Solutions to Tournament of Towns, Spring 2012, Senior

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O-Level

1. Each vertex of a convex polyhedron lies on exactly three edges, at least two of which have the same length. Prove that the polyhedron has three edges of the same length.

   Solution. Consider any vertex O. Now that there are two other vertices A, B such that there's an a with $OA = OB = a$, consider the side AB. If $AB = a$ we're done. Otherwise, let $AB = b \neq a$. For the lines originating from A, we have $AO = a$ and $AB = b$ hence the third line (say, AC) must have length either a or b. Similarly the third line from B (say, BD where D could be equal to C) must have length either a or b. If either of AC or BD have length a then there are three lines of length a. Otherwise both are b and since $AB = b$, this gives $AC = BD = AB = b$.

2. The cells of a $1 \times 2n$ board are labelled $1, 2, ..., , n, -n, ..., -2, -1$ from left to right. A marker is placed on an arbitrary cell. If the label of the cell is positive, the marker moves to the right a number of cells equal to the value of the label. If the label is negative, the marker moves to the left a number of cells equal to the absolute value of the label. Prove that if the marker can always visit all cells of the board, then $2n + 1$ is prime.

   Solution. Notice that the positions are $1, 2, \cdots, 2n$ and the numbers written on each cell is congruent to its position modulo $2n + 1$. Now consider a market at cell i. If $i \leq n$ then it will go to position $2i$; otherwise, $i > n$ and the cell has number $i - (2n + 1)$, which prompts it to jump to cell $i + (i - (2n - 1)) = 2i - (2n - 1)$. Thus with respect to modulo $2n + 1$, the position doubles. In other words, if the starting position is c then the position after k steps is $c \cdot 2^k \pmod{2n + 1}$.

   From the problem condition, being able to visit all cells of the board implies there exists a c with
   $$\{c \cdot 2^k : k \geq 0\} \pmod{2n + 1} = \{1, 2, \cdots, 2n\}$$
   If $p \mid 2n + 1$ for some prime $p < 2n + 1$, then since $2n + 1$ is odd, $p \leq n$. If $p \mid c$ then all numbers $c \cdot 2^k$ is divisible by p; otherwise $p \nmid c$ and all numbers $c \cdot 2^k$ are not divisible by p since p must be odd. Either way, we can't get a complete residue modulo $2n + 1$. Therefore $2n + 1$ has to be prime.

   Comment. A necessary and sufficient condition here would be that $2n + 1$ is a prime satisfying $ord_{2n+1}(2) = 2n$.

3. Consider the points of intersection of the graphs $y = \cos x$ and $x = 100 \cos(100y)$ for which both coordinates are positive. Let a be the sum of their x-coordinates and b be the sum of their y-coordinates. Determine the value of $\frac{a}{b}$.

   Answer. 100.

   Solution. The set of x satisfying this equation are the ones satisfying:
   $$x = 100 \cos(100 \cos x) : \cos x > 0; x > 0$$

Similarly, the set of y sasitfying this equation are the ones satisfying:

$$y = \cos(100\cos(100y)); y > 0; \cos(100y) > 0$$

We show that if x_0 is a solution to the x above then $y_0 := \frac{x_0}{100}$ is a solution to the y above. Now, given that $x_0 = 100\cos(100\cos x_0)$, we have

$$\cos(100\cos(100y_0)) = \cos(100\cos(x_0)) = \frac{x_0}{100} = y_0$$

and both $x_0, y_0 > 0$. Similarly, suppose that y_0 is a solution to the y above then if $x_0 := 100y_0$:

$$100\cos(100\cos x_0) = 100\cos(100\cos(100y_0)) = 100y_0 = x_0$$

and both x_0, y_0. Thus there's a natural pairing of $(100y_0, y_0)$ of the solutions, which then gives $a = 100b$.

4. A quadrilateral $ABCD$ with no parallel sides is inscribed in a circle. Two circles, one passing through A and B, and the other through C and D, are tangent to each other at X. Prove that the locus of X is a circle.

   Solution. Let AB and CD intersect at P. The common tangent of the circles at X, AB and CD are the radical axes of the pairs of circles determined by circle $ABCD$, and the two tangent circles passing through AB and CD. Thus, $PA \cdot PB = PC \cdot PD = PX^2$ by the power of point theorem. Therefore, X lies on a circle with center P and radius $\sqrt{PA \cdot PB}$.

   Conversely, if X satisfies this condition, then from $PX^2 = PA \cdot PB$ we deduce that the PX is tangent to the circumcircle of ABX. Similarly PX is tangent to the circumcircle of CDX. Thus, these two circles are tangent to each other at X.

A-Level

1. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least $3:1$. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

   Answer. No.

   Solution. Let $a_1 > a_2 > \cdots > a_k$ be the number assigned to the guards. We show that for each $i = 1, 2, \cdots, k$, among $a_1, \cdots, a_i$ there's a block of a_i consecutive days such that none of $a_1, \cdots, a_i$ are on duty.

