Solution to IMO 2016 shortlisted problems.

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Preface: The IMO 2016 shortlisted problems were published on July 19, 2017: after the second day of IMO 2017. As I am not involved in any IMO training, I have no access to the shortlisted problems until that day. Thereafter, I worked on the problems to the best ability possible, and decided to compile my solution to the problems here.

A shortcoming of providing only the solutions to the readers is that, it does not show the process of generating insights needed to solve the problems. For problems involving functional equations (A4, N6, and the currently unolved A7), the solutions are usually intuitive because the majortiy of it is to substitute variables. However, for problems like C7 (Problem 6: lines and endpoints) it may not be suggestive at all on how do we come out with the notion of "bearings" in the solution. In view of this, I have a writeup on my own thought proess before each solution, in effort of making it easy for readers to understand how did I arrive at the ideas needed for the solution.

To conclude: the thought process contains the intuitive part of the writing, while the solution contains the part where mathematical rigor is maintained at the highest possible level.

1 Algebra

A1 Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Thoughts. Non-homogeneous inequality, with the relation $\min(ab, bc, ca) \geq 1$ that we aren't sure how to use it (at its first sight). How should we solve it, then? The way, when desperate, is to brute force the whole thing by cubing the left-hand side and trying to expand $\left(\frac{a+b+c}{3}\right)^6 + 3\left(\frac{a+b+c}{3}\right)^4 + 3\left(\frac{a+b+c}{3}\right)^2 + 1$, but let's not make our lives miserable with (probably) hundreds of terms on the right.

Instead, we start with the following observation:

1. What happens when $a = b = c$? Then the equality holds! This motivates us to show that whenever $a+b+c$ is fixed, the maximal possible value of $(a^2+1)(b^2+1)(c^2+1)$ can be attained whenever $a = b = c$. With this in mind...
2. We want to see what happens to $(a^2+1)(b^2+1)$ by ranging all possible pairs of (a, b) such that $ab \geq 1$ and $a+b$ is fixed. As it turns out, $(a^2+1)(b^2+1) = a^2b^2+a^2+b^2+1$ (there are only four terms so it doesn't hurt to expand). Since $a+b$ is fixed in our context, we can write this in terms of $a+b$, giving $a^2b^2+a^2+b^2+1 = (a+b)^2+(ab-1)^2$. Now it's easy to see that this value increases with ab (as $ab \geq 1$).
3. Now with $a+b$ fixed, ab increases when $|a-b|$ decreases. The next step is to compare $f(a, b, c)$ with $f(k, k, k)$ where k is the average of a, b, c. (Here we denote $f(a, b, c)$ as $(a^2+1)(b^2+1)(c^2+1)$). Ideally, we want to find x and y with average k such that $f(a, b, c) \leq f(k, x, y) \leq f(k, k, k)$. The right inequality is easy to establish for any x, y with sum $2k$, given that $\min(kx, ky, xy) \geq 1$. This, however, requires us to maintain this invariant for the right inequality. Additionally, for left inequality to work we need one of k, x, y to be the same as one of a, b, c, wjich requires some case-by-case analysis (i.e. $m(a, b, c) \leq k$ and $m(a, b, c) \geq k$, where m denotes the median of the three variables here). These aren't hard, just some work needed.

Solution. We start with a preliminary observation: given that $k \geq 2$, and given the set of pairs $K = \{(a, b) : a+b = k, ab \geq 1\}$, then for any $(a_1, b_1), (a_2, b_2) \in K$ we have

$$(a_1^2+1)(b_1^2+1) \geq (a_2^2+1)(b_2^2+1) \text{ iff } |a_1 - b_1| \leq |a_2 - b_2|.$$

Indeed, for $(a, b) \in K$, $(a^2+1)(b^2+1) = (a+b)^2 + a^2b^2 - 2ab + 1 = k^2 + (ab-1)^2$. In addition, $ab = \frac{(a+b)^2-(a-b)^2}{2} = \frac{k^2-(a-b)^2}{2}$ so we have

$$(a_1^2+1)(b_1^2+1) - (a_2^2+1)(b_2^2+1) = (a_1b_1-1)^2 - (a_2b_2-1)^2 = (a_1b_1+a_2b_2-2)(a_1b_1-a_2b_2)$$

$$= (a_1b_1+a_2b_2-2)\left(\frac{(k^2-(a_1-b_1)^2)-(k^2-(a_2-b_2)^2)}{2}\right) = (a_1b_1+a_2b_2-2)\left(\frac{(a_2-b_2)^2-(a_1-b_1)^2)}{2}\right).$$

Now that $a_1b_1, a_2b_2 \geq 1$, $(a_1^2+1)(b_1^2+1) - (a_2^2+1)(b_2^2+1) \geq 0$ iff $a_1b_1 - a_2b_2 \geq 0$. (Technically we need to consider the case where $a_1b_1 = a_2b_2 = 1$, which gives $a_1b_1 + a_2b_2 - 2 = 0$. However this will also give $a_1b_1 - a_2b_2 = 0$ so the claim is valid.) This means $(a_1^2+1)(b_1^2+1) - (a_2^2+1)(b_2^2+1) \geq 0$ iff $(a_2-b_2)^2 - (a_1-b_1)^2 \geq 0$ iff $|a_2 - b_2| \geq a_1b_1|$. In other words, In addition, with $a+b$ fixed, this function is also dereasing in $(a-b)^2$, which turns out to also be decreasing in $|a-b|$.

Now let $a+b+c = 3k$, and let $f(a, b, c) = (a^2+1)(b^2+1)(c^2+1)$. Notice that the left hand side is $\sqrt[3]{f(a, b, c)}$ while the right hand side is $(k^2+1) = \sqrt[3]{f(k, k, k)}$. W.l.o.g. assume that $a \leq b \leq c$. We want to show that

1. If $b \le k$ then $f(a,b,c) \le f(a,k,b+c-k) \le f(k,k,k)$.
2. If $b \ge k$ then $f(a,b,c) \le f(a+b-k,k,c) \le f(k,k,k)$.

In the first case, we have $a \le k$ so $b+c \ge 2k$, meaning that $b+c-k \ge k$. Moreover, $b+c-k+k = b+c$, and $(b+c-k)-k = (b+c)-2k \ge b+c-2b = c-b$. Therefore $k(b+c-k) \ge bc \ge 1$ and by above, $(b^2+1)(c^2+1) \le (k^2+1)((b+c-k)^2)+1$, thus $f(a,b,c) \le f(a,k,b+c-k)$. In addition, $a \le k \le b+c-k$, so $\min\{ak, a(b+c-k), k(b+c-k)\} = ak \ge ab \ge 1$. We also have $a+(b+c-k) = 3k = k = k+k$, so by the first paragraph again $(a^2+1)((b+c-k)^2+1) \le (k^2+1)^2$, which gives $f(a,k,b+c-k) \le f(k,k,k)$.

In the second case, we have (similarly) $b+a \le 2k$ ($a \le k$ and $c \ge k$), which means $2k-(a+b) \ge 0$, or $k \ge a+b-k$. In addition, $k-(a+b-k) = 2k-(b+c) \le 2b-(a-b) = b-a$ (since $b \ge k$), and $ab \le k(a+b-k)$ for this reason. Therefore, $(a^2+1)(b^2+1) \le ((a+b-k)^2+1)(k^2+1)$ and we have $f(a,b,c) \le f(a+b-k,k,c)$. Additionally, $a+b-k \le k \le c$ so $\min\{(a+b-k)k, kc, (a+b-k)c\} = (a+b-k)k \ge ab \ge 1$ (as proven above). By the fact that $(a+b-k)+c = k+k$ we have, by first paragraph, $((a+b-k)^2+1)(c^2+1) \le (k^2+1)^2$ so $f(a+b-k,k,c) \le f(k,k,k)$.

A2 Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

Thoughts. By arranging $a_i \leq a_j$ and $a_k \leq a_l$ we know that C is bounded above by 1. A baby step, but a great start.

The next sensible thing we can do is to sort the numbers in order: $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. Also it's important to realize that $|\frac{a_1}{a_2} - \frac{a_4}{a_5}| = |\frac{a_1a_5 - a_2a_4}{a_2a_5}| \geq |\frac{a_1a_5 - a_2a_4}{a_4a_5}| = |\frac{a_1}{a_4} - \frac{a_2}{a_5}|$ so we just have to consider the latter. In a similar fashion let's consider $|\frac{a_1}{a_3} - \frac{a_2}{a_4}|$, $|\frac{a_2}{a_4} - \frac{a_3}{a_5}|$. As seen below, this is just a comparison among $\frac{1}{bc}|\frac{1}{a} - \frac{1}{d}|$, $\frac{1}{b}|\frac{1}{a} - \frac{1}{c}|$, $\frac{1}{c}|\frac{1}{b} - \frac{1}{d}|$ (as below $a = \frac{a_2}{a_1}$, $b = \frac{a_3}{a_2}$, $c = \frac{a_4}{a_3}$, $d = \frac{a_5}{a_4}$). It's difficult to see how great the minimum of the three numbers can go, but in light of the factors $\frac{1}{b}$ and $\frac{1}{c}$ we can try some simple cases like $b = c = 1$, giving $|\frac{1}{a} - \frac{1}{d}|$, $|\frac{1}{a} - 1|$, $|1 - \frac{1}{d}|$. Considering $0 \leq \frac{1}{a}, \frac{1}{d} \leq 1$ we have $(1 - \frac{1}{a}) + (\frac{1}{a} - \frac{1}{d}) = (1 - \frac{1}{d})$ so considering that $0 \leq 1 - \frac{1}{d}, 1 - \frac{1}{a} \leq 1$ and assuming that $\frac{1}{a} - \frac{1}{d} \geq 0$ we have $1 \geq 1 - \frac{1}{d} \geq 2\min\{1 - \frac{1}{a}, \frac{1}{a} - \frac{1}{d}\}$, which means one of the elements in the set must be at most $\frac{1}{2}$. A similar conclusion can be reached for the case $\frac{1}{a} - \frac{1}{d} \leq 0$. Moreover, this motivates the following equality case: by setting $a = 2$ and d approaching infinity (and yeah, this is how the example $1, 2, 2, 2, n$ can be conjured). The job now reduces to proving that $\min\{\frac{1}{bc}|\frac{1}{a} - \frac{1}{d}|, \frac{1}{b}|\frac{1}{a} - \frac{1}{c}|, \frac{1}{c}|\frac{1}{b} - \frac{1}{d}|\} \leq \frac{1}{2}$, which is no longer hard by case-by-case analysis as below.

Answer. $C = \frac{1}{2}$.

Solution. First, we show that $C \geq \frac{1}{2}$ is necessary. Suppose that there exists $\epsilon > 0$ such that for each a_1, a_2, a_3, a_4, a_5 we can choose i, j, k, l with $\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq \frac{1}{2} - \epsilon$.

First, let $a_1 = 1, a_2 = a_3 = a_4 = 2$, $a_5 = n$ for some extremely large real number n. The possible numbers of $\frac{a_i}{a_j}$ are $\frac{1}{n}, \frac{2}{n}, \frac{1}{2}, 1, 2, \frac{n}{2}, n$. Observe that the ratios $\frac{1}{n}, \frac{2}{n}, \frac{n}{2}, n$ all involve a_5, so there cannot exist distinct i, j, k, l satisfying $\frac{a_i}{a_j}, \frac{a_k}{a_l} \in \{\frac{1}{n}, \frac{2}{n}, \frac{n}{2}, n\}$. In addition, both $\frac{1}{2}$ and 2 will involve a_1, meaning that there cannot exist distinct i, j, k, l satisfying $\frac{a_i}{a_j}, \frac{a_k}{a_l} \in \{\frac{1}{2}, 2\}$. Since there are three i's satisfying $a_i = 2$, there cannot be distinct i, j, k, l satisfying $\frac{a_i}{a_j} = \frac{a_k}{a_l} = 1$. We therefore know that $\frac{a_i}{a_j} = \frac{a_k}{a_l}$ is impossible, and same goes to $\frac{a_i}{a_j} = \frac{1}{n}, \frac{a_k}{a_l} = \frac{2}{n}$. We also have $n - \frac{n}{2} > \frac{n}{2} - 2 > 2 - 1 > 1 - \frac{1}{2} > \frac{1}{2} - \frac{2}{n}$ for sufficiently large real n, and if $\frac{1}{2} - \frac{2}{n} > C = \frac{1}{2} - \epsilon$ then $C < n - \frac{n}{2}, \frac{n}{2} - 2, 2 - 1, 1 - \frac{1}{2}$, so $\frac{1}{2} - \frac{2}{n} \leq \frac{1}{2} - \epsilon$ for all n¡ which does not hold for $n > 2\epsilon$. Therefore $C \geq \frac{1}{2}$.

Now, we show that $C = \frac{1}{2}$ fits in all situations. W.L.O.G. let $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, and let $a = \frac{a_2}{a_1}$, $b = \frac{a_3}{a_2}$, $c = \frac{a_4}{a_3}$, $d = \frac{a_5}{a_4}$. Observe that $a, b, c, d \geq 1$. Suppose that for some a, b, c, d, $C = \frac{1}{2}$ does not fit for any of the four distinct subscripts. Now, considering $|\frac{a_1}{a_4} - \frac{a_2}{a_5}| = \frac{1}{bc}|\frac{1}{a} - \frac{1}{d}|$ and from $0 \leq \frac{1}{a}, \frac{1}{d} \leq 1$ we have $\frac{b}{c} \geq \frac{b}{c}|\frac{1}{a} - \frac{1}{d}| > C = \frac{1}{2}$ so $b, c < 2$. Next, $C < |\frac{a_1}{a_3} - \frac{a_2}{a_4}| = \frac{1}{b}|\frac{1}{a} - \frac{1}{c}| \leq |\frac{1}{a} - \frac{1}{c}|$ and from $\frac{1}{c} < \frac{1}{2}$ we must have $\frac{1}{a} > \frac{1}{2}$. Similarly, $C < |\frac{a_2}{a_4} - \frac{a_3}{a_5}| = \frac{1}{c}|\frac{1}{b} - \frac{1}{d}| \leq |\frac{1}{b} - \frac{1}{d}|$ and from $\frac{1}{b} < \frac{1}{2}$ we must have $\frac{1}{d} > \frac{1}{2}$. Looking back, we have $\frac{1}{2} < |\frac{a_1}{a_4} - \frac{a_2}{a_5}| = \frac{b}{c}|\frac{1}{a} - \frac{1}{d}| \leq |\frac{1}{a} - \frac{1}{d}|$, yet $\frac{1}{2} < \frac{1}{a}, \frac{1}{d} \leq 1$, contradiction.

A3 Find all positive integers n such that the following statement holds: Suppose real numbers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \dots, n$. Then there exists $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, each of which is either -1 or 1, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \le 1.$$

Thoughts. One good counterexample construction can be found using the triangle inequality: $|\sum_{i=1}^n \varepsilon_i a_i| + |\sum_{i=1}^n \varepsilon_i b_i| \ge |\sum_{i=1}^n \varepsilon_i a_i + \sum_{i=1}^n \varepsilon_i b_i|$. Therefore, for the case where $a_i, b_i \ge 0$ we have: $a_i + b_i = 1$ so the right hand side now becomes $|\sum_{i=1}^n \varepsilon_i|$. When n is even, we know that the condition holds only when $\sum_{i=1}^n \varepsilon_i$ is 0, which means exactly hald of the ε_i'is is 1 and the other -1. This motivates us to think of the counterexample $(1, 0, 0, \cdots, 0)$ on one side $(0, 1, 1, \cdots, 1)$ on the other. For odd n, we have $\sum_{i=1}^n \varepsilon_i = \pm 1$ to consider. Some experimentation yields that we can have $\sum_{i=1}^n \varepsilon_i = 1$ and $0 \le \sum_{i=1}^n \varepsilon_i a_i \le 1$, and $0 \le \sum_{i=1}^n \varepsilon_i b_i \le 1$, giving $|\sum_{i=1}^n \varepsilon_i a_i| + |\sum_{i=1}^n \varepsilon_i b_i| \le 1$. (Well, the proof for the existence of ε_i's isn't that straightforward, but so does the construction of the counterexamples a_i's and b_i's which is guarateed to fail. The proof of the existence of such ε_i's is better explained in the solution.

In view of this we likely know that the answer *might* be positive even extends to the cases that a_i's and b_i's might have opposite signs for odd n. And yes, the invariant of $|a_i - b_i| = 1$ or $|a_i + b_i| = 1$ further motivates us to split the pairs (a_i, b_i) into those with same signs and different signs. The simplified case of $a_i, b_i \ge 0$ for all i also motivates us to find such ε_i's such that $\sum \varepsilon_i a_i$ and $\sum \varepsilon_i b_i$ are both in $[-1, 1]$ for $a_i, b_i \ge 0$, which we can further narrow the interal down to $[0, 1]$ for both when m (the total number of summands) is odd, and $[0, 1]$ for sum of a_i's and $[-1, 0]$ for sum of b_i's when m is even. This is all we need to prove the answer is "yes".

Answer. All odd n.

Solution. We first find a counterexample for all $n = 2k$ for some $k \ge 1$ an integer. Consider $a_i = 1$, $b_i = 0$ for all $i \in [1, 2k-1]$, and $a_{2k} = 0, b_{2k} = 1$. Then $\sum_{i=1}^n \varepsilon_i b_i = \varepsilon_{2k}$, which has absolute value of 1. Also $\sum_{i=1}^n \varepsilon_i a_i = \sum_{i=1}^{2k-1} \varepsilon_i$. If x of the indices $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2k-1}$ are 1 and the rest -1, then the value would be $x - (2k-1-x) = 2x - 2k + 1$, which is an odd integer. Thus it has absolute value at least 1 too. Therefore we have $|\sum_{i=1}^n \varepsilon_i a_i| + |\sum_{i=1}^n \varepsilon_i b_i| \ge 1 + 1 = 2$.

