_1 < x_2 < \cdots < x_k$ be first k numbers such that $b_i \ne a_j$, $\forall i \ge 1$, $\forall 1 \le j \le k$. It follows that there are exactly $x_k - k$ such i's such that $b_i < x_k$ (1). If $k > 2015$, then from the fact $n < b_n \le n + 2015$ we know that for $i \le x_k - k$, $b_i \le i + 2015 \le x_k - k + 2015 < x_k$. Therefore, $b_1, b_2, \cdots, b_{x_k - k} \le x_k$ so from point 1, $\{b_1, b_2, \cdots b_{x_k-k}\} = \mathbb{N} \cap [1, x_k] \backslash \{x_i \mid 1 \le i \le k\}$. Now $b_{x_k-k+1} \ge x_k + 1$, and $a_{x_k-k+1} = b_{x_k-k+1} - (x_k - k + 1) \ge x_k + 1 - (x_k - k + 1) = k > 2015$, contradiction. Therefore k is finite (at least 1, since $b_n > 1$ for all n.

   We show that $b = k$ and $N = x_k$ satisfies the conclusion we want to prove. Now, we can further assert that for any $N > x_k$, there are exactly $N - k$ indices i such that $b_i \le N$. (2) Denote by S_n the sum $\sum_{i=1}^{n} b_i$, the sum of elements in the set $B_n = \{b_i \mid 1 \le k \le n\}$. Now,

$\sum_{j=m+1}^{n}(a_j-b)$ is precisely $S_n-S_m-\sum_{j=m+1}^{n}j-(n-m)b$. We consider S_n, in general, and prove that $T_n \le S_n \le T_n+(k-1)(2015-k)$, where T_n is taken as $\sum_{i=1}^{n+k}i-\sum_{j=1}^{k}x_k$, the elements in the set $C_n=\{1,2,\cdots,n+k\}\backslash\{x_i \mid 1\le i\le k\}$. This is the n smallest possible sequence that appears in set $\{b_1,b_2,\cdots\}$ so $S_n \ge T_n$ follows from here.

Denote by X' the set containing all elements not in set X. To prove the right inequality, from point 1.1 we know that if $i\in B_n\cap C_n'$, then $n+k+1\le i\le n+2015$ and $\mid B_n\cap C_n' \mid\le 2015-k$; if $j\in C_n\cap B_n'$ then $n+2\le i\le n+k$. and $\mid C_n\cap B_n'\mid\le k-1$. But since $\mid B_n\cap C_n'\mid=\mid C_n\cap B_n'\mid$ we must have this number at most $p=\min(k-1,2015-k)$. Now, we have S_n-T_n=sum of elements in $B_n\cap C_n'$-sum of elements in $C_n\cap B_n' \le (n+2015)+(n+2014)\cdots+(n+2016-p)-((n+2)+(n+3)+\cdots+(n+p+1)) = 2013+2011+\cdots+(2013-2(p-1))=\frac{p}{2}\cdot(4028-2p)$=$p(2014-p)$, which is precisely $(k-1)(2015-k)$.

Finally, T_n-T_m=$(m+k+1)+(m+k+2)+\cdots+(n+k)$=$k(n-m)+\sum_{j=m+1}^{n}j$, so $\sum_{j=m+1}^{n}(a_j-k)=S_n-S_m-\sum_{j=m+1}^{n}j-(n-m)k=(S_n-T_n)-(S_m-T_m)$. From above, $0\le S_n-T_n, S_m-T_m\le(k-1)(2015-k)$, so $-(k-1)(2015-k)\le(S_n-T_n)-(S_m-T_m)\le(k-1)(2015-k)$. Now $(k-1)(2015-k)\le(\frac{k-1+2015-k}{2})^2=1007^2$ by AM-GM inequality. Q.E.D.

3 Geometry

1. **G1.** Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ=AH$.

   Solution. Denote P as $AC\cap HG$. Now $IP=PH$, from H beign the orthocentre $\angle ACH=\angle HBA$, from $HG\parallel AB$ we have $\angle HBA=\angle BHG$, from $CH\perp AB, HG$ and $BC\perp AH, BG$ we have $\angle CHG=\angle CBG=90^\circ$, so $CHBG$ is cyclic and $\angle BHG=\angle BCG$. Moreover $\triangle IPJ\sim\triangle CPG$. So $IJ=CG\cdot\frac{IP}{CP}=CG\cdot\frac{PH}{CP}=CG\cdot\sin\angle PHC=CG\cdot\sin\angle ACH=CG\cdot\sin\angle HBA=CG\cdot\sin\angle BHG=CG\cdot\sin\angle BCG=BG=AH$.

