Solution to APMO 2014 Problems

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1. For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

Solution. We first show that for all $a < b$, there exists integer x with $S(x) = 2^a$ and $P(x) = 2^b$. Fill x with b 2's and c 1's for some c; the product of digits will always be 2^b regardless of c. Now for all $b \geq 1$, $2^a \geq 2b$ so $2^a - b \geq 2^a - 2^b > 0$, meaning that we can chooce $c = 2^a - 2b > 0$ and therefore $S(x) = 2b + c = 2b + (2^a - 2b) = 2^a$.

Now, we first choose a_1 such that there exists x and y such that $S(a_1) = 2^x$, $P(a_1) = 2^y$ and $y - x > n$. If we fill a_1 with 2^k 2's, then $S(a_1) = 2^k \cdot 2 = 2^{k+1}$ while $P(a_1) = 2^{2^k}$. The function $2^x - (x+1)$ is unbounded above, so for k sufficiently large, we will be able to choose k such that $2^k - (k+1) > n$. Next, we consider the following construction of integers $b_1 < b_2 < \cdots < b_n$ with $b_1 = x$ ans $b_n = y$. Since $y - x > n$, there exist integers $b_2, b_3, \cdots, b_{n-1}$ that satisfies the inequality constraint. Now that we have $S(a_1) = 2^{b_1}$ and $P(a_1) = 2^{b_n}$, all we need is for $i > 1$, $S(a_i) = 2^{b_i}$ and $P(a_i) = 2^{b_i - 1}$, which is possible by the first lemma.

2. Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of one of A, B, C.

Answer. $108 \cdot 2014!$.

Solution. We denote this representative function as $f : (\mathcal{P}(S) \backslash \emptyset) \to S$ such that:

- for all $T \subseteq S$ and T nonempty, $f(T) \in T$.
- for all $A, B, C \subseteq S$ that are disjoint and nonempty, $f(A \cup B \cup C) \in \{f(A), f(B), f(C)\}$.

We first start with a lemma: if $T \subseteq S$, then for each $U \subseteq T$ satisfying $|U| \leq |T| - 2$, if $f(T) \in U$ then $f(U) = f(T)$. Indeed, if U and T satisfy such a relation then we can find V and W that are nonempty such that T is a disjoint union of U, V, W. Since U, V, W are disjoint and $f(T) \in U$, $f(T) \notin V$ and W. But then $f(T) \in \{f(U), f(V), f(W)\}$ and therefore $f(U) = f(T)$.

Conversely, once this condition is fulfilled, then any $D = A \cup B \cup C$ with A, B, C nonempty will have $|D| \geq |A| + 2, |B| + 2, |C| + 2$ and by above, if $f(D) \in A$ then $f(D) = f(A)$, and similarly so for the cases $f(D) \in B$ or $f(D) \in C$. Thus here we have $f(D) \in \{f(A), f(B), f(C)\}$.

Next, we claim that if $T \subseteq S$, and $|T| \geq 5$, $U \subseteq T$ and $|U| = |T| - 1$ with $f(T) \in U$, then $f(U) = f(T)$. Suppose otherwise, consider the set $V = \{f(T), f(U)\}$. By above, $f(U) \in V$ and $f(T) \in V$. Also, $f(V) = 2 = 4 - 2 = 5 - 3 \leq |T| - 3 = |U| - 2$. This means that $f(V) = f(U)$ and $f(V) = f(T)$ must both hold, which is impossible.

Having this determined, we will have $a_5, a_6, \cdots, a_{2014} \subseteq S$, each distinct, such that the following holds:

- $f(S) = a_{2014}$, and so for any subset T containing a_{2014} we have $f(S) = f(T)$.
- Now for each $i \geq 5$, denote $S_i = S \backslash \{a_j : j > i\}$. Then we can pick $f(S_i) = a_i$ and therefore all $T \subseteq S_i$ containing a_i must have $f(T) = a_i$.

This gives $2014 \cdots 5 = \frac{2014!}{4!}$ choices to choose the sequence $a_5, \cdots, a_{2014}$.

We're now left with $S_4 = S \backslash \{a_5, \cdots, a_{2014}\}$. Suppose, now, that $f(S_4) = a_4$, then any subset T with size ≤ 2 and containing a_4 must satisfy $f(T) = a_4$. We're now left with:

- 3-element subets of S_4.
- 2-element subsets of S_4 not containing a_4.
- Other 1-element subsets of S_4.