Now let n be odd. We start with the following lemma: Let $(a_i, b_i), i \in [1, m]$ satisfy $0 \le a_i, b_i \le 1$ and $a_i + b_i = 1, \forall i \in [1, m]$. Then there exists $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m \in \{-1, 1\}$ satisfying

$$\begin{array}{rcll} 0 \le \sum_{i=1}^m \varepsilon_i a_i, \sum_{i=1}^m \varepsilon_i b_i \le 1 & \text{and} & \sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = 1 & \text{if } m \text{ is odd,} \\ -1 \le \sum_{i=1}^m \varepsilon_i a_i \le 0 \le \sum_{i=1}^m \varepsilon_i b_i \le 1 & \text{and} & \sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = 0 & \text{if } m \text{ is even.} \end{array}$$

In the first case, we notice that the second condition can be achieved whenever exactly $\frac{m+1}{2}$ of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ are 1 and the rest -1. For any combination of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ with exactly $\frac{m+1}{2}$ of them as 1 we have $\sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = \sum_{i=1}^m \varepsilon_i = \frac{m+1}{2} - \frac{m-1}{2} = 1$. The aim now is to assign $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ in a way that exactly $\frac{m+1}{2}$ of them are 1, and $0 \le \sum_{i=1}^m \varepsilon_i a_i \le 1$ (if this is true, $0 \le \sum_{i=1}^m \varepsilon_i b_i \le 1$ holds true too). W.L.O.G. assume that $a_1 \le a_2 \le \cdots \le a_m$, and consider the numbers $x_0, x_1, \cdots, x_{\frac{m-1}{2}}$ such that

$$x_k = \sum_{i=1}^{k} -a_i + \sum_{i=k+1}^{k+\frac{m+1}{2}} a_i + \sum_{i=k+\frac{m+1}{2}+1}^{m} -a_i$$

Notice, first, that for each x_i, exactly $\frac{m+1}{2}$ of the a_i's has coefficient 1 and the rest -1. Therefore if $0 \leq x_k \leq 1$ for some k we are done. Observe also that,

$$x_0 = \sum_{i=1}^{\frac{m+1}{2}} a_i + \sum_{i=\frac{m+1}{2}+1}^{m} -a_i \leq \sum_{i=1}^{\frac{m+1}{2}} a_{\frac{m+1}{2}} + \sum_{i=\frac{m+1}{2}+1}^{m} -a_{\frac{m+1}{2}} = a_{\frac{m+1}{2}} \leq 1;$$

$$x_{\frac{m-1}{2}} = \sum_{i=1}^{\frac{m-1}{2}} -a_i + \sum_{i=\frac{m+1}{2}}^{m} a_i \geq \sum_{i=1}^{\frac{m-1}{2}} -a_{\frac{m+1}{2}} + \sum_{i=\frac{m+1}{2}}^{m} a_{\frac{m+1}{2}} = a_{\frac{m+1}{2}} \geq 0.$$

If $x_0, x_{\frac{m-1}{2}} \notin [0,1]$ (observe that we are done if either of them is in the interval), then $x_0 < 0$ and $x_{\frac{m-1}{2}} > 1$. This allows us to choose a k such that $x_k > 1$ and $x_{k-1} \leq 1$. Moreover, $x_k - x_{k-1} = a_{k+\frac{m+1}{2}} - a_k - a_k + a_{k+\frac{m+1}{2}} = 2(a_{k+\frac{m+1}{2}} - a_k)$, so $x_{k-1} \leq a_k \leq x_{k-1}+2$ (since $|a_i - a_j| \leq 1$). If $x_{k-1} \geq 0$ we are done. Otherwise, we have $x_k - x_{k-1} > 1$ and therefore $a_{k+\frac{m+1}{2}} - a_k > \frac{1}{2}$, meaning that $a_{k+\frac{m+1}{2}} > \frac{1}{2}$. Now let $y_k = x_k - 2a_{k+\frac{m+1}{2}}$ we have $y_k \leq x_{k-1} < 0$ but $y_k \geq x_k - 2 > -1$. This means $0 \leq -y_k \leq 1$ and

$$-y_k = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{k+\frac{m+1}{2}-1} -a_i + \sum_{i=k+\frac{m+1}{2}}^{m} a_i.$$

Among them, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k, \varepsilon_{k+\frac{m+1}{2}}, \cdots, \varepsilon_m$ are 1 ($k+m-(k+\frac{m+1}{2}-1) = \frac{m+1}{2}$ of them) with the rest -1. Hence we are done.

The second case isn't much different from the first. The second condition can be achieved whenever exactly half of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ are 1 and the rest -1. To see why, we have $\sum_{i=1}^{m} \varepsilon_i a_i + \sum_{i=1}^{m} \varepsilon_i b_i = \sum_{i=1}^{m} \varepsilon_i = \frac{m}{2} - \frac{m}{2} = 0$. With this fixed, all we need to make sure is that $\sum_{i=1}^{m} \varepsilon_i a_i \in [0,1]$. Again w.l.o.g. let $a_1 \leq a_2 \leq \cdots \leq a_m$, and consider the numbers $x_0, x_1, \cdots, x_{\frac{m}{2}}$ such that

$$x_k = \sum_{i=1}^{k} -a_i + \sum_{i=k+1}^{k+\frac{m}{2}} a_i + \sum_{i=k+\frac{m}{2}+1}^{m} -a_i$$

Again each x_k satisfies exactly $\frac{m}{2}$ have coefficient 1 and the rest -1. Thus if $-1 \leq x_k \leq 1$ for some k we are done. Otherwise, notice that

$$x_0 = \sum_{i=1}^{\frac{m}{2}} a_i + \sum_{i=\frac{m}{2}+1}^{m} -a_i \leq \sum_{i=1}^{\frac{m}{2}} a_{\frac{m}{2}} + \sum_{i=\frac{m}{2}+1}^{m} -a_{\frac{m}{2}} = 0;$$

$$x_{\frac{m}{2}} = \sum_{i=1}^{\frac{m}{2}} -a_i + \sum_{i=\frac{m}{2}+1}^{m} a_i \geq \sum_{i=1}^{\frac{m}{2}} -a_{\frac{m}{2}} + \sum_{i=\frac{m}{2}+1}^{m} a_{\frac{m}{2}} = 0.$$

This forces $x_0 < -1$ and $x_{\frac{m}{2}} > 1$, which allows us to pick a k satisfying $x_k > 1$ and $x_{k-1} \leq 1$. By the similar logic as in case 1 we have $0 \leq x_k - x_{k-1} \leq 2$. This means $x_{k-1} > -1$, which gives $-1 \leq x_{k-1} \leq 1$. If $x_{k-1} \leq 0$ we are done. Otherwise, $-1 \leq -x_{k-1} \leq 0$ and we have

$$-x_{k-1} = \sum_{i=1}^{k-1} a_i + \sum_{i=k}^{k+\frac{m}{2}-1} -a_i + \sum_{i=k+\frac{m}{2}}^{m} a_i.$$

Again $\varepsilon_i = 1$ for $i = 1, 2, \cdots k-1$ and $k+\frac{m}{2}, \cdots, m$, which isn't that hard to verify that there are exactly $\frac{m}{2}$ of them.

To finish up the solution, we split the a_i's and b_i's into two groups: one with $a_ib_i \geq 0$, and the other one with $a_ib_i \leq 0$ (if $a_ib_i = 0$ then we assign them arbitrarily). W.L.O.G. let the first group be (a_i, b_i); $i \in [1, m]$ and the second group be (a_i, b_i): $i \in [m+1, n]$. For $i \in [1, m]$, we can w.l.o.g. assume that $a_i, b_i \geq 0$ (indeed, if ε_i is a solution for (a_i, b_i) then $-\varepsilon_i$ is a solution for $(-a_i, -b_i)$). Similarly we can also assume that $i \in [m+1, n]$ we have $a_i \geq 0$ and $b_i \leq 0$. Also notice that $|x| + |y| \in \{x+y, x-y, y-x, -x-y\}$ so to prove that $|x| + |y| \leq 1$ all w need to do is to prove that $-1 \leq x+y \leq 1$ and $-1 \leq x-y \leq 1$. (Bonus: try to show that this is an if and only if condition: convince yourself that $|x| + |y| = \max\{x+y, x-y, y-x, -x-y\}$.)

We split into two cases:

Case 1. m is odd. From our lemma, by some careful choices of ε_i's we have

$$\begin{array}{rcr} 0 \leq \sum_{i=1}^m \varepsilon_i a_i, \sum_{i=1}^m \varepsilon_i b_i \leq 1 & \text{and} & \sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = 1 \ ; \\ -1 \leq \sum_{i=m+1}^n \varepsilon_i a_i \leq 0 \leq \sum_{i=m+1}^n \varepsilon_i(-b_i) \leq 1 & \text{and} & \sum_{i=m+1}^n \varepsilon_i a_i + \sum_{i=m+1}^n \varepsilon_i(-b_i) = 0. \end{array}$$

This means, $\sum_{i=m+1}^n \varepsilon_i a_i = \sum_{i=m+1}^n \varepsilon_i b_i = c$ for some $c \in [-1, 0]$. Also let $\sum_{i=1}^m \varepsilon_i a_i = a$ and $\sum_{i=1}^m \varepsilon_i b_i = b$, from which we know $a + b = 1$ and $0 \leq a, b \leq 1$. Now our term of interest $|\sum_{i=1}^n \varepsilon_i a_i| + |\sum_{i=1}^n \varepsilon_i b_i|$ becomes $|a+c| + |b+c| = |a+c| + |1-a+c|$ Now we have $a+c+b+c = 1+2c$ and from $-1 \leq c \leq 0$ we have $-1 \leq c \leq 1$. Also, $a+c-b-c = a-b = a-(1-a) = 2a-1$ and from $0 \leq a \leq 1$ we have $-1 \leq 2a-1 \leq 1$. This settles our case 1.

Case 2. m is even. Again by our lemma we have

$$\begin{array}{rcr} -1 \leq \sum_{i=1}^m \varepsilon_i a_i \leq 0 \leq \sum_{i=1}^m \varepsilon_i b_i \leq 1 & \text{and} & \sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = 0 \ ; \\ 0 \leq \sum_{i=m+1}^n \varepsilon_i a_i, \sum_{i=m+1}^n \varepsilon_i(-b_i) \leq 1 & \text{and} & \sum_{i=m+1}^n \varepsilon_i a_i + \sum_{i=m+1}^n \varepsilon_i(-b_i) = 1. \end{array}$$

Now, let $\sum_{i=1}^m \varepsilon_i a_i = a$ and we have $\sum_{i=1}^m \varepsilon_i b_i = -a$ $(-1 \leq a \leq 0)$, let $\sum_{i=m+1}^n \varepsilon_i a_i = c$ and we have $\sum_{i=m+1}^n \varepsilon_i b_i = c - 1$ $(0 \leq c \leq 1)$. Again, we need to consider $|a+c| + |-a+c-1|$. Observe that $a+c-a+c-1 = 2c-1$ and from $0 \leq c \leq 1$ we have $-1 \leq 2c-1 \leq 1$. $a+c+a-c+1 = 2a+1$, and from $-1 \leq a \leq 0$ we have $-1 \leq 2a+a \leq 1$, which serves our purpose.

A4 Find all functions $f:(0,\infty)\to(0,\infty)$ such that for any $x,y\in(0,\infty)$,

$$xf(x^2)f(f(y))+f(yf(x))=f(xy)\left(f(f(x^2))+f(f(y^2))\right)\cdots(*).$$

Solution. The only function is $f(x)\equiv\frac{1}{x}$, which works because $xf(x^2)f(f(y))+f(yf(x))=x\frac{1}{x^2}y+\frac{1}{y\frac{1}{x}}=\frac{x}{y}+\frac{y}{x}=\frac{x^2}{xy}+\frac{y^2}{xy}=f(xy)\left(f(f(x^2))+f(f(y^2))\right)$.

For the rest of the solution we proceed with the normal functional algorithmic procedure to show that $f(x)\equiv\frac{1}{x}$ is the only function:

Step 1. Plugging $x=y=1$ gives $f(1)f(f(1))+f(f(1))=2f(1)f(f(1))$, and since $f>0$, we can factorize $f(f(1))$ out to get $f(1)+1=2f(1)$, giving $\boxed{f(1)=1}$.

Step 2. Plugging $x\leftarrow 1$ (and substituting $f(1)\leftarrow 1$ due to step 1) gives $f(f(y))+f(y)=f(y)(1+f(f(y^2)))$, giving $\boxed{f(f(y))=f(y)f(f(y^2))}$.

Step 3. Plugging $y\leftarrow 1$, on the other hand, gives $xf(x^2)+f(f(x))=f(x)(f(f(x^2))+1)$. From step 2, $f(f(x))=f(x)f(f(x^2))$, which gives rise to $\boxed{xf(x^2)=f(x)}$.

Step 4. Substitute $xf(x^2)\leftarrow f(x)$ (step 3), $f(f(y))\leftarrow f(y)f(f(y^2))$ (step 2), and $yf(x)\leftarrow xyf(x^2)$ (step 3) into (*) gives:

$$f(x)f(y)f(f(y^2))+f(xyf(x^2))=f(xy)\left(f(f(x^2))+f(f(y^2))\right)\cdots(**).$$

In the special case where $xy=1$ we have $f(x)f(y)f(f(y^2))+f(f(x^2))=1\left(f(f(x^2))+f(f(y^2))\right)$, so $f(x)f(y)=1$ whenever $xy=1$. In other words, for all $x\in\mathbb{R}^+$, $\boxed{f(\frac{1}{x})=\frac{1}{f(x)}}$.

Step 5. Having the results in Step 4 in mind, we do the following substitution:

5a. Substitute $\frac{1}{x}$ and $\frac{1}{x}$ in place of x and y into (**) (step 4) we have

$$f\left(\frac{1}{x}\right)f\left(\frac{1}{x}\right)f\left(f\left(\frac{1}{x^2}\right)\right)+f\left(\frac{1}{x^2}f\left(\frac{1}{x^2}\right)\right)=f\left(\frac{1}{x^2}\right)\left(f\left(f\left(\frac{1}{x^2}\right)\right)+f\left(f\left(\frac{1}{x^2}\right)\right)\right)$$

$$\frac{1}{f(x)}\cdot\frac{1}{f(x)}\cdot f\left(\frac{1}{f(x^2)}\right)+f\left(\frac{1}{x^2f(x^2)}\right)=\frac{1}{f(x^2)}\left(2f\left(\frac{1}{f(x^2)}\right)\right)$$

$$\frac{1}{f(x)}\cdot\frac{1}{f(x)}\cdot\frac{1}{f\left(f(x^2)\right)}+\frac{1}{f\left(x^2f(x^2)\right)}=\frac{1}{f(x^2)}\cdot\frac{2}{\left(f\left(f(x^2)\right)\right)}$$

$$\frac{1}{f(x)^2f(f(x^2))}+\frac{1}{f(x^2f(x^2))}=\frac{2}{f(x^2)f(f(x^2))}$$

.

5b. Subsitute $x=y$ into (**) we get $f(x)^2f(f(x^2))+f(x^2f(x^2))=2f(x^2)f(f(x^2))$.

5c. From now on denote $a=f(x)^2f(f(x^2))$ and $b=f(x^2f(x^2))$. Substituting $f(x^2)f(f(x^2))\leftarrow\frac{a+b}{2}$ gives $\frac{1}{a}+\frac{1}{b}=\frac{2}{f(x^2)f(f(x^2))}=\frac{4}{a+b}$. which we can cross multiply to get $(a+b)^2=4ab$, or $(a-b)^2=a^2+b^2-ab=a^2+b^2+2ab-4ab=(a+b)^2-4ab=0$. This yields $a-b=0$, and hence $f(x)^2f(f(x^2))=a=b=f(x^2f(x^2))$. Now looking back to 5(b) again we have $2f(x^2)f(f(x^2))=f(x)^2f(f(x^2))+f(x^2f(x^2))=f(x)^2f(f(x^2))+f(x)^2f(f(x^2))=2f(x)^2f(f(x^2))$, so $f(x^2)f(f(x^2))=f(x)^2f(f(x^2))$, or simply $f(x^2)=f(x)^2$ (after factorizing $f(f(x^2))$ out which we assumed its non-zero. Finally from step 3 again we have $f(x)=xf(x^2)=xf(x)^2$, so $xf(x)=1$ and $f(x)=\frac{1}{x}$.

A5 Consider fractions $\frac{a}{b}$ where a and b are positive integers.

(a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}+1$.

(b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}$.

Thoughts. Part (b) should very well give a hint on part (a) : $b = \lfloor \sqrt{n}+1 \rfloor$ is necessary. Together with another trick of arranging the natural numbers n according to their integer square root (i.e. $\lfloor \sqrt{n} \rfloor$) the case of $b = \lfloor \sqrt{n}+1 \rfloor$ covers half of the required numbers (i.e. those numbers in the form $n = k^2+1, k^2+3, \cdots k^2+2k-1$). Therefore we only need to worry about the case $n = k^2, k^2+2, \cdots, k^2+2k$, which turns out all to be comfortably settled by the case $b = \lfloor \sqrt{n} \rfloor$ (and how do we know that? Experimenting with small numbers!).

Part (b) requires some experimentation, which is not too hard is we carry on with the partition of the natural numbers according to their integer square roots. Notice, also, that there is no suitable b for $n = k^2+1$ for some k, which can be proven using the inequality $(k+\frac{1}{k})^2 > k^2+2$. This is precisely what we need to solve the problem.

Solution. For part (a), we partion the set of positive integers according to their integer square roots, that is, the sets $S_1 = \{1,2,3\}$, $S_2 = \{4,5,6,7,8\}$, $S_3 = \{9,10,11,12,13,14,15\}$, etc. Consider $S_k = \{k^2, k^2+1, \cdots k^2+2k\}$, and we claim that $b = k$ and $b = k+1$ alone will jointly work for the sets. (That is, for every positive integer $n \in S_k$, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \{k, k+1\}$). Indeed, let $n = k^2+a$ with $0 \leq a \leq 2k$. If $a = 2x$ for some integer x then $(k+\frac{x}{k})^2 = k^2 + 2k\left(\frac{x}{k}\right) + \left(\frac{x}{k}\right)^2$ $= k^2 + 2x + \frac{x^2}{k^2}$ and since $x = \frac{a}{2} \leq k$ we have $\frac{x^2}{k^2} \leq 1$. Therefore, $\sqrt{n} = \sqrt{k^2+a} \leq k + \frac{x}{k} \leq \sqrt{k^2+a+1} = \sqrt{n+1}$. On the other hand, if $a = 2x-1$ for some integer $x \in [1,k]$ we have $(k+\frac{x}{k+1})^2 = k^2 + \frac{2xk}{k+1} + (\frac{x}{k+1})^2 = k^2+2x-\frac{2x}{k+1}+(\frac{x}{k+1})^2$. Notice that $-\frac{2x}{k+1}+(\frac{x}{k+1})^2 = \frac{x^2-2x(k+1)}{(k+1)^2} = \frac{(x-(k+1))^2-(k+1)^2}{(k+1)^2} = \frac{(x-(k+1))^2}{(k+1)^2} - 1$, and with $0 \leq x \leq k+1$ we have $-1 \leq \frac{(x-(k+1))^2}{(k+1)^2} - 1 \leq 0$. Therefore $\sqrt{n} = \sqrt{k^2+2x-1} \leq k + \frac{x}{k+1} \leq \sqrt{k^2+2x} = \sqrt{n+1}$.

