Solution to APMO 2022 Problems

Anzo Teh

Problem 1. Find all pairs (a, b) of positive integers such that a^3 is multiple of b^2 and $b-1$ is multiple of $a-1$.

Answer. All pairs in the form $a = b$, and also $b = 1$.

Solution. We can easily show that the combinations above would work, so it remains to show that these are the only combinations.

Now, let $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ (i.e. prime factorize), then $b = p_1^{\beta_1} \cdots p_k^{\beta_k}$ with $0 \le \beta_i \le \frac{3}{2}\alpha_i$ to satisfy the condition $b^2 | a^3$. In addition, $\gcd(p_i, a-1) = 1$ for all the primes p_i, so we have

$$p_1^{\beta_1} \cdots p_k^{\beta_k} \equiv 1 \pmod{a-1}$$

and since we already have

$$p_1^{\alpha_1} \cdots p_k^{\alpha_k} \equiv 1 \pmod{a-1}$$

we can take multiplicative inverse to get

$$p_1^{\beta_1-\alpha_1} \cdots p_k^{\beta_k-\alpha_k} \equiv 1 \pmod{a-1}$$

which then becomes

$$\prod_{\beta_i \ge \alpha_i} p_i^{\beta_i-\alpha_i} \equiv \prod_{\beta_i < \alpha_i} p_i^{\alpha_i-\beta_i} \pmod{a-1}$$

Now the LHS is at most $\prod_i p_i^{\alpha_i} = \sqrt{a}$, and RHS is at most a (both sides are at least 1). Also the gcd of the two sides is 1 so the only possibilities are:

- LHS=RHS=1, which implies $\alpha_i = \beta_i$ for all i and so $a = b$
- $|LHS - RHS| \ge a - 1$, the only possibility is when $b = 1$.

Problem 2. Let ABC be a right triangle with $\angle B = 90°$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the seconf intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.

Solution. Let EF intersect lineBC at G, and we claim that $BC = BG$, i.e. EF will always pass through the reflection of C in B regardless of D.

We see that by focusing on the triangles ACD and BDE, the point F is the center of spiral similarity that brings FEB to FAC, so

$$\frac{FE}{FB} = \frac{EA}{BC} = \frac{EB}{BC}$$

where the equality $EA = EB = ED$ follows from that $\angle B = 90°$. Now, the inversion with center E and radius $ED = BE$ maps line CG to circle EDB (and vice versa), so F and G are mapped to each other in this inversion, resulting in

$$\frac{EF}{EB} = \frac{EB}{EG}$$

i.e. we have triangles EFB and EBG similar. Therefore

$$\frac{FE}{FB} = \frac{BE}{BG} = \frac{BE}{BC}$$

i.e. $BG = BC$ as desired.

Problem 3. Answer. 1, 100, 101, 201.

Solution. Let's first give the constructions for those: we have these working pairs (k, n):

$$(1, 101), (100, 2), (101, 99), (201, 1)$$

The cases for $k = 1, 100, 201$ are pretty clear, let's show for $k = 101$, where, the numerators (in that order) are

$$99, 198, 95, 194, \cdots, 3, 102, 201, 98, \cdots, 105, 2, 101$$

which can be rearranged into

$$2+\cdots+6+\cdots+198+(3+\cdots+99+(101+105+\cdots+201) = 100\cdot 50+51\cdot 25+\cdot 151\cdot 26 = 101^2$$

as desired.

To show that the others won't work, we first notice that the condition implies

$$n + 2n + \cdots + kn \equiv 101k$$

so we need $n\frac{k(k+1)}{2}$ to be divisible by 101. The only k with $101 \mid k(k+1)$ are 100, 101, 201, so we may now assume that $101 \mid n$, which then leaves us with $n = 101$. From here, we see that only $k = 1$ works since the sequence alternates in $\frac{1}{2}, 0, \frac{1}{2}, 0, \cdots$.

Problem 4. Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labelled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i + 1$. Cathy wins if at any point there is a box containing only marble n. Determine all pairs of integers (n, k) such that Cathy can win this game.

