Putnam 2019

A1 Determine all possible values of $A^3+B^3+C^3-3ABC$ where A, B, and C are nonnegative integers.

Answer. Any integer that's nonnegative and have remainders 0, 1, 2, 4, 5, 7, 8 modulo 9.

Solution. We first have $A^3+B^3+C^3-3ABC = \frac{1}{2}(A+B+C)((A-B)^2+(B-C)^2+(A-C)^2)$, so the fact that A, B, C are all nonnegative means that $A^3+B^3+C^3-3ABC$ is also nonnegative. Next, consider the numbers k with $A = B = k$ and $C = k+1$ for $k \geq 0$, we have:

$$A^3+B^3+C^3-3ABC = \frac{1}{2}(A+B+C)((A-B)^2+(B-C)^2+(A-C)^2) = \frac{1}{2}(3k+1)(0+1+1) = 3k+1$$

so all numbers in terms of $3k+1$ can be expressed in the terms above (i.e. 1, 4, 7 modulo 9). Meanwhile, $A = k$ and $B = C = k+1$ gives us

$$\frac{1}{2}(A+B+C)((A-B)^2+(B-C)^2+(A-C)^2) = \frac{1}{2}(3k+2)(1+1+0) = 3k+2$$

which covers all nonnegative numbers congruent to 2, 5, 8 modulo 9. and finally, setting $A = k, B = k+1, C = k+2$ gives

$$\frac{1}{2}(3k+3)(1^2+1^2+4^2) = 3(3k+3) = 9(k+1)$$

which gives us all multiples of 9 that's at least 9. The number 0 can be achieved by setting $A = B = C = 0$, therefore giving us the representation of all nonnegative integers with remainders 0, 1, 2, 4, 5, 7, 8 modulo 9.

To show that numbers with remainders 3 and 6 modulo 9 cannot be represented in the form we desire, it suffices to show that if $A^3+B^3+C^3-3ABC$ is divisible by 3, then it's divisible by 9. We first notice that $x^3 \equiv x \pmod 3$ (since $x^3 - x = x(x-1)(x+1)$ and one of $x, x-1, x+1$ is divisible by 3). Therefore we need $A+B+C$ to be divisible by 3. This gives us one of the following two scenarios:

$$A \equiv B \equiv C \qquad \{A, B, C\} = \{0, 1, 2\} \pmod 3$$

in the first case, we have $(A-B)^2+(B-C)^2+(A-C)^2 \equiv 0 \pmod 3$; in the second case, we have $(A-B)^2+(B-C)^2+(A-C)^2 \equiv 1+1+1 \equiv 0 \pmod 3$. This means that $(A-B)^2+(B-C)^2+(A-C)^2$ is divisible by 3 and so is $A+B+C$. Therefore the product $\frac{1}{2}(A+B+C)((A-B)^2+(B-C)^2+(A-C)^2)$ will be divisible by 9.

A2 In the triangle $\triangle ABC$, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

Answer. $90^\circ = \frac{\pi}{2}$.

Solution. Let CI intersect AB at M, and the circumcircle of ABC at $D \neq C$. From $IG \parallel AB$ we have $CI : IM = 2 : 1$, and using the well-known fact $DA = DI = DB$ and $DM \cdot DC = DI^2$, we have $IM = MD$.

Now, $\angle ABC = \angle ADC = 2\tan^{-1}(1/3)$ given that A, B, C, D are concyclic. Let N be the midpoint of AI and P be the midpoint of IM, then $NP \parallel AM$. Moreover, $IN/ND = \tan\angle IND = \frac{1}{3}$ since DN bisects $\angle IDA$ (well, N is midpoint of IA and $DI = DA$). But since M is midpoint of DI and P is midpoint of IM, we also have $IP : PD = 1 : 3$. Therefore $\frac{IP}{PD} = \frac{IN}{ND}$ and we have by angle bisector theorem, NP bisects $|angle IND$ (which is in fact a 90°), so $\angle INP = \angle IAM = 45^\circ$ (the first equality is because $NP \parallel AM$). Thus $\alpha = 90^\circ$.

A3 Given real numbers $b_0, b_1, \dots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \dots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \dots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \dots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

Answer. $M = 2019^{-\frac{1}{2019}}$.