   Base case $i = 1$ is clear. Now suppose this is true for some i. Consider the block of a_i days where none of $a_1, \cdots, a_i$ is on duty and consider the schedule of a_{i+1}. W.l.o.g. name these days as $1, 2, \cdots, a_i$. Let $x \leq a_{i+1}$ be the number of days left where the guard with label a_{i+1} changes status (that is, changing from duty to rest, or vice versa). Then among the two blocks $[x+1, \cdots, x+a_{i+1}]$ and $[x+a_{i+1}+1, \cdots, x+2a_{i+1}]$, exactly one of them is a "rest block" (that is, the guard a_{i+1}) is resting. Since $x + 2a_{i+1} \leq 3a_{i+1} \leq a_i$, this rest block also coincides with the resting time of $a_1, \cdots, a_i$, giving a consecutive block of a_{i+1} days where $a_1, \cdots, a_{i+1}$ are resting.

   In particular, given the k guards, there's a block of a_k consecutive days when all k guards are not working. Hence the result.

3. Let n be a positive integer. Prove that there exist integers $a_1, a_2, ..., a_n$ such that for any integer x, the number $(...(((x^2 + a_1)^2 + a_2)^2 + ...)^2 + a_{n-1})^2 + a_n$ is divisible by $2n - 1$.

Solution. Denote S_k as $\{(...(((x^2 + a_1)^2 + a_2)^2 + ...)^2 + a_k)^2 : x \in \mathbb{Z}\} \pmod{2n-1}$. We show that for each $k \le n-1$ we can choose $a_1, \cdots, a_k$ in a way such that $|S_k| \le n - k$.

Base case: $S_0 = \{x^2\} \pmod{2n-1}$, the quadratic residue set. We have $i^2 \equiv (2n-1-i)^2 \pmod{2n-1}$ and therefore

$$\{x^2\} = \{0^2, 1^2, \cdots, (n-1^2)\} \pmod{2n-1}$$

Inductive step: we show that if $S_k \ge 2$, a suitably choosen a_{k+1} will yield $|S_{k+1}| \le |S_k| - 1$. Observe that

$$S_{k+1} = \{(c + a_{k+1})^2 : c \in S_k\}$$

so $|S_{k+1}| \le |S_k|$ with equality if and only if $(c + a_{k+1})^2$ gives different residues in mod $2n-1$ for any different $c \in S_k$. Now, if $|S_k| \ge 2$, we can choose $c_0, c_1 \in S_k$ and $c_0 \ne c_1 \pmod{2n-1}$. Choose a_{k+1} such that $2n-1 \mid c_0 + c_1 + 2a_{k+1}$ (this is possible since $2n-1$ is odd), then $c_0 + a_{k+1} = -(c_1 + a_{k+1})$ and therefore $(c_0 + a_{k+1})^2 \equiv (c_1 + a_{k+1})^2$. In particular, if $2 \le |S_k| \le n-k$ then $1 \le |S_{k+1}| \le n-k-1$.

Now we have $|S_{n-1}| = 1$, there's a number y such that $...(((x^2+a_1)^2+a_2)^2+...)^2+a_{n-1})^2 \equiv y$ for all x. Therefore we can choose $a_n = -y$.

4. Alex marked one point on each of the six interior faces of a hollow unit cube. Then he connected by strings any two marked points on adjacent faces. Prove that the total length of these strings is at least $6\sqrt{2}$.

Solution. Denote the six faces as $x_0, x_1, y_0, y_1, z_0, z_1$ where x_i denotes the face with x-coordinate equal to i (similar for y_i, z_i). We consider the quadrilateral formed by the points on x_0, y_0, x_1, y_1 in that order. We show that the total length is at least $2\sqrt{2}$. Indeed, let the coorsinates of the 4 points be

$$(0, b_1, c_1), (a_2, 0, c_2), (1, b_3, c_3), (a_4, 1, c_4)$$

then the total distance is

$$\sqrt{a_2^2 + b_1^2 + (c_1 - c_2)^2} + \sqrt{(1 - a_2)^2 + b_3^2 + (c_2 - c_3)^2}$$

$$+\sqrt{(1 - a_4)^2 + (1 - b_3)^2 + (c_3 - c_4)^2} + \sqrt{a_4^2 + (1 - b_1)^2 + (c_1 - c_4)^2}$$

$$\ge \sqrt{a_2^2 + b_1^2} + \sqrt{(1 - a_2)^2 + b_3^2} + \sqrt{(1 - a_4)^2 + (1 - b_3)^2} + \sqrt{a_4^2 + (1 - b_1)^2}$$

$$\ge \sqrt{\frac{1}{2}}[(a_2 + b_1) + ((1 - a_2) + b_3) + ((1 - a_4) + (1 - b_3)) + (a_4 + (1 - b_1))] \ge 2\sqrt{2}$$

where the QM-AM inequality gives $\sqrt{a^2 + b^2} \ge \frac{a+b}{\sqrt{2}}$ for all real a, b.

Analogously, the quadrilateral of the points on the planes y_0, z_0, y_1, z_1 and x_0, z_0, x_1, z_1 are each $\ge 2\sqrt{2}$, giving the total of at least $6\sqrt{2}$.

