Solutions to Tournament of Towns, Fall 2015, Senior

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O-Level

1. Let p be a prime number. Determine the number of positive integers n such that pn is a multiple of $p + n$.

   Answer. 1.

   Solution. We have $p(p+n) - pn = p^2$ so this is equivalent to saying $p + n \mid p^2$. The only positive divisors of p^2 are $1, p, p^2$, and only $p^2 > p + n$. Thus $n = p(p-1)$ is the only answer (and it works).

2. Suppose that ABC and ABD are right-angled triangles with common hypotenuse AB (D and C are on the same side of line AB). If AC = BC and DK is a bisector of angle ADB, prove that the circumcenter of triangle ACK belongs to line AD.

3. Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

   Solution. Keep another table that adds 0 for any occurence of scissor, 1 for any occurence of rock and 2 for any occurence of paper. Then after each round, the number added to the table is congruent to the score added modulo 3. Since $3 \mid 0 + 1 + 2$, the conclusion follows.

4. In a country there are 100 cities. Every two cities are connected by direct flight (in both directions). Each flight costs a positive (not necessarily integer) number of doubloons. The flights in both directions between two given cities are of the same cost. The average cost of a flight is 1 doubloon. A traveller plans to visit any m cities for m flights, starting and ending at his native city (which is one of these m cities). Can the traveller always fulfil his plans given that he can spend at most m doubloons if

   (a) $m = 99$;

   (b) $m = 100$?

5. An infinite increasing arithmetical progression is given. A new sequence is constructed in the following way: its first term is the sum of several first terms of the original sequence, its second term is the sum of several next terms of the original sequence and so on. Is it possible that the new sequence is a geometrical progression?

A-Level

1. A geometrical progression consists of 37 positive integers. The first and the last terms are relatively prime numbers. Prove that the 19^{th} term of the progression is the 18^{th} power of some positive integer.

   Solution. Consider a prime p that divides some term in the sequence, and let a_i be the highest power of p dividing the i-th term. Then either $\{a_i\}$ is an arithmetic sequence. Since the first and last term given are relatiely prime, either $a_1 = 0$ or $a_{37} = 0$. It then follows that $a_{19} = \frac{\max(a_1, a_{37})}{2}$ and since all a_i are integers, a_{19} is divisible by 18.

2. A 10×10 square on a grid is split by 80 unit grid segments into 20 polygons of equal area (no one of these segments belongs to the boundary of the square). Prove that all polygons are congruent.

   Solution. There are $10 \times 10 \times 2 - 10 - 10 = 180$ unit grid segments that are not boundary of the square. This means 100 of those segments are in the interior of the polygons.

   Now, each of the polygons has area 5 (hence comprises 5 square grids). Consider each square grid as a graph with edge iff they are adjacent on the 10×10 square. Consider, also, coloring the square in checkerboard fashion. Then here's some properties of this graph:

   - it's connected;
   - separating into white and black squares would show that this graph is bipartite;
   - two squares (of the same colour) that's distance 2 apart and on the same row/column, has exactly one common neighbour: the square in between
   - the only two squares with ≥ 2 common neighbours are those share a common vertex: the two common neighbours also have common vertex.

   We now show that the number of edges in each polygon cannot exceed 5. Given our black vs white partition, the only possible way when we have this edge count > 5 is when we have $K_{2,3}$: the sides have 2, 3 squares, and every pair of squares (of different color) are neighbours. W.l.o.g. let our polygon to have 2 whites, W_1, W_2, each with the same 3 black neighbours, B_1, B_2, B_3. Then two of the black squares, say, B_1, B_2, must lie on opposite sides of W_1. But this would mean that the only common neighbour of B_1 and B_2 can be W_1, contradiction.

   On the other hand, the fact that the 20 polygon graphs have total edge count 100 means each graph has average edge count 5. With the upper bound shown before, each graph must have edge count 5. Thus each of them must have a cycle (having > 4 edges), and with the bipartiteness, this cycle must have length 4. If A_1, A_2, A_3, A_4 are the cycle in that order then A_i and A_{i+2} have the other two as common neighbours for each i (indices taken modulo 2). Then A_1 and A_3 share a common vertex, and so do A_2 and A_4. This means, A_1, A_2, A_3, A_4 must be a 2×2 square.

   We conclude that each polygon is 2×2 square, plus another square attached to this 2×2 square. It therefore follows that all those polygons are congruent.

3. Each coefficient of a polynomial is an integer with absolute value not exceeding 2015. Prove that every positive root of this polynomial exceeds $\frac{1}{2016}$.

   Solution. By dividing by x^k for some k, i.e. removing all the root at $x = 0$, we may assume that the constant term is nonzero. Let y be a positive root. Then

$$0 = P(y) := \sum_{k=0}^{n} a_k y^k$$

Thus we have

$$1 \leq |a_0| = | - \sum_{k=1}^{n} a_k y^k | \leq \sum_{k=1}^{n} |a_k y^k|$$

Suppose, now, $|y| \leq \frac{1}{2016}$. Then with $|a_k| \leq 2015$ for each k we have

$$\sum_{k=1}^{n} |a_k y^k| \leq \sum_{k=1}^{n} 2015 \cdot \frac{1}{2016^k} = 2015 \left(\frac{1}{2016} \frac{1-(1/2016)^n}{1-1/2016} \right) < 1$$

i.e. contradiction.

4. Let $ABCD$ be a cyclic quadrilateral, K and N be the midpoints of the diagonals and P and Q be points of intersection of the extensions of the opposite sides. Prove that $\angle PKQ + \angle PNQ = 180$.

   Solution. W.l.o.g. let rays BA and CD intersect at P and rays DA and CB intersect at Q. Also consider K as midpoint of BD and N the midpoine of AC. Then triangles PAC and PDB are similar, so $\angle APK = \angle DPN$, i.e. PK and PN are reflections of each other w.r.t. the internal angle bisector of $\angle BPC$. Similarly, QK and QN are reflections of each other w.r.t. the internal angle bisector of $\angle CQD$. Therefore, we have $\angle PKQ + \angle PNQ = \angle PAQ + \angle PCQ = \angle PBQ + \angle PDQ = 180°$, the last two equality is because $ABCD$ cyclic.