Since $\angle CHS - \angle CSB = 90^\circ$, if K is the circumcentre of $\triangle CHS$ then reflex angle $\angle CKS = 2\angle CHS$ hence non-reflex angle $\angle CKS = 360^\circ - 2\angle CHS$, and $\angle KSC = \angle KCS = 90^\circ - \frac{\angle CKS}{2} = \angle CHS - 90^\circ = \angle CSB \Rightarrow \angle CSB = \angle CSK$, and K lies on line AB. Similarly let L ~~st~~ the circumcentre of THC, then L lies on line AD.

This implies $K =$ perpendicular bisector of $CH \cap$ $\cancel{K}$ AB and $L =$ perpendicular bisector of $CH \cap AD$.

Now, if H' is the perpendicular from C to BD, then C and C' are symmetric to each other w.r.t. midpoint of BD as $\angle CBA = \angle CDA = 90^\circ$. Let O to be circumcentre of $\triangle HST$ and we are to prove that O lies on AH, so with $AH \perp BD$, BD will be tangent to this circle. Now, with Ceva's theorem (trig version) applied on $\triangle AKL$ and points O, H, C, we have $1 = \frac{\sin\angle AKH}{\sin\angle HKL} \times \frac{\sin\angle KLH}{\sin\angle HLD} \times \frac{\sin\angle LAH}{\sin\angle KAH} = \frac{\sin 2\angle SCH}{\sin\angle CSH} \times \frac{\sin\angle HTC}{\sin 2\angle HCT} \times \frac{\sin\angle BDC}{\sin\angle BCD}$,

and $1 = \frac{\sin\angle AKC}{\sin\angle \cancel{HCL}\, CKL} \times \frac{\sin\angle \cancel{CLH}\, KLC}{\sin\angle CLD} \times \frac{\sin\angle LAC}{\sin\angle KAC} = \frac{\sin 2\angle(180^\circ - \angle CHS)}{\sin\angle CSH} \times \frac{\sin\angle HTC}{\sin 2(180^\circ - \angle CHT)} \times \frac{\sin\angle CBD}{\sin\angle CDB}$ (as $ABCD$ is cyclic). Therefore,

$\frac{\sin 2\angle SCH}{\sin\angle CSH} \times \frac{\sin\angle HTC}{\sin 2\angle HCT} \times \frac{\sin 2(180^\circ - \angle CHS)}{\sin\angle CSH} \times \frac{\sin\angle HTC}{\sin 2(180^\circ - \angle CHT)} = 1$

Therefore, $\frac{\sin^2\angle HTC}{\sin^2\angle CSH} \times \left(\frac{\sin\angle SCH \cos\angle SCH \sin\angle CHS \cos(180^\circ-\angle CHS)}{\sin\angle HCT \cos\angle HCT \sin\angle CHT \cos(180^\circ-CHT)}\right) = 1.$

Now, $1 = \frac{\sin\angle AKO}{\sin\angle LKO} \times \frac{\sin\angle KLO}{\sin\angle ALO} \times \frac{\sin\angle LAO}{\sin\angle KAO}$, and we are to prove that

$\frac{\sin\angle LAO}{\sin\angle KAO} = \frac{\sin\angle LAH}{\sin\angle KAH}$. Observe that, we have

$$\frac{\sin\angle AKO \times \sin\angle KLO}{\sin\angle LKO \times \sin\angle ALO} = \frac{\sin\angle SCH \times \sin\angle CHT}{\sin\angle CHS \times \sin\angle TCH}.$$

It then suffices to prove that $\frac{\sin\angle SCH \times \sin\angle CHT}{\sin\angle CHS \times \sin\angle TCH} = \frac{\sin 2\angle SCH \times \sin\angle HTC}{\sin\angle CSH \times \sin 2\angle HCT}$

or $\frac{\sin\angle CHT}{\sin\angle CHS} = \frac{\cos\angle SCH \times \sin\angle HTC}{\sin\angle CSH \times \cos\angle HCT}$. Notice that by above,

$$\frac{\cos\angle SCH \times \sin\angle HTC}{\sin\angle CSH \times \cos\angle HCT} \times \frac{\sin^2\angle CSH \sin\angle HCT \cos\angle HCT \sin\angle CHT \cos(180^\circ-CHS)}{\sin^2\angle HTC \sin\angle SCH \cos\angle SCH \sin\angle CHS \cos(180^\circ-\angle CHT)}$$

$$= \frac{\sin\angle CSH \sin\angle HCT \cos(180^\circ-CHS) \sin\angle CHT}{\sin\angle HTC \sin\angle SCH \cos(180^\circ-\angle CHT) \sin\angle CHS},$$ so the equality

in turn ~~befo~~ becomes $\frac{\sin\angle CSH \sin\angle HCT \cos(180^\circ-CHS)}{\sin\angle HTC \sin\angle SCH \cos(180^\circ-\angle CHT)} = 1$