We claim that these subsets can take on any representatives as long as they belong to the subsets themselves (while still fulfilling the problem condition). Consider, now, $A, B, C \subseteq S$, disjoint and nonempty, with union D. If any of A, B, C is not a subset of S_4 (say A), neither will D and therefore by our construction there will be $i \geq 5$ such that $f(D) = f(A) = a_i$. If A, B, C is a subset of S_4 then there are two cases:

- $|A| = |B| = |C| = 1$, then if $A = \{a\}, B = \{b\}, C = \{c\}$ then the only way is $f(A) = a, f(B) = b, f(C) = c$. It follows that $f(D) \in D = \{a, b, c\}$.
- Otherwise, $|D| \geq 4$ and the only possible choice would be $D = S_4$. If $f(D) \notin \{f(A), f(B), f(C)\}$, then this could happen only when $f(D) \in A$ and $f(D) \neq f(A)$. This means $|A| = 2$ which contradicts our construction that $f(A) = f(D)$.

Finally, to finalize the computation, there are 4 ways to choose a_4, 3^4 ways to choose the representsatives for 3-element subsets of S_4, 2^3 ways to choose representative for 2-element subsets of S_4 not containing a_4, and 1 way for the 1-element subsets. Combined with the $\frac{2014!}{4!}$ ways described for sets that are not subsets of S_4 this gives:

$$\frac{2014!}{4!} \cdot 4 \cdot 3^4 \cdot 2^3 = 2014! \cdot 3^3 \cdot 2^2 = 108 \cdot 2014!$$

3. Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

   Answer. $n = 3^b, b \geq 0$.

   Solution. This is equivalent as finding n such that $\{a^3 + a : a = 0, 1, \cdots, n-1\} \equiv \{0, 1, 2, \cdots, n-1\} \pmod{n}$. We first show that $n = 3^b$ will fulfill this condition. Now suppose that $a^3 + a \equiv c^3 + c \pmod{n}$, then:

$$(a^3 + c) - (c^3 + c) = (a - c)(a^2 + ac + c^2 + 1)$$

   We see that there's no (a, c) that can make $a^2 + ac + c^2 + 1$ divisible by 3 (just try all 9 pairs of $a = 0, 1, 2$ and $c = 0, 1, 2$), therefore forcine $3^b \mid a - c$.

   Now we show that other numbers won't work, and in fact, it suffices to show that for all primes $p \neq 3$. For prime $p = 2$ we have $0^3 + 0 \equiv 1^3 + 1 \equiv 0 \pmod{2}$. For primes p satisfying $p \equiv 1 \pmod{4}$ we can choose a with $p \mid a^2 + 1$ and therefore $a^3 + a = a(a^2 + 1) \equiv 0 = 0^3 + 0 \pmod{p}$.

   It remains to settle for cases where $p \neq 3$ and $p \equiv 3 \pmod{4}$. As above, we need to find a, c such that

$$a \neq c \pmod{p} \qquad p \mid a^2 + ac + c^2 + 1$$

   There are now two cases:

- $p \equiv 1 \pmod 3$ (or $7 \pmod{12}$). Here, consider the case $(a, c) = (2x, -x)$, giving the equation $p \mid 3x^2 + 1$. Here, we need 3 to not be a quadratic residue mod p (since -1 is not a quadratic residue). We also have

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{(p-1)}{2}\frac{(3-1)}{2}} = (-1)$$

Here, $\left(\frac{p}{3}\right) = \left(\frac{1}{3}\right) = 1$ and therefore $\left(\frac{3}{p}\right) = -1$. This means 3 is not quadratic residue of p, and therefore $p \mid 3x^2 + 1$ has an integer solution.

- Consider, now, $p \equiv 2 \pmod 3$ (or $11 \pmod{12}$). Consider the equation $a^2 + ab + b^2$ as a binary quadratic form, with discriminant -3.

Recall a lemma that says that if m is odd and relatively prime to D, then m is properly represented by a primitive form of discriminant D iff D is a quadratic residue mod m. Here, $D = -3$ and we focus on m prime and congruent to $-1 \pmod n$. Again we have

$$\left(\frac{-3}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{3}{m}\right) = (-1)^{\frac{m-1}{2}}\left(\frac{3}{m}\right) = (-1)^{\frac{m-1}{2}}(-1)^{\frac{(3-1)(m-1)}{4}}\left(\frac{m}{3}\right) = \left(\frac{m}{3}\right)$$

so a prime m can be properly represented by a binary quadratic form of discriminant -3 iff $m \equiv 1 \pmod 3$. Since $\gcd(1, 3) = 1, \gcd(-1, n) = 1$ and $\gcd(3, n) = 1$, by Dirichlet's theorem there exists a prime m such that $m \equiv -1 \pmod n$ and $m \equiv 1 \pmod 3$.

