Solutions to Tournament of Towns, Spring 2017, Senior

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O-Level

3. Each cell of a square 1000×1000 table contains a number. It is known that the sum of the numbers in each rectangle of area S with sides along the borders of cells, contained in the table, is the same. Find all values of S which guarantee that all the numbers in the table are equal.

   Answer. $S = 1$ is the only possibility.

   Solution. The case $S = 1$ is clear. We now show that for any $S > 1$ we can construct a counterexample.

   Let p be any prime divisor of S. Denote $a_{i,j}$ the squares of table, $1 \leq i, j \leq 1000$. Let $a_{i,j}$ be filled with $i + j \pmod{p}$, taking values $0, 1, \cdots, p-1$. Then any rectangle with area S will have either its row or column size divisible by p. In the first case (row size divisible by p), for each column there's equal representation of $0, 1, \cdots, p-1$; for the second case (column size divisible by p), for each row there's equal representation of $0, 1, \cdots, p-1$. It therefore follows that the average of the squares in S is $\frac{p-1}{2}$, independent of S.

A-Level

1. On the plane, there is a triangle and ten lines. Every line is equidistant from two of the triangle's vertices. Prove that either at least two of these lines are parallel or at least three of them pass through a common point.

   Solution. Let A, B, C be the vertices of a triangle. We show that if no two lines are parallel and no three lines are concurrent then there can be at most 9 lines.

   Consider any line ℓ equidistant to A and B. If A and B are on the same side of AB, then $AB \parallel \ell$. Otherwise, AB passes through midpoint of AB. Thus we see that there's at most one line parallel to AB and two lines through the midpoint of AB. Doing the same analysis for BC and AC, we get at most $3 \times 3 = 9$ lines.

2. From given positive numbers, the following infinite sequence is defined: a_1 is the sum of all original numbers, a_2 is the sum of the squares of all original numbers, a_3 is the sum of the cubes of all original numbers, and so on (a_k is the sum of the k-th powers of all original numbers).

   (a) Can it happen that $a_1 > a_2 > a_3 > a_4 > a_5$ and $a_5 < a_6 < a_7 < \ldots$?

   (b) Can it happen that $a_1 < a_2 < a_3 < a_4 < a_5$ and $a_5 > a_6 > a_7 > \ldots$?

   Answer. Yes for (a), no for (b).

   Solution. For (a), let the numbers given be k copies of 2 and ℓ copies of 0.5. Then $a_n = k2^n + \ell \cdot \frac{1}{2^n}$ and therefore

$$a_{n+1} - a_n = k(2^{n+1} - 2^n) + \ell(\frac{1}{2^{n+1}} - \frac{1}{2^n}) = k \cdot 2^n - \ell \cdot \frac{1}{2^{n+1}}$$

which is strictly increasing. Now choose k, ℓ such that

$$2^4 k - \frac{1}{2^5}\ell < 0 < 2^5 k - \frac{1}{2^6}\ell$$

or, in other words, $2^9 < \frac{\ell}{k} < 2^{11}$. We can, for example, take $k = 1$ and $\ell = 2^{10}$.

For (b), we see that the sequence $\{a_n\}_{n\geq 1}$ is bounded. Given that $\{a^n\}_{n\geq 1}$ is unbounded for any given $a > 1$, we have all numbers given as ≤ 1. But then $\{a^n\}_{n\geq 1}$ is nonincreasing for any $a \leq 1$, so we cannot have $a_1 > a_2$.

5. In a triangle ABC with $\angle A = 45°$, AM is a median. Line b is symmetrical to line AM with respect to altitude BB_1, and line c is symmetrical to line AM with respect to altitude CC_1. Lines b and c meet at point X. Prove that $AX = BC$.

   Solution. Let H, O be the orthocenter and circumcenter of triangle ABC, respectively. Then HM and AO intersect at a point Z diametrically opposite A on circumcircle of ABC, with M being midpoint of ZH and O midpoint of AZ, so $AH \parallel OM$, with $AH = 2OM$. The 45° condition also implies $OM = MA = \frac{1}{2}AB$ so $AH = AB$. We henceforth turn our attention to proving $AX = AH$. If $AB = AC$ we're done since $X = H$ here so we may assume, say, $AB > AC$.

   Next, ABC and AC_1B_1 are similar. Thus AM being the median w.r.t. triangle ABC means AM is a symmedian w.r.t. triangle AB_1C_1. Additionally, MB_1 and MC_1 are the points of tangency to circle AB_1C_1.