As for part (b) we show that there's no fraction $\frac{a}{b}$ (with $b \leq k$) lying in the interval $[\sqrt{k^2+1}, \sqrt{k^2+2}]$. Notice that, $k < \sqrt{k^2+1} < \sqrt{k^2+2} < \sqrt{k^2+2k+1} = k+1$. Assume that $\frac{a}{b}$ satisfies this property, then from $\frac{a}{b} > k$ and $b \leq k$ we have $(\frac{a}{b})^2 \geq (k+\frac{1}{k})^2 = k^2+2+\frac{1}{k^2} > k^2+2$, contradiction.

A6/IMO 5 The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

Thoughts. 2016 is the minimum–the first thing that must jump out immediately of your mind. The construction should also be intuitive: we need to either have $LHS > RHS$ all the times or vice versa. W.L.O.G. let's make $LHS < RHS$. One observation is that, $RHS < 0$ must imply $LHS < 0$, meaning that on each side there are oddly many factors with roots greater than x (hence contributing to negative factors).

Thus this gives the "mod 4" construction as detailed below, because whenever RHS is negative we have $x \in (4i-2, 4i-1)$ for some integer i, which also guarantees the negativity of LHS. Next, it is not hard to prove that $|LHS| < |RHS|$ whenever both of them are positive, so the only hard part is to prove that $|LHS| > |RHS|$ when both negative. Thankfully $|\frac{(x-(4i-2))(x-(4i-1))}{(x-4i)(x-(4i-3))}| \leq \frac{1}{9}$ whenever $x \in (4i-2, 4i-1)$, and we can finish off by using appropriate sum telescoping, again as detailed below.

Answer. 2016.

Solution. For each i, the factor $x-i$ cannt appear on both sides (otherwise i will itself be a root), so $x-i$ must be erased on on of the sides for each $i \in \{1, 2, \cdots, 2016\}$, forcing at least 2016 factors to be erased. It remains to show that 2016 is good to go.

We claim that the equation

$$\prod_{i=1}^{504}(x-(4i-3))(x-(4i)) = \prod_{i=1}^{504}(x-(4i-2))(x-(4i-1))$$

has no real solution by showing that the left-hand side is always strictly smaller than the right hand side, realized by the following simpler cases:

Case 1. $x \in \{1, 2, \cdots, 2016\}$. Now, if $x = 4i$ or $x = 4i+1$ then LHS=0 while RHS has $2i$ negative factors (while the rest positive) hence positive, so $LHS = 0 < RHS$. If $x = 4i-1$ or $x = 4i-2$ then the right is 0 while the left has $2i-1$ negative factors (while the rest positive) hence negative, giving $LHS < 0 = RHS$.

Case 2. $x \in (4i+1, 4i+2)$ for some integer $i \in [0, 503]$. Now there are $2i+1$ negative factors (and the rest $1007-2i$ positive) on the left (hence negative) while $2i$ negative factors (and the rest $1008-2i$ positive) on the right (hence positive). This gives $LHS < 0 < RHS$.

Case 3. $x \in (4i-1, 4i)$ for some integer $i \in [1, 504]$. Similar to case 2, there are $2i-1$ negative factors (and the rest $1009-2i$ positive) on the left (hence negative) while $2i$ negative factors (and the rest $1008-2i$ positive) on the right (hence positive). Again $LHS < 0 < RHS$.

Case 4. $x > 2016$, $x < 1$, or $x \in (4j, 4j+1)$ for some integer $j \in [1, 503]$. Observe the following relation:

$$(x-(4i-2))(x-(4i-1)) - (x-(4i-3))(x-(4i)) = (4i-1)(4i-2) - (4i-3)(4i) = 2 \cdots (*)$$

We claim that for each integer $i \in [1, 504]$ we have $(x-(4i-2))(x-(4i-1)), (x-(4i-3))(x-(4i)) > 0$. If $x > 2016$ we have $x-(4i-2), x-(4i-1), x-(4i-3), x-(4i) > 0$. If $x < 1$ we have $x-(4i-2), x-(4i-1), x-(4i-3), x-(4i) < 0$ (and recall that the product of two negative numbers are positive). If $x \in (4j, 4j+1)$ for some integer $j \in [1, 503]$ we have:

$$\begin{cases} x-(4i-2), x-(4i-1), x-(4i-3), x-(4i) > 0 & \text{if } i \leq j \\ x-(4i-2), x-(4i-1), x-(4i-3), x-(4i) < 0 & \text{if } i > j \end{cases}$$

Therefore we always have

$$|(x-(4i-2))(x-(4i-1))| > |(x-(4i-3))(x-(4i))|.$$

This implies

$$\prod_{i=1}^{504} |(x-(4i-3))(x-(4i))| < \prod_{i=1}^{504} |(x-(4i-2))(x-(4i-1))|$$

and since each side is positive,

$$\prod_{i=1}^{504} (x-(4i-3))(x-(4i)) < \prod_{i=1}^{504} (x-(4i-2))(x-(4i-1)).$$

This case requires a little more manipulation than that of cases 4 and 5, but the idea is not much differen

We are now left with the trickiest case:

Case 5. $x \in (4i-2, 4i-1)$ for some $i \in [1, 504]$, whereby both sides are negative. The goal is therefore to show that $|LHS| > |RHS|$. By (*) in case 4 we always have $(x-(4j-2))(x-(4j-1)) - (x-(4j-3))(x-(4j)) = 2$, and since $x \in (4i-2, 4i-1)$, for $j > i$ we have $x-(4j-2), x-(4j-1), x-(4j-3), x-(4j) < 0$ and for $j < i$ we have $x-(4j-2), x-(4j-1), x-(4j-3), x-(4j) > 0$. This allows us to conclude that whenever $j \neq i$, we have $(x-(4j-2))(x-(4j-1)), (x-(4j-3))(x-(4j)) > 0$. First, from $(x-(4i-2))(x-(4i-1)) = (x-(4i-1.5)) - \frac{1}{4} \geq -\frac{1}{4}$ we get

$$\frac{|(x-(4i-2))(x-(4i-1))|}{|(x-(4i))(x-(4i-3))|} = \frac{c}{c+2} = 1 - \frac{2}{c+2} \leq 1 - \frac{2}{2+\frac{1}{4}} = \frac{1}{9}$$

where $c = |(x-(4i-2))(x-(4i-1))|$. Next, let's investigate $\dfrac{|(x-(4j-2))(x-(4j-1))|}{|(x-(4j))(x-(4j-3))|}$ for some $j < i$. We know that $x > 4i+1$, so $(x-(4j-2))(x-(4j-1)) > (4i-4j-1)(4i-4j) = 4(i-j)(4(i-j)-1)$. Again letting $c = (x-(4j-2))(x-(4j-1))$ we have

$$\frac{|(x-(4j-2))(x-(4j-1))|}{|(x-(4j))(x-(4j-3))|} = \frac{c}{c-2} = 1 + \frac{2}{c-2} < 1 + \frac{2}{4(i-j)(4(i-j)-1)-2}$$

$$= 1 + \frac{1}{2(i-j)(4(i-j)-1)-1} < 1 + \frac{1}{(i-j+1)^2-1},$$

the last inequality holds since for $i-j \geq 1$ we have $2(i-j)(4(i-j)-1) - 1 - ((i-j+1)^2-1) = 8(i-j)^2 - 2(i-j) - ((i-j)^2 + 2(i-j) + 1) = 7(i-j)^2 - 4(i-j) - 1 \geq 7-4-1 = 2$. and therefore $2(4(i-j)-1)(i-j) - 1 > (i-j+1)^2 - 1$ for $j \leq i-1$. Thus

$$\frac{\prod_{j=1}^{i-1}(x-(4j-2))(x-(4j-1))}{\prod_{j=1}^{i-1}(x-(4j-3))(x-(4j))} < \prod_{j=1}^{i-1}\left(1 + \frac{1}{(i-j+1)^2-1}\right) = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \dots \cdot \frac{i^2}{i^2-1}$$

$$= \frac{2(2)}{1(3)} \cdot \dots \cdot \frac{i(i)}{(i-1)(i+1)} = 2 \times \frac{i}{i+1} < 2$$

(notice that we dropped the modulus sign since each product is positive, as proven before). Likewise, $\dfrac{\displaystyle\prod_{j=i+1}^{504} (x-(4j-2))(x-(4j-1))}{\displaystyle\prod_{j=i+1}^{504} (x-(4j-3))(x-(4j))} < 2$. Thus $\dfrac{\displaystyle\prod_{i=1}^{504} |(x-(4i-2))(x-(4i-1))|}{\displaystyle\prod_{i=1}^{504} |(x-(4i-3))(x-(4i))|}$

$< 2 \times \frac{1}{9} \times 2 = \frac{4}{9} < 1$,

and now we are done (omg the long proof...)

2 Combinatorics

C1 The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leaders in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leaders string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Thoughts. This is equivalent to finding the number of possible strings said by the leader given the strings said by the deputy leader. Intuitively, for each digit, there is $\frac{k}{n}$ probability of it being changed by the deputy leader (so there will be $\frac{k}{n}$ of the strings with the digit being changed). From here, the string can be uniquely determined if $\frac{k}{n} \neq \frac{1}{2}$: for each digit we know that it is 0 or 1. In the case where $\frac{k}{n} = \frac{1}{2}$, then for each digit there are equally many strings with 0 on it as those wih 1 on it. Nevertheless, if we only look at the strings with leading 0, then among those strings, for each of the rest of the digits there are $\frac{k-1}{n-1}$ of the strings with that digit being changed (and yeah $\frac{k-1}{n-1} \neq \frac{1}{2}$) so the string can be uniquely determined like above (same goes for the strings with leading 1). Now we have two candidates, and the last step is to prove that it works. The verification might sound difficult, but again all we need is to show that if one string works we can find another string that works too (as of below).

Answer. The answer is 2 for $n = 2k$ and 1 otherwise.

Solution. Notice that there are $\binom{n}{k}$ strings the deputy leader can write. For the i-th digit (for any $i \in [0, n-1]$), there are $\binom{n-1}{k-1}$ such strings with i-th digit differing from the original, $\binom{n-1}{k}$ such strings with i-th digit equal to the original. If $\frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1} \neq \binom{n-1}{k} = \frac{(n-1)!}{(k)!(n-k-1)!}$, the contestant can determine that digit by counting the number of strings with 0 in it (and the number of strings with 1 in it). This happens when $(k-1)!(n-k)! \neq k!(n-k-1)!$, or $n - k \neq k$ (fatorizing factors out) or $n \neq 2k$. No further guesses is needed and the contestant can get it in one try.

If $n = 2k$, then for each digit, half of the strings have one's and half have zero's. The student then considers the strings with 0 on the leading digit. If, the correct string has 0 on that leading digit, then for each of the written strings (with leading 0), among the remaining $2k-1$ digits there are $k-1$ being changed from the original. By the claim above the student can determine the remaining $2k-1$ digits. Similar conclusion can be reached for he case with 1 as leading digit. This gives the student the correct answer after 2 guesses. To see why 2 guesses is necessary, let $a_0a_1 \cdots a_{2k-1}$ be the string given by the leader, $b_0b_1 \cdots b_{2k-1}$ be a string with $b_i = 1 - a_i$ for each i, $c_0c_1 \cdots c_{2k-1}$ be any string written by the deputy leader. Now, we have $c_i = a_i$ or $c_i = b_i$ but not both. With $c_0c_1 \cdots c_{2k-1}$ having k same digits and k different digits as $a_0a_1 \cdots a_{2k-1}$, it must have $2k - k = k$ same digits and $2k - k = k$ different digits as $b_0b_1 \cdots b_{2k-1}$ too. Thus $b_0b_1 \cdots b_{2k-1}$ is actually another possibility.

C2 Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:
- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

Thoughts. Prime numbers can't work (too obvious to explain), but how to generalize this idea to the general case? Since the divisors are all distinct, there cannot be exactly one row or one column, which means the sum of each column and each row must be greater than n. It might jump directly out of you that the sum of divisors of n ($\sigma_1(n)$) in this case must be greater $2n$ (meaning that the sum of divisors must be greater than $2n$); we can prove even more: $\sigma_1(n)$ must be greater than both rn and cn where r, c are the row count and column count of the rectangle, respectively. Given also that $rc = \sigma_0(n)$ is the number of divisors of n, we know that $\sigma_1(n) > n\sqrt{\sigma_0(n)}$. This turned out to be enough to produce a contradiction for those n which are not powers of 2.

Solution. The answer is $n = 1$, which works with 1 being placed in a 1×1 table. To show that this fails for other n, first prime factorize it into $\prod_{i=1}^{k} p_i^{a_i}$. If r is the number of rows and c is the number of columns then $rc = \prod_{i=1}^{k}(a_i + 1)$, the number of divisors of n. W.l.o.g. $r \geq c$ and therefore $r \geq \sqrt{\prod_{i=1}^{k}(a_i + 1)} = \prod_{i=1}^{k} \sqrt{(a_i + 1)}$. We have also known that the sum of divisors is $\prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}$. Knowing that one of the cells contains n, the sum of each row must be greater than n, (n cannot be the only cell in that row, otherwise all cells would have to contain the same number which is absurd for $n > 1$). This means that the sum of each column is greater than rn, giving the following inequality:

$$\prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1} > rn \geq \prod_{i=1}^{k} \sqrt{(a_i + 1)}p_i^{a_i}$$

or equivalently, $\prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{(p_i - 1)\sqrt{(a_i + 1)}} > 1$. Now for each prime p and positive integer a we let $f(p, a) = \frac{p - \frac{1}{p^a}}{(p-1)\sqrt{(a+1)}} = \frac{1}{\sqrt{(a+1)}}\left(1 + p^{-1} + \cdots + p^{-a}\right)$, and we show that

1. $f(p, a) < f(q, a)$ whenever $p > q$.
2. $f(2, a) \leq f(2, 1) = \sqrt{\frac{9}{8}}$ and $f(p, a) \leq f(3, 1) = \sqrt{\frac{8}{9}}$ for all $p \geq 3$.

(1) is easy: for $p > q$ we have

$$f(p, a) = \frac{1}{\sqrt{(a + 1)}}\left(1 + p^{-1} + \cdots + p^{-a}\right) < \frac{1}{\sqrt{(a + 1)}}\left(1 + q^{-1} + \cdots + q^{-a}\right) = f(q, a).$$

Now for (2), we notice that: $f(2, 1) = \frac{2-\frac{1}{2}}{\sqrt{(1+1)}} = \frac{3}{2\sqrt{2}} = \sqrt{\frac{9}{8}}$, $f(2, 2) = \frac{2-\frac{1}{2^2}}{\sqrt{(2+1)}} = \frac{7}{4\sqrt{3}} = \sqrt{\frac{49}{48}} < \sqrt{\frac{9}{8}}$, and for all $a \geq 3$ we have $f(2, a) = \frac{2-\frac{1}{2^a}}{\sqrt{(a+1)}} < \frac{2}{\sqrt{(3+1)}} = 1 < \sqrt{\frac{9}{8}}$. $f(3, 1) = \frac{3-\frac{1}{3}}{2\sqrt{(1+1)}} = \frac{8}{6\sqrt{2}} = \sqrt{\frac{8}{9}}$, and for all $a \geq 2$ we have $f(3, a) = \frac{3-\frac{1}{3^a}}{2\sqrt{(a+1)}} < \frac{3}{2\sqrt{2+1}} = \sqrt{\frac{3}{4}} < \sqrt{\frac{8}{9}}$. By (1) we have $f(p, a) \leq f(3, a) \leq f(3, 1) = \sqrt{\frac{8}{9}}$ whenever $a \geq 1$ and $p \geq 3$.

Summing up, recall that we always have $\prod_{i=1}^{k} f(p_i, a_i) > 1$. If $p_1 < p_2 < \cdots < p_k$ then we have $p_i \geq 3$ for $i \geq 2$. If $k \geq 2$ we have $\prod_{i=1}^{k} f(p_i, a_i) \leq f(2, a_1) \cdot \prod_{i=2}^{k} f(3, a_i) \leq \sqrt{\frac{9}{8}} \times \sqrt{\frac{8}{9}}^{i-1} \leq$ 1, which is a contradiction. Therefore we must have $k = 1$, with $f(p_1, a_1) > 1$. By (2) we have $p = 2$. However, this implies n is a power of 2 and from $a_i \geq 1$, at least two rows must be used (we assumed $r \geq c$). The row containing n must therefore have sum at least $2n$, but for n a power of two the sum of divisors is $2n - 1$, contradiction.

C3 Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

Thoughts. This problem is hard for a C3: how are we going to use the fact that an odd number of vertices is painted for each colour? Nevertheless, the fact that $3 \neq n$ generated some insight for us: for each fixed segment AB with endpoint of different colours there are three vertices C_1, C_2, C_3 such that the resulting triangle is isoceles. This should motivate the double-counting solution: assuming a contradiction and we consider those isoceles triangles with exactly two colours being used, then each of these triangles has two polychromatic lines while each polychromatic line belongs to three such triangles, so we can play around with the parity to produce contradiction!

Solution. Suppose that the conclusion is false, and let a, b, c be the three colours used. Let Γ be the circumcircle of the regular n-gon. Let N be the set of isoceles triangles such that each a and b is used at least once and the colour c is not used, and let M be the set of unordered pairs of vertices $\{A, B\}$ such that A is of colour a and B is of colour b (vice versa)

We start with an observation: each member of M is one side of exactly three triangles in N. To see why, let (A, B) be one member in M. From $2 \nmid n$ we know that AB cannot be a diameter of Γ. This implies that there exists one such vertex C_1 satisfying $AB = AC_1$, and another vertex C_2 satisfying $AB = BC_2$. In addition, since n is odd, the perpendicular bisector of AB will hit exactly one vertex in the n-gon, so there is exactly one such C_3 with $AC_3 = BC_3$. C_1, C_2, C_3 are also pairwise different; otherwise $C_1 = C_2 = C_3$ and ABC_1 is equilateral, contradicting the fact that $3 \nmid n$. Also notice that ABC_1, ABC_2, $ABC_3 \in N$ because the colour c is not used at C (otherwise we are done since A, B are of colour a and b), and each colour a and b is used at least once. This gives us the required three triangles.

Next, also notice that for each triangle in N, since no colour of c is used, among each a and b, one colour is used twice and the other once. This implies that each triangle in N has two sides in M. If we consider all such pairs (x, y) where $x \in M, y \in N$ and x is a side of y then we count each element in M for 3 times and each element in N for 2 times. Thus $2|N| = 3|M|$. This means M is even. On the other hand we also have $|M| = |a| \cdot |b|$ where $|a|$ and $|b|$ are the number of sides with colour a and b, respectively. This means $|a|$ or $|b|$ is even, contradiction.