Solution. We know that the product of roots is $-\frac{b_0}{b_{2019}}$ here. By AM-GM inequality,

$$\mu \geq \sqrt[2019]{(|z_1| \cdot \cdots \cdot |z_{2019}|)} = \sqrt[2019]{\frac{b_0}{b_{2019}}} \geq \sqrt[2019]{\frac{1}{2019}}$$

To show that equality can hold, consider $b_k = 2019^{\frac{k}{2019}}$, which satisfies $1 \leq b_0 < b_1 < \cdots < b_{2019} \leq 2019$. Consider ϵ as one of the 2020-th root of unity that is not 1 (that is, $\epsilon^{2020} = 1$). Then we have

$$P(2019^{-\frac{1}{2019}}\epsilon) = \sum_{k=0}^{2019} 2019^{\frac{k}{2019}} 2019^{-\frac{k}{2019}} \epsilon^k = \sum_{k=0}^{2019} \epsilon^k = \frac{\epsilon^{2020} - 1}{\epsilon - 1} = 0$$

and therefore the 2019 roots are indeed all the 2020th root of unity that isn't 1 multiplied by $2019^{-\frac{1}{2019}}$, hence $\mu = 2019^{-\frac{1}{2019}}$.

A5 Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by $q(x) = \sum_{k=1}^{p-1} a_k x^k$ where $a_k = k^{(p-1)/2} \bmod p$. Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Answer. $\frac{p-1}{2}$.

Solution. We first claim that for each $0 < n < p$, if n is the highest n such that $(x-1)^n$ divides $P(x)$, then $n-1$ is the highest power such that $(x-1)^{n-1}$ divides $P'(x)$ (we do differentiation the same manner as how we do it in $\mathbb{R}[x]$). To see this, let $P(x) = (x-1)^n Q(x)$ where Q is not divisible by $x-1$. Then

$$P'(x) = (x-1)^n Q'(x) + n(x-1)^{n-1} Q(x) = (x-1)^{n-1}((x-1)Q'(x) + nQ(x))$$

and since $0 < n < p$, we have $P'(x)$ divisible by $(x-1)^{n-1}$ but $(x-1)Q'(x) + nQ(x) \equiv nQ(x) \pmod{(x-1)}$, and therefore $n-1$ is the highest power of $x-1$ dividing $P'(x)$, as claimed. This means that, if n is the highest power of $x-1$ dividing P and $n < p$, then $P(x), P'(x), \cdots, P^{(n-1)}(x)$ are divisible by $x-1$ but not $P^{(n)}(x)$ (where $P^{(n)}(x)$ denotes the n-th derivative).

Now, consider our polynomial q and the derivatives. For each $n < p$, the n-th derivative is

$$q^{(n)}(x) = \sum_{k=1}^{p-1} a_k k(k-1)\cdots(k-n+1)x^k = \sum_{k=1}^{p-1} k^{(p-1)/2} k(k-1)\cdots(k-n+1)x^k$$

Notice that $x-1$ divides $q^{(n)}(x)$ iff $q^{(n)}(1) = 0$. Therefore we're interested in the value of the sum

$$\sum_{k=1}^{p-1} k^{(p-1)/2} k(k-1)\cdots(k-n+1)$$

when evaluated in $\mathbb{F}_p$.

Denote, now, $f(x) = x^{(p-1)/2}x(x-1)\cdots(x-n+1)$, which is a degree $(p-1)/2+n$ polynomial. This means it can be written in the form

$$\sum_{k=(p-1)/2}^{(p-1)/2+n} b_k x^k$$

Then we're looking at the term

$$f(1)+f(2)+\cdots+f(p-1) = \sum_{k=(p-1)/2}^{(p-1)/2+n} b_k(1^k+\cdots+(p-1)^k)$$

If g is a primitive root of p, then provided $p-1$ does not divide k,

$$1^k+\cdots+(p-1)^k = g^0+g^k+\cdots+g^{(p-2)k} = \frac{g^{(p-1)k-1}}{g^k-1} = 0$$

so if $(p-1)/2+n < p-1$, $\displaystyle\sum_{k=(p-1)/2}^{(p-1)/2+n} b_k(1^k+\cdots+(p-1)^k) = 0$. This would mean that $q(n)(x)$ is divisible by $(x-1)$ for all $n = 0, 1, \cdots, \frac{(p-3)}{2}$.

