Solution to APMO 2011 Problems

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1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2 + b + c, b^2 + c + a, c^2 + a + b$ to be perfect squares.

   Solution. W.l.o.g. let $a \geq b \geq c$, then $a^2 < a^2 + b + c \leq a^2 + a + a = a^2 + 2a < a^2 + 2a + 1 = (a + 1)^2$. This means that $a < \sqrt{a^2 + b + c} < a + 1$ so $\sqrt{a^2 + b + c}$ cannot be an integer.

2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

   Answer. 36°.

   Solution. Denote $\theta = \min\{\angle A_iA_jA_k : 1 \leq i, j, k \leq 5\}$ Consider, now, the convex hull formed by the 5 points on the plane. Let A_1 be part of the convex hull (WLOG). Also let $A_1A_2, A_1A_3, A_1A_4, A_1A_5$ to be in that order counterclockwise; these four lines define the outermost angle $\angle A_2A_1A_5$ divided into 3 subangles by the lines A_1A_3 and A_1A_4. It follows that each of the interior angles of the convex hull is divided into 3 subangles. If k is the number of vertices of this convex hull then the total interior angles of this convex hull has sum $(k - 2)180^\circ$ and since each of the k interior angles are divided into 3 subangles, there are $3k$ subangles in total and so $\theta \leq \frac{k-2}{3k}180^\circ = 60^\circ \frac{k-2}{k}$. Since $k \leq 5$ (we only have 5 points here), we have $\frac{k-2}{k} \leq \frac{3}{5}$. Therefore $\theta \leq 60^\circ \times \frac{3}{5} = 36^\circ$.

   This θ is achievable by having the 5 points to form a regular pentagon. Since the pentagon is cyclic, the angle $A_iA_jA_k$ is the angle subtended by A_iA_k. The 5 points divide the circle into 5 equal arcs, each subtending an angle of 36°. It then follows that $A_iA_jA_k$ must be a multiple of 36°.

3. Let ABC be an acute triangle with $\angle BAC = 30^\circ$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90^\circ$.

4. Let n be a fixed positive odd integer. Take $m + 2$ distinct points $P_0, P_1, \ldots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following three conditions are satisfied:

   1) $P_0 = (0, 1), P_{m+1} = (n + 1, n)$, and for each integer $i, 1 \leq i \leq m$, both x- and y-coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).

   2) For each integer $i, 0 \leq i \leq m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.

   3) For each pair i, j with $0 \leq i < j \leq m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

   Determine the maximum possible value that m can take.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:

1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.

2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.

Answer. There are two such functions, the zero function $f \equiv 0$ and the function given by

$$f(x) = \begin{cases} 0 & x \geq 0 \\ 2x & x < 0 \end{cases}$$

Solution. We'll use the following identity: If $h(x) \leq 2M$ for all x and $g(x) \to \infty$ as $x \to A$, then $h(x) - g(x) \to -\infty$ as $x \to A$. We're interested in the behavior when A is $+\infty$ or $-\infty$.

Now suppose that