However, $180^\circ-\angle CHS = \angle SHX$, $180^\circ-\angle CHT = \angle THX$ where $X = ST \cap CH$.

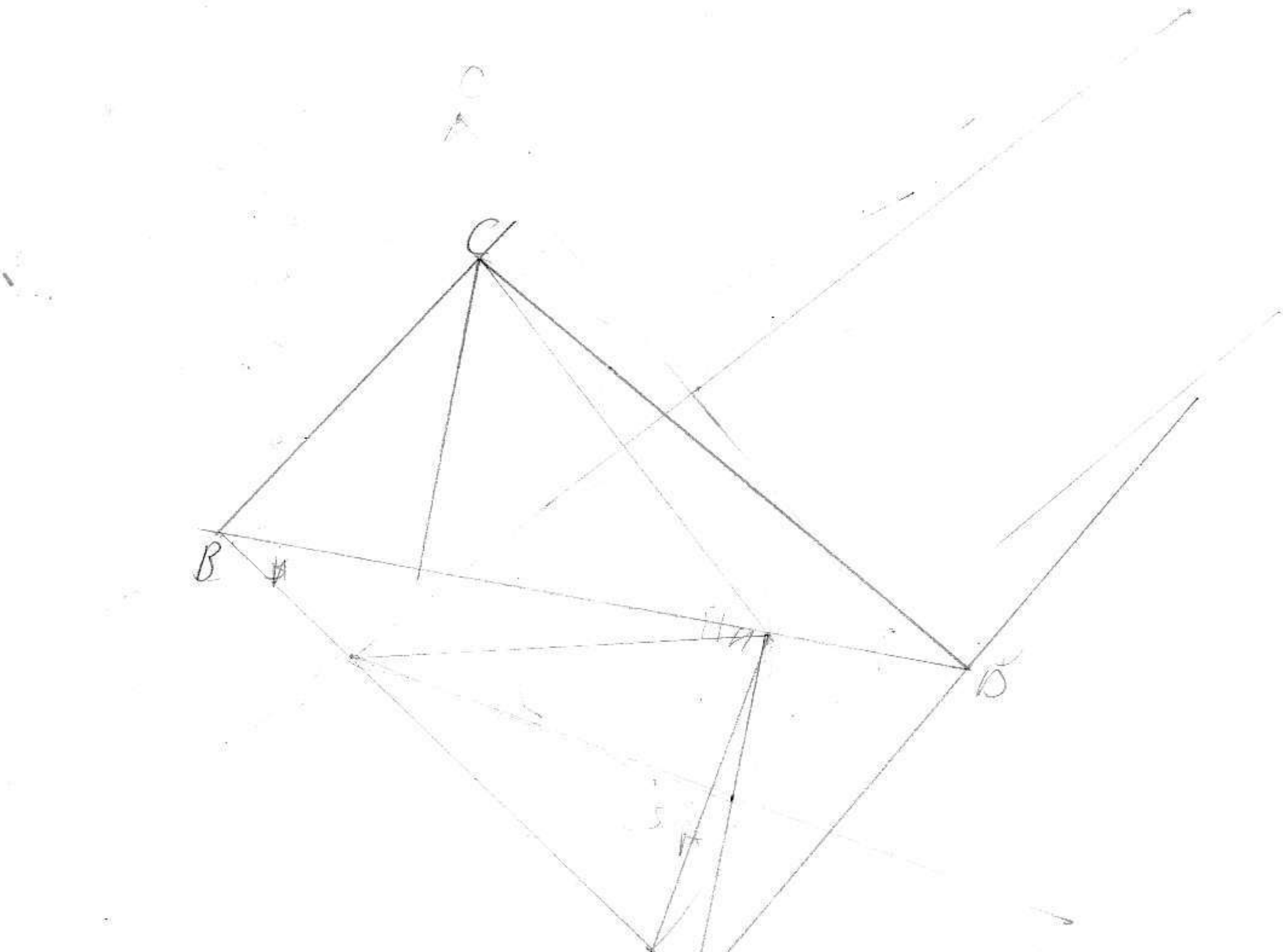
Moreover, considering Ceva in $\triangle SCT$ we have $\frac{\sin\angle CSH \times \sin\angle HCT}{\sin\angle SCH \sin\angle HTC} \times \frac{\sin\angle HTS}{\sin\angle HST} = 1$,