When $n = \frac{p-1}{2}$, we have the leading term, $b_{(p-1)/2}$ as 1, so in this case $q^{(n)}(1) \equiv 1 \pmod{p}$. We thus conclude that the highest power of n with $q^{(n)}(x)$ divisible by $(x-1)$ is $\frac{p-3}{2}$, and therefore the highest power we're looking for is $\frac{p-1}{2}$.

B1 Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2+y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Answer. $5n+1$.

Solution. We first claim that all pairs (x, y) with $x^2+y^2 = 2^k$ are of the following:

$$\begin{cases} (\pm 2^{\frac{k-1}{2}}, \pm 2^{\frac{k-1}{2}}) & k \text{ odd} \\ (0, \pm 2^{\frac{k}{2}}), (\pm 2^{\frac{k}{2}}, 0) & k \text{ even} \end{cases}$$

To see why, let ℓ be the highest power of 2 dividing both x and y (which exists so long as x and y are not both zero). This means, $x = 2^\ell x_0$ and $y = 2^\ell y_0$, where at least one of x_0 and y_0 is odd. Given that $x^2+y^2 = 2^{2\ell}(x_0^2+y_0^2)$, we need $x_0^2+y_0^2$ to be power of 2. If one of x_0 is odd and the other is even, then $x_0^2+y_0^2$ is odd and the only possibility here will be $x_0^2+y_0^2 = 1$, which menas $(x_0, y_0) = (\pm 1, 0)$ or $(0, \pm 1)$. Otherwise, both are odd and both $x_0^2, y_0^2 \equiv 1 \pmod{4}$. Therefore, $x_0^2+y_0^2 \equiv 2 \pmod{4}$. This means that $x_0^2+y_0^2 = 2$ is the only possibility, and therefore $x_0, y_0 = (\pm 1, \pm 1)$.

Now to first show that we have $5n+1$ such constructions, we first notice that when $n = 0$, the only possibility is $(1, 0), (0, 1), (-1, 0), (0, -1)$, and that the 5 new constructions consisting at least one point in $P_n \backslash P_{n-1}$ for all $n > 0$ are the following:

- For n even, we can have $(2^{n/2}, 0), (2^{n/2-1}, 2^{n/2-1}), (0, 0), (2^{n/2-1}, -2^{n/2-1})$ as one square, and have this square rotate by $90°, 180°, 270°$ around the origin. Finally, we have one big square at $(2^{n/2}, 0), (0, 2^{n/2}), (-2^{n/2}, 0), (0, -2^{n/2})$.
- The case where n odd is similar. We have $(2^{(n-1)/2}, 2^{(n-1)/2}), (2^{(n-1)/2}, 0), (0, 0), (0, 2^{(n-1)/2})$, and again have it rotate by $90°, 180°, 270°$ around the origin. Finally, we have one big square at $(2^{(n-1)/2}, 2^{(n-1)/2}), (2^{(n-1)/2}, -2^{(n-1)/2}), (-2^{(n-1)/2}, 2^{(n-1)/2}), (-2^{(n-1)/2}, -2^{(n-1)/2})$.

We're therefore left to show that these are the only solutions. We first notice that a square is a parallelogram, which means that the two opposite vertices have the same coordinate-wise sum as the other two opposite vertices. Now, let k be the minimum power of 2 that divides all the x-coordinates of the square, i.e. one of them is $\pm 2^k$. By considering the sum of opposite vertices, we must have other one vertex that is $\pm 2^k$ (either the side, or diagonal). This means that this square must have 2 points from the following 6: $(\pm 2^k, \pm 2^k)$ (4 points here), and $(\pm 2^k, 0)$.

B2 For all $n \geq 1$, let $a_n = \sum_{k=1}^{n-1} \frac{\sin(\frac{(2k-1)\pi}{2n})}{\cos^2(\frac{(k-1)\pi}{2n})\cos^2(\frac{k\pi}{2n})}$. Determine $\lim_{n\to\infty} \frac{a_n}{n^3}$.

