1. Throughout the question we name E as our set of interest.

Term	Definition
Open ball	$B_r(\vec{x})(r>0) = \{\vec{y} : \|\|\vec{y}-\vec{x}\|\| < r\}$.
Open set	A set E is open iff $\forall \vec{x} \in E \; \exists \; r > 0$ with $B_r(\vec{x}) \subseteq E$.
Closed set	Any set is closed iff its complement is open.
Connected set	That is, there's no disconnecting pairs U and V satisfying the following conditions: • U and V are open. • $U \cap E \neq \phi$ and $V \cap E \neq \phi$. • $U \cap V \cap E \neq \phi$. • $E \subseteq U \cup V$.
Compact set	For any open cover of E there exists a finite subcover whose union contains E.

2. Again denote E as set of interest.

Term	Definition
Interior	$E^\circ = \{\vec{x} : \exists r > 0 : B_r(\vec{x}) \subseteq E\}$.
Boundary	$\partial E = \{\vec{x} : \forall r > 0 : B_r(\vec{x}) \not\subseteq E, B_r(\vec{x}) \not\subseteq E^c\}$.
Closure	$\overline{E} = E^\circ \cup \partial E$.
Limit point	$\vec{x}_0$ is a limit point of E iff $\exists$ a sequence $\{x_k\} \subseteq E$ s.t. $\vec{x}_k \to \vec{x}_0$.
Cluster point	$\vec{x}_0$ is a cluster point of E iff for all $r > 0$, $B_r(\vec{x}_0) \cap E$ has inifnitely many elements.

3. Heine-Borel Theorem states that a set is compact iff it's both closed and bounded.

4.

E	Opn	Clsd	Cmpct	E°	∂E	$\overline{E}$ / Limit	Cluster	Disc_
ϕ	1	1	1	ϕ	ϕ	ϕ	ϕ	DNE
$\mathbb{R}^n$	1	1	0	$\mathbb{R}^n$	ϕ	$\mathbb{R}^n$	$\mathbb{R}^n$	DNE
$B_r(\vec{a})$	1	0	0	$B_r(\vec{a})$	$\{\vec{x} : \|\|\vec{x}-\vec{a}\|\| = r\}$	$\overline{B_r(\vec{a})}$	$\overline{B_r(\vec{a})}$	DNE
$\{\vec{a}\}$	0	1	1	ϕ	$\{\vec{a}\}$	$\{\vec{a}\}$	ϕ	DNE
$\{\frac{1}{n} : n \in \mathbb{N}\}$	0	0	0	ϕ	$E \cup \{0\}$	$E \cup \{0\}$	$\{0\}$	$(-\infty, 0.23)$ $(0.23, \infty)$
$\bigcup_{n\in\mathbb{Z}} (n-1, n)$	1	0	0	$\cup(n-1,n)$	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, 0)$ $(0, \infty)$
$\bigcup_{n-1}^{\infty} (-n, n)$	1	1	0	$\mathbb{R}^n$	ϕ	$\mathbb{R}^n$	$\mathbb{R}^n$	DNE
$\bigcup_{n\in\mathbb{N}} (\frac{1}{n+1}, \frac{1}{n})$	1	0	0	$\bigcup_{n\in\mathbb{N}} (\frac{1}{n+1}, \frac{1}{n})$	$\{\frac{1}{n}\} \cup \{0\}$	$[0,1]$	$[0,1]$	$(-\infty, \frac{1}{2})$ $(\frac{1}{2}, \infty)$
$[2,\infty) \cup \{-1\}$	0	1	0	$(2, \infty)$	$\{2, -1\}$	$[2,\infty) \cup \{-1\}$	$[2,\infty)$	$(-\infty, \frac{1}{2})$ $(\frac{1}{2}, \infty)$
$\mathbb{Z} \cap (-10, 10)$	0	1	1	ϕ	E	E	ϕ	$(-\infty, \frac{1}{2})$ $(\frac{1}{2}, \infty)$
$\mathbb{Q}$	0	0	0	ϕ	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, \sqrt{2})$ $(\frac{1}{2}, \sqrt{2})$
$\{(x,y) :$ $x - y \neq 2\}$	2	0	0	E	$\{(x,y) :$ $x-y=2\}$	$\mathbb{R}^2$	$\mathbb{R}^2$	$x - y < 2$ $x - y > 2$

5. Covered above.

6. Any union of open sets is open.

Proof: Let $\vec{x}$ be in the union U of open sets, meaning that $\vec{x} \in U_i$ for some open set U_i (where U_i is an open set that is 'part of' U). Then there exists an open ball $B_\epsilon(\vec{x}) \subseteq U_i \subseteq U$; the former follows from the definition of openness and the latter follows from the definition of union.

Corollary: Any intersection of closed sets is closed.

Any *finite* intersection of open sets is open.

Proof: Consider the sets $U_1, \cdots, U_n$ and let $\vec{x}$ be in their intersection. This means for each $i \in [1, n]$ there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(\vec{x}) \subseteq U_i$. Letting $m = \min\{\epsilon_i : i \in [1, n]\} > 0$ and we have $B_m(\vec{x}) \subseteq U_i$ for all i, thus $B_m \subseteq \cap U_i$.

Corollary. Any *finite* union of closed sets is closed.

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11. (a) The interval $(-2, 3]$ is neither open nor closed.
    (b) Yes, only $\mathbb{R}^n$ and ϕ are both open and closed.
    (c) Not true. For example, the set $(B_{\frac{1}{n}}(\vec{O}))^c$ is closed but their union (when $n = 1, 2, \cdots$) is $\{\vec{O}\}^c$ which is not closed.
    (d) Yes, every open set is the union of open balls, as proven in assignment 1 before.
    (e) Nope, $B_r(\vec{x})$ doesn't have its boundary points.
    (f) Yes, E° is the biggest open set contained in E.
    (g) Yes, $\overline{E}$ is the smallest closed set containing E.
    (h) Not really, 2 is a limit point of $(2, 3]$.
    (i) Not true, $\overline{A^\circ}$ is always closed but not necessarily so for A.
    (j) Not true, Consider $[1, 5]$ and $[2, 4]$ for example.
    (k) Nope, nope, nope. Take $A = \mathbb{Q}$ and $B = \mathbb{R}\backslash\mathbb{Q}$. Then $(A \cup B)^\circ = \mathbb{R}$ because $A \cup B = \mathbb{R}$, but $A^\circ \cup B^\circ = \phi \cup \phi = \phi$.
    (l) Yes, let's prove that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$; the other direction is simpler. Suppose that $\vec{x} \in A^\circ \cap B^\circ$, we have $B_a\vec{x} \subseteq A$ and $B_b\vec{x} \subseteq B$ for some $a, b > 0$. Then if $c = \min\{a, b\}$ we have $B_c\vec{x} \subseteq A \cap B$, so $\vec{x} \in (A \cap B)^c$.
    (m) Yes: A is open iff ∂A is contained in A^c, or disjoint from A at all.
    (n) No: what if $B = \mathbb{R}^n$?
    (o) Yes: B is closed and bounded $\to$ A is also bounded. Since A is already closed, A is compact.