Solution to APMO 2015 Problems

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1. Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y . The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point Z distinct from point B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $AB = VW$

   Solution. It suffices to show that the angle subtended by AB and VW with respect to the circumcircle of ABC are equal. In particular, we'll show that $\angle ACB$ and $\angle VZW$ are either equal or supplementary.

   We first show that D, C, Y, Z are concyclic. Below, denote (1) as A, B, Z, C concyclic and (2) as B, X, D, Z concyclic. Using the notion of directed angles we have

   $$\angle(ZC, CY) = \angle(ZC, CA) \overset{(1)}{=} \angle(ZB, BA) = \angle(ZB, BX) \overset{(w)}{=} \angle(ZD, DX) = \angle(ZD, DY)$$

   and therefore D, C, Y, Z are concyclic. Therefore,

   $$\angle(VZ, ZW) = \angle(DZ, ZY) = \angle(DC, CY) = \angle(BC, AC)$$

   showing that angles VZW and $\angle ACB$ are indeed either equal or supplementary.

2. Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

   $$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

   Answer. No.

   Solution. We'll focus on the following setup:

   $$f(2^k)f(2^\ell) = f(2^{2(k+\ell)}) \text{ for all } k, \ell \geq 1 \text{ with } k \neq \ell$$

   Consider any prime p and denote $v_p(a)$ as the highest power of p dividing a. Now,

   $$v_p(f(2^{2(k+\ell)})) = v_p(f(2^k)f(2^\ell)) = v_p(f(2^k)) + v_p(f(2^\ell))$$

   so if $g : \mathbb{N} \to \mathbb{N}_0$ are defined as $g(k) = v_p(f(2^k))$ we have

   $$g(k) + g(\ell) = g(2(k + \ell)), \forall k \neq \ell$$

   Let $n \geq 3$ be arbitrary. This means we have

   $$g(n) + g(2) = g(2(n + 2)) = g(1) + g(n + 1)$$

   and so $g(n + 1) - g(n) = g(2) - g(1)$ for all $n \geq 3$. Next, we have

   $$g(3) + g(4) = g(2(7)) = g(2) + g(5)$$

   and therefore $g(3) - g(2) = g(5) - g(4) = g(2) - g(1)$ and so $g(n + 1) - g(n) = g(2) - g(1)$ for all $n \geq 1$. This means g is linear and there exists m, c with $g(n) = mn + c$ for all $n \geq 1$. However for all $a \neq b$ we also have $g(a) + g(b) = g(2(a + b))$ which translates into

   $$ma + c + mb + c = m(2(a + b)) + c$$

which gives $c = m(a+b)$. Plugging $a = 1, b = 2$ gives $c = 3m$; plugging $a = 2, b = 3$ gives $c = 5m$. This means $m = c = 0$ and consequently g is a zero function. In other words, $p \nmid f(2^k)$ for all $k \geq 1$. Since this works for any p, $f(2^k) = 1$ for all $k \geq 1$, which contradicts $f(2^k) \in S$.

3. A sequence of real numbers $a_0, a_1, \ldots$ is said to be good if the following three conditions hold.

   (i) The value of a_0 is a positive integer.

   (ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \frac{a_i}{a_i+2}$

   (iii) There exists a positive integer k such that $a_k = 2014$.

   Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

   Answer. $n = 60$.

   Solution. Given that $a_0 > 0$, this sequence will comprise of positive rational numbers. This means, for each i, if $a_{i+1} = 2a_i + 1$ then $a_{i+1} > 1$ and if $a_{i+1} = \frac{a_i}{a_i+2}$ then $a_{i+1} < 1$. This means given a_{i+1}, a_i is determined uniquely given by:

$$a_i = \begin{cases} \frac{a_{i+1}-1}{2} & a_{i+1} > 1 \\ \frac{2a_{i+1}}{1-a_{i+1}} & a_{i+1} < 1 \end{cases}$$

   Denoting sequence b as $b_i = a_{n-i}$, we have $b_0 = 2014, b_n$ an integer, and let (p_i, q_i) be pairs of integers satisfying $\gcd(p_i, q_i) = 1$ and $b_i = \frac{p_i}{q_i}$. Then:

$$b_{i+1} = \begin{cases} \frac{p_i - q_i}{2q_i} & p_i > q_i \\ \frac{2p_i}{q_i - p_i} & p_i < q_i \end{cases}$$

   Note that $(p_0, q_0) = (2014, 1)$. We first show that p_i and q_i will always have different parity, and that the form of b_{i+1} above is already in its lowest form. The base case $(2014, 1)$ satisfies this. Now given (p_i, q_i) there are two cases as detailed above. If $p_i > q_i$ then we're looking at $(p_i - q_i, 2q_i)$ and now $p_i - q_i$ is odd (as p_i and q_i had different parity) and $2q_i$ even. Moreover, if any prime r divides $p_i - q_i$ and $2q_i$ simultaneously, since $p_i - q_i$ is odd this r has to be odd and therefore r divides p_i and q_i simultaneously, contradicting that $\gcd(p_i, q_i) = 1$. Therefore $\gcd(p_i - q_i, 2q_i) = 1$ and similarly, $\gcd(2p_i, q_i - p_i) = 1$. In addition, $(p_i - q_i) + 2q_i = 2p_i + (q_i - p_i) = q_i + p_i$ so the sum of $p_i + q_i$ is the same across $i = 0, 1, \cdots, n$, hence equal to $p_0 + q_0 = 2015$.

