Solution to APMO 2017 Problems

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1. We call a 5-tuple of integers arrangeable if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, ..., n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

   Answer. $n_1 = \cdots = n_{2017} = 29$ is the only sequence.

   Solution. We've already supplied an answer above so it remains showing that this answer is unique. More generally, consider the solution matrix $Mx = b$ (i.e. solve for x) where M is 2017×2017 and x, b are vectors of length $n = 2017$, and all entries in b are 29, and M is defined by

$$M_{ij} = \begin{cases} (-1)^{j-i} & i \leq j \leq i+4 \\ 0 & \text{otherwise} \end{cases}$$

   (throughout the solution we let the indices to be modulo n). It then suffices to show that M has nonsingular, thereby $Mx = b$ has a unique solution for x.

   First, consider the matrix A of shape $n \times n$ where

$$M_{ij} = \begin{cases} 1 & i \leq j \leq i+1 \\ 0 & \text{otherwise} \end{cases}$$

   Then if $B = AM$, then the i-th row of B is simply the sum of i-th and $i+1$-th rows of M (again indices modulo n). This means:

$$B_{ij} = \begin{cases} 1 & i = j \text{ or } j = i+5 \\ 0 & \text{otherwise} \end{cases}$$

   To show that B is nonsingular, we consider the system of equations $Bx = c$ with c fixed. This means $c_i = x_i + x_{i+5}$ for all $i = 1, 2, \cdots, n$. This means that the value of x_i uniquely determines x_{i+5} based on the value of c_i for all $i = 1, 2, \cdots, n$ and since $\gcd(5, 2017) = 1$, all values $x_2, \cdots, x_{2017}$ can be uniquely determined by x_1. Thus, writing everything in term of x_1 we have:

$$x_6 = c_1 - x_1; x_{11} = c_6 - x_6 = (c_1 - c_6) + x_1, \cdots$$

   which generalizes to: the coefficient of x_1 in x_{1+5k} is $(-1)^k$. However, since we're taking indices modulo n, the coefficient of x_1 in $x_1 = x_{1+5n}$ is $(-1)^n = -1$ since n is odd. Therefore x_1 is itself uniquely determined too, and therefore $Bx = c$ has unique solution. Therefore $\det(B) \neq 0$, and since $B = AM$ and A is $n \times n$ square matrix, $\det(M) \neq 0$.

2. Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

   Solution. The internal and external angle bisectors are perpendicular to each other (known fact!!!), so $\angle ZAD$ is $90°$. This motivates us to think of the Pythagoras' theorem, where $AD^2 + AZ^2 = DZ^2$. If we can prove that, given M as the midpoint of A we have $ZM^2 + MD^2 = AD^2 + AZ^2$ then we have $ZM^2 + MD^2 = DZ^2$, which gives $\angle ZMD = 90°$, and Z, A, M, D concyclic.

By cosine rule we have $MZ^2 = AM^2 + AZ^2 - 2 \cdot AM \cdot AZ \cdot \cos\angle MAZ$, $DM^2 = AD^2 + AM^2 - 2 \cdot AD \cdot AM \cdot \cos\angle MAD$. Summing these two up gives $AD^2 + AZ^2 + 2AM^2 - 2AM(AZ \cdot \cos\angle MAZ + AD \cdot \cos\angle MAD)$. A little bit of algebra yields that it is enough to prove that $2AM^2 - 2AM(AZ \cdot \cos\angle MAZ + AD \cdot \cos\angle MAD)$ (you the readers will verify this, not me!). We claim that $AM - AZ \cdot \cos\angle MAZ - AD \cdot \cos\angle MAD = 0$ (which will be enough to prove our hypothesis). Now, $\cos\angle MAZ = \cos(90° + \frac{1}{2}\angle A) = -\sin\frac{1}{2}\angle A$, and $\cos\angle MAD = \cos\frac{1}{2}\angle A$. So now we need $AM + AZ \cdot \sin\frac{1}{2}\angle A - AD \cdot \cos\frac{1}{2}\angle A = 0$. Denote N as midpoint of AC, then $AN = AZ\cos\angle ZAN = AZ\sin\angle DAC = AZ \cdot \sin\frac{1}{2}\angle A$. By Ptolemy's theorem, we also have $BD \cdot AC + AB \cdot DC = BC \cdot AD$. In view of the fact that $BD = DC$ and $BC = 2DC\cos\angle BCD = 2DC\cos\angle BAD = 2DC\cos\frac{1}{2}\angle A$. This transforms the equality above into: $CD \cdot (AB + AC) = 2DC\cos\frac{1}{2}\angle A \cdot AD$, i.e. $AB + AC = 2AD\cos\frac{1}{2}\angle A$. Therefore $AM + AZ \cdot \sin\frac{1}{2}\angle A - AD \cdot \cos\frac{1}{2}\angle A = AM + AN - \frac{1}{2}(AB + AC)$ which is obviously zero (midpoints!!!)