C4/IMO 2 Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Thoughts. The first condition already says that $3|n$, but sadly $n = 3$ doesn't work (and it's extremely easy to prove that it doesn't). One way to start with is, therefore, to check the smallest n that works. How about...making sure that if we partition the table into 3×3 grids, each diagonal of the grids have exactly one of each I, M, O? Wait a minute...we have to make sure that among the center cells of the 3×3 grids, there are equally many I, M, and O. Best to have that condition for the columns in the 3×3 grids too, and we can sort out the rows later. This gives $n = 9$ works, and for $n = 9k$ all we need to do is to replicate the table.

The proof of $9|n$ being necessary seems a little bit difficult, but the insights can be seen as we try to prove why $n = 3$ fails: it seems like the fault lies on the center cell. As it turns out, the center cells of each 3×3 grid lie on exactly two diagonals with size a multiple of 3 (the corner: 1 and the sides: 0). This motivates us to consider just the columns and rows with indices congruent to 2 modulo 3, which works. (Alternatively one can also consider the columns and rows with indices not congruent to 2 modulo 3).

Answer. Any n divisible by 9.

Solution. Throughout the solution we denote the We start by showing an example for $n = 9$, given below:

I	M	O	M	O	I	O	I	M
M	M	M	O	O	O	I	I	I
I	M	O	M	O	I	O	I	M
O	I	M	I	M	O	M	O	I
I	I	I	M	M	M	O	O	O
O	I	M	I	M	O	M	O	I
M	O	I	O	I	M	I	M	O
O	O	O	I	I	I	M	M	M
M	O	I	O	I	M	I	M	O

For $n = 9k$ for some k we just have to split the grid into k^2 9×9 grids, and fill each one with the letters above. (Formally, if we let $(i, j) : 1 \le i \le j \le n$ be the table coordinates then for each $1 \le i, j \le 9$ and $0 \le a, b \le k-1$, (i, j) and $(9a+i, 9b+j)$ contain the same letter). For sake of verification, observe that there are exactly 3 I's, 3 M's and 3 O's in each column or each row of a single 9×9 grid. Also, each diagonal is in the form of either $R_m = \{(i, j) : i+j = m\}$, or $L_m = \{(i, j) : i-j = m\}$, for some m satisfying $1 \le (i, j) \le n$. Now for R_m, the size $|R_m|$ is $m-1$ for $m \le n+1$, and $2n+1-m$ for $m \ge n+1$. Notice that 3 divides $|R_m|$ iff $m \equiv 1 \pmod{n}$ (first case), or iff $m \equiv 1 \pmod{n}$ (second case). Thus it is not hard to see that the diagonals are in the form of $(1, m-1), (2, m-2), \cdots (m-1, 1)$ in the first case, and $(m-n, n), (m-n+1, n-1), \cdots (n, m-n)$ in the second case. In each of the cases we can group them into groups of three, such that, if we further split each 9×9 grids into 3×3 grids, each group contains three cells along the main diagonal. Nevertheless, from the construction above we see that each main diagonal in the 3×3 grids have one I, one M and one O. Thus this set of diagonal works too. A similar conclusion can be yielded for diagonals in the form of L_m.

To show that $9|n$ is necessary, observe from the first condition that $3|n$. Let $n = 3k$ and let's split the table into k^2 3×3 cells. Notice from the logic (of diagonals characterization) as of above, the center of each 3×3 cell ((i, j) where $i, j \equiv 2 \pmod 3$) lie on both R_m and L_m with both size divisible by 3; the four corners (((i, j) where $i, j \not\equiv 2 \pmod 3$) lie on exactly one of the sets satifying the properties; the four sides (((i, j) where exactly one of i and j is congruent to 2 mod 3) lie on none of them. Thus, when we mark the cells in each column, each row, and each diagonal with size divisible by 3, the center cells are marked 4 times, the corners thrice, and the sides twice (as illustrated below).

3	2	3
2	4	2
3	2	3

Let c be the number of M's on the center cells. Considering just the $3i - 1$-th column for $i \in [1, k]$ and the $3j - 1$-th row for $j \in [1, k]$ yields $2k^2$ M's being counted. Each cell on the "side" is being counted once, each cell on the "center" twice, an each cell on the "corner" none. This gives the number of M's on the side as $2k^2 - c$, which follows that there must be $k^2 + c$ M's at the corner. Now let's see what happens as we consider all such markings (all columns, all rows, and all diagonals of size divisible by 3). Observe that for each 3×3 cells we have $3+2+3+2+4+2+3+2+3 = 24$ markings, so each letter (M, in particular) has $8k^2$ markings. This means $8k^2 = 4c + 2(2k^2 - c) + 3(k^2 + c) = 3c + 7k^2$, or $c = \frac{k^2}{3}$. Hence $3|k^2$, or $3|k$, or $9|n$.

C5 Let $n \geq 3$ be a positive integer. Find the maximum number of diagonals in a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

Thoughts. A few things must come into mind: no perpendicular diagonals for odd n (proving this needs some work but not hard to think of), and for even case we will probably need perpendicular and intersecting diagonals to maximize our selection. It's also worth noting that, the more regions the diagonals split the circular arc into, the fewer available diagonals there are for us to use. There are intuitively true, and it's not difficult for us to arrive at the answer (or rather, conjecture) of $n-2$. The difficult part is to make the proof rigorous—and writing down.

So how do we proceed? The first thing is to notice that those intersecting diagonals belong to just one pair of opposite class: those in the same class will be parallel to each other and perpendicular to everyone in the opposite class (not hard to see—just draw it yourself). Next, we want to find those diagonals that will intersect the most diagonals that are of the opposite class–which we can show that they will intersect all diagonals of the opposite class. And that's basically it! Simply form consider the relationship between the number of diagonals and the number of regions they split the circle into, and show that the rest of the diagonals must have diagonals lying entirely in the regions (which reduces to our simplified case of triangulation).

Answer. $n-3$ for n odd, and $n-2$ for n even. (Alternatively, $2\lfloor \frac{n}{2} \rfloor - 2$).

Solution. For simplicity we denote the circumcircle of the n-gon as Γ, and our n-gon be $A_0A_2\cdots A_{n-1}$, with A_i having coordinates $C\left(\frac{2i\pi}{n}\right) = \left(\cos\frac{2i\pi}{n}, \sin\frac{2i\pi}{n}\right)$.

We first show that no two diagonals can be perpendicular for n odd. Now on the circle Γ, let one diagonal joining $C(a)$ and $C(b)$, and the other joining $C(c), C(d)$, with $0 \leq a, b, c, d < 2\pi$. Then the diagonals have gradients $\frac{\sin b - \sin a}{\cos b - \cos a}$ and $\frac{\sin d - \sin c}{\cos d - \cos c}$. If they are perpendicular then the gradient has product -1 (one of them might be infinity, but this means that the other one has gradient 0). Therefore, $\frac{(\sin b - \sin a)(\sin d - \sin c)}{(\cos b - \cos a)(\cos d - \cos c)} = -1$, or $(\cos b - \cos a)(\cos d - \cos c) + (\sin b - \sin a)(\sin d - \sin c) = 0$. Using the identity of $\cos b - \cos a = -2\sin\frac{b-a}{2}\sin\frac{b+a}{2}$ and $\sin b - \sin a = 2\sin\frac{b-a}{2}\cos\frac{b+a}{2}$, we have $\sin\frac{b-a}{2}\sin\frac{d-c}{2}(\sin\frac{b+a}{2}\sin\frac{d+c}{2} + \cos\frac{b+a}{2}\cos\frac{d+c}{2}) = 0$. Since $0 \leq a, b, c, d < 2\pi$, and $a \neq b, c \neq d$, we have $0 < \frac{|a-b|}{2}, \frac{|c-d|}{2} < \pi$, and therefore $\sin\frac{b-a}{2}, \sin\frac{d-c}{2} \neq 0$. We necessarily have $\sin\frac{b+a}{2}\sin\frac{d+c}{2} + \cos\frac{b+a}{2}\cos\frac{d+c}{2} = 0$, or $\cos(\frac{(b+a)-(d+c)}{2}) = 0$, which forces $\frac{(b+a)-(d+c)}{2} = \frac{k\pi}{2}$ for some odd k. Now, knowing that $a, b, c, d = \frac{2w\pi}{n}, \frac{2x\pi}{n}, \frac{2y\pi}{n}, \frac{2z\pi}{n}$ for some integers $0 \leq w, x, y, z < n$ we have $kn = 2(w+x) - 2(y+z)$. Knowing that k is odd, n has to be even. Thus the case where n is odd reduces to finding the number of diagonals without any two of them intersecting in the interior. (The fact that a triangulation has exactly $n-3$ diagonals is well-known, but let's prove it anyway). Denoting $f(n)$ be this number and we show that $f(n) = n-3, \forall n \geq 3$. Base case when $n = 3$, where $f(3) = 0$ as there is no diagonal in a triangle. Next, let $f(n) = n-3$ for $n = 3, 4, \cdots k$ for some $k \geq 3$. For $n = k+1$, let one diagonal split our n-gon into an x-gon and a y-gon, where $x+y = n+2$ and $x, y \geq 3$. Then there are extra $f(x)+f(y) = x-3+y-3 = x+y-6 = n-4$ diagonals that can be constructed (each diagonal must belong to only one of the polygons because it cannot intersect our first diagonal in the interior). This gives $n-4+1 = n-3$ diagonals, at most. Equality can be achieved by taking diagonals A_1A_k, $k = 3, 4, \cdots, n-1$.

Now n be even. Let F be the set of diagonals intersecting at least one other diagonals. We show that:

1. Each two lines in F are perpendicular or parallel to each other.
2. There are two lines in F such that each of them intersects every other lines in F that are perpendicular to itself.

3. Let A and B be two consecutive endpoints of lines in F. Then for any selected diagonal not in F with an endpoint on the minor arc AB (that is different from A and B), the other the other endpoint must also lie on this minor arc too (possibly equal to A or B).

First claim: Now let AC and BD be two intersecting perpendicular chords on Γ. We claim that the four endpoints split the circles into four minor arcs (that is, arcs that are less than a semicircle). Let the intersection be P, then $\angle ACB = \angle ADB < \angle APB = 90°$, since P lies on both segment AC and BD. This means that the arc AB not containing C and D ais minor, and a similar conclusion can be achieved for other three arcs. Suppose that AC and BD are in F, and let another diagonal with endpoints E, G be in F and perperndicular to neither of AC nor BD. This means that there necessarily exists another diagonal (with endpoints H, J) that intersects EG in its interior (and perpendicular to it). Since they do not intersect chords AC and BD, they must lie on a minor arc (say, AB). This instantly contradicts the fact that E, F, G, H must split Γ into four minor arcs.

Second claim: Since every two lines in F are either parallel or perpendicular to each other, we can split them into two sets F_1 and F_2 such that any two diagonals are parallel to each other if and only if they belong to the same partition. An example would be $F_1 = \{(A_aA_b) : a + b \equiv 0 \pmod{n}\}$ and $F_2 = \{(A_aA_b) : a + b \equiv \frac{n}{2} \pmod{n}\}$, valid by the identity derived in the first paragraph. We claim that the longest line in F_1 intersects every line in F_2 in its interior, and similarly the longest line in F_2 intersects every line in F_1 in its interior. Indeed, let AB and CD be parallel lines on in F_1 satisfying $CD \geq AB$, then $ABCD$ is an isoceles trapezoid satisfying $\angle A = \angle B \geq 90°$ and $\angle C = angle D \leq 90°$. Denote also E and F as the perpendiculars from A and B on CD. It follows that E and F are on the closed segment CD itself. Let l be a line perpendicular to both AB and CD, intersecting open segment AB at G and line CD at H. With $AE \parallel BF \parallel GH$, the fact that G lies on segment AB means that H lies on open segment EF too, therefore lying on open segment CD as well. This means, any perpendicular line intersecting AB in its interior will intersect CD in its interior too. Now let l be the longest line in F_1, and m be any line in F_2. Since there exists n in F_1 such that m intersects n in its interior, and n is no longer than n, m must intersect l in its interior too. This proves the claim that every line in F_2 intersects l in its interior, and similarly every line in F_1 intersects the longest line in F_2 in its interior too.

Third claim: denote the endpoints of the longest line in F_1 as C and D, and in F_2 as E and F (which might or might not be completely distinct from A and B). Let G be a point on AB and presumably it's an endpoint of a selected diagonal (with $G \neq A$ and $G \neq B$). W.L.O.G. we assume that AB lies on minor arc CE (so same goes for G, and the other endpoint, obviously, must also be on the minor arc CE). W.L.O.G. also that C, A, G, B, E are on Γ in that order. By the choice of A and B (consecutive endpoints in F) we know this diagonal cannot be in F_1 or F_2, so it intersects none of the segment CD or EF. Suppose H is a point on the minor arc BE, and $H \neq B$. If the line in F with endpoint B does not intersect GH, then (by drawing) it intersects neither EF nor CD, contradicting that every line in F intersects either of them. Therefore H cannot be the endpoint of the selected diagonal, and similarly, any point on the minor arc CA that is not A cannot be the endpoint of the selected diagonal. Thus that point must be on minor arc AB. Notice that the case would be trickier if G coincides with A or B. However, in this case if A, B, C are consecutive points of any diagonal in F in that order, B can be treated as on arc AC, thus any diagonal with B as endpoint must have another endpoint either in minor arc AB or in minor arc BC.

To conclude the proof, if $A_{t_1}A_{t_2}\cdots A_{t_k}$ are consecutive endpoints of F then any remaining diagonals must have both endpoints lying on $A_{t_i}A_{t_{i+1}}$ for some $i \in [1, k]$, indices taken modulo k. Moreover, each endpoints belong to at most two diagonals, with at least four of them belonging exclusively to l_1 and l_2, the longest lines in F_1 and F_2 (they intersect every other lines in the opposite family of F so they cannot share endpoint with other lines in F). Since each diagonal has two endpoints, the number of elements in F cannot exceed $\frac{1}{2}(1+1+1+1+2+\cdots+2) = \frac{1}{2}(4+2(k-4)) = k-2$. Considering the polygon $A_{t_i} \cdots A_{t_{i+1}}$, which is a $t_{i+1} - t_i + 1$-gon, we know that $t_{i+1} - t_i - 2$ diagonals not intersecting each other can be drawn, plus one line $A_{t_i} A_{t_{i+1}}$ to be drawn. This gives $t_{i+1} - t_i - 1$ lines, resulting in $\displaystyle\sum_{i=1}^{k}(t_{i+1} - t_i - 1) = t_{k+1} - t_1 - k = n - k$ extra diagonals, thus the upper bound is $n - k + k - 2 = n - 2$.

To achieve this bound, select $A_0 A_k$, where $k = \frac{n}{2}$, and $A_1 A_{n-1}$. Now take $A_1 A_i$ with $i = 3, 4, \cdots, k$ and $A_{n-1} A_i$ with $i = n-3, n-4, \cdots k+1$. This gives $2 + 2(k-2) = 2k - 2 = n - 2$. Q.E.D.

C7/IMO 6 There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Feridun has to choose an endpoint of each segment and place a goose on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each goose will immediately jump forward to the next intersection point on its segment. Geese never change the direction of their jumps. Feridun wishes to place the geese in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Feridun can always fulfill his wish if n is odd.

(b) Prove that Feridun can never fulfill his wish if n is even.

Thoughts. We wish to play parity game to prove that the geese cannot coincide in part (a). Intuitively, this makes sense: for each point segment ℓ there are even number of intersection points, which means that for each intersection point we can chacterize one side of having even number of points and the other odd (excluding the point itself), giving rise of the notion "even sides" and "odd sides" as below (inspired by the case $n = 3$). This means for each two lines we can arrange pick one endpoints such that one endpoint is on even side and the other endpoint is on odd side (w.r.t. the intersection). What if these selections conflict with each other when n lines are considered together? They won't, based on the Menelaus' logic below.

In part (b), things are trickier: unfortunately we cannot say that the geese will meet just because both of them come from even side or odd side: the number of intersections must be the same. This motivates another version of the solution: to find cases where the other lines such that for selected rays of some two selected lines, any other lines either intersect both rays or none of them. This means the geese coming from both rays will intersect (same for the case when the geese coming not from the rays). Denoting the angles of the line to x-axis as θ_1 and θ_2, then this happens whenever either all lines have angles in (θ_1, θ_2) or all lines have angles not in this range. This shows that, if we rank the lines according to their angles, then any two lines neighbouring in ranking must have endpoints at different directions being chosen. Don't forget to include the comparison between the first and the last line to arrive at contradiction! (Notice that this approach works for part (a) too; the intention is to show another soltuion that works uniquely for part (a) which gives an opporunity for some partial marks).

Solution. (a) Let the segments be $\ell_1, \ell_2, \cdots, \ell_n$. Let P_{ij} be the intersection of line ij. For each segment ℓ_i we aim to investigate the number of points on each side of P_{ij} (other than P_{ij}). Since there are $n-2$ such points (which is odd), one side has even number of points and the other side odd. We call this even and odd side of ℓ_i w.r.t. point P_{ij}, respectively.

Now place the first goose arbitrarily on ℓ_1. For $i \in [2, n]$ we do the following: if the goose corresponding to ℓ_1 is place on the odd side of ℓ_1 w.r.t. P_{1i}, Feridun places one goose at the even side of ℓ_i w.r.t. P_{1i} (and vice versa). We now proceed to the following claim: using the procedure detailed above, for each two distinct integers $i, j \in [1, n]$, the geese corresponding to l_i and l_j lie on different parity of l_i and l_j, respectively, both w.r.t. P_{ij}. Indeed, consider the triangle formed by lines ℓ_1, ℓ_i and ℓ_j. Menelaus' theorem says that any line either intersects none or two of the segments $P_{ij}P_{1i}$, $P_{1j}P_{1i}$, $P_{ij}P_{1j}$. Thus considering lines l_k with $k \notin \{1, i, j\}$ we know that it has even number of total intersection points with segments $P_{ij}P_{1i}$, $P_{1j}P_{1i}$, $P_{ij}P_{1j}$. If this number is even on $P_{1j}P_{1i}$, then each endpoint is on the odd side of l_1 w.r.t. one of P_{1j} and P_{1i}, and even on the other. Thus according of our choice of placing the geese , either one goose come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}, or vice versa. The intersection with $P_{ij}P_{1i}$ and $P_{ij}P_{1j}$ will be both odd or both even. If it's both odd and in the first case (one goose come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}), then the goose corresponding to i come from the odd side of l_i w.r.t. P_{1i}

and the other from even side of l_j w.r.t. P_{1j}, which works for this pair of (i, j). The other three subcases can be treated equally. If this number is odd on $P_{ij}P_{1i}$, then each endpoint is on the odd side of l_i w.r.t. both P_{1i} and P_{1j}, or vice versa (both even). According to our choice again, both geese come from the odd side of l_i w.r.t. P_{1i} and the and of l_j w.r.t. P_{1j}, or both from the even side of their respective lines. The intersection with $P_{ij}P_{1i}$ and $P_{ij}P_{1j}$ will be one odd and one even, for the same endpoint w.r.t the lines l_i and l_j, exactly one of them will change sign when switching from P_{1i} to P_{ij} and from P_{1j} to P_{ij}. Again this (i, j) works.

