Putnam 2017

A1 Let S be the smallest set of positive integers such that

(a) 2 is in S,

(b) n is in S whenever n^2 is in S, and

(c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

(The set S is "smallest" in the sense that S is contained in any other such set.)

Answer. 1 and all integers divisible by 5.
Solution. To show that all numbers not in the above category must be in S, we note the following lemma: if n is in S for some n, then by (c), $(n+5)^2$ is in S and by (b), $n+5$ is in S. Hence by repeated iteration of this process, we get

$$n \in S \to n + 5k \in S, \forall k \geq 0 \cdots (d)$$

Thus starting from $2 \in S$ as of (a), we get $2 + 5k \in S \forall k \geq 0$. Now (a) and (c) combined imply that $7^2 = 49 \in S$, too. By (c) again, $(49+5)^2 = 54^2 \in S$ too. Notice that $56^2 - 54^2 = 2 \times 110$ is divisible by 5 and is nonnegative, so $56^2 \in S$ by (d) again. By (b), $56 \in S$ and by (d) again, $9^2 = 81 = 56 + 5(5) \in S$ and $11^2 = 121 = 56 + 5(13) \in S$, so by (b), $9, 11 \in S$. By (b) again, $\sqrt{9} = 3 \in S$. Finally, since $11 \in S$, by (d) again, $11 + 5 = 16 \in S$, so by (b), $\sqrt{16} = 4 \in S$. Similarly, $11 + 5(5) = 36 \in S$, by (d) again. Thus $\sqrt{36} = 6 \in S$. Since $2, 3, 4, 6 \in S$ so by (d), $2 + 5k, 3 + 5k, 4 + 5k, 6 + 5k \in S$. These are all the numbers that are not 1 and not divisible by 5.

To show that $S_1\{a : a > 1, 5 \nmid a\}$ is valid, let a be arbitrary integer in S_1. Clearly, $2 \in S_1$, so (a) is satisfied. If $a = k^2$ for some k, then from $a > 1$ then $k = \sqrt{a} > 1$. Since $5 \nmid a, 5 \nmid \sqrt{a} = k$ too. So $5 \nmid k$. Hence (b) is fulfilled. Finally, $(a+5)^2 > a > 1$, and from $5 \nmid a$, we have $5 \nmid a + 5$. As 5 is a prime number, $5 \nmid (a+5)^2$ too. Thus (c) is also fulfilled.

A2 Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Solution. We show that $Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x)$ for all $n \geq 2$ via induction. For $n = 2$ (base case), we have $Q_2(x) = x^2 - 1 = x(x) - 1 = xQ_1(x) - Q_0(x)$. Now suppose that $Q_{n-1}(x) = xQ_{n-2}(x) - Q_{n-3}(x)$ for some $n \geq 3$. We consider the following:

$$\begin{aligned} Q_{n-1}^2(x) - 1 &= (xQ_{n-2}(x) - Q_{n-3}(x))Q_{n-1} - 1 \\ &= xQ_{n-2}(x)Q_{n-1}(x) - Q_{n-3}(x)Q_{n-1}(x) - 1 \\ &= xQ_{n-2}(x)Q_{n-1}(x) - (Q_{n-3}(x)Q_{n-1}(x) + 1) \\ &= xQ_{n-2}(x)Q_{n-1}(x) - Q_{n-2}^2(x) \\ &= Q_{n-2}(x)(xQ_{n-1}(x) - Q_{n-2}(x)) \end{aligned}$$

notice the use of the fact $Q_{n-3}(x)Q_{n-1}(x) + 1 = Q_{n-2}^2(x)$ as followed form the definition $Q_{n-1}(x) = \frac{(Q_{n-2}(x))^2-1}{Q_{n-3}(x)}$¿ Therefore we have $Q_n(x) = \frac{(Q_{n-1}(x))^2-1}{Q_{n-2}(x)} = xQ_{n-1}(x) - Q_{n-2}(x)$. By inductive hypothesis, we get $Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x)$ for all $n \geq 2$. Since Q_0 and Q_1 are

A3 Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

Solution. First, we notice the following use of the Cauchy-Schawz inequality in the form of integrals:

$$I_{n-1} \cdot I_{n+1} = \int_a^b \frac{(f(x))^n}{(g(x))^{n-1}}\,dx \cdot \int_a^b \frac{(f(x))^{n+2}}{(g(x))^{n+1}}\,dx \geq \left(\int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx\right)^2 = I_n^2$$