3. Let $A(n)$ denote the number of sequences $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers for which $a_1 + \cdots + a_k = n$ and each $a_i + 1$ is a power of two ($i = 1, 2, \cdots, k$). Let $B(n)$ denote the number of sequences $b_1 \geq b_2 \geq \cdots \geq b_m$ of positive integers for which $b_1 + \cdots + b_m = n$ and each inequality $b_j \geq 2b_{j+1}$ holds ($j = 1, 2, \cdots, m-1$). Prove that $A(n) = B(n)$ for every positive integer n.

   Solution. Denote the mapping f from A to B by the following: consider a sequence $a_1, \cdots, a_k \in A$, and let $a_i = 2^{c_i} - 1$ with $c_1 \geq \cdots \geq c_k$. Then $f(a_1, \cdots, a_k) = b_1, \cdots, b_m$ where:

   - $m = c_1$.
   - For $1 \leq i \leq c_k$, we have: $b_i = \sum\limits_{j:c_j \geq i} 2^{c_j - i}$

   We now need to show that f is valid, and is a bijection. The validity hinges on two things: $b_i \geq 2b_{i+1} \geq 1$ for each $i \leq c_1 - 1$, and that $\sum b_i = n$. The first condition is due to that:

$$b_i = \sum_{j:c_j \geq i} 2^{c_j - i} \geq \sum_{j:c_j \geq i+1} 2^{c_j - i} = 2 \sum_{j:c_j \geq i+1} 2^{c_j - i - 1} = 2b_{i+1}$$

   and the positivity of each b_i is guaranteed since $c_1 \geq i$ for each $i = 1, 2, \cdots, c_1 = m$. The second condition is guaranteed by the following:

$$\sum_{i=1}^{m} b_i = \sum_{i=1}^{m} \sum_{j:c_j \geq i} 2^{c_j - i} = \sum_{j=1}^{k} \sum_{i=1}^{c_j} 2^{c_j - i} = \sum_{j=1}^{k} \sum_{i=0}^{c_j - 1} 2^i = \sum_{j=1}^{k} 2^{c_j} - 1 = \sum_{j=1}^{k} a_j = n$$

   as desired.

   Now to prove that f is a bijection, we need two things too: that it's injective and surjective. Even easier, we define the inverse f^{-1} for f. Now given $b_1, \cdots, b_m$, we define the following algorithm to find $a_1, a_2, \cdots, a_k$, with each $a_i = 2^{c_i} - 1$. We set $k = b_m + \sum\limits_{i=1}^{m-1}(b_i - 2b_{i-1}) = b_1 - b_2 - \cdots - b_m$ and we have b_m copies of m, $b_{m-1} - 2b_m$ copies of $m-1$, $\cdots$, $b_1 - 2b_2$ copies of 1 in the sequence $c_1, \cdots, c_k$. To show that this is indeed the inverse of f, we need to consider the following:

   - Here, c_1 is the largest among c_i's, and given that we have $b_m \geq 1$ copies of m, we have $m = c_1$.

