Solutions to Tournament of Towns, Fall 2014, Senior

Anzo Teh

O-Level

1.

A-Level

3. Gregory wrote 100 numbers on a blackboard and calculated their product. Then he increased each number by 1 and observed that the product didn't change. He increased the numbers in the same way again, and again the product didn't change. He performed this procedure k times, each time having the same product. Find the greatest possible value of k.

   Answer. $k = 99$.

   Solution. Let the numbers be $a_1, \cdots, a_{100}$ and the product be P. Then the problem statement implies

   $$(a_1 + x)(a_2 + x)\cdots(a_{100} + x) - P = 0, \forall x = 0, 1, \cdots, k$$

   The left hand side is a monic polynomial of degree 100, hence having at most 100 roots. Thus $k \leq 100 - 1 = 99$.

   Equality is attained when the numbers are $0, -1, \cdots, -99$. Then for $k = 1, 2, \cdots, 99$, exactly one of the numbers $a_i + k$ is 0, giving the product 0 overall.

4. The circle inscribed in triangle ABC touches the sides BC, CA, AB at points A', B', C' respectively. Three lines, AA', BB' and CC' meet at point G. Define the points C_A and C_B as points of intersection of the circle circumscribed about triangle $GA'B'$ with lines AC and BC, different from B' and A'. In similar way define the points AB, AC, BC, BA. Prove that the points C_A, C_B, A_B, A_C, B_C, and B_A belong to the same circle.

   Solution. Given that $AB' = AC'$ and A_B, A_C, B', C', G lie on the same circle, we gave $AA_B = AA_C$ and similarly $BB_A = BB_C$ and $CC_A = CC_B$. Additionally, the circle $GA'C'$ and $GA'B'$ have radical axis GA', we have $AC' \cdot AB_A = AB' \cdot AC_A$ and therefore $AB_A = AC_A$, too. A series of angle chasing gives

   $$\angle C_A B_A B_C = 180^\circ - \angle AB_A C_A - \angle BB_A BB_C = \frac{\angle BAC + \angle\angle ABC}{2} = 90^\circ - \frac{\angle ACB}{2} = \angle C_A C_B C$$

   so B_A, B_C, C_A, C_B are indeed concyclic, with center of circle the intersection of perpendicular bisectors of $B_A B_C$ and $C_A C_B$. However, since $BB_A = BB_C$, we the perpendicular bisector of $B_A B_C$ as the angle bisector of $\angle ABC$, and similarly the other perpendicular bisector as the angle bisector of $\angle ACB$. Thus the center of this circle $B_A B_C C_A C_B$ is actually I, the incenter of ABC. Therefore, I is equidistant from B_A, B_C, C_A, C_B and with the similar logic we deduce that I is equidistant from A_B, A_C, B_A, B_C. So I is equidistant from the six points and so the six points lie on the same circle.

5. Pete counted all possible words consisting of m letters, such that each letter can be only one of T, O, W or N and each word contains as many T as O. Basil counted all possible words consisting of $2m$ letters such that each letter is either T or O and each word contains as many T as O. Which of the boys obtained the greater number of words?

   Answer. The two sets have the same number of words.

   Solution. Let A be the set counted by Pete and B the set by Basil. We define a mapping $f : A \to B$ by the following: if $a = a_1 a_2 \cdots a_m \in A$, then denote $f(a) = b = b_1 \cdots b_{2m}$ where for each $i = 1, \cdots, m$:

   - If $a_i = T$, then $b_i = b_{m+i} = T$
   - If $a_i = O$, then $b_i = b_{m+i} = O$
   - If $a_i = W$, then $b_i = T, b_{m+i} = O$
   - If $a_i = N$, then $b_i = O, b_{m+i} = T$

   To show that the mapping is valid, we notice that since the frequency of i with $a_i = T$ is equal to that of $a_i = O$, the frequency of i with $b_i = b_{m+i} = T$ is equal to that of $b_i = b_{m+i} = O$ and therefore the resulting b has equal T and O. In addition, the fact that $b_i = b'_i$ and $b_{m+i} = b'_{m+i}$ implies that $a_i = a'_i$ (with b' as the image of a under f) so f is also injective. Finally, for any $b \in B$, an inverse f^{-1} can be defined where a_i is defined according to the combinations of b_i and b_{m+i} above. This guarantees that a has as many T as O's by a previous reasoning.

   Thus f is actually a bijection, which then implies $|A| = |B|$.