Finally, to see why the geese won't intersect at the same time, observe that if this happens for some of (i, j), then the geese must have encountered the same number of points before. This implies that they have to come both from the odd side or the even side of the line, contradiction.

(b) Let $\ell_1, \ell_2, \cdots, \ell_n$ be the segments, and let $\theta_i \in [0, \pi)$ be the angle need for x-axis to rotate counterclockwise to reach ℓ_i. W.L.O.G let $0 = \theta_1 < \theta_2 < \cdots < \theta_n$. For convinience we introduce $\ell_{n+1} = \ell_1$ with $\theta_{n+1} = \pi$.

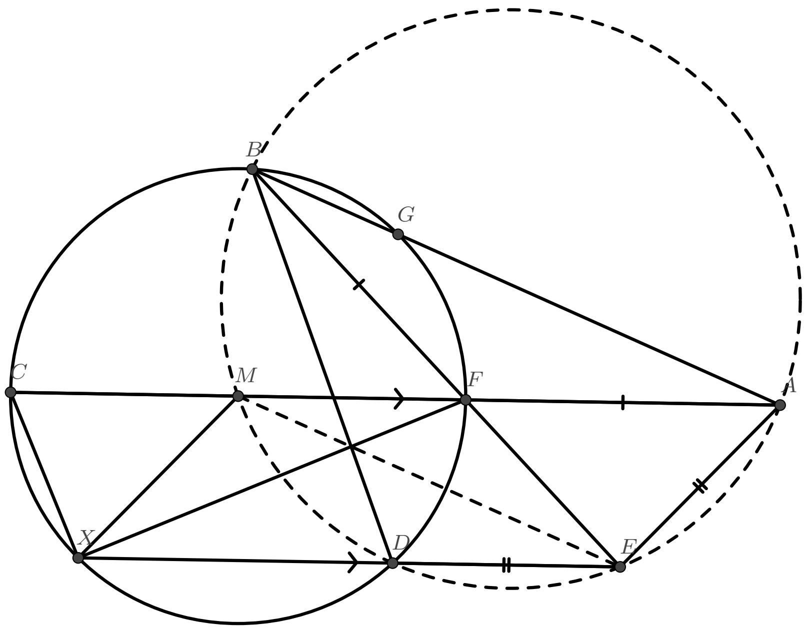
For each segment Feridun has the choice of placing the goose in the direction of θ_i or $\theta_i + \pi$ compared to the positive x-direction. Let's investigate ℓ_i and ℓ_{i+1} together. Now, suppose that ℓ_i and ℓ_j intersect at P_{ij} for some j; define $\ell_{(i+1)j}$ similarly. We know that for each j different from i, $i + 1$, ℓ_i, ℓ_{i+1} and ℓ_j are in that order (in anticlockwise angle, cycles allowed). This means that $P_{ij}P_{i(i+1)}, P_{i(i+1)}P_{(i+1)j}, P_{(i+1)j}P_{ij}$ must also be in that order, forcing $P_{ij}, P_{i(i+1)}, P_{(i+1)j}$ to be in clockwise order. From here we infer that the vectors $P_{i(i+1)}P_{ij}$ and $P_{i(i+1)}P_{(i+1)j}$ either have directions $(\theta_i,\, \theta_{i+1})$ or $(\theta_i + \pi,\, \theta_{i+1} + \pi)$, and considering all such j's, we know that there are equal number of intersection points lying on the half-line starting from $P_{i(i+1)}$ and extending in the θ_i direction, and on the half-line starting from $P_{i(i+1)}$ and extending in the θ_{i+1} direction. This means that the geese will collide when both placed in the θ_i, θ_{i+1} direction, or $\theta_i + \pi, \theta_{i+1} + \pi$ direction, forcing the directions to be $\theta_i, \theta_{i+1} + \pi$ or $\theta_i + \pi, \theta_{i+1}$.

Summarizing above, if we let θ_1 to be the direction headed by the first goose the directions must be $\theta_1, \theta_2 + \pi, \theta_3, \theta_4 + \pi, \cdots, \theta_n + \pi$ (n is even). Recall that we can also compare it with the "n + 1-th" line (which is the first) and it has to have direction $\theta_{n+1} = \theta_1 + \pi$, which contradicts our choice of making the first *goose* facing the direction θ_1. (The case where $\theta_1 + \pi$ is chosen as the direction is completely analogous: if one configuration works, then the completely opposite configuration works too).

3 Geometry

G1/IMO 1 Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Thoughts. Right angle at B, and midpoint of CF is given in the problem. What else could be better than drawing the circumcircle of BCF? And that's pretty much all we need for idea generation: as you draw the diagram you will soon realize tha D and X both lie on this circumcircle, and that ED is parallel to CF (which practically means that M, D, E are collinear). The rest of the job is to prove the claims above (and there are many ways to do it; I have found two–one as detailed below, the other one using trigonometric bashing). And finally the use of the collinearity of B, F, E and the fact that $BFDX$ is isoceles trapezoid is just one of the many possible ways to finish the solution.



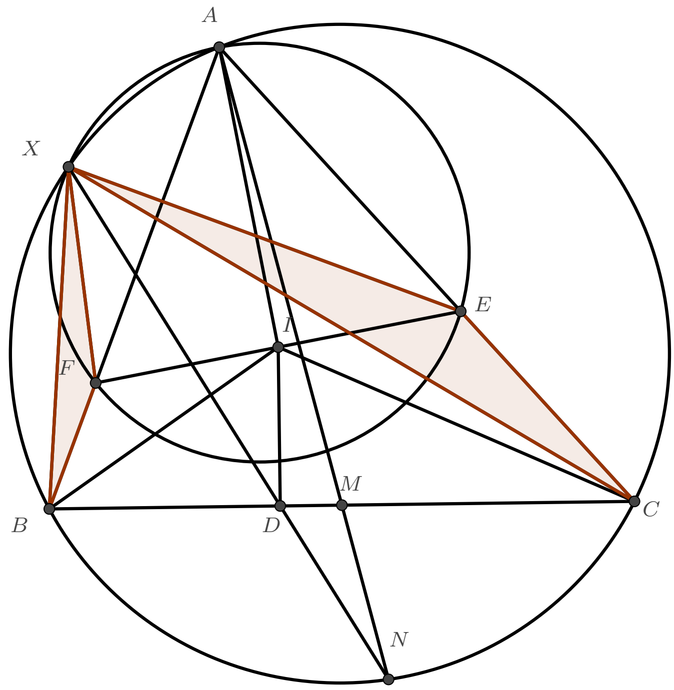
Solution. Let Γ be the circumcircle of triangle BCF, which we know that M is its center and CF its diameter. Denote G as the intersection of AB and Γ other than B (with $G \neq B$ unless AB is tangent to Γ). Since $FA = FB$, we have $\angle GCA = \angle GCF = \angle GBF = \angle ABF = \angle BAF = \angle GAC$, hence $GA = GC$. Now, denote D' as the reflection of G in AC, and we know that D' lies on Γ. Notice also that $AD' = AG = CG = CD'$, and $\angle BAC = \angle GAC = \angle D'AC$, so D' fulfills $D'A = D'B$ and AC is the bisector of $\angle D'AB$. We therefore have $D = D'$ since there is only one such point fulfilling such property (i.e. the intersection of perpendicular bisector of AC and the reflection of AB in CF, which cannot be the same unless AB is parallel or perpendicular to AC, which forces triangle FAB and BCF to be degenerate). Now that we established that D is on Γ, we claim that $MDEA$ is an isoceles trapezoid. Indeed, $\angle AMD = \angle FMD = \angle FMG = 2\angle GCF = 2\angle GAF = 2\angle CAD = \angle CAE = \angle MAE$, and $\angle DEA = 180^\circ - \angle DAE - \angle ADE = 180^\circ - 2\angle DAE = 180^\circ - \angle MAE = 180^\circ - \angle AMD$ (proven). This also gives $AC \parallel ED$, and D, E, X collinear. Moreover $\angle MXD = \angle MXE = \angle MAE = \angle MDX$ so $MD = MX$ and X is on Γ. Therefore $CFDX$ is also an isoceles trapezoid.

To finish up the solution, we claim that EM is the perpendicular bisector of both DF and BX. Indeed, M is on the perpendicular bisector of the two lines because $MD = MF = MB = MX$ (all four points lie on Γ). Now, notice that $\angle BMD + \angle BAD =$

$2\angle BCD + 2\angle BAC = 2(\angle BCF + \angle DCF + \angle BAC) = 2(\angle BCF + \angle GCF + \angle GAC) = 2(\angle BCF + \angle GAF + \angle GAC) = 2(\angle BCF + \angle BFC) = 2(90^\circ) = 180^\circ$, meaning that B, D, M, A are concylic. Nevertheless, knowing that $AMDE$ is isoceles trapezoid, E lies on this circle too. Thus $\angle BED = \angle BAD = 2\angle BAM = \angle BAF + \angle ABF = \angle BFM$, and coupled with the fact that $AC \parallel EX$ we have B, F, E collinear. Finally, $\angle BFD = \angle BFC + \angle CFD = \angle BFC + \angle CFG = 2\angle GAC + 90^\circ - \angle GCF = 2\angle GAC + 90^\circ - \angle GAF = 90^\circ + \angle GAC = 90^\circ + \angle GCF = 90^\circ + (90^\circ - \angle CFG) = 180^\circ - \angle CFG = 180^\circ - \angle CFD = \angle FDX$, showing that $BFDX$ is also an isoceles trapezoid. With BF and DX intersecting at E, we conclude that EM is the perpendicular bisector of both DF and BX, and DB and FX will intersect on this perpendicular bisector too.

G2 Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Thoughts. The first thing that must naturally jump out of your mind is the spiral similarity between XFB and XEC (come on, two circles intersecting at A and X). All's remaining is to trigonometric-bash the problem.



Solution. W.L.O.G. let $AB < AC$. First, well-known spiral similarity property should dictate the similarity of triangles BXF an CXE, so $\frac{CX}{CE} = \frac{BX}{BF}$. Also, let's also invoke an identity for triangles (feel free to verify it; I'm not gonna do this):

$$\frac{BX}{XC} \cdot \frac{\sin \angle BXD}{\sin \angle CXD} = \frac{BD}{DC}.$$

Denoting N_1 as the other intersection of XD and Γ gives $\frac{\sin \angle BXD}{\sin \angle CXD} = \frac{BN_1}{CN_1}$. Similarly we have $\frac{AB}{AC} \cdot \frac{\sin \angle ABM}{\sin \angle ACM} = \frac{BM}{CM} = 1$. ALso let N_2 as the other intersection of AM and Γ and we have $\frac{\sin \angle ABM}{\sin \angle ACM} = \frac{BN_2}{CN_2}$. Therefore all we need is $\frac{\sin \angle ABM}{\sin \angle ACM} = \frac{\sin \angle BXD}{\sin \angle CXD}$, and it's not hard to see that $\frac{\sin \angle ABM}{\sin \angle ACM} = \frac{AC}{AB}$, so we are left with proving the fact $\frac{BF}{EC} \cdot \frac{AC}{AB} = \frac{BD}{DC}$.

Now, $\frac{BD}{DC} = \frac{\tan \frac{1}{2}\angle C}{\tan \frac{1}{2}\angle B}$, $\frac{AC}{AB} = \frac{\sin \angle B}{\sin \angle C} = \frac{2 \sin \frac{1}{2}\angle B \cos \frac{1}{2}\angle B}{2 \sin \frac{1}{2}\angle C \cos \frac{1}{2}\angle C}$. Also $IE = IF$, and by angle chasing we have $\angle FIB = \angle ICE = \frac{1}{2}\angle C$, $\angle EIC = \angle IBF = \frac{1}{2}\angle B$. Therefore BIF and ICE similar, yielding $\frac{BF}{EC} = (\frac{BF}{FI})^2 = (\frac{\sin \frac{1}{2}\angle C}{\sin \frac{1}{2}\angle B})^2$, now it's no longer difficult to prove that $(\frac{\sin \frac{1}{2}\angle C}{\sin \frac{1}{2}\angle B})^2 \cdot \frac{2 \sin \frac{1}{2}\angle B \cos \frac{1}{2}\angle B}{2 \sin \frac{1}{2}\angle C \cos \frac{1}{2}\angle C} = \frac{\tan \frac{1}{2}\angle C}{\tan \frac{1}{2}\angle B}$.

G3 Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be nice if

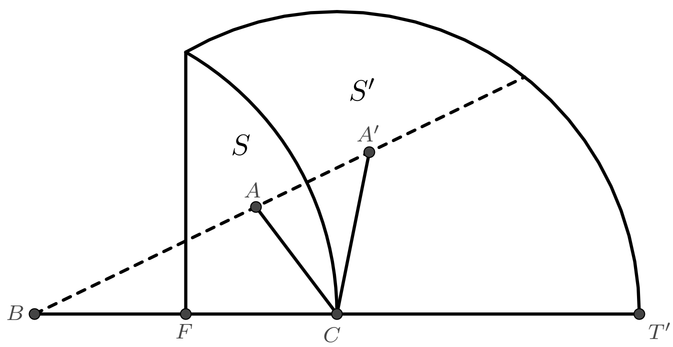
(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Thoughts. By letting $P_1P_2P_3$ equilateral we know that the only viable point A with ABC equilateral and with non-negative coordinates for A is $(0, \sqrt{3})$. This also forces $BA = 2$, and if for set S and S', the product $BA \cdot BA'$ is fixed across all such triangles $P_1P_2P_3$, then we must have $BA \cdot BA' = 2 \times 2 = 4 = BC^2$.

Now this gives us an intuition on how S and S' *might* be constructed: if A, A', B are collinear, A and A' are lying on the same side as B, and $BA \cdot BA' = BC^2$, then triangles BCA and $BA'C$ are similar. What if...we try out S containing all points A with $BA \leq 2$ (and having non-negative coordinates), and S' containing the points A' such that for each point A in S we have $BA \cdot BA' = 4$. That would work, but again we need to verify it. Notice that a sensible question to ask would be: what exactly is S' (given the condition above)? Now that $BA \leq 2$ for $A \in S$, we know that it is bounded by a circle, and the line $y = 0$, which means that S' has something to do with the inversion of y-axis w.r.t. the circular arc $X : BX = 2$ and radius 2. And yeah, $S \cup S'$ is precisely the part of this image of inversion that has nonnegative coordinates.



Solution. We show that the following works: $S = \{(x, y) : x \geq 0, y \geq 0, (x+1)^2 + y^2 \leq 4\}$ and $S' = \{x \geq 0, y \geq 0, (x+1)^2 + y^2 \geq 4, (x-1)^2 + y^2 \leq 4\}$. We claim that $BA \cdot BA' = 4$ for those choices. Denote triangles ABC and DEF as quasi-similar if there exists a permutation σ of $\{D, E, F\}$ with ABC and $\sigma(D)\sigma(E)\sigma(F)$ as similar.

We first start with the following claim: for every point A above x-axis, $A \in S$ iff $AC \leq AB \leq BC$ and $A' \in S'$ iff $A'C \leq BC \leq A'B$.
Proof: let's first investigate all points $A \in S$ and $A' \in S'$. Let $A = (x, y)$ be arbitrary. Now, $AC \leq AB \leftrightarrow (x-1)^2 + y^2 \leq (x+1)^2 + y^2 \leftrightarrow 0 \leq x$. $BC \geq AB \leftrightarrow 2^2 \geq (x+1)^2 + y^2 \leftrightarrow (x+1)^2 + y^2 \leq 4$. Therefore $AC \leq AB \leq BC$ iff (x, y) satisfies both $x \geq 0$ and $(x+1)^2 + y^2 \leq 4$ iff $(x, y) \in S$. Next, $AC \leq BC$ iff $AC \leq 2$ or $(x-1)^2 + y^2 \leq 4$ and $BC \leq AB$ iff $(x-1)^2 + y^2 \geq 4$, therefore $AC \leq BC \leq AB$ iff the two conditions are satisfied (and from here $(x-1)^2 + y^2 \leq (x-1)^2 + y^2$ so $x \geq 0$ is implied), and this is equivalent to $A \in S'$.

We are now ready to justify our selection:

(i) In S we can simply take any point as T, since the boundaries x-axis with $x \in [0, \sqrt{3}]$, y-axis with $y \in [0, 1]$, and the arc of the circle $(x-1)^2 + y^2 = 4$ with $x \geq 0$ are convex. In S' we take $T' = (3, 0)$. Let Q' be in S' and we want to show that the whole segment $T'Q'$ lies on S'. Consider the region $S \cup S'$, i.e. the region bounded by $x-$ and $y-$ axes, together with the circular arc of the circle $(x-1)^2 + y^2 = 4$. This region is also convex, so with T', Q' lying in the region $S \cup S'$, the entire segment is in this region too. The aim is therefore to show that no point of the segment lies in the interior of S or on the y-axis. Suppose on the contrary we have $R \in S$ belonging in the segment $T'Q'$. From the convexity of the region $S \cup S'$ we only need to consider the part of the line $T'Q'$ lying in this region. With R lying in S, we know that segment $T'Q'$ intersects the boundary separating S and S' (i.e. the circular arc) at least once, and since Q' is outside (or on the boundary of) the region S, it must intersect the boundary for another time, entailing the fact that this segment has to intersect the circular arc for exactly twice. Let X and X' be the intersections, and we have $\angle BX'T' = 180° - \angle BXT'$. Denoting X' as the further point from T' and we have $\angle BX'T' < 90°$. Now consider the point $A = (0, \sqrt{3})$, and A is on both S and S', which we have $\angle BAT' = 90°$. Therefore all X on the arc must sastisfying $\angle BXT' \geq \angle BAT' = 90°$ since X lies inside the triangle BAT' (contradiction).

(ii) Observe that the objective is equivalent to: for each triangle $P_1P_2P_3$ there is a unique point A with triangle ABC quasisimilar to $P_1P_2P_3$ satisfying $AC \leq AB \leq BC$, and another unique point A' with $A'BC$ quasisimilar to $P_1P_2P_3$ and $A'C \leq BC \leq A'B$. Moreover we want to prove that $BA \cdot BA' = BC^2 = 4$. We will also use the fact that for each triangle DEF there is a unique A above x-axis that is similar to DEF (with B, C fixed). We split into the following cases:

Case 1. $P_1P_2P_3$ equilateral. The only point A with $y \geq 0$ satisfying this is $A = (0, \sqrt{3})$. Since we have $(x+1)^2 + y^2 = (x-1)^2 + y^2 = 1 + 3 = 4$, A lies in both S and S' and we have $BA \cdot BA = 2 \times 2 = 4$.