   Let $k = 2015$. If $p_i > q_i$ then $p_{i+1} = p_i - q_i = p_i - (k - p_i) = 2p_i - k \equiv 2p_i$ and $2q_i \equiv 2q_i \pmod{k}$. Similarly if $p_i < q_i$ then $2p_i \equiv 2p_i \pmod{k}$ and $q_i - p_i = q_i - (k - q_i) = 2q_i - k \equiv 2q_i \pmod{k}$. Therefore speaking in modulo k, both the denominator and numerator doubled for each iteration. In order for b_n to be an integer, we need $q_n = 1$ and given $q_n \equiv 2^n q_0 = 2^n \pmod{k}$, it remains to find the smallest n with $2^n \equiv 1 \pmod{k}$.

   This leads to us finding the order of 2 modulo 2015=$5 \times 13 \times 31$. The minimum positive n with $2^n \equiv 1$ modulo $5, 13$, and 31 are 4, 12, 5 and therefore the n we're looking for here is $lcm(4, 12, 5) = 60$.

4. Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n - 1$ points, and also intersects $\mathcal{R}$ in exactly $2n - 1$ points.

   Solution. The goal can be restated as to find a circle that is tangent to exactly one red line and exactly one blue line, and intersect all other lines at two points.

To start with, consider the following terminology governing a pair of lines ℓ_1 and ℓ_2, separting the plane into four regions. Let the two points intersect at C. Let a line ℓ_3 intersects ℓ_1 at A and ℓ_2 at B. We call the region containing segment AB "incident to ℓ_3", the region that's exclusive of line AB as "opposite to ℓ_3, and the other two regions as "at the side of ℓ_3". Now, any circle tangent to both ℓ_1 and ℓ_2 must lie in exactly one region. We consider such circles and whether it will intersect any line ℓ_3 that does not concur with ℓ_1 and ℓ_2, with the following claims:

- If the circle lies in the region opposite to ℓ_3, then it won't intersect ℓ_3.
- If the circle ω lies in the region incident to ℓ_3, then there exists $r_1 < r_2$ such that ℓ_3 intersects the circle at two points if and only if the radius of ω is in the interval (r_1, r_2).
- If the circle ω lies in a region at the side of ℓ_3, then there exists r such that ℓ_3 intersects the circle at two points if and only if the radius of ω is greater than r.

The proof for the first one (opposite) is trivial (the region has no intersection with ℓ_3), and for the second one, if we denote A, B, C as before then r_1 is the inradius of triangle ABC and r_2 is the radius of excircle of ABC opposite C. It's the third one that deserves our attention. Now, the excircles opposite B and opposite A lie in the two different regions at the side of r_3, we consider just the region corresponding to the one opposite A. Consider all such circles in the region, and their radius r. If $r \to 0$ the it will approach C and has no intersection with ℓ_3. We're therefore interested to investigate what happens when r varies. Between the shift from not intersecting ℓ_3 to intersecting ℓ_3 at two points (and vice versa) there must be an intermediate point r_0 where the circle intersects ℓ_3 at exactly one point (tangent), which happens only once when this circle is the excircle of ABC opposite A. We could then infer that any bigger circle will intersect ℓ_3 at two points.

(Another way to justify it is to identify the center of all such circles which all lie on an angle bisector of ℓ_1 and ℓ_2, and using some trigonometry we see that the radius grows faster than the distance from ℓ_3 if and only if the circle lies in the region a the side of ℓ_3).

Now, consider the bearings of the $2n$ lines with, say, the x-axis ℓ_0 and we sort these lines ℓ according to the value $\angle(\ell, \ell_0)$. Since we are taking modulo $180°$. these lines (after being sorted) can be arranged in a circle using this sorting algorithm (therefore giving a cyclic relation). This means that we can choose a red line r and a blue line b adjacent to each other on this circle (that is, adjacent to each other on this sorting system).

This means that for r and b, by the nature of the positions in the rankings of the lines, there will be two regions out of 4 defined by r and b that are on the side of all other $2n-2$ lines (and the other two regions are either incident or opposite each of the $2n-2$ lines). Therefore, there exists a threshold r_0 such that for all circles tangent to r and b and in the regions on the side of all other $2n-2$ lines and have radius greater than r_0, these circles must all intersect each of the $2n-2$ lines in two points.

5. Determine all sequences $a_0, a_1, a_2, \dots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

   (i) a_{n+2} is divisible by a_n ;

   (ii) $|s_{n+1} - (n+1)a_n| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0$.