2. **G2/IMO 4.** Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

   Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

   Solution. Since AO is the perpendicular bisector of FG (why? $OF=OG$ and $AF=AG$), we only need the fact $\angle AFK=\angle AGL$. First, we show that $\angle AFD-\angle AGE=\angle ABC-\angle ACB$. Indeed, let FG intersect BC at R, and for sake of simplicity assume that R lies on ray CB beyond B. Taking the triangle RGB and the exterior angle at B yields $\angle GBC=\angle FGB+\angle GRB$; taking the triangle RGD and exterior angle at D yields $\angle GDE=\angle FGD+\angle GRD$. Now $\angle ABC-\angle ACB=(\angle ABG+\angle GBC)-(\angle ACF+\angle FCB)=\angle GRB$ (bearing in mind that $\angle ABG=\angle ACF$ since $AG=AF$)$=\angle GRD=\angle GDE-\angle FGD$$=\frac{1}{2}\angle GAE-\frac{1}{2}\angle FAD=(90^\circ-\angle AGE)-(90^\circ-\angle AFD)=\angle AFD-\angle AGE$ (since $AF=AD$ and $AG=AE$). Now, $\angle AFK-\angle AGL=(\angle AFD-$

$\angle KFD$) – ($\angle AGE - \angle LGE$) = ($\angle ABC - \angle ACB$) - ($\angle ABC - \angle ACB$)=0, since $BFKD$ and $CGLE$ are both cyclic and $\angle KFD = \angle KBD = \angle ABC$, $\angle LGE = \angle LCE = \angle ACB$.

3. **G3.** Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

   Solution. By Menelaus' theorem applied on triangle DHB and line CH we have $\frac{AH}{HB} = \frac{PD}{PB}$. Now, let AD intersect ω at T, and let CT intersect ω again at Q'. We are then left to prove that PQ' is tangent to ω, so $Q' \equiv Q$. Since $\angle DTB = \angle ATB = \angle ACB = 90°$, $ACTB$ is cyclic and so $\angle CBA = \angle CTA = \angle Q'TD = \angle Q'BD$. With $\angle DQ'B = 90°$ we have $\triangle ABC \sim \triangle DBQ'$. Finally, if the tangent to Q' intersects BD at P', then $\frac{P'D}{P'B} = (\frac{DQ'}{Q'B})^2 = (\frac{AC}{AB})^2 = \frac{AH}{HB} = \frac{PD}{PB}$, yielding $P \equiv P'$ and thus PQ' is tangent to ω.

4. **G4.** Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

   Solution. Let S be the common midpoint of BT and PQ. We claim that, if $BMQP$ is cyclic (regardless of whether T, A, B, C are concyclic) then S lies on a fixed line. Indeed, consider any P, Q, P', Q' with $BMQP$ and $BMQ'P'$ cyclic, and P, P' on AB, Q, Q' on BC, then it is not hard to see that $\triangle MPP' \sim \triangle MQQ'$. Denoting S' as midpoint of $P'Q'$ we know that there exists a spiral similarity centred at M that brings P to P', Q to Q' and S to S'. Moreover, $\angle(PP', SS') = \angle(MP, MS)$, i.e. if there is another point S'' with this property then S, S', S'' are collinear. So S lies on a fixed line. Denote P_0, Q_0, S_0 as P, Q, S in the case when $PQ \parallel AC$. We know that S_0 is on BM, so $\angle PMS = \angle P_0M_0S_0$ $= \angle P_0Q_0B = \angle ACB$. Similarly $\angle QMS = \angle BAC$. In degenerate case where Q coincides with B, let P_1 be the midpoint of BP and we have $\angle P_1MB = \angle QMS = \angle BAC$, meaning $BM^2 = BP_1 \cdot AB$. Similarly, $BM^2 = BQ_1 \cdot BC$, where Q_1 is defined as S when $P \equiv B$. This means that if O is the circumcentre of triangle ABC, then $BO \perp P_1Q_1$ since P_1Q_1 is antiparallel to BC.