In particular, substituting $n = 0$ we get $I_{-1}I_1 \geq I_0$. Now $I_0 = \int_a^b f(x)dx$ and $I_{-1} = \int_a^b g(x)dx$, so $I_0 = I_{-1}$, and thus $I_1 \geq I_0$. Since $f(x)$ and $g(x)$ are both continuous on $[a, b]$, so is the function $\frac{f(x)^2}{g(x)}$, so equality can only hold if and only if $\frac{f(x)^2}{g(x)} \div g(x)$ is constant on $[a, b]$. This requires $|f(x)| = |g(x)|$ on $[a, b]$, which becomes $f(x) = g(x)$ since both positive returns only positive values. However, this is not true since $f \neq g$.

So $I_1 > I_0$, and denote the ratio $\frac{I_1}{I_0} = c > 1$. We will in fact claim that $\frac{I_{n+1}}{I_n} \geq c$ for all $n \geq 0$, which will finish the proof since $I_n \geq c^n I_0$ and $\lim_{n\to\infty} c^n = \infty$ as $c > 1$. The base case is given as $\frac{I_1}{I_0} = c$. If $\frac{I_n}{I_{n-1}} \geq c$ for some $n \geq 1$, then from the Cauchy-schawrz inequality we had before, $I_{n-1}I_{n+1} \geq I_n^2$ means that $\frac{I_{n+1}}{I_n} \geq \frac{I_n}{I_{n-1}} = c$. Hence we completed our inductive hypothesis, and concludes the proof.

A4 A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

Solution. The total score of the group is $14.8N = \frac{74N}{5}$, which is an integer since all individual scores are integers. Since $\gcd(74, 5) = 1$, we have $N = 5k$ for some integer k. This means $14.8N = 14.8(5k) = 74k$, which is even. Thus the goal now becomes finding a group of N students where the total score is $7.4N$, which is $37k$, an integer.

Let $x_1 \leq \cdots \leq x_{2N}$ be the scores of students. Let $m = x_1 + \cdots + x_N$ and $M = x_{N+1} + \cdots x_{2N}$. Since $m + M = 14.8N$ and from $x_i \leq x_{N+i}$ we have $m \leq M$, we have $m \leq x \leq M$. We will show that for any integer $x \in \{m, m+1, \cdots, M-1, M\}$ it is possible to choose a group of N students such that the total score in this group is x, thereby showing that it is possible to choose a group of N students with the total group score of $7.4N$.

We first notice that for all $1 \leq i < N$, $0 \leq x_{i+1} - x_i \leq 1$. The left inequality is obvious by our sorting algorithm. Suppose that $x_{i+1} - x_i \geq 2$ for some i. By our sorting algorithm, again, nobody scored $x_i + 1, \cdots, x_{i+1} - 1$. We now define a sequence of $N^2 + 1$ numbers $y_0, y_1, \cdots, y^{N^2}$ as follows:

- $y_0 = x_1 + x_2 + \cdots + x_N$
- For some $i < N^2$, denote $y_i = x_{a_1} + x_{a_2} + \cdots + x_{a_N}$ for some $1 \leq a_1 < a_2 < \cdots < a_N \leq 2N$. If there exists $j < N$ such that $a_{j+1} - a_j > 1$, then denote $y_{j+1} = x_{a_1} + x_{a_2} + \cdots + x_{a_j+1} + x_{a_{j+1}} + \cdots x_{a_N}$ (basically, shift one of the indices to the right by 1). Otherwise, denote $y_i = x_{a_1} + x_{a_2} + \cdots + x_{a_N+1}$.

We first show that this construction sequence is legitimate: that is, when $i < 2N$, either such j can be found or $a_N < 2N$ (so x_{a_N+1} exists). To see why, we consider the sum of indices $S(i) = a_1 + a_2 + \cdots + a_N$ when $y_i = x_{a_1} + \cdots + x_{a_N}$. When $i = 0$ then sum is

$S(0) = 1 + \cdots + N = \frac{N(N+1)}{2}$, and whenever the sequence y_i and y_{i+1} are both legitimate, $S(i+1) - S(i) = 1$. Thus, the recursion from y_i to y_{i+1} is legitimate if and only if y_i is not $x_{N+1} + \cdots + x_{2N}$. If such i exists, then $S(i) = (N+1) + \cdots + (2N) = \frac{N(3N+1)}{2} = S(0) + N^2$. It then follows that such i must be at least N^2 for this to happen. The converse is also true: we have $y_{N^2} = x_{N+1} + \cdots + x_{2N}$.