- A less straightforward part would be to show that $b_i = \sum_{j:c_j \geq i} 2^{c_j - i}$ holds. This can be done by induction from $i = m$ to 1. Now, b_m is the number of times m appears in $c_1, \cdots, c_k$. Considering $m = \max\{c_1, \cdots, k\}$, we have $b_m = \sum_{j:c_j=m} 1 = \sum_{j:c_j=m} 2^{c_j-m} = \sum_{j:c_j \geq m} 2^{c_j-m}$ as desired. As per the induction step, we will consider the following for all $i \leq m-1$:

$$\sum_{j:c_j \geq i} 2^{c_j-i} = \sum_{j:c_j \geq i+1} 2^{c_j-i} + \sum_{j:c_j=i} 2^{c_j-i} = 2 \sum_{j:c_j \geq i+1} 2^{c_j-i-1} + \sum_{j:c_j=i} 1$$

and by induction hypothesis, $2 \sum_{j:c_j \geq i+1} 2^{c_j-i-1} = b_{i+1}$. By our construction, we have $b_i - 2b_{i+1}$ copies of i in c_i, therefore $2 \sum_{j:c_j \geq i+1} 2^{c_j-i-1} + \sum_{j:c_j=i} 1 = 2b_{i+1} + (b_i - 2b_{i+1}) = b_i$ as desired.

4. Call a rational number r powerful if r can be expressed in the form $\dfrac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful.

Solution. We extend the definition of v_p (p prime) to rationals such that: if $a = \frac{x}{y}$ with $\gcd(x, y) = 1$ (both integers), then:

$$v_p(a) = \begin{cases} v_p(x) & p \mid x \\ -v_p(y) & p \mid y \\ 0 & \text{none of the above} \end{cases}$$

The task is to show that $\gcd\{p : v_p(a) \geq 0\} > 1$ (and similarly for b, c).

We now have the following properties:

- Just like the integers, $v_p(ab) = v_p(a)v_p(b)$, $v_p(a^x) = xv_p(a)$ for all rationals a, b and integer x.
- Suppose that $a_1, a_2, \cdots, a_n$ are rationals such that for some p, $v_p(a_i) > v_p(a_1)$ for all $i > 1$. Then $v_p(a_1 + \cdots a_n) = v_p(a_1)$.

The second identity can be justified by simply multiplying by the lowest common multiple of the denominators of $a_1, \cdots, a_n$.

Suppose that $v_p(a) > 0$ for some prime p. Then $v_p(b) < 0$ or $v_p(c) < 0$ since $abc = 1$ (which means that $v_p(a) + v_p(b) + v_p(c) = 0$ for all primes p). But if $v_p(b^y) < v_p(c^z)$ then $v_p(a^x + b^y + c^z) = v_p(b^y) = yv_p(b) < 0$ and therefore $a^x + b^y + c^z$ cannot be integer. This is also the case if $v_p(b^y) > v_p(c^z)$ except we have $v_p(a^x + b^y + c^z) = v_p(c^z) = zv_p(c) < 0$. Therefore, $yv_p(b) = zv_p(c)$ must hold, and there exists integer ℓ such that $v_p(b) = -\ell \cdot \frac{z}{\gcd(y,z)}$ and $v_p(c) = -\ell \cdot \frac{y}{\gcd(y,z)}$. Therefore

$$v_p(a) = -(v_p(b) + v_p(c)) = \ell \cdot \frac{z}{\gcd(y, z)} + \ell \cdot \frac{y}{\gcd(y, z)} = \ell \cdot \frac{y + z}{\gcd(y, z)}$$

which gives $\frac{y+z}{\gcd(y,z)} \mid v_p(a)$. With $\gcd(y, z) \mid y$ and $\gcd(y, z) \mid z$, $\frac{y+z}{\gcd(y,z)}$ is an integer and since $\frac{y+z}{\gcd(y,z)} \leq y$ and $\frac{y+z}{\gcd(y,z)} \leq z$, we have $\frac{y+z}{\gcd(y,z)} \geq 1 + 1 = 2$. This means a is powerful since all its positive exponents of primes are divisible by $\frac{y+z}{\gcd(y,z)} \geq 2$. In a similar manner, we can show that b and c are powerful.

5. Let n be a positive integer. A pair of n-tuples $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ with integer entries is called an exquisite pair if

$$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

Solution. This is hard. Will be back later.