   Now T, A, B, C concyclic iff $\angle BSO = 90°$. From $S \in P_1Q_1$ and $P_1Q_1 \perp BO$, if h_1 is the disance from B to P_1Q_1 we have $BS^2 = h_1 \cdot R$ (with R the circumradius of $\triangle ABC$). Notice also that $\triangle BP_1Q_1$ and $\triangle BCA$ are similar with similitude $\frac{BM^2}{BA \cdot BC}$. Therefore if h is the perpendicular distance from B to AC then $BS^2 = h \cdot R(\frac{BM^2}{BA \cdot BC})$. Since $\frac{hR}{BA \cdot BC}$ is $\left(\frac{BS}{BM}\right)^2$, this ratio is what we sought for.

   It is not hard to notice that $\frac{hR}{BA \cdot BC} = \frac{2R|\triangle ABC|}{BA \cdot BC \cdot AC}$, where $|\triangle ABC|$ is the area of triangle ABC. Indeed, it is well-known that $|\triangle ABC| = \frac{BA \cdot BC \cdot AC}{4R}$. Therefore $\frac{2R|\triangle ABC|}{BA \cdot BC \cdot AC} = \frac{1}{2}$. Finally, $BT = 2BS = 2\left(\sqrt{\frac{1}{2}}\right) BM = \sqrt{2}BM$.

5. **G5.** Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

   Solution. Denote the centre of Γ as O, and W.L.O.G. we have $CA < CB$. Denote also the angle α and $\angle CDA = \angle CID = \frac{1}{2}\angle COD$. We know that $AC^2 = AD^2 + DC^2 - 2 \cdot AD \cdot$

$CD{\cdot}\cos\alpha$ and $BC^2 = BD^2+DC^2-2{\cdot}BD{\cdot}CD{\cdot}\cos(180^\circ-\alpha)$. With $\cos(180^\circ-\alpha) = -\cos\alpha$ and $AD = BD$, we have $\cos\alpha = \frac{BC^2-AC^2}{4\cdot AD\cdot CD}$ (1), and $AC^2 + BC^2 = 2AD^2 + 2DC^2$. (2)

Notice that $CQ = QP$ iff $OQ \perp CM$. Also name the point E as the midpoint of FG, and X the intersection of FG and the tangent to Γ at C. Since $FG \parallel AB$ and E lies on CD, it's not hard to notice that $XC = XE$. Moreover, $OD \perp FG$, $OE \perp CD$, and $\angle CEX = \alpha$. What we need now reduces to the fact $\angle MCE = \angle QOE$ (or $\sin\angle MCE = \sin\angle QOE$) and equivalently $\angle CME = 180^\circ - \angle QOD$ (also $\sin\angle CME = \sin\angle QOD$.) However, with $\angle MCE + \angle CME = 180^\circ - \angle MEC = \angle EOD = \angle QOD - \angle QOE$ we only need the fact $\frac{ME}{CE} = \frac{\sin\angle MCE}{\sin\angle CME} = \frac{\sin\angle QOE}{\sin\angle QOD} = \frac{QE}{QD} \div \frac{OE}{OD}$. The first equality follows by sine rule, and the last equality is well-known in trigonometry. The second equality left to be proven.

Now $\frac{OE}{OD} = \cos\alpha = \frac{BC^2-AC^2}{4\cdot AD\cdot CD}$ by (1). We need the equivalence $CH' = HA = \frac{AD^2}{AC}$ and $CI' = BI = \frac{AD^2}{BC}$ (bearing in mind that $AD = BD$). If we let R to be $H'I' \cap AB$ then the relation $\frac{RA}{RB} = \frac{AH'}{H'C} \cdot \frac{CI'}{I'B}$ holds by Menelaus' theorem. Changing $H'C$ into $AC - CH'$ and BI' into $BC - CI'$ yields $\frac{AH'}{H'C} \cdot \frac{CI'}{I'B} = \frac{AC^2-AD^2}{BC^2-AD^2}$. This, in turn, means $\frac{RA}{RD} = \frac{2(AC^2-AD^2)}{AC^2-AD^2+BC^2-AD^2}$ (since D is the midpoint of AB), so by Menelaus' theorem on $\triangle ACD$ and line $H'I'$ we have $\frac{2(AC^2-AD^2)}{AC^2-AD^2+BC^2-AD^2} = \frac{AH'}{H'C} \cdot \frac{CQ}{QD}$. With $\frac{AH'}{H'C} = \frac{AC^2-AD^2}{AD^2}$ we have $\frac{CQ}{QD} = \frac{2AD^2}{AC^2+BC^2-2AD^2}$. But $\frac{CQ}{CD} = \frac{CQ}{CQ+QD} = \frac{2AD^2}{AC^2+BC^2}$ and $CE = \frac{1}{2}CD$, $\frac{CQ}{CE} = \frac{4AD^2}{AC^2+BC^2}$, $\frac{CQ}{QE} = \frac{CQ}{CE-CQ} = \frac{4AD^2}{AC^2+BC^2-4AD^2}$ and $\frac{QE}{QD} = \frac{AC^2+BC^2-4AD^2}{2AC^2+2BC^2-4AD^2}$, and therefore $\frac{QE}{QD} \div \frac{OE}{OD} = \frac{AC^2+BC^2-4AD^2}{2AC^2+2BC^2-4AD^2} \cdot \frac{4\cdot AD\cdot CD}{BC^2-AC^2}$. (3)