In addition, for each $0 \leq i < 2N$, by the construction above there exists index j such that $y_{i+1} - y_i = x_{j+1} - x_j$. By an earlier lemma, $0 \leq x_{j+1} - x_j \leq 1$, so $0 \leq y_{i+1} - x_i \leq 1$. We also have $y_0 = x_1 + \cdots + x_N = m$ and $y_{N^2} = X_{N+1} + \cdots + x_{2N} = M$, which means:

$$m = y_0 \leq y_1 \leq \cdots \leq y_{N^2} \leq M$$

which means the set $\{y_0, \cdots, y_{N^2}\}$ is precisely the set of integers in the interval $[m, M]$, inclusive.

B1 Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

Solution. Fix P, and let A_1 be a variable point on L_1. Let A_2 be such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$. Then as A_1 varies, the locus of A_2 is a line L_3 that's parallel to L_1, and satisfying $d(P, L_3) = |\lambda| d(P, L_1)$, where d is the distance of P to lines.

If L_1 and L_2 intersect (i.e. nonparallel), then L_2 and L_3 intersect and we can take A_2 as the unique intersection of L_2 and L_3 and A_1 as the point satisfying $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$, which will be on L_1 since A_2 is on L_3. Conversely, if L_1 and L_2 are parallel, then L_2 and L_3 coincide for a particular $\lambda_0 \neq 0$, and do not intersect for the other $\lambda \neq 0$. It then follows that such points A_1 and A_2 do not exist for all $\lambda \neq \lambda_0$.

B2 Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

Answer. $a = 16$

Solution. N can be written as sum of k consecutive positive integers if and only if $N = \frac{k(2a+k-1)}{2}$ for some positive integer a. This means N need to satisfy the following properties:

(a) $N \geq \frac{k(k+1)}{2}$

(b) $k|N$ for k odd, and $k|N - \frac{k}{2}$ when k is even.

The second condition is due to the fact that, when considering mod k, $a, a+1, \cdots, a+k-1$ is congruent to $1, 2, \cdots, k$ in some order, and thus $N \equiv \frac{k(k+1)}{2} \pmod{k}$. If k is odd then this is divisible by 0; converse ly if k is even, then $k+1$ is odd so it's congruent to $\frac{k}{2}$.

Coming back to the problem, we need one such N that can be written as sum of k consecutive integers. Denote $N = 2017 \cdot m$ with $m \geq 1009$. Now consider the case when $m \leq 1024$. If m has an odd divisor that's greater than 1, say q, then $q|N$ too, and since $N \geq \frac{2017(2018)}{2} ge \frac{q \cdot (q+1)}{2}$ (since $q \leq m < 2017$), it can be written as the sum of q integers, too. This m will then not be valid. This happens when $m \leq 1024$ and has an odd divisor > 1, which is equivalent to the fact that it is not a power of 2. Hence $m \geq 1024$.

To show that $m = 1024$ is good, observe that its only odd divisors are 1 and 2017, so if q is odd and it can be written as sum of q consecutive numbers, then $q = 1$ or $q = 2017$. Now suppose that q is even, whereby we have $N \equiv \frac{q}{2} \pmod{q}$. This means that $2N \equiv 0 \pmod{q}$, i.e. $q = 2^k 2017^\ell$ with $1 \leq k \leq 11$ and $0 \leq \ell \leq 1$. With $q \nmid N$ we must have $k = 11$, so the only choice is $q = 2^{11}$ and $q = 2^{11} \cdot 2017$. However, $q \geq 2048$ so $N \geq \frac{2048(2049)}{2} = 1024 \cdot 2049 > 1024 \cdot 2017$, contradiction. Hence $k = 2017$ is the only possibility here. Since $2a + k - 1 = 2048$ in this case, $a = 16$.

