Solutions to Tournament of Towns, Fall 2021, Senior

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O-Level

1.

A-Level

1. The wizards A, B, C, D know that the integers 1, 2, . . . , 12 are written on 12 cards, one integer on each card, and that each wizard will get three cards and will see only his own cards. Having received the cards, the wizards made several statements in the following order.

   A: "One of my cards contains the number 8".

   B: "All my numbers are prime".

   C: "All my numbers are composite and they all have a common prime divisor".

   D: "Now I know all the cards of each wizard".

   What were the cards of A if everyone was right?

   Answer. 1, 8, 9.

   Solution. We first claim that A must have all composite numbers or 1. Notice that there are 5 primes 2, 3, 5, 7, 11. If A has at least one prime numbers, then D will not be able to tell which of the prime numbers belong to A, and which one belong to D based on information from A, B, C.

   Thus D will have two of the remaining prime numbers (after B). We show that the other cards by D must either be 6 or 12. Now, we see that the cards of C could be one of the following:

   $$(6, 9, 12), (4, 6, 10), (4, 6, 12), (6, 10, 12), (4, 10, 12)$$

   Thus for D to uniquely determine the cards of C, this non-prime card of D must fall into all but one of the triples above, and hence can either be 6 or 12. If it's 6, then C has $4, 10, 12$; if it's 12, then C has $4, 6, 10$. Thus, the cards of B, C, D are the 5 primes and $4, 6, 10, 12$, leaving A with $1, 8, 9$.

2. There was a rook at some square of a 10 ×10 chessboard. At each turn it moved to a square adjacent by side. It visited each square exactly once. Prove that for each main diagonal (the diagonal between the corners of the board) the following statement is true: in the rook's path there were two consecutive steps at which the rook first stepped away from the diagonal and then returned back to the diagonal.

3. Grasshopper Gerald and his 2020 friends play leapfrog on a plane as follows. At each turn Gerald jumps over a friend so that his original point and his resulting point are symmetric with respect to this friend. Gerald wants to perform a series of jumps such that he jumps over each friend exactly once. Let us say that a point is achievable if Gerald can finish the

2020th jump in it. What is the maximum number N such that for some initial placement of the grasshoppers there are just N achievable points?

Answer. $\binom{2020}{1010}$.

Solution. We'll use the following identity: the reflection of a in b is $2b-a$. Thus if a_0 is the original location of Gerald and $a_1, \cdots, a_n$ are the locations of other $n = 2020$ grasshoppers (which Gerald jumps over in that order) then Gerald's final position is

$$a_0 + 2\left(\sum_{i=1}^{n/2} a_{2i} - \sum_{i=1}^{n/2} a_{2i-1}\right)$$

(note the use that n is even here). Note that Gerald is allowed to permute $(a_1, \cdots, a_n)$, and also this sum depends only on the $n/2$ points Gerald chooses to jump on odd (respectively even) time. This means $N \le \binom{n}{n/2}$.

To show equality, we just need to show that it's possible to have $\sum_{i=1}^{n/2} a_{2i} \neq \sum_{i=1}^{n/2} b_{2i}$ whenever $\{a_{2i}\}$ and $\{b_{2i}\}$ are two different subsets of $\{a_j\}_{j\le n}$. This can be accomplished by, say, having the n points to be $a_i = 2^i$.

4. What is the minimum k for which among any three nonzero real numbers there are two numbers a and b such that either $|a-b| \le k$ or $|\frac{1}{a} - \frac{1}{b}| \le k$?

Answer. $\frac{3}{2}$.

Solution. For any triple (a_1, a_2, a_3) nonzero real numbers, denote

$$f(a_1, a_2, a_3) = \min_{i\neq j}\left\{|a_i - a_j|, |\frac{1}{a_i} - \frac{1}{a_j}|\right\}$$

Then $f(-1, \frac{1}{2}, 2) = \frac{3}{2}$.

To show that $f \le \frac{3}{2}$ in all cases, consider $a_1 < a_2 < a_3$. Consider the following cases:

Case 0. $\min\{a_2 - a_1, a_3 - a_2\} \le \frac{3}{2}$. This is obvious. So for each of the following cases, assume $\min\{a_2 - a_1, a_3 - a_2\} \ge \frac{3}{2}$.