Now $H'F = CF - CH' = \frac{1}{2}AC - \frac{AD^2}{AC} = \frac{AC^2-2AD^2}{2AC}$. Similarly $I'G = \frac{BC^2-2AD^2}{2BC}$. Therefore by Menelaus' theorem again $\frac{MF}{MG} = \frac{H'F}{H'C} \cdot \frac{CI'}{I'G} = \frac{AC}{BC} \cdot (\frac{AC^2-2AD^2}{2AC} / \frac{BC^2-2AD^2}{2BC}) = \frac{AC^2-2AD^2}{BC^2-2AD^2}$. (It is easy to prove that $\frac{CI'}{CH'} = \frac{AC}{BC}$.) This means $\frac{MF}{FG} = \frac{AC^2-2AD^2}{BC^2-AC^2}$ $(FG = MG - MF)$, so $ME = MF + \frac{1}{2}FG = FG(\frac{AC^2-2AD^2}{BC^2-AC^2} + \frac{1}{2}) = FG(\frac{BC^2+AC^2-4AD^2}{2(BC^2-AC^2)})$. Therefore, $\frac{ME}{CE} = \frac{FG}{CE} \cdot \frac{BC^2+AC^2-4AD^2}{2(BC^2-AC^2)} = \frac{AD}{CD} \cdot \frac{BC^2+AC^2-4AD^2}{(BC^2-AC^2)}$, since $AB = 2AD = 2FG$ and $CD = 2CE$. (4)

Finally, combining (2), (3) and (4), we have the relation $2CD^2 = AC^2 + BC^2 - 2AD^2$ that implies $\frac{AC^2+BC^2-4AD^2}{2AC^2+2BC^2-4AD^2} \cdot \frac{4\cdot AD\cdot CD}{BC^2-AC^2} = \frac{AD}{CD} \cdot \frac{BC^2+AC^2-4AD^2}{(BC^2-AC^2)}$ This proves $\frac{ME}{CE} = \frac{QE}{QD} \div \frac{OE}{OD}$.

6. **G6/IMO 3.** Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

   Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

   Solution. Let QH intersect Γ again at U, we know that $\angle AQU = \angle ABU = \angle ACU = 90^\circ$, so from $BU, CH \perp AB$ and $CU, BH \perp AC$ we have $BHCU$ a parallelogram, so HU bisects BC and we have Q, H, M collinear.

   Now denote by X the midpoint of QH. First, the circumcircle of MFH is tangent to the circumcircle of QKH, because they intersect at H and from $\angle HFM = \angle QKH = 90^\circ$, the centres of the circles must lie on midpoints of HM and QH, respectively, and H lies on the line joining the centres. Next, denote by R the radical centre of circumcircles of HFM, QKH, MKF, The radical axis of HFM and MKF is FM (i.e. BC) and the radical axis of HFM and QKH is HR, with $HR \perp QH$. It suffices to prove that KR

is tangent to the circumcrcle of QKH since KR is the radical axis of the circles that we want to prove them tangent.