B3 Suppose that

$$f(x) = \sum_{i=0}^{\infty} c_i x^i$$

is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Solution. Consider $c = f(1/2)$, and consider the binary representation of c. We know that if c is rational, then the binary digits (after decimal point) is eventually periodic. We show that this is the same for the sequence $\{c_i\}$ too.

Clearly, if $c < 2$ and $c = \overline{d_0.d_1d_2d_3\cdots}$, is the binary representation, then putting $c_i = d_i$ we have $\sum_{i=0}^{\infty} c_i x^i = c$, too (The case $c \geq 2$ happens only when $c \geq 2$, but then $c = f(1/2) \leq 1 + 1/2 + 1/4 + \cdots = 2$, so equality must hold and we have $c_i = 1$ for all i, and therefore $\{c_i\}$ is periodic). If $\{c_i\}$ is indeed the binary representation we are done. Now, suppose that $\{c_i\}$ is not the binary representation: this means c has more than 1 way to be represented as the power series. Let $\{c_i\}$ and $\{d_i\}$ to be two different representations and let n_0 be the minimum index such that $c_{n_0} \neq d_{n_0}$. WLOG let $c_{n_0} = 0$ and $d_{n_0} = 1$. Then

$$\sum_{i=n_0+1}^{\infty} c_i/2^i = c - \sum_{i=0}^{n_0-1} c_i/2^i = 1/2^{n_0} + \sum_{i=n_0+1}^{\infty} d_i/2^i$$

But then

$$\sum_{i=n_0+1}^{\infty} c_i/2^i \leq \sum_{i=n_0+1}^{\infty} 1/2^i = 1/2^{n_0} \leq 1/2^{n_0} + \sum_{i=n_0+1}^{\infty} d_i/2^i$$

therefore equality must hold: $c_i = 0$ and $d_i = 1$, both for all $i > n_0$. Thus both $\{c_i\}$ and $\{d_i\}$ is eventually periodic with period 1 (and we are done).

Now, given that $\{c_i\}$ is eventually periodic: there is an $n_0 \geq 0$ and $m \geq 1$ such that for all $n \geq n_0$ we have $c_n = c_{n+m}$. We now have

$$\begin{aligned}
f(2/3) &= \sum_{i=0}^{\infty} \frac{2^i c_i}{3^i} \\
&= \sum_{i=0}^{n_0-1} \frac{2^i 3^{n_0-1-i} c_i}{3^{n_0-1}} + \sum_{i=n_0}^{\infty} \frac{2^i c_i}{3^i} \\
&= \sum_{i=0}^{n_0-1} \frac{2^i 3^{n_0-1-i} c_i}{3^{n_0-1}} + \sum_{i=n_0}^{n_0+m-1} c_i \left(\frac{2^i}{3^i} + \frac{2^{i+m}}{3^{i+m}} + \frac{2^{i+2m}}{3^{i+2m}} + \cdots \right) \\
&= \sum_{i=0}^{n_0-1} \frac{2^i 3^{n_0-1-i} c_i}{3^{n_0-1}} + \sum_{i=n_0}^{n_0+m-1} c_i \cdot \frac{2^i}{3^i} \cdot \frac{3^m}{3^m - 2^m}
\end{aligned}$$

and since for each i and m, all $3^{n_0-1}, 3^i$ and $3^m - 2^m$ are odd, the quantity $f(2/3)$ can be written as p/q with q odd. However, given that $f(2/3) = 1/2$ and $1/2$ doesn't have this property (2 is even and $1/2$ is irreducible: if q is odd then $\frac{q}{2}$ is not an integer), this is a contradiction. Thus $\{c_i\}$ cannot be eventually periodic and the conclusion follows.