Case 1a. $a_1 > 0$. Then $a_2 \ge \frac{3}{2}$ and

$$\frac{1}{a_2} - \frac{1}{a_3} < \frac{1}{a_2} \le \frac{2}{3} < \frac{3}{2}$$

Case 1b. $a_3 < 0$. This case is symmetric to 1a.

case 2a. $a_1 < 0 < a_2$. Let $x = -a_1$. Then we have $a_2 \ge a_1 + \frac{3}{2} = \frac{3}{2} - x$, and $\frac{1}{a_3} \ge \frac{3}{2} + \frac{1}{a_1} = \frac{3}{2} - \frac{1}{x}$, i.e. $a_3 \le \frac{1}{\frac{3}{2} - \frac{1}{x}}$. Now consider the function

$$\frac{1}{\frac{3}{2} - \frac{1}{x}} - (\frac{3}{2} - x)$$

which has desirvative $1 - \frac{1}{(\frac{3}{2}x-1)^2}$, hence decreasing for $x \ge 1$. We thus see that if $a_1 \le -1$,

$$a_3 - a_2 \le \frac{1}{\frac{3}{2} - \frac{1}{x}} - (\frac{3}{2} - x) \le \frac{1}{\frac{3}{2} - 1} - (\frac{3}{2} - 1) = \frac{3}{2}$$

Similarly, consider the function

$$\frac{1}{\frac{3}{2} - x} - \frac{1}{\frac{3}{2} - \frac{1}{x}}$$

similar to before cases, when $-1 \le a_1 < 0$, we have $0 \le x \le 1$

$$\frac{1}{a_2} - \frac{1}{a_3} \le \frac{1}{\frac{3}{2} - x} - \frac{1}{\frac{3}{2} - \frac{1}{x}} \le \frac{1}{\frac{3}{2} - 1} - (\frac{3}{2} - 1) = \frac{3}{2}$$

Thus $f(a_1, a_2, a_3) \le \frac{3}{2}$.

case 2b. $a_2 < 0 < a_3$. It's symmetric to 2a.

5. Let $ABCD$ be a parallelogram and let P be a point inside it such that $\angle PDA = \angle PBA$. Let ω_1 be the excircle of PAB opposite to the vertex A. Let ω_2 be the incircle of the triangle PCD. Prove that one of the common tangents of ω_1 and ω_2 is parallel to AD.

   Solution. Let E be such that $AEPD$ is a parallelogram. Then $\angle AEP = \angle ADP = \angle ABP$, and B and E are on the same side of AP. Therefore A, E, B, P are concyclic. In addition, $PE \parallel AD \parallel BC$ and $PE = AD = BC$ (length) so $PEBC$ is also a parallelogram. Thus AEB and DPC are congruent triangles, and are in fact translation with axis parallel to AD. Therefore, if we denote ω_3 as the incircle of AEB then it suffices to show that ω_1 and ω_3 have common tangent parallel to PE (given also that $PE \parallel AD$).

   Now let the centers of ω_1 and ω_3 be I_1 and I_3, respectively. We first have $\angle I_1AI_3 = \frac{\angle EAP}{2}$ since I_1 is the incenter of AEB and I_3 the A-excenter of ABP. Also $\angle I_1BI_3 = 90° + \frac{\angle EBP}{2}$. Thus $\angle I_1AI_3 + \angle I_1BI_3 = 180°$, since $\angle EAP + \angle EBP = 180°$, $(A, E, B, P$ concyclic). So I_1BPI_3 also cyclic. This means:

$$\angle(AB, EP) = \angle(AB, AE) + \angle(AE, EP) = \angle(AB, AE) + \angle(AB, BP) = 2\angle(AB, AI_3) + 2\angle(AB, BI_1)$$

$$= 2\angle(AB, AI_3) + 2\angle(AI_3, I_1I_3) = 2\angle(AB, I_1I_3)$$

   and since AB is a common tangent to both ω_1, ω_3, it follows that so is AB's reflection in I_1I_3, namely ℓ. With $2\angle(AB, I_1I_3) = \angle(AB, EP) = \angle(AB, \ell)$ we have $\ell \parallel EP$, as desired.