Finally, the class number $h(-3) = 1$, so the only reduced form is $x^2 + xy + y^2$. If m is as chosen, it can be represented in the form of $x^2 + xy + y^2$, and so $n|m + 1 = x^2 + xy + y^2 + 1$. The only problem might be $x \equiv y \pmod n$; however, this won't happen in this case because $n \nmid 3x^2 + 1$ for any x (i.e. -3 is not a quadratic residue of n).

4. Let n and b be positive integers. We say n is b-discerning if there exists a set consisting of n different positive integers less than b that has no two different subsets U and V such that the sum of all elements in U equals the sum of all elements in V.

(a) Prove that 8 is 100-discerning.

(b) Prove that 9 is not 100-discerning.

Solution.

(a) Consider the set $A = \{3, 6, 12, 24, 48, 96, 97, 98\} = \{97, 98\} \cup \{3 \cdot 2^n : 0 \leq n \leq 5\}$. We show that for each integer k there's at most 1 way to find a subset of S with sum k. We partition A into $A_0 = \{3 \cdot 2^n : 0 \leq n \leq 5\}$ and $A_1 = \{97, 98\}$.

The subsets in A_0 have sum $3a$ for all $0 \leq a \leq 63$ with exactly one subset in A_0 fulfilling this sum for each of the numbers $0, 3, \cdots, 189$. The subset sums in A_1 are $0, 97, 98, 195$. Therefore:

- If $k \equiv 1 \pmod 3$, the only choice from A_1 is $\{97\}$ and the only choice in A_0 is the subset giving sum $k - 97$, if it ever exists.
- If $k \equiv 2 \pmod 3$, the only choice from A_1 is $\{98\}$ and the only choice in A_0 is the subset giving sum $k - 98$, if it ever exists.
- If $k \equiv 0 \pmod 3$, there are two cases: if $k \leq 189$ then since $97 + 98 = 195 > 189$, the only choice is to choose $\emptyset$ from A_1 and the subset (if exists) from A_0 giving sum k. Otherwise, if $k > 189$, then the only choice from A_1 is A_1 itself and the subset (if exists) from A_0 giving sum $k - 195$, if exists.

Thus regardless of the remainder of k modulo 3, its choice from A_0 and A_1 are both unique.

5. Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

   Solution. Let $\ell_P \cap \ell_Q = X$, $\ell_P \cap AB = Y$, and $\ell_Q \cap AB = Z$. We'll show that the circumcircle of XYZ is tangent to Ω at a point opposite Z. If $p(X)$ be the length of tangent of X to Ω, we'll need to show (due to Casey's theorem):

$$XY \cdot p(Z) = YZ \cdot p(X) + XZ \cdot p(Y)$$

   Now, Y and Z are on the radical axis AB of ω and Ω, and therefore have the same power of point to these circles. This means $p(Z) = ZQ = XQ - XZ$ and $p(Y) = YP = XP - XY$. In addition, $p(X) = XP$. This means we need to show that

$$XY \cdot (ZQ - XZ) = YZ \cdot XP + XZ \cdot (XP - XY)$$

   or equivalently

$$XY \cdot XQ = XP \cdot (YZ + XZ) \leftrightarrow \frac{XQ}{XP} = \frac{YZ + XZ}{XY}$$

   Now, $XQ > XP$ so we can name $\angle XPQ = b$ and $\angle XQP = a$, In addition, let AB intersect PQ at R. The line tangent to ω at M is parallel to AB, and therefore $ZQ = ZR$ and $\angle ZRQ = a$. This means $\angle X, Y, Z$ (on triangel XYZ) are $180° - a - b$, $b - a$, $2a$ respectively. Using sine rule we are looking at

$$\frac{XQ}{XP} = \frac{\angle P}{\angle Q} = \frac{\sin b}{\sin a}$$

$$\frac{YZ + XZ}{XY} = \frac{YZ + XZ}{XY} = \frac{\sin Y + \sin X}{\sin Z} = \frac{\sin(a + b) + \sin(b - a)}{\sin 2a} = \frac{2 \sin b \cos a}{2 \sin a \cos a} = \frac{\sin b}{\sin a}$$

   and therefore these two ratios are equal.