# Solutions to Tournament of Towns, Fall 2019, Senior

Anzo Teh

## O-Level

1.

## A-Level

1. The polynomial $P(x, y)$ is such that for any integer $n \geq 0$ each of the polynomials $P(n, y)$ and $P(x, n)$ either is the constant zero or has the degree not greater than $n$. Is it possible that the polynomial $P(x, x)$ has an odd degree?

   **Answer.** No.

   **Solution.** Suppose $P(x, x)$ is odd. If $x^m y^n$ is a term in $P$ that gives the highest degree, then $m+n$ is odd and therefore, $m \neq n$. Suppose w.l.o.g. that $m < n$. If $n_0$ is the highest exponent of $y$ that appears in $P$ then we can write

$$P(x, y) = \sum_{k=0}^{n_0} Q_k(x) y^k$$

   where $Q_k$ are polynomials. By the problem condition, $Q_k(\ell) = 0$ if $\ell$ is an integer with $0 \leq \ell < k$. In particular, $Q_n(x) = 0$ for $x = 0, 1, \cdots, n-1$. Since $x^m y^n$ gives the highest degree to $P$, $Q_n$ has degree $m < n$. But then $Q_n(x)$ has roots $0, 1, \cdots, n-1$ (i.e. at least $n$ roots), which is a contradiction unless $Q_n \equiv 0$.

4. Consider a increasing sequence of positive numbers

$$\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots$$

   infinite in both directions. For a positive integer $k$ let $b_k$ be the minimal integer such that the ratio of the sum of any $k$ consecutive elements of the sequence to the largest of those $k$ elements is not greater than $b_k$. Prove that the sequence $b_1, b_2, b_3, \ldots$ either coincides with the sequence $1, 2, 3, \ldots$ or is constant after some point.

   **Solution.** It's not hard to observe that

$$b_k = \lceil \sup\{\frac{a_{x-k+1} + \cdots + a_x}{a_x} : x \in \mathbb{Z}\} \rceil$$

   and since $\frac{a_{x-k}+\cdots+a_x}{a_x} - \frac{a_{x-k+1}+\cdots+a_x}{a_x} = \frac{a_{x-k}}{a_x} \in (0, 1)$, we have $b_{k+1} - b_k \in \{0, 1\}$.

   Suppose that $b_\ell \neq \ell$ for some $k$. Then given that $b_{k+1} - b_k \in \{0, 1\}$ and $b_1 = 1$, we have $b_\ell < b_\ell$, i.e. $b_\ell \leq b_\ell - 1$. This means that:

$$\forall x : \lceil \sup\{\frac{a_{x-\ell+1} + \cdots + a_\ell}{a_\ell} \leq \ell - 1$$

   and since the sequence is strictly increasing,

$$\frac{a_{x-\ell+1}}{a_x} \leq \frac{\ell - 1}{\ell}$$

which then gives that, if $y \le x - k(\ell-1)$ then $\frac{a_y}{a_x} \le (\frac{\ell-1}{\ell})^k$. Now consider the infinite sum

$$\frac{\sum_{k=0}^{\infty} a_{x-k}}{a_x} \le \frac{\sum_{k=0}^{\infty} a_x \cdot (\frac{\ell-1}{\ell})^{\lfloor \frac{k}{\ell-1} \rfloor}}{a_x} = \frac{\ell-1}{\ell}(\ell-1)\sum_{k=0}^{\infty}(\frac{\ell-1}{\ell})^k = \frac{\ell-1}{\ell}(\ell-1)\cdot\frac{1}{1-\frac{\ell-1}{\ell}} = (\ell-1)^2$$

which then shows that this infinite sum is bounded by $(\ell-1)^2$. We therefore have $\{b_k\}$ bounded above as well. However, given also that $b_k$ are integers that are either equal or the one more than the previous term, so boundedness of $\{b_k\}$ also implies that it's eventually constant.

5. The point $M$ inside a convex quadrilateral $ABCD$ is equidistant from the lines $AB$ and $CD$ and is equidistant from the lines $BC$ and $AD$. The area of $ABCD$ occurred to be equal to $MA \cdot MC + MB \cdot MD$. Prove that the quadrilateral $ABCD$ is both cyclic and circumscribed.

   **Solution.** Let $M_A, M_B, M_C, M_D$ be the projections from $M$ to $AB, BC, CD, DA$ respectively. Then $MM_A = MM_C$ and $MM_B = MM_D$. Let's now show that:

   $$[MM_AA] + [MM_CC] \le \frac{1}{2} MA \cdot MC$$

   Since $MM_A = MM_C$ and $\angle MM_AA = \angle MM_AC = 90°$, we can consider combining the two triangles with the common vertex $M$ coincide, and $M_A$ and $M_C$ coincide. Then the new triangle has area the sum of old triangles, $[MM_AA] + [MM_CC]$ with sides $M, A, C$. This also means that this triangle has area $\frac{1}{2} MA \cdot M_C \cdot \sin AMC \le \frac{1}{2} MA \cdot M_C$ with equality iff $\angle AMM_A + \angle CMM_C = \angle AMC = 90°$. Similarly we have

   $$[MM_DA] + [MM_BC] \le \frac{1}{2} MA \cdot MC$$

   and also

   $$[MM_AB] + [MM_CD] \le \frac{1}{2} MB \cdot MD \qquad [MM_BB] + [MM_DD] \le \frac{1}{2} MB \cdot MD$$

   which means that , summing all of these:

   $$[MM_AA]+[MM_CC]+[MM_DA]+[MM_BC]+[MM_AB]+[MM_CD]+[MM_BB]+[MM_DD]$$
   $$\le MA \cdot MC + MB \cdot MD$$

   The left hand side is the same as the area of $ABCD$, so for that to hold all inequalities must be equalities. This means

   $$\angle AMM_A + \angle CMM_C = \angle AMM_D + \angle CMM_B = 90°$$

   which then means $\angle A + \angle D = 180°$ and the quadrilateral is cyclic.

   To prove the other statement, consider the combined triangle $AMC$ above and we can also do the same by combining $M_B$ and $M_D$ instead (on the triangles $AM_DM$ and $CM_BM$). The two triangles $AMC$ formed will be congruent, and therefore the heights ($MM_A = MM_C$ for first; $MM_B = MM_D$ for second) will be equal too. This then shows that $ABCD$ is circumscribed.