5. Let ℓ be a tangent to the incircle of triangle ABC. Let ℓ_a, ℓ_b and ℓ_c be the respective images of ℓ under reflection across the exterior bisector of $\angle A, \angle B$ and $\angle C$. Prove that the triangle formed by these lines is congruent to ABC.

Solution. The first step is to show that the triangle formed is similar to ABC. If A_1, B_1, C_1 are the excenters opposite BC, CA, AB then:

$$\angle(\ell_a, \ell_b) = \angle(\ell_a, \ell) + \angle(\ell, \ell_b) = 2\angle(B_1C_1, \ell) + 2\angle(\ell, A_1C_1) = 2\angle(B_1C_1, A_1C_1)$$

and on the other hand we have

$$2\angle(AB, B_1C_1) = \angle(AB, AC) \quad 2\angle(AB, A_1C_1) = \angle(AB, BC)$$

and therefore

$$2\angle(B_1C_1, A_1C_1) = 2\angle(B_1C_1, AB)+2\angle(AB, A_1C_1) = \angle(AC, AB)+\angle(AB, BC) = \angle(AC, BC)$$

and similarly we can show that $\angle(\ell_a, \ell_c) = \angle(AB, BC)$ and $\angle(\ell_b, \ell_c) = \angle(AB, AC)$, proving the similarity between the two triangles.

Next, denote by A_0, B_0, C_0 the intersection of ℓ and B_1C_1, A_1C_1, A_1B_1. Also let ℓ_a and ℓ_b meet at C_2 (define A_1 and B_1 similarly). We now turn our attention to the triangle $A_0B_0C_2$. Now, A_0B_0 is ℓ, A_0C_2 is ℓ_A and B_0C_2 is ℓ_B. It follows that B_1C_1 bisects angle $C_2A_0B_0$ and similarly A_1C_1 bisects angle $C_2B_0A_0$. It then follows that C_1 is either the incenter or an excenter of this triangle $A_0B_0C_2$. If we consider the circle $A_0B_0C_2$, then the midpoint of arc A_0C_2 excluding B_0 is the center of triangle $C_1A_2A_0$, and hence on the angle bisector B_0C_1 of $\angle A_0B_0C_2$. This means:

$$90^\circ = \angle(A_0B_0, B_1C_1) + \angle(C_1C_2, C_1A_1) = \angle(\ell, B_1C_1) + \angle(C_1C_2, C_1A_1)$$

and therefore using the same logic we get

$$\angle(C_1C_2, A_1A_2) = \angle(C_1C_2, C_1A_1) - \angle(A_1A_2, C_1A_1)$$

$$= (90^\circ - \angle(\ell, B_1C_1)) - (90^\circ - \angle(\ell, A_1B_1)) = \angle(B_1C_1, A_1B_1)$$

which means that C_1C_2 and A_1A_2 intersect on the circumcircle of $A_1B_1C_1$ (namely X). Similarly, B_1B_2 will pass through X too. (Additionally, by directed angles, too, we can show that X is the incenter of $A_2B_2C_2$).

Now, let I be the incenter of ABC, which is the orthocenter of $A_1B_1C_1$. Denote A_3, B_3, C_3 as the reflection of I in lines B_1C_1, C_1A_1, A_1B_1, which lie on lines AA_1, BB_2, CC_1 and all lie on the circumcircle of $A_1B_1C_1$. Let's now claim that A_3X, B_3X, C_3X are parallel to ℓ_a, ℓ_b, ℓ_c, respectively. Indeed:

$$\angle(B_3X, \ell_b) = \angle(B_3X, XC_1) + \angle(XC_1, \ell_b) = \angle(B_3B_1, B_1C_1) + \angle(C_1C_2, \ell_b)$$

However, notice that $B_1B \perp C_1A_1$ so $\angle(B_3B_1, B_1C_1) = 90^\circ - \angle(B_1C_1, A_1C_1)$ and by the previous angle condition we have $\angle(C_1C_2, \ell_b) = 90^\circ - \angle(A_1C_1, B_1C_1)$. Therefore

$$\angle(B_3X, \ell_b) = 90^\circ - \angle(B_1C_1, A_1C_1) + 90^\circ - \angle(A_1C_1, B_1C_1) = 0$$

considering that all angles are modulo 180°. Similarly we can show that $A_3X \parallel \ell_a$ and $C_3X \parallel \ell_c$.

Finally, the distance of I to ℓ is r, the inradius of ABC. Since the images of I and ℓ when reflected in A_1C_1 are B_3 and ℓ_B, reslectively, the distance from B_3 to ℓ_B is also r. Since $B_3X \parallel \ell_B$, the distance of X to ℓ_B is also r. Thus X has distance r to A_2B_2, B_2C_2, C_2A_2, showing that the inradius of $A_2B_2C_2$ is indeed r (as briefly mentioned, we can use directed angles to establish that this is the incenter, not excenter). Since ABC and $A_2B_2C_2$ are similar and have equal inradius r, they are indeed congruent.