Now, $OX^2 = R^2 - QX \cdot XU$ (power of point)$=R^2 - XH \cdot (XH + 2HM) = R^2 - XH^2 - 2XH \cdot HM$ ($HM = MU$ and $XH = XQ$), $XR^2 = XH^2 + HR^2$ and $OR^2 = OM^2 + MR^2 = (R^2 - QM \cdot MU) + HM^2 + HR^2 = (R^2 - (2XH + HM) \cdot HM) + HM^2 + HR^2 = R^2 - 2XH \cdot HM + HR^2$. Combining the three yields $OX^2 + XR^2 = OR^2$, so $\angle OXR = 90°$. Finally, since OX is the perpendicular bisector of KQ, we know that XO is an angle bisector of lines XH and XK, With $\angle OXR = 90°$, XR is another angle bisector of these two lines. Therefore, $\angle HXR = \angle KXR$. With $XK = XH$, $\triangle XHR \cong \triangle XKR$, so $\angle XKR = \angle XHR = 90°$, and KR is indeed tangent to circumcircle of QKH.

7. **G7.** Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Solution. We first start with a lemma: let ℓ_0, ℓ_1, ℓ_2 be lines that are either concurrent or parallel. Let A, C be on ℓ_0, A_1, C_1 on ℓ_1 and A_2, C_2 on ℓ_2. Let C_2A_1 intersect AA_2 and CC_1 at S and Q, respectively. Let A_2C_1 intersect AA_1 and CC_2 at P and R, respectively. Then PQ, RS, AC will also be either concurrent and parallel.

Proof: the problem condition tells us that the triangles AA_1A_2 and CC_1C_2 are Desargues' perspective of each other. This means, if we denote $AA_1 \cap CC_1$ as X_1 and $AA_2 \cap CC_2$ as X_2 then the intersection of A_1A_2 and C_1C_2 will lie on X_1X_2 too (or possibly, X_1X_2, A_1A_2, C_1C_2 are all parallel).

By Menelaus' theorem we have $\frac{A_1A}{A_1X_1} \cdot \frac{A_2X_2}{A_2A} = \frac{C_1C}{C_1X_1} \cdot \frac{C_2X_2}{C_2C}$. Denoting the intersection of A_1C_2 as T (possibly point of infinity) and considering triangles AX_1X_2 and CX_1X_2 gives $\frac{A_1A}{A_1X_1} \cdot \frac{SX_2}{SA} \cdot \frac{X_1T}{X_2T} = -1 = \frac{C_2X_2}{C_2C} \cdot \frac{QC}{QX_1} \cdot \frac{X_1T}{X_2T}$, which gives $\frac{A_1A}{A_1X_1} \cdot \frac{SX_2}{SA} = \frac{C_2X_2}{C_2C} \cdot \frac{QC}{QX_1}$. Similarly, $\frac{A_2X_2}{A_2A} \cdot \frac{PA}{PX_1} = \frac{C_1C}{C_1X_1} \cdot \frac{RX_2}{RC}$. Combining everything above gives $\frac{PA}{PX_1} \cdot \frac{SX_2}{SA} = \frac{RX_2}{RC} \cdot \frac{QC}{QX_1}$. By Menelaus' theorem again PS and QR either intersect on X_1X_2 or both parallel to X_1X_2, so APS and CQR are also perspective of each other. Thus AC, PQ, RS are concurrent or parallel. ■

Denote the incircles of $APOS$, $BQOP$, $CROQ$, and $DSOR$ by ω_A, ω_B, ω_C, ω_D, respectively. Denote $T(W, XY)$ by the point of tangency of circle ω_W to line XY too.

We first prove that $PR = QS$. take E, the exsimilicenter of ω_B and ω_C and consider the triangle formed by $e(BC), O, Q$. Now, ω_B and ω_C are incircle and excircle of this triangle, so $OT(C, OQ) = QT(B, OQ)$ (*). In the case where $PR \parallel BC$, (*) follows by symmetry, We similarly have $PT(B, PR) = OT(A, PR) = OT(A, SQ)$, $OT(B.PR) = OT(B, SQ)$, $OT(C, PR) = OT(C, SQ) = QT(B, SQ)$ and $RT(C, PR) = OT(D, PR) = OT(D, QS) = ST(A, QS)$ Therefore, $PR = PO + OR = PT(B, PR) + OT(B.PR) + OT(C, PR) + RT(C, PR) = OT(A, SQ) + OT(B, SQ) + QT(B, SQ) + ST(A, QS) = SO + OQ = QS$. It then follows that 0$=SQ - PR = SO - OR + OQ - OP = SD - DR + BQ - BP$. Similarly $0 = SO - OP + OQ - OR = SA - AP + QC - RC$. Adding the two equations we have $0 = SD + SA + BQ + QC - DR - RC - BP - AP = AD + BC - DC - AB$, and this last relation yields $ABCD$ circumscribed.