B4 Evaluate the sum

$$\sum_{k=0}^{\infty} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5} \right)$$

$$= 3 \cdot \frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3 \cdot \frac{\ln 6}{6} - \frac{\ln 7}{7} - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3 \cdot \frac{\ln 10}{10} - \cdots.$$

Solution. (Cited from my post on AoPS) To avoid dealing with problems in absolute convergence, we deal with the n-th partial sum. That is,

$$\sum_{k=0}^{n} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right)$$

$$= 3 \cdot \frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + \cdots + 3 \cdot \frac{\ln(4n+2)}{4n+2} - \frac{\ln(4n+3)}{4n+3} - \frac{\ln(4n+4)}{4n+4} - \frac{\ln(4n+5)}{4n+5}.$$

Because the sum is finite here, we have no issue of convergence and therefore can do the following conversion:

$$\sum_{k=0}^{n} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right)$$

$$= \sum_{k=0}^{n} \left(\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} + \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right) + \sum_{k=0}^{n} 2\left(\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+4)}{4k+4}\right)$$

Also notice that

$$\sum_{k=0}^{n} 2\left(\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+4)}{4k+4}\right) = \sum_{k=0}^{n} \left(\frac{\ln 2 + \ln(2k+1)}{2k+1} - \frac{\ln 2 + \ln(2k+2)}{2k+2}\right)$$

So we have

$$\sum_{k=0}^{n} \left(\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} + \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right) + \sum_{k=0}^{n} 2\left(\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+4)}{4k+4}\right)$$

$$= \sum_{k=0}^{2n+1} \left(\frac{\ln(2k+2)}{2k+2} - \frac{\ln(2k+3)}{2k+3}\right) + \sum_{k=0}^{n} \left(\frac{\ln 2 + \ln(2k+1)}{2k+1} - \frac{\ln 2 + \ln(2k+2)}{2k+2}\right)$$

$$= \ln 2 \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) - \frac{\ln(2n+2)}{2n+2} + \sum_{k=n+1}^{2n+1} \left(\frac{\ln(2k+2)}{2k+2} - \frac{\ln(2k+3)}{2k+3}\right)$$

Now, notice that $\frac{\ln x}{x}$ is a decreasing sequence with limit 0 as $x \to \infty$. Thus $\sum_{k=0}^{2n+1} \left(\frac{\ln(2k+2)}{2k+2} - \frac{\ln(2k+3)}{2k+3}\right)$ is an alternating sum hence converges), which means that $\sum_{k=n+1}^{2n+1} \left(\frac{\ln(2k+2)}{2k+2} - \frac{\ln(2k+3)}{2k+3}\right) \to 0$ as $n \to 0$. It is also well known that $\sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) \to \ln 2$ as $n \to \infty$. Therefore $\lim_{n\to\infty} \ln 2 \sum_{k=0}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) - \frac{\ln(2n+2)}{2n+2} + \sum_{k=n+1}^{2n+1} \left(\frac{\ln(2k+2)}{2k+2} - \frac{\ln(2k+3)}{2k+3}\right) = (\ln 2)^2 + 0 + 0 = (\ln 2)^2$

B5 A line in the plane of a triangle T is called an equalizer if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

Answer. $9, 8, 7$

Solution. Throughout the solution we focus on lines that split T into equal perimeter. This line is only meaningful if it either passes through two of the sides of the triangle, or it passes through a vertex and its opposite side. In the second case, the fact that this line is an equalizer means that it has to be a median of a side, say having length c. Let m to be the length of median, then the perimeter of the first triangle is $a + \frac{c}{2} + m$ and the second, $b + \frac{c}{2} + m$. But since $a \neq b$, this cannot be an equalizer.

So now each equalizer must pass through exactly two of the sides (it has to be 2 or 0 by menelaus' theorem, and the case of 0 is impossible since it doesn't divide T at all). From now on, denote $s = \frac{a+b+c}{2}$, the semiperimeter. We consider each of the three cases (following $a > b > c$):