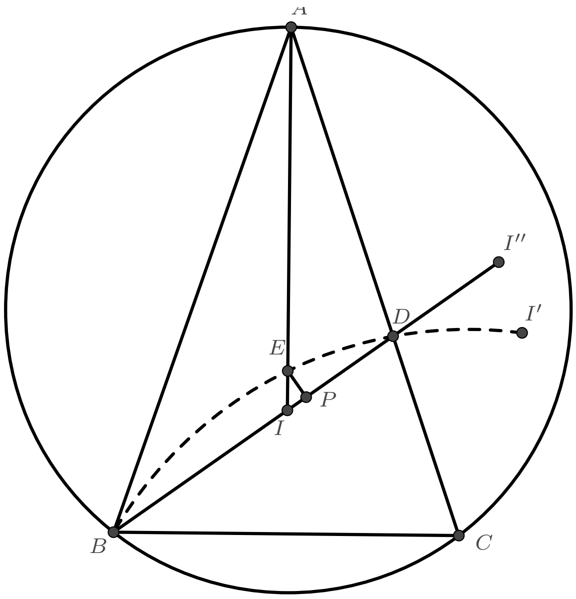
Case 2. $P_1P_2P_3$ is isoceles, with the two equal sides longer than the other. Now, let $P_1P_2 = P_1P_3 > P_2P_3$. This means, if AC is the shortest and if triangle ABC is quasisimilar to $P_1P_2P_3$ then AC corresponds to P_2P_3, and AB, BC correspond to P_1P_2 and P_1P_3, which implies that $AB = BC = 2$. Such A can be uniquely constructed, and with $AC < AB = BC = 2$ we have A lies in S. Similarly, if $A'C$ is the shortest side of $A'BC$ and if triangle $A'BC$ is quasisimilar to $P_1P_2P_3$ then $A'C$ corresponds to P_2P_3, and $A'B, BC$ corresponds to P_1P_2 and P_1P_3, so A' can also be uniquely constructed (which turns out to be equal to A in this case). Therefore, $A'C < BC = A'B = 2$, which implies that A' is in S', and moreover $BA \cdot BA' = 2 \times 2 = 4$.

Case 3. $P_1P_2P_3$ is isoceles with the two equal sides longer than the other. Now, let $P_1P_2 = P_1P_3 < P_2P_3$. This means AC and $A'C$ corresponds to P_1P_2. In ABC, we know that $AB \leq BC$ implies AB corresponds to P_1P_3, BC corresponds to P_2P_3; In ABC, we know that $BC \leq A'B$ implies BC corresponds to P_1P_3, $A'B$ corresponds to P_2P_3. Therefore $\frac{AB}{BC} = \frac{P_1P_3}{P_2P_3} = \frac{BC}{A'B}$ and $BC^2 = AB \cdot A'B$.

Case 4. $P_1P_2P_3$ scalene, and let $P_1P_2 < P_1P_3 < P_2P_3$. This means AC and $A'C$ corresponds to P_1P_2. In ABC, we know that $AB \leq BC$ implies AB corresponds to P_1P_3, BC corresponds to P_2P_3; In ABC, we know that $BC \leq A'B$ implies BC corresponds to P_1P_3, $A'B$ corresponds to P_2P_3. Therefore $\frac{AB}{BC} = \frac{P_1P_3}{P_2P_3} = \frac{BC}{A'B}$ and $BC^2 = AB \cdot A'B$.

G4 Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Thoughts. Focusing on the circumcircle of BDE (as per suggested by the problem) is itself a pain because there isn't much information we can take home regarding the point I'. Instead, the notion of reflection should immediately let us think of angle bisection: that AC bisects $\angle BDI'$. As DE is perpendicular to AC, line DE bisects the same angle too. This realization should let us focus on the circumcircle of BDI' instead: we know that E lies on this circumcircle if and only if $BE = EI'$. Now the equality $BE = EI'$ is something easier to establish; we chose the method of trigonometry and introducing the point I'' (although direct computation *should* work too).



Solution. Let I' be the reflection of I in AC. Observe that AC is an angle bisector of $\angle BDI'$ by the definition of I', and since $DE \perp AC$, DE is another angle bisector of this angle. This implies that the intersection of DE and the circumcircle of BDI' (other than D) is equidistant from B and I', i.e. on the perpendicular bisector of BI'. It therefore suffices to prove that E lies on this perpendicular bisector, or $BE = EI'$.

Let I'' be the image of I when reflected in D, then $DI = DI' = DI''$. Moreover, I'' lies on line BD, which entails that $I'I''$ is parallel to AC and perpendicular to DE. Therefore, DE is the perpendicular bisector of $I'I''$ and $EI'' = EI'$. The problem is now reduced to proving $BE = EI''$. Let P be the foot of perpendicular from E to BD, then the problem is now equivalent to proving that P is the midpoint of BI''. Knowing that $BP = BI + ID - PD$ and $PI'' = PD + DI'' = PD + DI$ it suffices to prove that $BI = 2PD$.

Denote the common angles $\angle ABI, \angle IBC, \angle ICB, \angle ACI$ as α. Then, $\angle ADB = 3\alpha$, and $\angle IDE = |90^\circ - 3\alpha|$ (as we will see, we are only interested in the cosine of this angle so don't worry about the sign). So

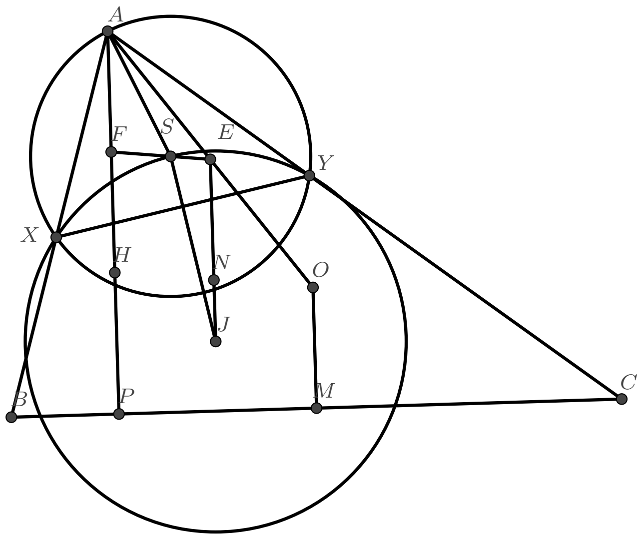
$$\frac{PD}{BI} = \frac{DE\cos\angle IDE}{BI} = \frac{(AD\tan\angle DAI)\cos\angle IDE}{BI} = \frac{AB\sin\angle ABD}{\sin\angle ADB}\cdot\frac{\tan\angle DAI\cos\angle IDE}{BI}$$

$$= \frac{BI\sin\angle AIB}{\sin\angle BAI}\cdot\frac{\tan\angle DAI\cos\angle IDE\sin\angle ABD}{BI\sin\angle ADB} = \frac{\tan(90^\circ-2\alpha)\cos|90^\circ-3\alpha|\sin\alpha\sin(90^\circ+\alpha)}{\sin(3\alpha)\sin(90^\circ-2\alpha)}$$

$$= \frac{\sin(90^\circ-2\alpha)\sin(3\alpha)\sin\alpha\cos\alpha}{\sin(3\alpha)\sin(90^\circ-2\alpha)\cos(90^\circ-2\alpha)} = \frac{\sin\alpha\cos\alpha}{\sin(2\alpha)} = \frac{\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha} = \frac{1}{2},$$

as $\cos|90^\circ - x| = \cos(90^\circ - x) = \sin x$, $\sin(90^\circ + \alpha) = \cos\alpha$ and $\tan x = \frac{\sin x}{\cos x}$.

G5 Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.



Solution 1. Denote by O the circumcenter and H the orthocenter of triangle ABC Denote also by E the midpoint of AO, F the midpoint of AH, N the midpoint of OH (the nine-point-center), and J the circumcenter of XSY. Observe that our goal is to prove that the circumcenter of triangle XSY (namely J) lies on the perpendicular bisector of PM. Considering the 9-point circle we know that P and M pass through the circle, so the center N is on the perpendicular bisector of PM. Thus is suffices to prove that J lies on the perpendicular from N to BC.

Now let's break the solution into the following:

Lemma 1. S lies on EF.
Proof: Obviously S passes through the perpendicular bisector of AD, so this locus is a line. In the case there the circle passes through H, from the fact that $\angle ADH = 90°$ we know that $S = F$. Similarly, if the circle passes through O, $S = E$ in this case with $\angle ADO = 90°$. Thus the locus is actually EF, i.e. parallel to OH.

Lemma 2. The claim holds through in the special cases where $S \equiv E$ or $S \equiv F$.
Proof: When $S \equiv E$, X is the midpoint of AB and Y is the midpoint of AC (imagine the homothety centered at A with factor $\frac{1}{2}$ which brings ABC to AXY and point O to the midpoint of AO). J lies on the perpendicular bisector of XY Notice that, with $XY \parallel BC$, this perpendicular bisector of XY is also perpendicular to BC. Moreover, nine-point-circle passes through the midpoints of AB and AC, so this perpendicular bisector passes through the nine-point center. Therefore the perpendicualr bisector of XY is the perpendicular bisector of PM itself, and with $SX = SY$, S (the midpoint of AO, a.k.a. E in this case) lies on this perpendicular bisector too. When $S \equiv F$, X and Y are going to be the altitude from C to AB, and B to AC, respectively. Since the nine-point circle passes through the midpoint of AH (S in this case), X, Y, P, M, the circumcenter of $XSYPM$ (i.e. J) is the nine-point center itself.

Now let's do the general case. Observe that with E (midpoint of AO) and N (midpoint of OH) both equidistant from PM the conclusion now becomes J lies on EN. We first need the following:

Lemma 3. $\frac{SJ}{AS} = \frac{AE}{EN}$

Proof: we have $AE = \frac{1}{2}AO$ and $EN = AF = \frac{1}{2}AH$, and $AS = SX = SY$, so we just have to prove that $\frac{SJ}{SX} = \frac{AO}{AH}$. First, it is well notice that SJ is the circumradius of SXY, so knowing that $\angle SXY = \angle XSY = 90° - \frac{1}{2}\angle XSY = 90° - \angle XAY = 90° - \angle BAC$ we have $SX = 2SJ\sin\angle SXY = 2SJ\cos\angle BAC$, yielding $\frac{SJ}{SX} = \frac{1}{2\cos\angle BAC}$. Let T be the reflection of H in M, then $HBMT$ is a parallelogram with $\angle ABT = \angle ABC + \angle CBT\ \angle ABC + \angle HCB = 90°$, and similarly $\angle ACT = 90°$. Therefore A, O, T collinear and with $AH \parallel OM$ we have $\frac{OM}{AH} = \frac{TH}{MT} = \frac{1}{2}$. We also know that $\frac{OM}{AO} = \frac{OM}{BO} = \cos\angle BOM = \cos\angle BAC$ so $\frac{AO}{AH} = \frac{AO}{2OM} = \frac{1}{2\cos\angle BAC} = \frac{SJ}{AS}$.

Lemma 4. $\angle(HA, AS) = \angle(SJ, AO)$. (This would also imply $\angle(AO, AS) = \angle(SJ, AH)$).

Proof: we use the well-known fact that the circumcenter and orthocenter of each triangle are the isogonal conjugates of each other. In particular, if ℓ is the perpendicular from A to XY then AS and ℓ are the images of each other in the reflection of the internal angle bisector of $\angle AXY$. This gives $\angle(AB, AS) = \angle(\ell, AC)$. Same goes for the relation between AH and AO, and therefore $\angle(AB, AH) = \angle(AO, AC)$. Moreover, $SJ \perp XY$ (since $SX = SY$ and $JX \perp JY$ we know that SJ must be the perpendicular bisector of XY). Therefore $SJ \parallel \ell$. Now we have $\angle(SJ, AO) = \angle(\ell, AO) = \angle(\ell, AC) + \angle(AC, AO) = \angle(AB, AS) + \angle(AH, AB) = \angle(AH, AS)$.

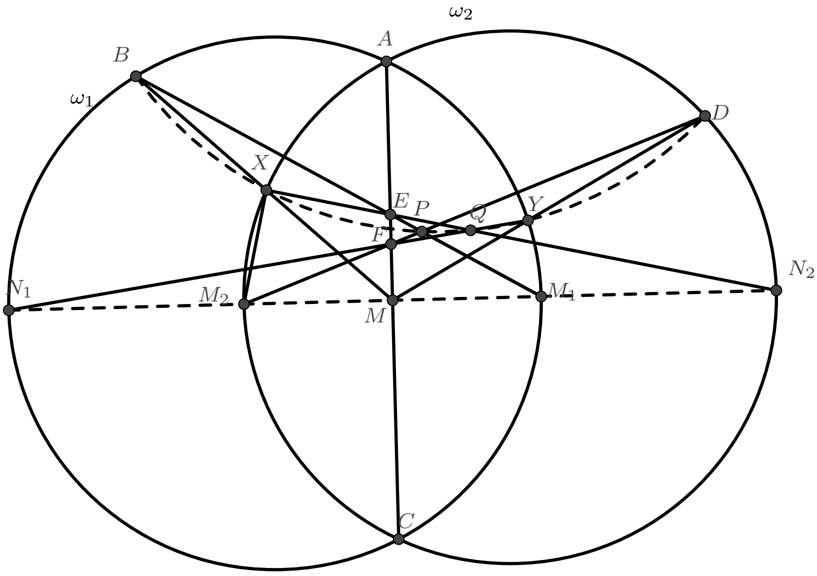
To complete the proof denote J' by the intersection of SJ and EN and we shall prove that $J = J'$ by proving that $SJ = SJ'$. From the first lemma it suffices to prove that $\frac{SJ'}{AS} = \frac{AE}{EN}$. Now $\frac{AS}{SE} = \frac{\sin\angle AES}{\sin\angle SAE} = \frac{\sin\angle AOH}{\sin\angle SAO}$ and $\frac{SJ'}{SE} = \frac{\sin\angle SEJ'}{\sin\angle EJ'S} = \frac{\sin\angle AFE}{\sin\angle EJ'S} = \frac{\sin\angle AHO}{\sin\angle EJ'S}$. Now, $\angle(EJ', J'S) = \angle(EN, SJ) = \angle(AH, SJ) = \angle(AO, AS)$ so the angles $\angle SEJ'$ and $\angle SAO$ are either equal or supplementary, hence $\sin\angle SEJ' = \sin\angle SAO$. Therefore, $\frac{SJ'}{AS} = \frac{SJ'}{SE} \div \frac{AS}{SE} = \frac{\sin\angle AHO}{\sin\angle EJ'S} \div \frac{\sin\angle AOH}{\sin\angle SAO} = \frac{\sin\angle AHO}{\sin\angle AOH} = \frac{AO}{AH} = \frac{EN}{AE}$, Q.E.D.

Solution 2. We present another proof of $NJ \perp BC$, this time involving the point D itself. Denote F as above (the midpoint of AH), X_H as the foot of perpendicular from C to AB, and Y_H the foor of perpendicular from B to AC.

We firstproe that triangles DXY and DX_HY_H are similar. Observe that with $ADXY$ cyclic, we have $\angle(DX, XY) = \angle(DA, AY) = \angle(DA, AC)$, $\angle(DY, XY) = \angle(DA, AX) = \angle(DA, AB)$ and finally $\angle(DX, DY) = \angle(AB, AC)$ for the similar reason. Now considering X_H and Y_H, we know that from $\angle AX_HH = \angle AY_HH = \angle ADH = 90°$ we know that A, X_H, Y_H, H, D concyclic. By the similar logic above, $\angle(DX_H, X_HY_H) = \angle(DA, AY_H) = \angle(DA, AC)$, $\angle(DY_H, X_HY_H) = \angle(DA, AX_H) = \angle(DA, AB)$ and finally $\angle(D_HX, DY_H) = \angle(AB, AC)$ for the similar reason. This gives $\angle(DX, XY) = \angle(DX_H, X_HY_H)$, $\angle(DY, XY) = \angle(DY_H, X_HY_H)$ and $\angle(DX_H, DY_H) = \angle(DX, DY)$, proving that the two triangles are similar. (A plausible question would be, since we are using directed angles, what if some two angles are not equal but supplementary? This wouldn't happen with the constraint that sum of angles of a triangle is 180°).

Now denote Φ as the spiral similarity sending triangle DX_HY_H to DXY, so $\Phi(X_H) = X$ and $\Phi(Y_H) = Y$. Notice that, Φ sends the corresponding circumcenters as well, thus $\Phi(F) = S$. This means that $\Phi(FX_HY_H) = SXY$, and this spiral similarity, again, maps their corresponding circumcenters too. But we have proven in solution 1 that the circumcenter of FX_HY_H is N, so $\Phi(N) = J$. With $\Phi(D) = D$, we have $\Phi(DX_HN) = \Phi(DXJ)$. This could, in turn, be manipulated to get the fact that not only DX_HN and DXJ are similar, DX_HX and DNJ are similar too (in a similar orientation). So $\angle(DN, NJ) = \angle(DX_H, X_HX) = \angle(DX_H, X_HA) = \angle(DH, HA) = \angle(OH, HA)$, the last two equality following from the fact that D lies on OH and D, X_H, A, H are concyclic. However we also have $\angle(DN, NJ) = \angle(OH, NJ)$ because D also lies on OH. Thus $\angle(OH, NJ) = \angle(OH, HA)$, meaning that $NJ \parallel HA$, i.e. HA and NJ are both perpendicular to BC.

G6 Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.



Solution. Let ω_1 be the circumcircle of ABC and ω_2 the circumcircle of ADC, then these two circles are symmetric w.r.t. AC. Also notice that BP passes through M_1, the midpoint of arc AC of ω_1 not containing B, and DP passes through M_2, the midpoint of arc AC of ω_2 not containing D.

We first start with a preliminary observation: X lies on ω_2 and Y lies on ω_1. W.L.O.G. for this section we assume that $AB \le AC$. Indeed, let X' be on BM satisfying $MX' \cdot MB = MA^2 = MC^2$. Then $\angle X'AC = \angle MBA$ and $\angle X'CA = \angle MBC$. Thus $\angle ADC = \angle ABC = \angle MBA + \angle MBC = \angle X'AC + \angle X'CA = \pi - \angle AX'C$, so X' lie on ω_2. In addition, let BM intersect ω_1 again at X'', then X' and X'' are symmetrical w.r.t. AC. Combining with the fact that M_1 and M_2 are also symmetrical w.r.t. AC (being the midpoint of arc) we have $X'M_2 = X''M_1$. Knowing that the two circles have the same radius further allows us to assert $\angle X'BP = \angle X''BM_1 = \angle X'DM_2 = \angle X'DP$, showing that D, B, P, X' cyclic hence $X' = X$. Similarly, Y lies on ω_1.