Now, let AD and BC intersect PR, at A_2 and C_1, respectively, AB, CD intersect QS at A_1 and C_2 respectively. Denote T as the exsimilicenter of ω_A and ω_C (possible point at infinity). Denote also ω_O as the incircle of $ABCD$ (which we proved exist). Then A is the exsimilicenter of ω_O and ω_A; B, of ω_O and ω_B; C, of ω_O and ω_C; and D, of ω_O and ω_D. Thus by Monge's theorem, A, C, T are collinear. Moreover, A_1 is the exsimilicenter of ω_A and ω_B; and C_1, of ω_B and ω_C. Again by Monge's theorem, A_1, C_1, T collinear. Similarly, A_2, C_2, T collinear. So AC, A_1C_1, A_2C_2 parallel or concurrent. Keeping in mind that $P = C_1A_2 \cap AA_1$, $Q = CC_1 \cap C_2A_1$, $R = C_1A_2 \cap CC_2$, $S = A_1C_2 \cap AA_2$, we can use the lemma aforementioned to finish our proof. Q.E.D.

4 Number Theory

1. **N1.** Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k\lfloor a_k\rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Solution. It's not hard to notice that if $a_k =$ integer+0.5, then a_{k+1} is either in the same form or is an integer. Notice also that, if a_k is even integer+0.5, then a_{k+1} is an integer. As such, all even M are solutions, and from now on we assume a_k as an odd integer+0.5.

We denote $v_2(x)$ as the highest power of 2 dividing x. If $v_2(a_k-1.5) = v_2(\lfloor a_k\rfloor-1) = c$ then $a_{k+1} = (a_k-0.5)^2+\frac{a_k-0.5}{2}$. Now $a_{k+1}-1.5 = (a_k-0.5)^2+\frac{a_k-0.5}{2}-1.5 \equiv 1+\frac{a_k-0.5}{2}-1.5 = \frac{a_k-1.5}{2} \pmod{2^c}$, since $a_k - 0.5 \equiv 1 \pmod{2^c}$. By assumption, $2^{c+1} \nmid a_k - 1.5$ so $\frac{a_k-1.5}{2} \equiv 2^{c-1} \pmod{2^c}$, or $v_2(a_{k+1} - 1.5) = v_2(a_k - 1.5) - 1$. This means that $v_2 a_{k+c} - 1.5 = 0$, or $\lfloor a_{k+c}\rfloor$ is even (hence satisfying the problem condition). This is particularly true for $k = 0$ and $M > 1$. For $M = 1$, the sequence yields $_k = 1.5$ for all k, so the answer is every number except 1.

2. **N2.** Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Solution. If $a = 1$ then $b!+1 \mid b!$, absurd. So we can assume that $a, b \geq 2$ and $3a < 2b+2$ implies $b > a$, which we can safely assume.

Now change the problem to be $a!(1+(a+1)(a+2)\cdots b) \mid a!b!$, or $1+(a+1)(a+2)\cdots b \mid b!$. If $b > \frac{3a}{2} - 1$, then $\underline{\ }a \geq \frac{a}{2}$ for a even and $\geq \frac{a-1}{2}$ otherwise. Let $p \mid b!$ for some prime p and $p \leq \frac{a-1}{2}$. Since $(a + 1)(a + 2)\cdots b$ consists of $b - a \geq \frac{a-1}{2}$ consecutive integer, p divides this number and therefore $p \nmid 1 + (a + 1)(a + 2)\cdots b$. Consequently, if prime p s.t. $p \mid \gcd(1 + (a + 1)(a + 2)\cdots b, b!)$ then $\frac{a+1}{2} \leq p \leq a$ and $p \nmid a + 1, a + 2, \cdots b$. Therefore, since $2p > a$ we must have $2p > b$ as well, yielding $p \parallel b!$. Considering all those primes yield the gcd of the two numbers is at most $(\frac{a}{2} + 1)(\frac{a}{2} + 2)\cdots a$, for a even, or $a(a-2)\cdot\cdots\cdot\frac{a+1}{2}$(or $\frac{a+3}{2}$) (notice that we eliminated all even factors since they cannot be prime). In both cases they are less than $1 + (a + 1)(a + 2)\cdots b$, so the problem condition cannot hold.

