Solutions to Tournament of Towns, Spring 2022, Senior

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A-Level

1. For each of the nine positive integers $n, 2n, 3n, \cdots, 9n$ Alice takes the first decimal digit (from the left) and writes it onto a blackboard. She selected n so that among the nine digits on the blackboard there is the least possible number of different digits. What is this number of different digits equal to?

   Answer. 4. This is achieved by taking $n = 29$, and the different digits are $1, 2, 5, 8$.

   Solution We now show that this cannot be improved. Let's change the setting and consider n not an integer, but a rational number with $1 \leq n < 10$, instead (this can be done by dividing n by 10^k for some integer k), and let $d = \lfloor n \rfloor$, the leading digit of n.

   Let m be the minimum integer such that $mn \geq 10$ (i.e. $2 \leq m \leq 10$). Then:

   - $n, 2n, \cdots, (m-1)n$ all have different leading digits, all at least d.
   - $mn, \cdots, 9n, 10n$ are all ≥ 10 and < 100, so the leading digits differ by at most 1 at each step. Considering $(m-1)n$, too, means leading digit of mn is 1. Since the leading digit of $10n$ is d, it follows that we must have the leading digits of $mn, \cdots, 9n$ as $1, \cdots, d-1$, and maybe d too.

   Thus the number of digits is $d - 1 + (m-1) = \lfloor n \rfloor + \lceil \frac{10}{n} \rceil - 2$, and in all cases, $\lceil \frac{10}{n} \rceil \geq 2$ since $n < 10$. So if $\lfloor n \rfloor \geq 4$ the number of digits is ≥ 4, and if $\lfloor n \rfloor = 1$ then $\lceil \frac{10}{n} \rceil = 6$, giving a total of $\geq 1 + 6 - 2 = 5$. Thus we only need to consider $d = 2, 3$.

   When $d = 2$, $\lfloor n \rfloor + \lceil \frac{10}{n} \rceil \geq \lceil \frac{10}{3} \rceil = 4$, when $d = 3$, $\lfloor n \rfloor + \lceil \frac{10}{n} \rceil \geq \lceil \frac{10}{4} \rceil = 3$, so in both cases the number of digits is ≥ 4.

2. On a blank paper there were drawn two perpendicular axes x and y with the same scale. The graph of a function $y = f(x)$ was drawn in this coordinate system. Then the y axis and all the scale marks on the x axis were erased. Provide a way how to draw again the y axis using pencil, ruler and compass if:

   (a) $f(x) = 3^x$;

   (b) $f(x) = \log_a x$, where $a > 1$ is an unknown number.

   Solution. The main goal of both part is to draw a distance of exactly 1. For $f(x) = 3^x$, this can be achieved via the following:

   - choose any point on the graph, say, $A = (x_0, 3^{x_0})$, with x_0 unknown.
   - Draw the line perpendicular from A to the x-axis, which intersects the x-axis at $B = (x_0, 0)$.
   - Let C be on line AB such that $\vec{BC} = 3\vec{BA}$, we know $C = (x_0, 3^{x_0+1})$.
   - Let D be the foot of perpendicular to AB to $f(x)$, we have $D = (x_0 + 1, 3^{x_0+1})$. Hence we have $CD = 1$.

Now that we know how to draw distance 1, we can draw the line $y = 1$ by referencing to the x-axis. Then the intersection of $y = 1$ and $f(x)$ is $(0, 1)$, and the y-axis can be taken as the perpendicular from $(0, 1)$ to the x-axis.

For $f(x) = \log_a x$, we may draw a line ℓ_1 parallel to the x-axis and above the x-axis, and ℓ_2 that's the x-axis reflected in ℓ_1. Now consider A_1, A_2 the intersections of ℓ_1 and ℓ_2 (respectively) with the $f(x)$, and B_1, B_2 their projection to the x-axis. We have $B_1 = (x_0, 0)$ and $B_2 = (x_0^2, 0)$ for some unknown $x_0 \geq 1$. We also have the point $(1, 0)$, which is just the intersection of x-axis and $f(x)$.

5. What is the maximal possible number of roots on the interval (0, 1) for a polynomial of degree 2022 with integer coefficients and with the leading coefficient equal to 1?

   Answer. 2021.

6. The king assembled 300 wizards and gave them the following challenge. For this challenge, 25 colors can be used, and they are known to the wizards. Each of the wizards receives a hat of one of those 25 colors. If for each color the number of used hats would be written down then all these numbers would be different, and the wizards know this. Each wizard sees what hat was given to each other wizard but does not see his own hat. Simultaneously each wizard reports the color of his own hat. Is it possible for the wizards to coordinate their actions beforehand so that at least 150 of them would report correctly?

   Answer. Yes.

   Solution. If the frequencies of the colours used are $c_0, \cdots, c_{24}$ in some order, then $\sum c_i = 300$. On the other hand, all c_i are different, which gives another bound:

$$300 = \sum c_i \geq 0 + 1 + \cdots + 24 = 300$$

so $\{c_i\} = \{0, \cdots, 24\}$ in some order. This means, when wearing the hats later, if the color worn by a wizard has frequency $i \geq 1$, then the wizard will see the colour frequency as $0, 1, \cdots, i-1, i-1, i+1, \cdots, 24$, i.e. they will only have to decide between the two.

Now, the wizards can do the following coordination: label the colours as $0, \cdots, 24$, and the wizards themselves as $1, \cdots, 300$. Let c_i be the frequency of colour i, which is a permutation σ of $0, \cdots, 24$, used later on. Then for each wizard:

- The wizard knows all c_i except for two values, x and y.
- The wizard also knows $\{c_x, c_y\} = \{j-1, j\}$, the hat they have is either x or y, with frequency j.
- One of the selection ($c_x = j$, or $c_y = j$) will yield σ (and therefore correct hat). Since swapping any two elements in a permutation changes its parity, the other permutation has parity different from σ.

Thus the wizard will choose the permutation with the same parity as their own label. Consequently, regardless of σ, the wizards can always achieve exactly accuracy count of 150.