   Let BB_1, CC_1 intersect AM at E, F, respectively. Then by the reflection condition and that $\angle EHF = 45°$ (i.e. acute), H is the excenter of triangle XEF and $\angle EXF = 90°$, with $\angle EXH = \angle FXH = 45°$. Let Y be the intersection of EF and HX, then we have the following equalities:

$$\frac{HE}{HF}\frac{\sin\angle EHX}{\sin\angle FHX} = \frac{EY}{FY} = \frac{XE}{XF}\frac{\sin\angle EXH}{\sin\angle FXH}$$

$$\frac{HE}{HF}\frac{\sin\angle EHA}{\sin\angle FHA} = \frac{EA}{FA} = \frac{XE}{XF}\frac{\sin\angle EXA}{\sin\angle FXA}$$

   Thus combining these two gives

$$\frac{\sin\angle EXA}{\sin\angle FXA} = \frac{\sin\angle EHA}{\sin\angle FHA}\cdot\frac{HE}{HF}\cdot\frac{XF}{XE} = \frac{\sin\angle EHA}{\sin\angle FHA}\cdot\frac{\sin\angle EXH}{\sin\angle FXH}\frac{\sin\angle FHX}{\sin\angle EHX}$$

   Notice that AB_1HC_1 is cyclic so $\angle EHA = \angle B_1HA = \angle B_1C_1A$ and similarly $\angle FHA = \angle C_1B_1A$. $\angle EXH = \angle FXH = 45°$, and finally since H is an excenter of XEF, HX passes through the circumcenter of HEF so $\angle EHX = 90° - \angle HFE = 90° - \angle C_1FA = \angle FAC_1 = \angle MAC_1$, since $\angle AC_1F = 90°$. Similarly, $\angle FHX = \angle MAB_1$. Thus putting this together, we get

$$\frac{\sin\angle EXA}{\sin\angle FXA} = \frac{\sin\angle B_1C_1A}{\sin\angle C_1B_1A}\frac{\sin\angle MAB_1}{\sin\angle MAC_1} = \frac{AB_1}{AC_1}\cdot\frac{AB_1}{AC_1} = \left(\frac{AB_1}{AC_1}\right)^2$$

   the first factor $\frac{AB_1}{AC_1}$ comes from sine rule on triangle AB_1C_1, the second comes from that AM is a symmedian of AB_1C_1.

   Finally, $\angle FXA = \sin\angle EXA + 90°$ so $\frac{\sin\angle EXA}{\sin\angle FXA} = \frac{\sin\angle EXA}{\cos\angle EXA} = \tan\angle EXA$. On the other hand, AM being a symmedian also means it cuts B_1C_1 into the ratio $\left(\frac{AB_1}{AC_1}\right)^2$, and with $MB_1 = MC_1$, $\left(\frac{AB_1}{AC_1}\right)^2 = \frac{\sin\angle B_1MA}{\sin\angle C_1MA} = \frac{\sin\angle B_1MA}{\cos\angle B_1MA} = \tan\angle B_1MA$. This gives

$$\angle EXA = \angle B_1MA$$

To finish off, recalling the identity $\angle MAC_1 = \angle EHX$ we have

$$\angle AXH = \angle AXE + \angle EXH = \angle B_1MA + \angle 45^\circ = \angle B_1MA + \angle MB_1C$$

$$= \angle M_AC_1 + \angle ACB_1 = \angle EHX + \angle AHB_1 = \angle AHX$$

so $AH = AX$, as desired.

6. Find all positive integers n such that for any integer $k \geq n$ there is a number divisible by n and with the sum of digits equal to k.

   Answer. All integers not divisible by 3.

   Solution. We see that if $3 \mid n$, then any number of sum of digits equal to $k = n+1$ will be congruent to 1 mod 3, and therefore impossible to be divisible by 3.

   Now consider n not divisible by 3. Since adding trailing zeros to an integer k will take care of the powers of 2 and 5 dividing n without changing the sum of digits, we may assume that $\gcd(n, 10) = 1$. Let's now claim the following:

   Lemma. For any $k \geq n$, there exist nonnegative integers a and b with $a + b = k$, and $n \mid a + 10b$.

   Proof: consider the sequence $x_a = a + 10(k-a)$ for $a = 0, 1, \cdots, n-1$. Then for any $a \neq b$, we have $x_a - x_b = 9(b-a)$. Since n is not divisible by 3, $\gcd(9, n) = 1$ and since $0 \leq a, b \leq n-1$, $n \nmid x_a - x_b$. It follows that $x_0, \cdots, x_{n-1}$ are different modulo n and since there are n of them, exactly one of them must be 0 mod n. $\square$

   Thus now we let (a, b) satisfying the lemma condition. Let $\phi(n)$ be the number of integers in $[0, n-1]$ and relatively prime to n. Then the number

$$\sum_{i=0}^{a-1} 10^{i\phi(n)} + \sum_{j=0}^{b-1} 10^{j\phi(n)+1} = 1 + 10^{\phi(n)} + \cdots + 10^{(a-1)\phi(n)} + 10 + 10^{\phi(n)+1} + \cdots + 10^{(b-1)\phi(n)+1}$$

   has sum of digits k and remainder $a + 10b \equiv 0$ modulo n.