Answer. $\frac{8}{\pi^3}$

Solution. We first notice the following:

$$\cos^2 A - \cos^2 B = (\cos A - \cos B)(\cos A + \cos B) = (-2\sin\frac{A+B}{2}\sin\frac{A-B}{2})(2\cos\frac{A+B}{2}\cos\frac{A-B}{2})$$
$$= \sin(A+B)\sin(B-A)$$

and therefore

$$\frac{1}{\cos^2(\frac{k\pi}{2n})} - \frac{1}{\cos^2(\frac{(k-1)\pi}{2n})} = \frac{\cos^2(\frac{(k-1)\pi}{2n}) - \cos^2(\frac{k\pi}{2n})}{\cos^2(\frac{(k-1)\pi}{2n})\cos^2(\frac{k\pi}{2n})} = \frac{\sin(\frac{(2k-1)\pi}{2n})\sin(\frac{\pi}{2n})}{\cos^2(\frac{(k-1)\pi}{2n})\cos^2(\frac{k\pi}{2n})}$$

which means

$$\sum_{k=1}^{n-1} \frac{\sin(\frac{(2k-1)\pi}{2n})}{\cos^2(\frac{(k-1)\pi}{2n})\cos^2(\frac{k\pi}{2n})} = \sum_{k=1}^{n-1} \frac{1}{\sin\frac{\pi}{2n}}(\frac{1}{\cos^2(\frac{k\pi}{2n})} - \frac{1}{\cos^2(\frac{(k-1)\pi}{2n})}) = \frac{1}{\sin\frac{\pi}{2n}}(\frac{1}{\cos^2(\frac{(2n-1)\pi}{2n})} - \frac{1}{\cos^2(\frac{\pi}{2n})})$$

when $n \to \infty$, $\sin(\frac{\pi}{2n}) \to \frac{\pi}{2n}$, $\frac{1}{\cos^2(\frac{\pi}{2n})} \to 1$ and $\frac{1}{\cos^2(\frac{(2n-1)\pi}{2n})} \to \frac{1}{\sin^2(\frac{\pi}{2n})} \to \frac{1}{(\frac{\pi}{2n})^2}$. Therefore,

$$\frac{1}{\sin\frac{\pi}{2n}}(\frac{1}{\cos^2(\frac{(2n-1)\pi}{2n})} - \frac{1}{\cos^2(\frac{\pi}{2n})}) \to \frac{2n}{\pi}((\frac{2n}{\pi})^2 - 1)$$

and so

$$\frac{a_n}{n^3} \to \frac{2}{\pi}((\frac{2}{\pi})^2 - \frac{1}{n^2}) = \frac{8}{\pi^3}$$

B4 Let $\mathcal{F}$ be the set of functions $f(x,y)$ that are twice continuously differentiable for $x \geq 1, y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$xf_x + yf_y = xy\ln(xy) \qquad x^2 f_{xx} + y^2 f_{yy} = xy$$

For each $f \in \mathcal{F}$, let

$$m(f) = \min_{s\geq 1}(f(s+1,s+1) - f(s+1,s) - f(s,s+1) + f(s,s))$$

Determine $m(f)$, and show that it is independent of the choice of f.

Answer. $2\ln 2 - \frac{1}{2}$.

Solution. We first note that

$$xf_{xy} + f_y + yf_{yy} = \frac{\partial}{\partial y}(xf_x + yf_y + xy\ln(xy)) = \frac{\partial}{\partial y}(xy\ln(xy)) = x(\ln xy + 1)$$

Multiplying this by y gives $xyf_{xy} + yf_y + y^2 f_{yy} = xy(\ln xy + 1)$. Similarly, taking this original first derivative condition, taking $\frac{\partial}{\partial x}$ and multipled by x again gives $xyf_{xy} + xf_x + x^2 f_{xx} = xy(\ln xy + 1)$. Thus adding them up we have

$$2xy(\ln xy + 1) = 2xyf_{xy} + xf_x + yf_y + x^2 f_{xx} + y^2 f_{yy} = 2xyf_{xy} + xy\ln xy + xy$$

Therefore, $f_{xy} = \frac{1+\ln(xy)}{2}$. This gives us functions $C_1(x)$ and $C_2(y)$ such that

$$f_x = \int f_{xy} dy = \int \frac{1+\ln x + \ln y}{2} dy = \frac{y\ln(xy)}{2} + C_1(x)$$

$$f_y = \int f_{xy} dx = \int \frac{1+\ln x + \ln y}{2} dy = \frac{x\ln(xy)}{2} + C_2(y)$$

By the first order derivative condition, this gives $xy\ln(xy) = xf_x + yf_y = xy\ln(xy) + xC_1(x) + yC_2(y)$, so there is a constant C such that $C_1(x) = \frac{C}{x}$ and $C_2(y) = -\frac{C}{y}$.

