Solutions to Tournament of Towns, Spring 2021, Senior

Anzo Teh

O-Level

1.

A-Level

1. In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each of them divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile.

   Solution. Consider the situation with n sweets, b boys and g girls. Then we show that the number of sweets boys get will always be

$$f(n,b,g) := b\lfloor \frac{n}{b+g} \rfloor + \min(n\%(b+g), b)$$

   where $0 \leq n\%(b+g) < b+g$ is the remainder after n is divided by $b+g$. We proceed by induction on the total number of people, $b+g$.

   Base case: $b+g=1$, in which case $f(n,1,0)=n$ and $f(n,0,1)=0$.

   Base case: either $b=0$ or $g=0$. Then $f(n,b,0) = b\lfloor \frac{n}{b} \rfloor + \min(n\%b, b) = n$ and $f(n,0,g)=0$.

   Inductive step: now consider the case where $b, g \geq 1$. Also let $n = k(b+g)+\ell$ with $0 \leq \ell < b+g$. We'll need to show that the boys will get $kb + \min(b,\ell)$ sweets.

   If $\ell = 0$, the first boy will take k sweets; else he will take $k+1$ (hence he will always take $k+\min(1,\ell)$ sweets). Thus, $n-k = k(b+g-1)+\max(0,\ell-1)$ and the boys will get

$$k + \min(1,\ell) + f(n-k, b-1, g)$$

$$= k + \min(1,\ell) + (b-1)\lfloor \frac{k(b+g-1)+\max(0,\ell-1)}{b+g-1} \rfloor + \min\{\max(0,\ell-1), b\}$$

$$= kb + \min(1,\ell) + \min\{\max\{0,\ell-1\}, b\}$$

   Depending on the cases $\ell = 0, 1 \leq \ell \leq b, b \leq \ell < b+g$, we will always have $\min(1,\ell) + \min\{\max\{0,\ell-1\}, b\} = \min(\ell, b)$.

   If girl is the first, then she will take k sweets regardless. Now our tally becomes

$$f(n-k, b, g-1) = b\lfloor \frac{k(b+g-1)+\ell}{b+g-1} \rfloor + \min((n-k)\%(b+g-1), b)$$

$$= kb + b\lfloor \frac{\ell}{b+g-1} \rfloor + \min(\ell\%(b+g-1), b)$$

So it suffices to show that $b\lfloor \frac{\ell}{b+g-1} \rfloor + \min((\ell\%(b+g-1), b) = \min(\ell, b)$. This is clear when $\ell < b+g-1$. If $\ell = b+g-1$ we get $b+0 = b = \min(\ell, b)$ instead. This solves the claim.

2. Does there exist a positive integer n such that for any real x and y there exist real numbers $a_1, \cdots, a_n$ satisfying
$$x = a_1 + \cdots a_n \quad y = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$$

   Answer. Yes, pick $n = 4$.

   Solution. Consider $x_1, x_2 > 0$ arbitrary such that $x_1 - x_2 = x$ (e.g. $x_1 = 2x, x_2 = x$). Cosider, also, $a_1, a_2 > 0$ with $a_1 + a_2 = x_1$, and $a_3, a_4 < 0$ such that $a_3 + a_4 = -x_2$. Then as we vary a_1, a_2, a_3, a_4 in this range with $0 < a_1, a_2 < x_1$ and $-x_2 < a_3, a_4 < 0$, we have
$$\text{Range}(\frac{1}{a_1} + \frac{1}{a_2}) = [\frac{x_1^2}{2}, \infty) \quad \text{Range}(\frac{1}{a_3} + \frac{1}{a_4}) = (-\infty, -\frac{x_2^2}{2}]$$

   Thus $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}$ has range $(-\infty, \infty)$, and therefore can satisfy the sum as y.

3. Let M be the midpoint of the side BC of the triangle ABC. The circle ω passes through A, touches the line BC at M, intersects the side AB at the point D and the side AC at the point E. Let X and Y be the midpoints of BE and CD respectively. Prove that the circumcircle of the triangle MXY touches ω.