Next, let N_1 be diametrically opposite M_1 w.r.t. ω_1 and define similarly for N_2. We claim that XE passes through N_2 by claiming that XE is the internal angle bisector of $\angle AXC$. Indeed, by angle bisector theorem we have $\frac{AE}{EC} = \frac{AB}{BC}$. Invoking our X'' from the previous section (i.e. the other intersection of BM and ω_1) gives $AXCX''$ parallelogram. Now invoking a little bit more trigonometric bashing we have $1 = \frac{AM}{CM} = \frac{AB}{BC} \cdot \frac{\sin \angle ABM}{\sin \angle CBM} = \frac{AB}{BC} \cdot \frac{AX''}{CX''} = \frac{AB}{BC} \cdot \frac{CX}{AX}$, so $\frac{AX}{CX} = \frac{AB}{BC} = \frac{AE}{EC}$, and the conclusion follows by the angle bisector theorem. Analogously, YF passes through N_1.

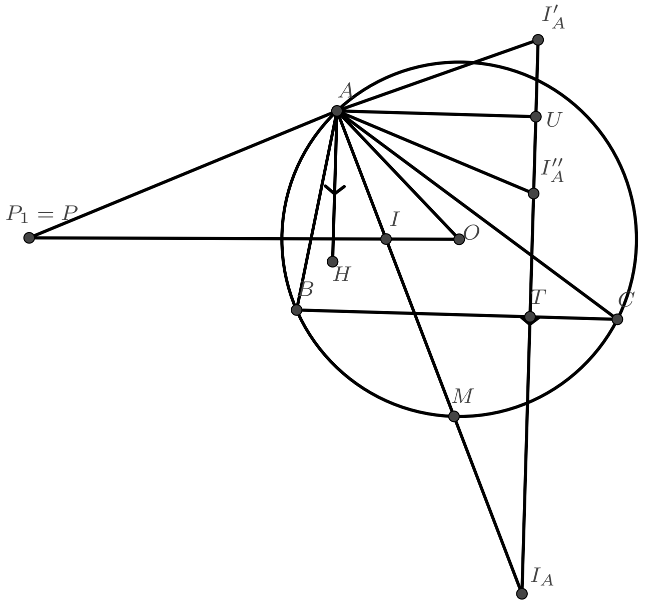
Finally, considering triangle PEF, and letting the perpendicular from P to reach AC at P_1 we have (considering signed length) $\frac{EP_1}{FP_1} = \frac{\cot \angle FEP}{\cot \angle EFP}$. Similarly if letting perpendcular from Q to reach AC at Q_1 we have $\frac{EQ_1}{FQ_1} = \frac{\cot \angle FEQ}{\cot \angle EFQ}$. Now $\cot \angle FEP = \cot \angle MEM_1 = \frac{MM_1}{EM}$, $\cot \angle EFP = \cot \angle MFM_2 = \frac{MM_2}{FM}$. Considering $MM_2 = MM_1$ we have $\frac{\cot \angle FEP}{\cot \angle EFP} = \frac{FM}{EM}$. Analogously, $\cot \angle FEQ = \cot \angle FEQ = \cot \angle MEN_2 = \frac{MN_2}{EM}$, and $\cot \angle EFQ = \cot \angle N_1FM = \frac{MN_1}{FM}$. Therefore we have $\frac{\cot \angle FEQ}{\cot \angle EFQ} = \frac{FM}{EM}$ since again it is not hard to verify that $MN_2 = MN_1$. (For signed convention we can say that $ME < 0$ if it's nearer to A than B, and > 0 otherwise). Therefore, $\frac{EP_1}{FP_1} = \frac{EQ_1}{FQ_1}$, so $P_1 \equiv Q_1$ and the two perpendicular lines coincide.

Note: We present another proof of $PQ \perp AC$ after establishing the claims that X on ω_2, Y on ω_1, XE passes through N_2 and YF passes through N_1. Now, consider the circle containing B, P, D, X, Y. Denoting Q_1 as the intersection of XE with this circle we have $\angle(PQ_1, XE) = \angle(PQ_1, XQ_1) = \angle(PD, XD) = \angle(M_2D, XD) = \angle(M_2N_2, XN_2) = \angle(M_2N_2, XE)$ (because P, D, M_2 collinear and X, M_2, D, N_2 lie on the same circle ω_2). Thus $PQ_1 \parallel M_2N_2$. In a similar way if Q_2 is the intersection of Y_F with the circle containing B, P, D, X, Y we also have $PQ_2 \parallel M_1N_1$. But with M_1, N_2, M_2, N_2 collinear we have: P, Q_1, Q_2 lying on the same line, and since all three lie on the circle BPD we have Q_1 coincides with Q_2 (P cannot be equal to either of them; otherwise $M_1 \equiv N_2$ or $M_2 \equiv N_1$ which is impossible). This gives $Q_1 \equiv Q_2 \equiv Q$ and since $PQ \parallel M_1N_1$ and $M_1N_1 \perp AC$, we have $PQ \perp AC$.

G7 Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

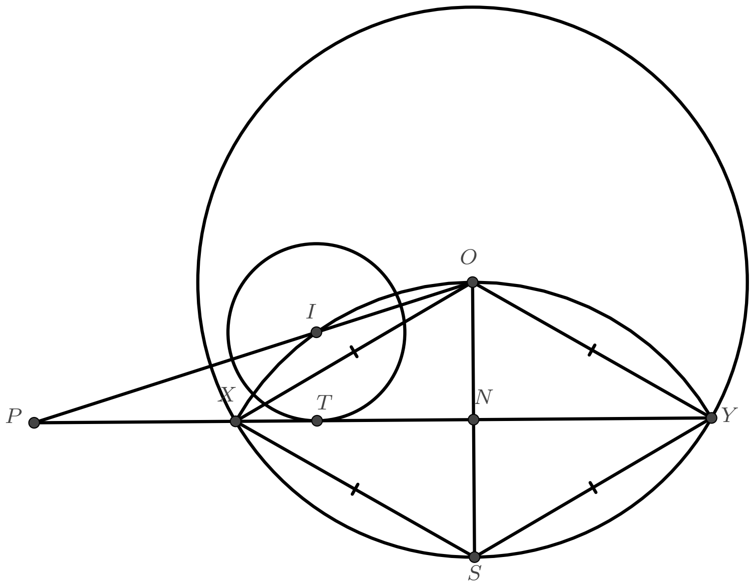


Solution.

(a) We first prove the following lemma: let I''_A the point on $I_AI'_A$ satisfying $AI''_A = AI'_A$, and $I''_A \neq I'_A$ unless AI'_A and $I_AI'_A$ are perpendicular to each other. Then triangles AOI and $I_AAI''_A$ are similar. Indeed, let H be the orthocenter of triangle ABC, then $AH \parallel I_AI''_A$. Also, $\angle OAI = \angle IAH = \angle AI_AI''_A$, so it suffices to prove that $\frac{AI}{AO} = \frac{I_AI''_A}{I_AA}$. To do so, define I_C has how I_A and I_B are defined. We first notice that I is the orthocenter of $I_AI_BI_C$, so I_AA is an altitude of triangle $I_AI_BI_C$. In addition, we know that triangles $I_AI_BI_C$ are I_ACB are similar with a similitude of $\frac{I_AB}{I_AI_B} = \cos\angle BI_AC$. Denote by T the altitude from I_A to BC, then by the similarities of the triangles mentioned above we have $\frac{I_AI'_A}{I_AA} = \frac{2I_AT}{I_AA} = 2\cos\angle BI_AC$. In addition, if we let U to be the perpendicular from A to $I_AI''_A$ we have I'_A and I''_A symmetric to each other w.r.t. U. Therefore, $I_AI''_A + I_AI'_A = 2I_AU = 2I_AA\cos\angle AI_AI'_A = 2I_AA\cos\angle AOI$. So $\frac{I_AI''_A}{I_AA} = \frac{2I_AA\cos\angle AOI - I_AI'_A}{I_AA}$ $= \frac{2I_AA\cos\angle AOI - 2I_AA\cos\angle BI_AC}{I_AA} = 2(\cos\angle AOI - \cos\angle BI_AC)$. On the other hand, denoting M as the other intersection of AI and BC we have $MB = MC = MI = 2AO\cos BAM$ (because AO is the circumradius of triangle ABC), and $AM = AO\sin\angle ABM$, so $\frac{AI}{AO} = \frac{AM - MI}{AO} = \frac{2AO\cos\angle ABM - 2AO\cos BAM}{AO} = 2(\cos\angle ABM - \cos\angle BAM) = 2(\cos\angle AOI - \cos\angle BI_AC)$ (the equality $\angle ABM = \angle AOI$ and $\angle BAM = \angle BI_AC$ can be established via angle chasing). This establishes the desired equality.

Now denote P_1 on OI such that I lies between O and P_1 and $OI \cdot OP_1 = AO$. Then, triangles AOI anf AP_1O are similar. Therefore, by the fact that $\triangle AOI \sim \triangle I_AAI''_A$ $\angle IAP_1 = \angle OAP_1 - \angle OAI = \angle AIO - \angle OAI = \angle I_AI''_AA - \angle OAI = 180^\circ - \angle I_AI'_AA - \angle HAI$ $= 180^\circ - \angle I_AI'_AA - \angle AI_AI'_A = \angle I'_AAI_A$. This means that l_A passes through P_1 and similarly, l_B passes through P_1. Hence $P_1 = P$ and lies on OI.

(b)



We first establish a relation between the length OI and the inradius r. Let the circumradius be $R = AO = BO = CO$. Using fresh new label than (a), we denote M as the second intersection of AI and the circumcircle of ABC. This M is also the midpoint of arc BC not containing A. Now, $MI = MB = MC = 2R\sin\angle BAI = 2R\sin\frac{\angle A}{2}$, $AM = AB\frac{\sin\angle ABI}{\angle AIB} = 2R\sin\angle C\frac{\sin\frac{1}{2}B}{\sin(90^\circ+\frac{1}{2}C)} = 2R(2\sin\angle\frac{\angle C}{2}\cos\angle\frac{\angle C}{2})\frac{\sin\frac{1}{2}\angle B}{\cos\frac{1}{2}\angle C}$ $= 4R\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$ so $MI\cdot AI = 8R^2\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$. Meanwhile, letting D be the point of tangency of the incircle to BC we have $r = ID = IB\sin\angle IBC = BC\frac{\sin\angle ICB}{\sin\angle CIB}\sin\frac{1}{2}\angle B = 2R\sin A\frac{\sin\frac{1}{2}C}{\sin(90^\circ+\frac{1}{2}A)}\sin\frac{1}{2}\angle B = 2R(2\sin\angle\frac{\angle A}{2}\cos\angle\frac{\angle A}{2})\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}A)}\sin\frac{1}{2}\angle B$ $= 4R\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$. Considering the fact that $-MI\cdot AI$ is the power of point of I w.r.t. the circumcircle we have $OI^2 = R^2 - MI\cdot AI = R^2 - 8R^2\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$, or $\frac{OI^2}{R^2} = 1 - 8\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$ while $\frac{r}{R} = 4\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$ so $\frac{OI^2}{R^2} = 1 - \frac{2r}{R}$, or $\frac{r}{R} = \frac{1}{2} - \frac{OI^2}{2R^2}$, or $r = \frac{1}{2}R(1 - \frac{OI^2}{R^2})$ (this is actually a well-known identity, the purpose of including the proof is to show the power of trigonometry in solving problems).

Now, let T be the tangency point of the incircle to XY, and N be the midpoint of XY. Keeping in mind that $OP\cdot OI = R^2$, we now have $\frac{PI}{OP} = 1 - \frac{OI}{OP} = 1 - \frac{OI}{R^2\div OI} = 1 - \frac{OI^2}{R^2}$. Therefore $ON = IT\frac{PO}{PI} = r\frac{1}{1-\frac{OI^2}{R^2}} = \frac{1}{2}R(1 - \frac{OI^2}{R^2})\frac{1}{1-\frac{OI^2}{R^2}} = \frac{1}{2}R$. Moreover, letting S be the midpoint of arc XY lying on the opposite side as I w.r.t. XY we have O, N, S collinear, $ON = NS$, and $ON \perp XY$. Therefore, $OX = OY = OS = XS = YS$, yielding OXS and OYS both equilateral and $\angle XOY = 60^\circ + 60^\circ = 120^\circ$. Additionally, $PI\cdot PO = PO^2 - (IO\cdot OP) = PO^2 - R^2$, which is the power of point of P w.r.t. the circumcircle. This, in turn, is equal to the value $PX\cdot PY$, so $IOYX$ is cyclic. Thus $\angle XIY = \angle XOY = 120^\circ$.

4 Number Theory

N1 For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Thoughts. The first thing to try is $c \cdot 10^k$ for $c \leq 9$, of course, because $S(c \cdot 10^k) = c$ and it's easier to manipulate. (Well I first tried $n = 10^k$ that gives $S(P(10^k)) = P(1)$, a constant). The next thing is that, if we space k big enough, the numbers are likely to be in the form $(a_k c^k)(0 \cdots 0)(a_{k-1}c^{k-1}(0 \cdots 0) \cdots (0 \cdots 0)a_0c^0$, provided $a_i \geq 0$ for all i. But we cannot simply make that assumption! Fortunately, the fact that $a_i < 0$ will cause tons of trailing 9's, which will be good for a contradiction. Having that in mind, we know that $P(c)$ is the sum of $a_kc^k, \cdots, a_0c^0$ and the sum of digits (as shown above) is $S(a_kc^k) + \cdots + S(a_0c^0)$. With the fact that $S(i) \leq i$ with equality iff $i \leq 9$, it's no longer difficult to complete the solution.

Solution. The answer is the constant polynomial $P(x) = c$ where $c \in \{1, 2, \cdots, 9\}$, or the identity polynomial $P(x) = x$. In the first case we have $S(P(n)) = S(c) = c = P(\text{anything}) = P(S(n))$, and in the second case $S(P(n)) = S(n) = P(S(n))$.

Now let $P(x) = \sum_{i=0}^{k} a_i x^i$. The first thing to do is to prove that $a_i \geq 0$, $\forall i \geq 0$. Indeed, let $n = c \cdot 10^m$ $(1 \leq c \leq 9)$ then we have $P(c) = P(S(n)) = S(P(n)) = S(P(c \cdot 10^m))$. Lt d be such that $10^d > max\{|a_i(9^i)| : i \in [0, k]\}$ For $m > d$ satisfying we have $P(c \cdot 10^m) = \sum_{i=0}^{k} a_i(c^i)(10^{mi})$. Let $a_j < 0$ for some j. Now notice that $\sum_{i=0}^{j-1} a_i(c^i)(10^{mi}) < \sum_{i=0}^{j-1} 10^d(10^{mi}) < \sum_{i=0}^{mj-m+d} 10^i < 10^{mj-m+d+1} \leq 10^{mj}$ so $P(c \cdot 10^m) = \sum_{i=0}^{k} a_i(c^i)(10^{mi}) = \sum_{i=0}^{j-1} a_i(c^i)(10^{mi}) + a_j(c^j)(10^{mi}) + \sum_{i=j+1}^{k} a_i(c^i)(10^{mi}) < 10^{mj} + a_j(c^j)(10^{mj}) + \sum_{i=j+1}^{k} a_i(c^i)(10^{mi}) \leq \sum_{i=j+1}^{k} a_i(c^i)(10^{mi})$ (the first inequality is due to our choice of m). As $P(x) > 0$ for all $x \geq 2016$, the leading coefficient is positive so we can choose j such that there exists an $l \geq 1$ satisfying $c_{j+l} > 0$ and $c_{j+1}, c_{j+2}, \cdots c_{j+l-1} = 0$. In a similar way we can also deduce that $P(c \cdot 10^m) > \sum_{i=j+l}^{k} a_i(c^i)(10^{mi}) - 10^{m(j+l)-m+d+1}$

Combining the inequalities and by assuming that $c_{j+1}, c_{j+2}, \cdots c_{j+l-1} = 0$, $c_j < 0$ and $c_{j+l} > 0$ we have

$$\sum_{i=j+l}^{k} a_i(c^i)(10^{mi}) - 10^{m(j+l)-m+d+1} < P(c \cdot 10^m) < \sum_{i=j+1}^{k} a_i(c^i)(10^{mi}) = \sum_{i=j+l}^{k} a_i(c^i)(10^{mi})$$

This means that there will be at least $m - d$ consecutive 9's as digit, meaning that $S(P(c \cdot 10^m))$ is at least $9(m - d)$. It follows that $P(S(c \cdot 10^m)) = P(c) \geq 9(m - d)$ for all sufficiently large m. However, this is contradicted by the fact that $\lim_{m \to \infty} 9(m-d) \to \infty$. Hence $c_i \geq 0$ for all i.

Since $a_i(c^i)(10^{ni}) < 10^{(n+1)i}$ (because $a_i(c^i) < 10^n$ by our choice of n), the number $P(c \cdot 10^n)$ are in the form of $(a_kc^k)(0 \cdots 0)(a_{k-1}c^{k-1})(0 \cdots 0) \cdots (0 \cdots 0)(a_0c^0)$ when laid in decimal form. Therefore $S(P(c \cdot 10^n)) = \sum_{i=0}^{k} S(a_i(c^i))$, and $P(S(c \cdot 10^n)) = P(c) = \sum_{i=0}^{k} a_i(c^i)$. Knowing that $S(x) \leq x$ with equality holds if and only if $0 \leq x \leq 9$ (indeed, if $k = \sum_{i=0}^{k} b_i(10^i)$ then $S(k) = \sum_{i=0}^{k} b_i$, so $k - S(k) = \sum_{i=0}^{k} b_i(10^i - 1) \geq 0$, with equality holds iff $b_i = 0$ for $i \geq 1$,) we have $a_i(c^i) \leq 9$ for all $c \in \{0, 1, \cdots 9\}$. This means $k \leq 1$ (if we assume that $a_k > 0$). If $k = 0$ then we get $a_0 \leq 9$, yielding the constant solution. If $k = 1$, then $9a_1 \leq 9$ (when $c = 9$) and $a_1 = 1$, yielding $P(x) = x + c$ for some constant c (and since $c = a_0$ we have $c = a_0 \leq 9$ too). This entails $S(P(n)) = S(n + c)$ and $P(S(n)) = S(n) + c$ for all $n \geq 2016$, and letting $n = 10^d - 1$ we have $S(n) = 9d$, and for $c \geq 1$, $S(n + c) = S(10^k - 1 + c) = c$, which doesn't hold for $d = 5$. Therefore $c = 0$ and we get the identity polynomial.