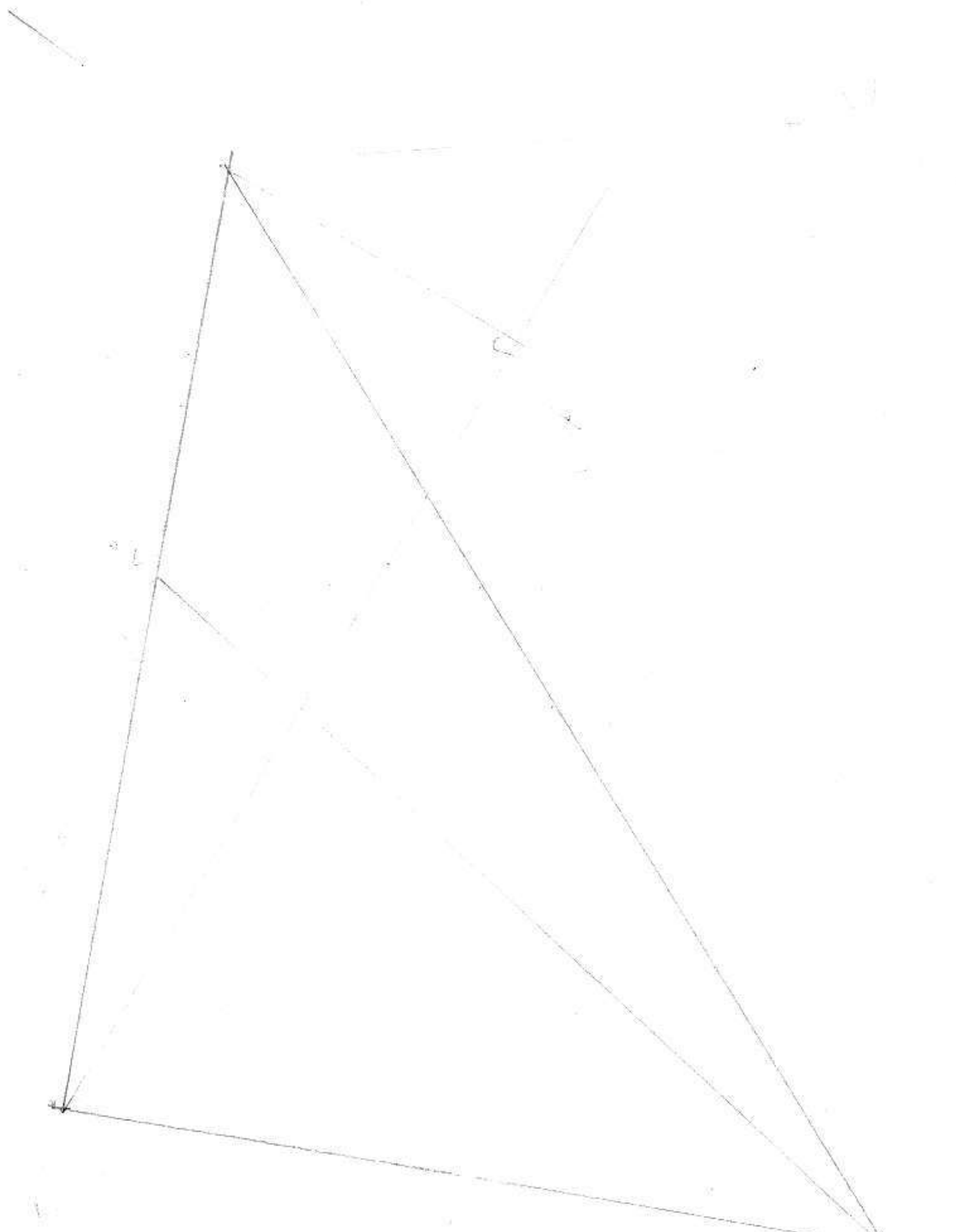
so we want to prove $\frac{\cos\angle SHX}{\cos\angle THX} = \frac{\sin\angle HTS}{\sin\angle HST}$. It suffices to prove