We now compute $m(f,s) \triangleq f(s+1,s+1) - f(s+1,s) - f(s,s+1) + f(s,s)$. Observe that:

$$f(s+1,s) - f(s,s) = \int_s^{s+1} f_x(x,s)dx = \int_s^{s+1} \frac{y\ln(xs)}{2} + \frac{C}{x}dx = \frac{xs\ln(xs) - xs}{2} + C(\ln x)|_s^{s+1}$$

$$= \frac{s(s+1)\ln(s+1) - s(s-1)\ln s - s}{2} + C\ln(\frac{s+1}{s})$$

Similarly, we get $f(s,s+1) - f(s,s) = \frac{s(s+1)\ln(s+1) - s(s-1)\ln s - s}{2} - C\ln(\frac{s+1}{s})$. Finally, we get

$$f(s+1,s+1) - f(s,s) = \int_s^{s+1} f_x(x,x) + f_y(x,x)dx = \int_s^{s+1} x\ln(x^2)dx$$

$$= x^2\ln x - \frac{x^2}{2}|_s^{s+1} = (s+1)^2\ln(s+1) - s^2\ln s - \frac{2s+1}{2}$$

Thus

$$m(f,s) = (s+1)^2\ln(s+1) - s^2\ln s - \frac{2s+1}{2} - s(s+1)\ln(s+1) + s(s-1)\ln s + 2$$

$$= (s+1)\ln(s+1) - s\ln s - \frac{1}{2}$$

and therefore, $m(f) = \min_{s\geq 1} m(f,s)$. Finally, differentiating w.r.t. s on $m(f,s)$ gives $\ln\frac{s+1}{s} > 0$, showing $m(f,s)$ is increasing in s. Thus $m(f) = m(f,1) = 2\ln 2 - \frac{1}{2}$.

B5 Let F_m be the m'th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

Answer. $j = 2019, k = 1010$.

Solution. Start with $p_0 = p$, and for each $i \geq 1$, $p_{i+1}(x) = p_i(x+2) - p_i(x)$. We notice the following:

- By considering the expansion $(x+2)^k - x^k = \sum_{i=0}^{k-1} \binom{k}{i} 2^{k-i} x_i$, we have $\deg(p_{i+1}) = \deg(p_i) - 1$ whenever $\deg(p_i) \geq 1$.
- For each $k \leq 1008$, we can inductively show that for all $n = 0, 1, \cdots, 1008 - k$, $p_k(2n+1) = F_{2n+1+k}$. Indeed, this is true for $k = 0$. If this is true for some $k \geq 0$ (but $k \leq 1007$) then for $n = 0, \cdots, 1007 - k$, we have

  $$p_{k+1}(2n+1) = p_{k+1}(2n+3) - p_k(2n+1) = F_{2n+3+k} - F_{2n+1+k} = F_{2n+2+k}$$

  as desired.

So from above, we have $p_{1008}(1) = F_{1009}$, but $\deg(p_{1008}) = 0$ so p_{1008} is a constant (i.e. F_{1009}).

Next, we'll recover $p(2019)$ by the following: we see that $p_{1008}(3) = F_{1009}$, and moreover for all $k \leq 1007$,

$$p_k(2(1008-k)+3) = p_{k+1}(2(1008-k)+1) + p_k(2(1008-k)+1)$$

$$= p_{k+1}(2(1008-k)+1) + F_{k+2(1008-k)+1} = p_{k+1}(2(1008-k)+1) + F_{2016-k+1}$$

So we can do telescoping sum to get

$$p(2019) - p_{1008}(3) = \sum_{k=0}^{1007} p_k(2(1008-k)+3) - p_{k+1}(2(1008-k)+1) = F_{2016-k+1}$$

That is,

$$p(2019) = F_{1009} + F_{1010} + \cdots + F_{2017} = F_{2017} + \sum_{k=505}^{1008} F_{2k-1} + F_{2k} = F_{2017} + \sum_{k=505}^{1008} F_{2k+1}$$

$$= F_{2017} + (F_{1011} + F_{1013} + \cdots + F_{2017})$$

But then $F_{1011}+F_{1013}+\cdots+F_{2017} = F_{2018}-F_{1010}$, so $F_{2017}+(F_{1011}+F_{1013}+\cdots+F_{2017}) = F_{2017} + F_{2018} - F_{1010} = F_{2019} - F_{1010}$.