Solution to APMO 2010 Problems

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1. Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be the diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

   Solution. Since ON is diameter and $OB = OC$, we have ON bisect $\angle BOC$ and $\angle BOC = 2\angle BAC$ means $\angle BON = \angle CON = \angle BAC$. Now P, B, O, N are on Γ in this order so $\angle APN = \angle BPN = 180^\circ - \angle BON = 180^\circ - \angle BAC$ and similarly $\angle AQN = 180^\circ - \angle BAC$. This means $AP \parallel NQ$ and $AQ \parallel NP$.

2. For a positive integer k, call an integer a *pure $k-th$ power* if it can be represented as m^k for some integer m. Show that for every positive integer n, there exists n distinct positive integers such that their sum is a pure 2009−th power and their product is a pure 2010−th power.

   Solution. Consider the numbers $ax^{2010}, x = 1, 2, \cdots, n$. Denote the sum $1^{2010} + \cdots + n^{2010}$ as S snd the product as $1^{2010} \cdot \cdots \cdot n^{2010}$ as P. The goal is to find a with aS a 2009-th power and $a^n P$ a 2010-th power (however, since P is already a 2010-th power, it suffices to have a as a 2010-th power).

   Consider, now, any prime p dividing S and the power $v_p(S)$ dividing it. Choose a such that $2010 \mid v_p(a)$ and $2009 \mid v_p(aS) = v_p(a) + v_p(S)$. This can be obtained by considering ℓ as any integer with $2009\ell - v_p(S) > 0$ and having $v_p(a) = 2010(2009\ell - v_p(S))$. Then we have $v_p(aS) = 2010(2009\ell - v_p(S)) + v_p(S) = 2009(2010\ell - v_p(S))$ which is now divisible by 2009. Since the set of prime numbers dividing S is finite, we can construct a suitable a satisfying the criterion for all such prime numbers, thereby fulfilling the problem condition.

3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

   Answer. $\binom{n-1}{2}$.

   Solution. The construction above can be made by having a single vertex (say, 1) to be adjacent to all other vertices 2, 3, $\cdots, n$. Any (i, j) with $2 \le i < j \le n$ would satisfy the criterion given.

   To prove that $\binom{n-1}{2}$ is indeed an upper bound, we notice that for every pair of vertices (i, j) satisfying the problem condition, (i, j) are not adjacent and there exists another vertex m with (i, m) and (j, m) adjacent. Here, we say edges (i, m) and (j, m) correspond to (i, j).

   Each pair of edges can only correspond to at most a pair of vertices (these edges must incident to a common vertex, and the other two vertices incident to the edges must not be adjacent). If k is the number of pairs of vertices fulfilling the problem condition, and ℓ is the number of edges in the graph, then:

$$k + \ell \le \binom{n}{2} \qquad k \le \binom{\ell}{2}$$

If $k > \binom{n-1}{2}$, then $\ell < \binom{n}{2} - \binom{n-1}{2} = n - 1$ but then

$$\binom{n-1}{2} < k \leq \binom{\ell}{2} < \binom{n-1}{2}$$

a contradiction.

4. Let ABC be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by O and H the circumcentre and orthocentre, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcentre of the triangle MNH lies on the line OH.

   Solution. TODO

5. Find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity

$$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz).$$

   Solution. TODO