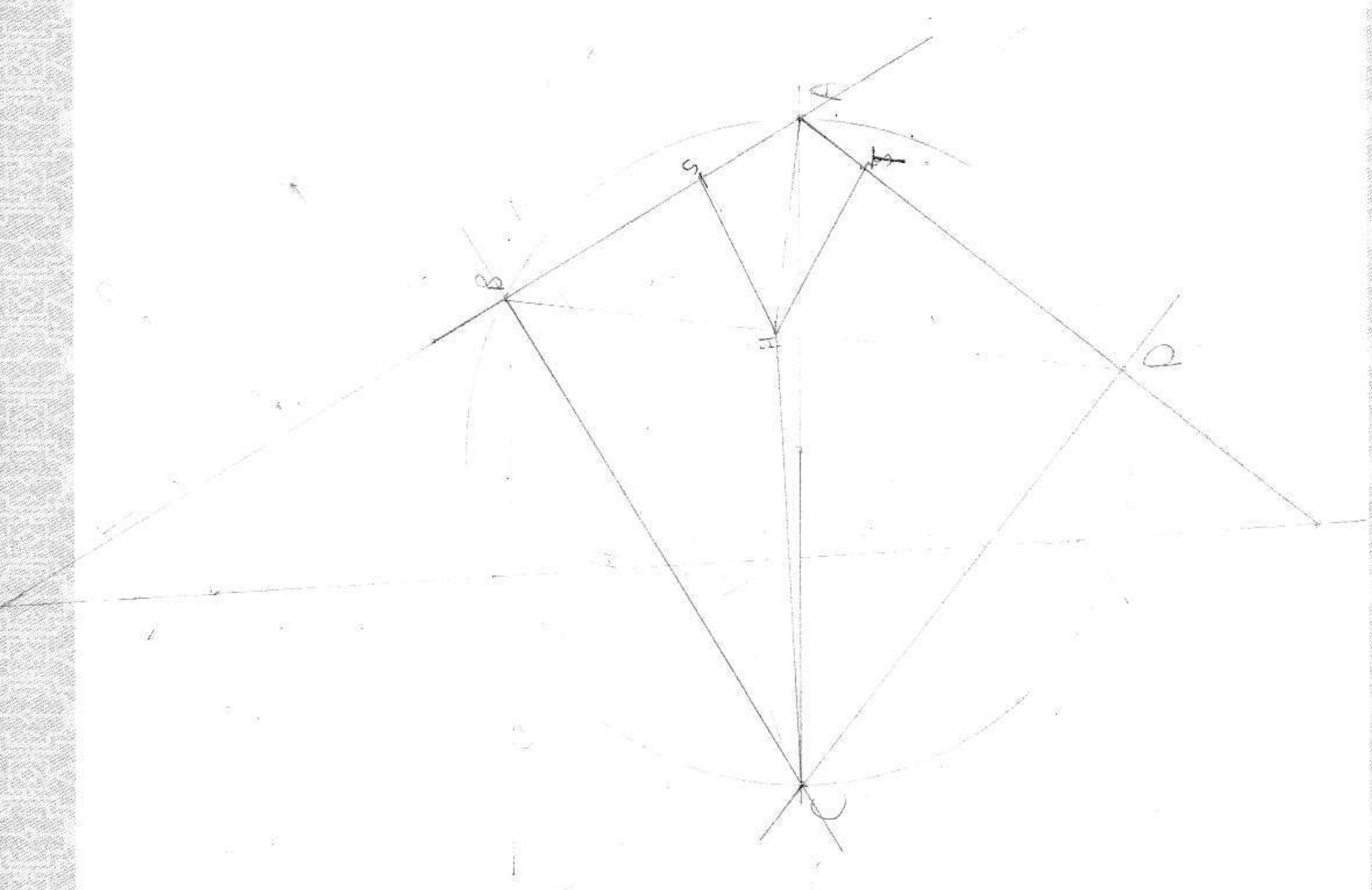
that $ST \perp CH$, or $KL \parallel ST$.