3. **N3.** Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1x_2\ldots x_{n+1} - 1$ is divisible by an odd prime.

Solution. Let $a_k = x_k-1 = \frac{m-n}{n+k}$, we know that $m-n$ is divisible by $n+1, n+2, \cdots 2n+1$ so it is divisible by $l = lcm(n + 1, n + 2, \cdots 2n + 1)$. Let $g(k) = \frac{x}{n+k}$ and we show that eactly one of $g(k)$ among $g(1), g(2), \cdots g(n + 1)$ is odd. Indeed, if $2^c \leq n < 2^{c+1}$ then $2^{c+1} \leq 2n < 2^{c+2}$ and we know that exactly one power of 2, which is 2^{c+1}, is in $n + 1, n + 2, \cdots 2n + 1$. Conversely, only one number is divisible by 2^{c+1} and $g(2^{c+1} - n)$ is odd, but $g(k)$ is even for other k.

Now, let $x = \frac{m-n}{l}$ and $a_k = g(k)\cdot x$. Now $P(x) = x_1x_2\cdots x_{n+1} - 1 = (\prod_{k=1}^{n+1}(g(k)\cdot x+1)) -$ 1$= x(\prod_{i=1}^{n+1} c_i x^{i-1})$, where c_i is $x(\prod_{b_1<b_2<\cdots b_i} g(b_1)g(b_2)\cdots g(b_i))$. If $P(x)$ is a power of 2, then x must be a power of 2, and from the fact that c_1 is odd but $c_2, c_3, \cdots c_{n+1}$ all even we conclude that $(\prod_{i=1}^{n+1} c_i x^{i-1})$ must be 1, which is impossible as $g(1), g(2), \cdots g(n+1)$ are all at least one and $(\prod_{i=1}^{n+1} c_i x^{i-1}) > c_i \geq n+1$. Contradiction is achieved and $P(x)$ is divisible by an odd prime.

4. **N4.** Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Solution. We partition the sequence (a_n) into *groups* of adjacent numbers such that a_{i+1} and a_i are in the same group if and only if $a_{i+1} > a_i$. Obviously, $a_{i+1} - 1 \mid a_i$, so $a_{i+1} > a_i \Leftrightarrow a_{i+1} = a_i + 1 \Leftrightarrow a_i \mid b_i \Leftrightarrow b_{i+1} = b_i - 1$. The tail of a group is defined as the last (and therefore the largest) number in that group; we claim that the sequence of numbers containing all tails of groups must be non-increasing. Indeed, let a_n be a tail, then $a_{n+1} = \gcd(a_n, b_n) + 1$, and for sake of simplicity denote this number by $g + 1$. We then have $b_{n+1} = \dfrac{a_n b_n}{g} - 1$. Now, suppose that the next tail is at least a_n, then $(a_{n+k}, b_{n+k}) = (g + k, \dfrac{a_n b_n}{g} - k)$, $\forall k \leq a_n - g$. We therefore have $a_{n+a_n-g} = a_n$ and $b_{n+a_n-g} = \dfrac{a_n b_n}{g} - a_n + g$. Notice that $g = \gcd(a_n, b_n)$ so $g \mid b_n$, and a_n divides both $\dfrac{a_n b_n}{g}$ and a_n, so $\gcd(a_{n+a_n-g}, b_{n+a_n-g}) = g < a_n$, and this a_{n+a_n-g} is a tail now (hence cannot exceed a_n).

Since the tail of the sequence cannot decrease forever, it must remain constant at one point. Now, for sufficiently large indices when the tail remains constant, from above we know that if a_n and a_m are both tails then $\gcd(a_n, b_n) = \gcd(a_m, b_m)$. It folows that the number succeeding each tail must be the same too, and that's the smallest number of the group thereafter. Summing up, the smallest and the biggest number (i.e. first and last number) in a period becomes constant, and $a_{i+1} = a_i + 1$ for a_i, a_{i+1} in the same group. We therefore conclude that the groups must be identical at one point, hence eventually periodic.