   Solution. Using power of point theorem,
$$BD \cdot BA = BM^2 = CM^2 = CE \cdot CA$$

   and since $\frac{BM}{BC} = \frac{1}{2} = \frac{BX}{BE}$ and $\frac{CM}{CB} = \frac{1}{2} = \frac{CY}{CD}$, we have triangles BXM and BEC similar, and CMY and CBD similar. Therefore,
$$MX = \frac{1}{2}CE \quad MY = \frac{1}{2}BD$$

   and therefore $\frac{MX}{MY} = \frac{CE}{BD} = \frac{BA}{CA}$. Moreover, $\angle BMX = \angle BCE$ and $\angle CMY = \angle CBD$, and therefore $\angle XMY = \angle ABC$. It therefore follows that ABC and MXY are similar. This means,
$$\angle YMC = \angle ABC = \angle MXY$$

   and so MXY touches ω.

4. There is a row of $100N$ sandwiches with ham. A boy and his cat play a game. In one action the boy eats the first sandwich from any end of the row. In one action the cat either eats the ham from one sandwich or does nothing. The boy performs 100 actions in each of his turns, and the cat makes only 1 action each turn; the boy starts first. The boy wins if the last sandwich he eats contains ham. Is it true that he can win for any positive integer N no matter how the cat plays?

   Answer. No.

   Solution. Consider the general setting where we have MN sandwiches and the boy making M actions at one turn (thus we have $M = 100$ here). We show that the cat can prevent the boy from winning when $N = 2^M$.

   Denote a_i as sandwich with position i, such that $a_i = 1$ if with ham, and $a_i = 0$ if without ham. Denote also S_i as the subsequence a_{Mk+i} for $k = 0, \cdots, N-1$. We claim the following claims:

Lemma 1. For each i with $1 \le i \le M$, the boy will eat exactly one sandwich in S_i in each of his turn.

Proof: since the boy takes M sandwiches in each turn, the number of sandwiches after each turn is multiple of M. Let $N' \le N$ be such that there are MN' sandwiches left, and that the index of the first sandwich at left end as ℓ. Then the remaining sandwiches are

$$a_\ell, \cdots, a_{\ell+MN'-1}$$

Now suppose the boy takes k sandwiches from left end and $M-k$ sandwiches from right end. The sandwiches taken have indices

$$\ell, \cdots, \ell+k-1, \ell+MN'-(M-k), \cdots, \ell+MN'-1$$

which, considering modulo M, have indicies

$$\ell, \cdots, \ell+k-1, \ell+k, \cdots, \ell+M-1$$

those containing all the remainders mod M exactly once. In particular, this holds for those with indices $\equiv i \pmod M$.

Lemma 2. Fix i with $1 \le i \le M$. Then starting with MN' sandwiches with N' even, the cat can perform actions such that after $\frac{N'}{2}$ turns from each player, all the remaining sandwiches in S_i have no ham (i.e. $a_{Mk+i} = 0, \forall k$ remaining).

Proof: By lemma 1, it suffices to focus on those N' sandwiches in the form S_i, in which the boy will take just one sandwich from one end. Now at first turn, after the boy makes his move, the cat considers the middle sandwich in S_i (i.e. $\frac{N'+1}{2}$-th from the left), and eats the ham from that sandwich (or do nothing, if the sandwich already didn't have ham). Now relabel the remaining $N'-2$ sandwiches into the following:

$$b_1, b_2, \cdots, b_{\frac{N'}{2}-1}, b_{\frac{N'}{2}+1}, \cdots, b_{N'-1}$$

and consider the pairing $(b_g, b_{g+\frac{N'}{2}})$ for $g = 1, \cdots, \frac{N'}{2}-1$. Then at each step, for each sandwich A the boy takes, the cat removes ham from the sandwich paired with A. We see the boy will never take the image later on within the $\frac{N'}{2}$ steps: if the boy takes b_g for some $g \le \frac{N'}{2}-1$, this means it already takes g sandwiches from S_i from the left, while $b_{\frac{N'}{2}+g}$ is the $\frac{N'}{2}-g$-th sandwich from the right. Thus for the boy to take that there needs to be at least $g + (\frac{N'}{2}-g) + 1 = \frac{N'}{2}+1$ steps (the $+1$ is for the very first step before the cat acts from the center). Consequently, after $\frac{N'}{2}$ steps, for each of the $\frac{N'}{2}-1$ pairs there's one sandwich being eaten by the boy and the other has ham taken out, establishing the lemma.

Now by Lemma 2, we can do the following: starting from $N = 2^M$, the cat removes ham from S_1, then $M2^{M-1}$ sandwiches left, and then removes ham from S_2, then $M2^{M-2}$ sandwiches left. Continuing this until the cat reaches S_M, and exactly M sandwiches left. But since $S_1, \cdots, S_M$ represents all sandwiches, we conclude that the all of the last M sandwiches do not have ham, so the boy loses.