To show $KL \parallel ST$, we need $\frac{KS}{LT} = \frac{KA}{LA}$, or $\frac{KH}{LH} = \frac{KA}{LA}$.

However this follows from the fact that H and H' are images of each other w.r.t. midpoint of BO.






A
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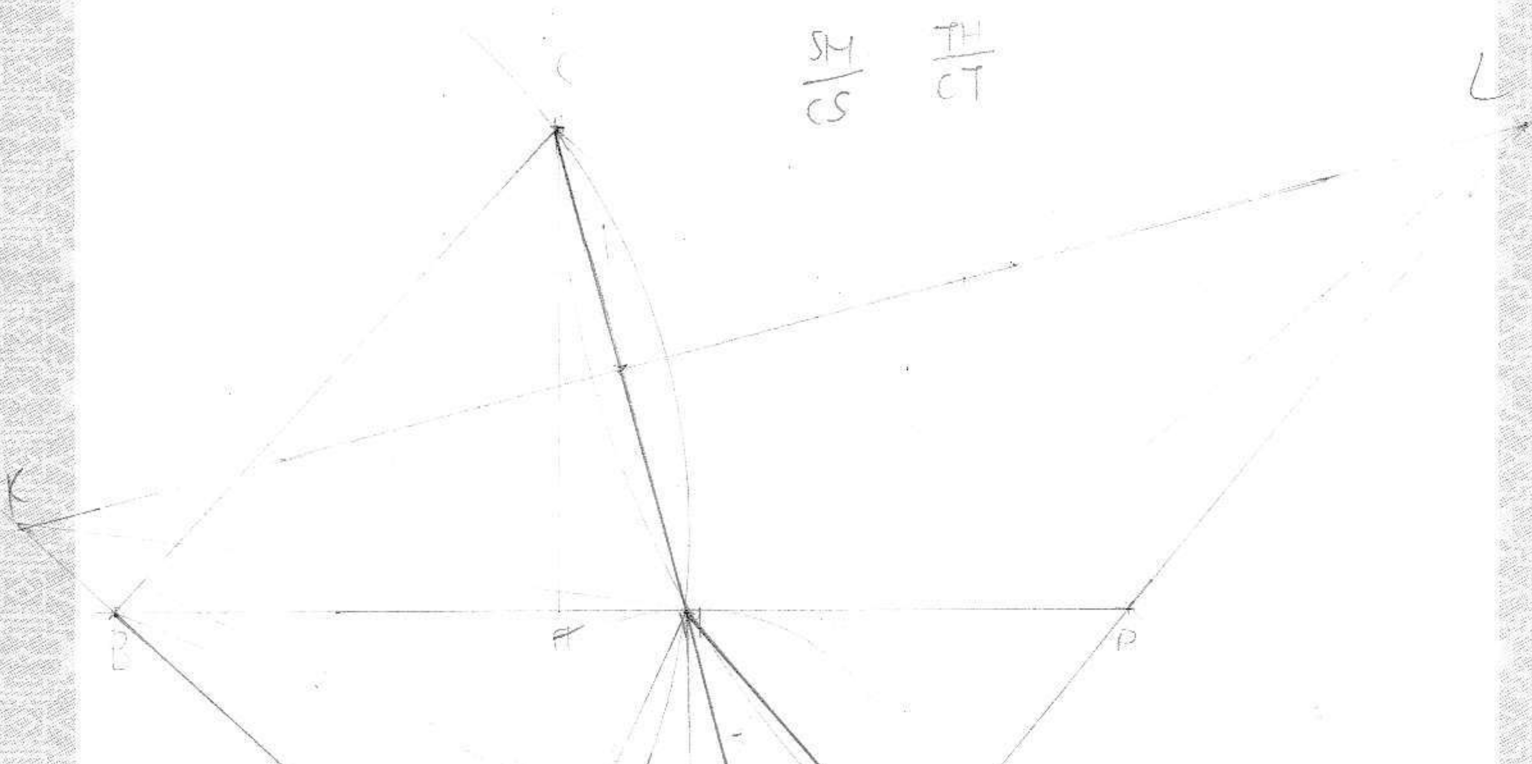

$\frac{SH}{CS}$ $\frac{TH}{CT}$

$\angle CKA = 2$

$\frac{KS}{LT} \times$ [illegible]