(a) If the line passes through sides with length b and c, let the line cut the first side into a smaller triangle of length b_1, c_1, m, with b_1 on the b-side and c_1 on the c-side. This splits T into a triangle of perimeter $b_1 + c_1 + m$ and a quadrilateral of length $a + m + (b - b_1) + (c - c_1)$, which means $b_1 + c_1 = \frac{a+b+c}{2} = s$, and the ratio of area of smaller triangle to the bigger one is $\frac{b_1 c_1}{bc}$ (for the case of equalizer, this ratio must be $\frac{1}{2}$). Given that $b - 1 + c_1 = s$, we have $b_1 c_1 = \frac{s^2-(b_1-c_1)^2}{4}$. Now considering all such lines on the two sides satisfying the perimeter constraint, we have $b_1 \leq b$ and $c_1 \leq c$, which means we have $c_1 \geq (s - b)$ and $b_1 \leq s - c$. Thus $b_1 - c_1$ has to lie in the interval $[s - 2c, 2b - s]$. Given that $b < a$ and $c < a$, when $b_1 = b$ we have $c_1 = s_b$ so the ratio of the triangle area is now $\frac{s-b}{c} = \frac{a+c-b}{2c} > \frac{1}{2}$ since $a > b$. Similarly when $c_1 = c$ we have $b_1 = s - c$ and the resulting ratio is $\frac{a+b-c}{2b} > \frac{1}{2}$ since $a > c$. Therefore we get $\frac{s^2-(b_1-c_1)^2}{4} > \frac{1}{2}bc$ when $b_1 - c_1 \in \{s - 2c, 2b - s\}$. For all $x \in [s - 2c, 2b - s]$ we either have $|x| \leq s - 2c$ or $|x| \leq 2b - s$, so we always have $\frac{s^2-(b_1-c_1)^2}{4} > \frac{1}{2}bc$. Hence no equalizer in this case.

(b) Similar to the case above we consider what happened when it passes through length a and c. Now denote a_1 and c_1 like above; we get that $a_1 - c_1$ is in the interval $[s - 2c, 2a - s]$. Now when $a_1 = a$ the resulting ratio is $\frac{(s-a)}{c} = \frac{b+c-a}{2c} < \frac{1}{2}$ while if $c_1 = c$ the ratio is $\frac{s-c}{a} = \frac{a+b-c}{2a} > \frac{1}{2}$. Thus the value $a_1 c_1 = \frac{s^2-(a_1-c_1)^2}{4} > \frac{1}{2}ac$ when $a_1 - c_1 = s - 2c$ while is $< \frac{1}{2}ac$ when $a_1 - c_1 = 2a - s$. Therefore considering x that satisfies $\frac{s^2-x^2}{4} = \frac{1}{2}ac$, we get $|x| < 2a - s$ while $|x| > s - 2c$. This implies that there's exactly one such x in the interval $[s - 2c, 2a - s]$, and has one equalizer.

(c) Finally, let the line cuts the sides a and b which forms a smaller triangle with length a_1 on side a and b_1 on side b, then $a_1 - b_1 \in [s - 2b, 2a - s]$. When $a_1 = a$ we have $b_1 = s - a$ and the area ratio becomes $\frac{s-a}{b} = \frac{b+c-a}{2b} < \frac{1}{2}$, and similarly for $b_a = b$ we get $a_1 = s - b$, so the ratio becomes $\frac{s-b}{a} = \frac{c+a-b}{2a} < \frac{1}{2}$. Thus $\frac{s^2-(a_1-c_1)^2}{4} < \frac{1}{2}ac$ when $a_1 - c_1$ is at these extreme points. If $s^2 < 2ac$ then there's no equalizer in this case; if $s^2 = 2ac$ then equalizer exists when $a_1 = c_1$ (here, $0 \in [s - 2b, 2a - s]$ since $s - 2b = \frac{a+c-3b}{2} < \frac{a-2b}{2} < 0$) as $2b > b + c > a$ by triangle inequality, and $2a - s = \frac{3a-b-c}{2} > \frac{3a-a-a}{2} > 0$); if $s^2 > 2ac$, denote x as the two solutions to $s^2 - x^2 = 2ac$. From our example we have $|x| < |s - 2b|$, $|x| < |2a - s|$ and $s - 2b < 0 < 2a - s$ so both solutions lie in the interval $[s - 2b, 2a - s]$. In this case we have two equalizers.

Now knowing all the cases above, there must be exactly 1 equalizer in the second case, and exactly 1 equalizer in the third case. The third case implies that $a_1 = b_1 = \frac{s}{2}$, which entails (by the equality of area) $\frac{s^2}{4} = \frac{1}{2}ab$, or $(a+b+c)^2 = 8ab$. For $8ab$ to be a square, we need $ac = 2 \cdot k^2$ for some k, bearing in mind that $2a > 2b > a$. Considering $k = 1, 2, \cdots$, the smallest k that has this property is when $a = 9, b = 8$, forcing $c = 7$. For $k \geq 7$ we have $a > k\sqrt{2} = 7\sqrt{2} > 9$, so $a = 9$ is the smallest possible answer.