Note: I guess that the answer should be all $n \geq 2^{k-1}$. What I know is that the equality case $n = 2^{k-1}$ does have an iterative algorithm, and the numbers in a box must be consecutive. The rest are TODO.

Problem 5. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Determine the minimum value of $(a - b)(b - c)(c - d)(d - a)$ and determine all values of (a, b, c, d) such that the minimum value is achived.

Answer. $\frac{-1}{8}$, realized by the following:

$$\left(-\frac{\sqrt{3}}{4} - \frac{1}{4}, -\frac{\sqrt{3}}{4} + \frac{1}{4}, +\frac{\sqrt{3}}{4} - \frac{1}{4}, +\frac{\sqrt{3}}{4} + \frac{1}{4}\right)$$

assigned to $(a, b, c, d), (d, c, b, a)$, and all their cyclic shifts.

Solution. It now remains to show that $-\frac{1}{8}$ is the optimal value. Since it's negative, among $a - b, b - c, c - d, d - a$, either exactly 1 or 3 are positive. This means that there's a cyclic shift among (a, b, c, d) that's either monotonically increasing, or monotonically decreasing. These cases are symmetric, so we may assume $a < b < c < d$.

Next, let's show that we can consider only the case $a + b + c + d = 0$. Indeed, let m be such that $a + b + c + d = 4m$, and consider the mapping $f : x \to \frac{x-m}{\sqrt{1-4m^2}}$. Then

$$f(a)^2 + f(b)^2 + f(c)^2 + f(d)^2 = 1$$

$$(f(a)-f(b))(f(b)-f(c))(f(c)-f(d))(f(d)-f(a)) = \frac{(a-b)(b-c)(c-d)(d-a)}{(1-4m^2)^2} \leq (a-b)(b-c)(c-d)(d-a)$$

since $(a-b)(b-c)(c-d)(d-a) < 0$ and $1-4m^2 \leq 1$.

We now show that we can consider only the case where $|d-c| = |b-a|$, which now becomes $d-c = b-a$ since $a < b < c < d$. Indeed, consider, now, the tuples (a', b', c', d') with sum 0, $c-b = c'-b'$ and $d'-c' = b'-a' = \frac{b-a+d-c}{2}$ Then $d'-a' = d-a$. In addition we have $a = -d$ and $c = -b$, so

$$a^2+b^2+c^2+d^2 \geq \frac{(d-a)^2}{2} + \frac{(c-b)^2}{2} = a'^2+b'^2+c'^2+d'^2$$

while

$$(a-b)(b-c)(c-d)(d-a) = (a-b)(b'-c')(c-d)(d'-a') \geq (a'-b')(b'-c')(c'-d')(d'-a')$$

where we used $(a-b)(c-d) \leq (\frac{d-c+b-a}{2})^2 = (a'-b')(c'-d')$ (since $a-b$ and $c-d$ have the same sign). Thus by rescaling we can actually attain

$$\frac{1}{(a'^2+b'^2+c'^2+d'^2)^2}(a'-b')(b'-c')(c'-d')(d'-a') \leq (a'-b')(b'-c')(c'-d')(d'-a') \leq (a-b)(b-c)(c-d)(d-a)$$

proving the claim.

Now that $a+b+c+d = 0$ and $d-c = b-a$, we have $a = -d$ and $b = -c$, so $(a,b,c,d) = (-y, -x, x, y)$ for some $x, y > 0, y > x$ with $x^2+y^2 = \frac{1}{2}$. This gives

$$(a-b)(b-c)(c-d)(d-a) = -4(x-y)^2xy = -4\left(\frac{1}{2}-2xy\right)xy = 8\left((xy-\frac{1}{8})^2 - \frac{1}{64}\right) \geq -\frac{1}{8}$$

which shows $-\frac{1}{8}$ is indeed optimal. Equality holds when we have $x^2+y^2 = \frac{1}{2}$ and $xy = \frac{1}{8}$, i.e. $(x+y)^2 = \frac{3}{4}$ and $(x-y)^2 = \frac{1}{4}$. Using $y > x > 0$ we can solve these to get $y = \frac{\sqrt{3}}{4} + \frac{1}{4}$ and $x = \frac{\sqrt{3}}{4} - \frac{1}{4}$.