5. **N6.** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$; (ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

Solution. If $f(m) = f(k)$, then $f^n(m) = f^n(k), \forall n \in \mathbb{N}$. By (i), n divides both $f^n(m) - m$ and $f^n(m) - k$, so n divides $m - k$ for all positive integers n. This means $m - k = 0$ and f is injective. From (i) again we have $f(m) - m > 0, \forall m \in \mathbb{N}$. This allows us to partition the set of positive integers into groups such that for one group $A = \{a_i \mid i \geq 0\}$, $a_{k+1} = f(a_k)$, $\forall k \geq 1$ and we name a_0 as the element such that there exists no integer i such that $f(i) = a_0$. By (ii), the number of set A is finite, so name them as $A_1, A_2, \cdots A_p$.

We proceed with this lemma:
For any set $A_k = \{a_i \mid i \geq 0\}$ *such that there eists a positive integer* s *with* $a_{i+1} - a_i < s$ *for infinitely many* i*, the number* $a_0, a_1, \cdots$ *necessarily form an arithmetic progression.*
Proof: For every i and every M, we can find infinitely many $j > i + M$ s.t. j satisfies this property $a_{j+1} - a_j < s$. Now, take $M > |a_{i+1} - a_i| + s$ and we have $j - i$ divides both $a_j - a_i$ and $a_{j+1} - a_{i+1}$. It therefore divides $(a_{j+1} - a_{i+1}) - (a_j - a_i) = (a_{j+1} - a_j) - (a_{i+1} - a_i)$ and $|(a_{j+1} - a_j) - (a_{i+1} - a_i)| \leq |(a_{j+1} - a_j)| + |(a_{i+1} - a_i)| \leq |(a_{j+1} - a_j)| + s < j - i$, meaning that $(a_{j+1} - a_j) - (a_{i+1} - a_i) = 0$. Taking this for infinitely many j yields $a_{j+1} - a_j = x$, a constant for infinitely many j. This, in turn, allows us to take any i and j with $a_{j+1} - a_j = x$ and $j - i > |(a_{j+1} - a_j)| + x$. Repeating the above yields the same thing, whereby $a_{i+1} - a_i = a_{j+1} - a_j = x$. Thus, $a_{i+1} - a_i$ is indeed a constant for all i, i.e. the elements form an arithmetic progression.

We prove, inductively, that elements in all sets satisfy the lemma condition, hence form arithmetic progressions. First we prove that this is true for at least one set. Now, among any interval $[k(pm + 1) + 1, (k + 1)(p + 1)]$, there must exist two integers i and j in this interval such that $i, j \in A_c$ for some integer c. We record such c, and consider different $c \in [1, p]$ for all integers k. This means, there exist a number c that is recorded infinitely many times. W.l.o.g. let $c = 1$, so this is fulfilled for A_1 since we can substitute $p + 1$ into s in the lemma. Suppose that this is true for groups $A_1, A_2, \cdots A_i$, and let $D=$ lcm $(d_1, d_2, \cdots, d_i)$. Also name $B = A_{i+1} \cup A_{i+1} \cup \cdots \cup A_p$ (the unselected). This means that, if $D \mid a - b$ for $a, b \in \mathbb{N}$, we have $a \in A_k \Leftrightarrow b \in A_k$, $\forall k \in [1, i]$. More importantly, this means $a \in B \Leftrightarrow b \in B$. With this, we conclude that B contains at least an element in the interval $[a + 1, a + D]$ for each a (otherwise $A_1, A_2, \cdots A_i$ jointly contains all positive integers), and at least $p - i + 1$ elements among $[a + 1, a + (p - i + 1)D]$. In other words, there are at least two elements in the interval belong to the same set, say A_c, and record this c. Repeat this for infinitely many disjoint intervals of length $(p - i + 1)D$, and since the choice of c is finite (in the set $[i + 1, p]$), at least one such c recorded infinitely many times, hence such c (say, $i + 1$) fulfills the condition in the lemma (this time we have $s = (p - i + 1)D$).

Finally, since every element in each set form an arithmetic progression, we denote D=lcm $(d_1, d_2, \ldots d_p)$ (whereby d_i is the common difference of A_i) and prove that $f(i) - i$ has period D. Indeed, for every $i \in \mathbb{N}$. i and $i + D$ are in the same group, say, A_c. Hence $f(i + D) - (i + D) = f(i) - i = d_c$. Q.E.D.