Solution to APMO 2020 Problems

Anzo Teh

Problem 1. Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Solution. Now A, B, C, F all lie on circle ω, and let the circle passing through B, D, F, E be ω_1. Then BF is the radical axis of ω and ω_1. Given that DE parallel to AB and AE tangent to ω, we have $\angle EAC = \angle ABD = \angle EDC$, so E, A, D, C are also concyclic, and let the circumcircle be ω_2. Thus, ω_1 and ω_2 has radical axis DE, and ω and ω_2 has radical axis AC. Therefore AC, BF, DE concur at the radical center of $\omega, \omega_1, \omega_2$.

Problem 2. Show that $r = 2$ is the largest real number r which satisfies the following condition:

If a sequence a_1, a_2, $\ldots$ of positive integers fulfills the inequalities

$$a_n \leq a_{n+2} \leq \sqrt{a_n^2 + ra_{n+1}}$$

for every positive integer n, then there exists a positive integer M such that $a_{n+2} = a_n$ for every $n \geq M$.

Solution. First, let $r > 2$. We show that, if a is sufficiently large that $2 + \frac{1}{a} < r$ and the sequence $a, a, a+1, a+1, a+2, a+2, \cdots$ fulfills the condition. For each $n \geq 1$, $a_{n+2} = a_n + 1$ so the left inequality is satisfied. Now, for a_{n+1} is either a_n or $a_n + 1 = a_{n+2}$ for each n. We now have

$$a_n^2 + ra_{n+1} \geq a_n^2 + ra_n > a_n^2 + (2 + \frac{1}{a})a_n \geq a_n^2 + (2 + \frac{1}{a_n})a_n = (a_n + 1)^2 = a_{n+2}^2$$

so the right inequality is also satisfied by this sequence.

Now suppose that $r \leq 2$. If $a_n = a_{n+2}$ for all n then we're done. Now consider any n with $a_{n+2} > a_n$. Claim: $a_{n+1} \geq a_{n+2}$.

Proof: $a_{n+2}^2 \leq a_n^2 + ra_{n+1} \leq a_n^2 + 2a_{n+1}$, i.e. $a_{n+1} \geq \frac{a_{n+2}^2 - a_n^2}{2}$. If $a_{n+1} < a_{n+2}$, then $\frac{a_{n+2}^2 - a_n^2}{2} \leq a_{n+1} \leq a_{n+2} - 1$, or $a_n^2 \geq a_{n+2}^2 - 2a_{n+2} + 2 > (a_{n+2} - 1)^2$. This gives $a_n \geq a_{n+2}$, contradiction. In other words, if $a_{n+1} \leq a_n$ we have $a_{n+2} = a_n$.

Now let a_n be any number with $a_n < a_{n+2}$. Then $a_{n+1} \geq a_{n+2}$. If $a_{n+3} > a_{n+1}$ then $a_{n+2} \geq a_{n+3} > a_{n+1}$, contradiction, so $a_{n+3} = a_{n+1}$. If $a_{n+2k} \leq a_{n+1}$ for all $k \geq 1$, then the indices $a_n, a_{n+2}, a_{n+4}, \cdots$ become constant, and by the previous argument we also have $a_{n+1} = a_{n+3} = a_{n+5} = \cdots$, and we're done. Otherwise, let k be the minimal index with $a_{n+2k} > a_{n+1}$. Then from $a_{n+2k-2} \leq a_{n+1}$ and we $a_{n+2k-1} = a_{n+1}$ by the similar argument. Using a similar argument, we have $a_{n+2k-2} = a_{n+2k}$ instead and the sequence a_{n+2k} will never exceed a_{n+1} (hence staying eventually constant), and so does the sequence a_{n+2k+1}.

Problem 3. Determine all positive integers k for which there exist a positive integer m and a set S of positive integers such that any integer $n > m$ can be written as a sum of distinct elements of S in exactly k ways.

Answer. $k = 2^a$ for any nonnegative integer a.

Solution. The case $k = 1$ can be achieved by the set $S = \{2^a : a \geq 0\}$. If this statement works for $k = 2^a$ for some $a \geq 0$ with corresponding threshold m_a and set S_a, then consider the set $S_{a+1} = 4S_a \cup \{1, 2, 3\}$ where $4S_a = \{4x : x \in S_a\}$. Then for all $n \geq 4(m + 2)$ we have:

- If $n = 4b$, $b \geq m + 2$ then $n = 4(b - 1) + 1 + 3$ and $n = 4b$ where $b - 1, b$ each has 2^a ways to be represented as sum in S_a.
- If $n = 4b + 1$, $b \geq m + 2$ then $n = 4(b - 1) + 2 + 3$, and $n = 4b + 1$.
- If $n = 4b + 2$, then $n = 4(b - 1) + 1 + 2 + 3$ and $n = 4b + 2$.
- If $n = 4b + 3$, $n = 4b + 3 = 4b + (1 + 2)$.

The above shows that there are 2^a ways to determine the appropriate sum coming from $4S_a$, and 2 ways coming from $\{1, 2, 3\}$, completing the proof.

Problem 4.