N2 Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Solution. The answer is 2 and all composite numbers. Let $m = 10n$, with $m = 3^y \cdot \prod_{i=1}^{k} p_i^{a_i} \cdot \prod_{i=1}^{l} q_i^{b_i}$ with $p_i \equiv 1 \pmod 3$ and $q_i \equiv 2 \pmod 3$. Notice that $\tau(m) = (y+1) \cdot \prod_{i=1}^{k}(a_i+1) \cdot \prod_{i=1}^{l}(b_i+1)$.

Now we want to investigate all the divisors that is congruent to 1 mod 3, observe that such divisors fulfill $\prod_{i=1}^{k} p_i^{c_i} \cdot \prod_{i=1}^{l} q_i^{d_i}$ with $c_i \le a_i$, $d_i \le b_i$ and $\sum_{i=1}^{l} d_i$ even. We proceed with the following claim: the number of combinations $(d_1, d_2, \cdots, d_l)$ satisfying $\sum_{i=1}^{l} d_i$ even and $d_i \le b_i$ is $\lfloor \frac{\prod_{i=1}^{l}(b_i+1)}{2} \rfloor$.

Case 1. b_i odd for some i, and w.l.o.g. let this i be l. Now, let x be the number of combinations $(d_1, d_2, \cdots, d_{l-1})$ $(d_i \le b_i)$ satisfying $\sum_{i=1}^{l-1} d_i$ even, and z be the number of combinations with corresponding odd sums. Considering $d_i = \{0, 2, \cdots, b_l - 1\}$ and $d_i = \{1, 3, \ldots, b_l\}$ we have: the number of combinations $(d_1, d_2, \cdots, d_{l-1})$ $(d_i \le b_i)$ satisfying $\sum_{i=1}^{l} d_i$ even is $x + z + x + z + \cdots + x + z = (x+z) \cdot \frac{b_i+1}{2}$, and similarly $z + x + + \cdots + z + x = (x+z) \cdot \frac{b_i+1}{2}$ for odd-sum combinations. Therefore there is equally many odd and even sum combinations, and we are done.

Case 2. Now let b_i even for all i. Let O be number of combinations with $\sum_{i=1}^{l} d_i$ odd and E be combinations with $\sum_{i=1}^{l} d_i$ even. The claim is $E - O = 1$. We induct on l. Base case $l = 0$ yield 1 combination for even sum and 0 combination for odd sum, vacuously. Now let $l = k$ for some k and we have O' as the number of combinations $(d_1, d_2, \cdots, d_k)$ with $\sum_{i=1}^{k} d_i$ odd, and E' as the number of combinations with $\sum_{i=1}^{l} d_i$ even. Now that b_{k+1} is even, using the logic above the number of even combination is $E' + O' + E' + O' + \cdots + E' = E'(\frac{b_{k+1}}{2} + 1) + O'(\frac{b_{k+1}}{2})$, and similarly the number of combinations yielding odd sum is $O'(\frac{b_{k+1}}{2} + 1) + E'(\frac{b_{k+1}}{2})$. This yields $E - O = E' - O'$ and by induction hypothesis this number is 1, so we are done.

Summing above, $\tau_1(m) = \prod_{i=1}^{k}(a_i+1)\lfloor \frac{\prod_{i=1}^{l}(b_i+1)}{2} \rfloor$, so the ratio now becomes $(y+1)\frac{\prod_{i=1}^{l}(b_i+1)}{\lfloor \frac{\prod_{i=1}^{l}(b_i+1)}{2} \rfloor}$. Equivalently, $2(y+1)$ when b_i odd for some b_i, or $(y+1)\frac{2k+1}{k}$ otherwise (where $2k+1 = \prod_{i=1}^{l}(b_i+1)$ here). The first case yields that the ratio must be even; in the second case, we have $\gcd(2k+1, k) = 1$ so $k | y+1$. In other words, the ratio must be divisible by $2k+1$. Notice, also, that $l \geq 2$ ($m = 10n$ contains prime factors 2 and 5) so $2k+1 = \prod_{i=1}^{l}(b_i+1)$ must be composite. So our integer ratio cannot be an odd prime.

It remains to show that any even or composite numbers work. For even numbers $2k$, simply take $10 \cdot 3^{k-1}$ and by our proof the ratio is $2k$. For odd composite number xz with $x, z \geq 3$, take $m = 2^{x-1}5^{z-1}$.

N3/IMO 4 A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

Thoughts. This is problem requiring no more than number experimentation, and it's immediate to see why $P(n)$ and $P(n+1)$ are relatively prime (otherwise the answer is 2 and it's too trivial to be on the IMO), which directly gives away the fact that $b = 3$ doesn't work either. How about $P(n)$ and $P(n+2)$? The investigation of the first case also shows that $b = 4$ fails, and further checking on $P(n)$ and $P(n+3)$ shows how $b = 5$ fails too. Finally, for $b = 6$ the relation between $\gcd(P(n), P(n+2))$, $\gcd(P(n), P(n+3))$ and $\gcd(P(n), P(n+4))$ will be good to construct an example (which cannot be determined by brute force since it's rather big!!!) Ps: basic combinatorics skill needed to construct an example.

Solution. The answer is $b = 6$. Observe that this solution works because the set $\{P(197), P(198), P(199), P(200), P(201), P(202)\}$ has $P(199) \equiv P(202) \equiv P(1) = 3 \equiv 0 \pmod 3$, $P(198) \equiv P(2) = 7 \equiv 0 \equiv 21 = P(4) \equiv P(200) \pmod 7$, $P(197) \equiv P(7) = 57 \equiv 0 \equiv 133 = P(11) \equiv P(201) \pmod{19}$.

First, notice that $P(n) - P(n-1) = n^2 + n + 1 - (n^2 - n - 1) = 2n$, and knowing that $n^2 + n + 1 \equiv n + n + 1 = 2n + 1 \equiv 1 \pmod 2$, we know that if $p|P(n)$ and $p|2n$ then $p|n$ (since $P(n)$ is relatively prime to 2), and consequently $p|n^2 + n$ and $p|1$, showing that $P(n)$ and $P(n-1)$ are relatively prime. This means, $b = 2$ fails, and $b = 3$ fails too sine $P(a+1)$ and $P(a+3)$ are both relatively prime to $P(a+2)$. (We will use profusely the fact that $P(a)$ and $P(a+1)$ cannot have any common prime factor throughout the solution).

Now, for $b = 4$ and $b = 5$ our strategy is to determine an upper bound for $\gcd(P(n), P(n+c))$ for $c = 2, 3$. Observe that $P(n+c) - P(n) = 2cn + c^2 + c = c(2n + c + 1)$. For $c = 2$ this is the same as $2(2n+3)$. If $p|P(n+2)$ and $p|P(n)$ then $p|2(2n+3)$, and therefore $p|2n+3$ with P being odd at all times. This entails $2n \equiv -3 \pmod p$, and $0 \equiv 4P(n) = 4n^+4n + 1 = (2n)^2 + 2(2n) + 1 \equiv (-3)^2 - 3 + 1 = 7 \pmod 7$. Hence $p = 7$ and $n \equiv 2 \pmod 7$. Now for $b = 4$, knowing that $P(a+2)$ is relatively prime with $P(a+1)$ and $P(a+3)$ it must have a common prime factor with $P(a+4)$, and by the previous step this prime factor has to be 7. Similarly $P(a+1)$ and $P(a+3)$ must both be divisible by 7. This means $P(a+1), P(a+2), P(a+3), P(a+4)$ are all divisible by 7 for some a, contradicting that any two neighbouring elements are coprime.

Finally for $b = 5$ we investigate $c = 3$ as in the previous paragraph. Now $3(2n+3+1) = 3(2n+4) = 3(2)(n+2)$. If a prime p satisfies $p|P(n)$ and $p|P(n+3)$ simultaneously then either $p = 3$ or $p|n+2$ (again p must be relatively prime to 2 so this can be easily factored out). In the second case we have $n \equiv = 2 \pmod p$, so $P(n) \equiv P(-2) = 4 - 2 + 1 = 3 \equiv \pmod p$, forcing $p = 3$ (no choice!) Thus viewing the set $\{P(a+1), \cdots, P(a+5)\}$ we know that $P(a+3)$ must have a common factor with $P(a+1)$ or $P(a+5)$, and by previous paragraph this common factor has to be 7. Thus neither of $P(a+2)$ nor $P(a+4)$ can be divisible by 7, and they cannot have common prime factor (again by previous paragraph). This entails $P(a+1)$ and $P(a+4)$ must have common factor, and by what we established earlier this factor must be 3. Similarly, $P(a+2)$ and $P(a+5)$ must both be divisible by 3. However, $P(a+1)$ and $P(a+2)$ are both divisible by 3, contradiction.

N4 Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l | k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

Solution. We split our solution into two cases:

Case 1. $l \leq k$. Now from $n^k + mn^l + 1 | n^{k+l} - 1$, and from the fact that $(n^l - 1)(n^k + mn^l + 1) = n^{k+l} + mn^{2l} + n^l - n^k - mn^l - 1$ we have $(n^{k+l} + mn^{2l} + n^l - n^k - mn^l - 1) - (n^{k+l} - 1) = mn^{2l} + n^l - n^k - mn^l = n^l(mn^l + 1 - n^{k-l} - m) = n^l(m(n^l - 1) - (n^{k-l} - 1))$ is divisible by $n^k + mn^l + 1$. Knowing that $\gcd(n, n^k + mn^l + 1) = \gcd(n, 1) = 1$ we have $\gcd(n^l, n^k + mn^l + 1) = 1$ so $m(n^l - 1) - (n^{k-l} - 1)$ is itself divisible by $n^k + mn^l + 1$. Now, $m(n^l - 1) < mn^l < n^k + mn^l + 1$ and $n^{k-l} - 1 \leq n^k - 1 < n^k + mn^l + 1$, meaning that $0 < \frac{m(n^l-1)}{n^k+mn^l+1}, \frac{(n^{k-l}-1)}{n^k+mn^l+1} < 1$. Therefore $|\frac{m(n^l-1)-(n^{k-l}-1)}{n^k+mn^l+1}| < 1$, and therefore has to be 0. We thus have $m(n^l - 1) = (n^{k-l} - 1)$ and since $n > 1$, $m = \frac{n^{k-l}-1}{n^l-1}$. Let $k - l = cl + d$ with $0 \leq d < l$, then $n^{k-l} = n^{cl} \cdot n^d \equiv (n^l)^c \cdot n^d \equiv 1 \cdot n^d = n^d \pmod{n^l - 1}$, and from $n^d < n^l$ we have $n^d \not\equiv 1 \pmod{n^l - 1}$ unless $d = 0$. Therefore $l | k - l$, or $l | k$.

Case 2. $l \geq k$. Similar to above we have $(n^k - 1)(n^k + mn^l + 1) - (n^{k+l} - 1) = n^{2k} + mn^{k+l} + n^k - n^k - mn^l - 1 - (n^{k+l} - 1) = n^{2k} + mn^{k+l} - mn^l - n^{k+l} = n^k(n^k - mn^{l-k} + (m-1)n^l)$ is divisible by $n^k + mn^l + 1$. Again by the logic above, $\gcd(n^k, n^k + mn^l + 1) = 1$, which very well means that $n^k + mn^l + 1 | n^k - mn^{l-k} + (m-1)n^l$. Again we have $n^k + (m-1)n^l < n^k + mn^l + 1$ and $mn^{l-k} < n^k + mn^l + 1$ so by the logic above, again, $n^k - mn^{l-k} + (m-1)n^l = 0$. Rearranging the terms give: $m = \frac{n^l - n^k}{n^l - n^{l-k}}$. Now, if $m \geq 2$, then we have $n^l - n^k \geq 2n^l - 2n^{l-k}$, or $2n^{l-k} \geq n^l + n^k > n^l = n^k(n^{l-k})$, or $2 > n^k$, forcing $n = 1$ (contradiction since $k \geq 1$). Thus $m = 1$ (m must be positive) and we have $k = l - k$, or $l = 2k$.

N5 Let a be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let A be the set of positive integers k for which the equation admits a solution with $x > \sqrt{a}$, and let B be the set of positive integers for which the equation admits a solution with $0 \le x < \sqrt{a}$. Show that $A = B$.

Solution. To show that $k \in A \to k \in B$, let $x > \sqrt{a}$ for some x satisfying the equation. It follows that $|y| < |x|$. Denote $y = x - c$ and we have $a = x^2 - k(x^2 - y^2) = x^2 - kc(2x - c) = x^2 - 2kcx + kc^2$. Let $x_1 = x - 2kc$ and $y_1 = x_1 + c$ and we have $\frac{x_1^2 - a}{x_1^2 - y_1^2} = \frac{(x - 2kc)^2 - (x^2 - kc(2x - c))}{(x_1 - y_1)(x_1 + y_1)} = \frac{-2kc(2x - 2kc) + kc(2x - c)}{-c(2(x - 2kc) + c)} = \frac{-kc(4x - 4kc - 2x + c)}{-c(2x - 4kc) + c)} = k$. This means k admits (x_1, y_1) as well, and from $x_1 = y_1 + c < y_1$ we have $x_1 < \sqrt{a}$. Also notice that $x \ge 2kc$ because... so $x_1 \ge 0$. Therefore $k \in B$ too.

Conversely, we want to show that $k \in B \to k \in A$. Let $x < \sqrt{a}$ for some x satisfying the equation. It follows that $|y| > |x|$. Denote $y = x + c$ and we have $a = x^2 - k(x^2 - y^2) = x^2 - k(-c)(2x + c) = x^2 + 2kcx - kc^2$. Let $x_2 = x + 2kc$ and $y_2 = x_1 0c$ and we have $\frac{x_2^2 - a}{x_1^2 - y_2^2} = \frac{(x + 2kc)^2 - (x^2 + kc(2x + c))}{(x_2 - y_2)(x_2 + y_2)} = \frac{2kc(2x + 2kc) - kc(2x + c)}{c(2(x + 2kc) - c)} = \frac{kc(4x + 4kc - 2x - c)}{c(2x + 4kc) - c)} = k$. This means k admits (x_2, y_2) as well, and from $x_2 = y_2 + c > y_2$ we have $x_1 > \sqrt{a}$. Therefore $k \in A$ too.

N6 Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Solution. The only function is $f(x) = x^2, \forall x \in \mathbb{N}$. In fact, $mf(m) + nf(n) = m^3 + n^3 = (m+n)(m^2 - mn + n^2) = (m+n)(f(m) - mn + n^2)$.

Substituting $m = n = 1$ gives $2f(1)-1|2f(1)$, so $2f(1)-1 = \pm 1$. Since $f(1) > 0$ ($f(1) \in \mathbb{N}$) we have $f(1) = 1$. Next, letting $n = 1$ gives $f(m) - (m-1)|mf(m) + 1 = m(f(m) - (m-1)) + m^2 - m + 1$, so with $f(m) - (m-1)|m^2 - m + 1$ and $m^2 - m + 1 = (m - \frac{1}{2})^2 + \frac{3}{4} > 0$ we have $|f(m) - (m-1)| \leq m^2 - m + 1$ and $f(m) \leq m^2$.

The next step is to show that $f(p) = p^2$ for all sufficiently large prime p. Substituting $m = n = p$ gives $2f(p) - p^2|2pf(p) = p(2f(p) - p^2) + p^3$, so $2f(p) - p^2|p^3$ and from $f(p) \leq p^2$ we have $2f(p) - p^2 \in \{p^2, p, 1, -1, -p\}$ (again it this value cannot be $-p^2$ or lower because $f(p) > 0$. Therefore $f(p) \in \{p^2, \frac{p^2+p}{2}, \frac{p^2+1}{2}, \frac{p^2-1}{2}, \frac{p^2-p}{2}\}$. Now we check $n = 1, m = p$ again and we have (from above) $f(p) - (p-1)|p^2 - p + 1$. We investigate the following cases:

(a) $f(p) = \frac{p^2+p}{2}$, then $\frac{p^2+p}{2} - (p-1)|p^2 - p + 1 = 2(\frac{p^2+p}{2} - (p-1)) - 1$, so $\frac{p^2+p}{2} - (p-1) \leq 1$, which doesn't hold for $p \geq 2$.

(b) $f(p) = \frac{p^2+1}{2}$, then $\frac{p^2+1}{2} - (p-1)|p^2 - p + 1 = 2(\frac{p^2+1}{2} - (p-1)) + p - 2$, which means $\frac{p^2+1}{2} - (p-1) \leq p - 2$, not true for $p \geq 3$.

(c) $f(p) = \frac{p^2-1}{2}$, then $\frac{p^2-1}{2} - (p-1)|p^2 - p + 1 = 2(\frac{p^2-1}{2} - (p-1)) + p$, meaning $\frac{p^2+1}{2} - (p-1) \leq p$, not true for $p \geq 3$.

(d) $f(p) = \frac{p^2-p}{2}$, then $\frac{p^2-p}{2} - (p-1)|p^2 - p + 1$. Observe that $2(\frac{p^2-p}{2} - (p-1)) = p(p-1) - 2(p-1) = (p-1)(p-2)$, so $p - 1|2(p^2 - p + 1)$. Now $2(p^2 - p + 1) \equiv 2(1^2 - 1 + 1) = 2 \pmod{p-1}$, so $p - 1 \leq 2$¡ or $p \leq 3$.

We therefore know that all four cases cannot hold for $p \geq 5$, so $f(p) = p^2$ for $p \geq 5$.

Now, let m be arbitrary and let $n = p$ for some prime p we have $f(m) + p^2 - mp|mf(m) + p^3 = m(f(m) + p^2 - mp) + p^3 - mp^2 + m^2p = m(f(m) + p^2 - mp) + p(p^2 - pm + m^2)$. Consider the ratio $\dfrac{p(p^2 - pm + m^2)}{f(m) + p^2 - mp} = p(1 + \dfrac{m^2 - f(m)}{f(m) + p^2 - mp})$, and therefore $\dfrac{p(m^2 - f(m))}{f(m) + p^2 - mp}$ must be an integer. Choosing any $p > f(m)$ gives $p \nmid f(m)$, and hence $p \nmid f(m) + p^2 - mp$, hence p and $f(m) + p^2 - mp$ are relatively prime. Therefore $\dfrac{m^2 - f(m)}{f(m) + p^2 - mp}$ is itself an integer, and with $f(m) + p^2 - mp$ approaching infinity as p approaching infinity we know that $f(m) - m^2$ must be zero. (Remember, there are infinitely many primes).