Putnam 2016

A1 Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left.\frac{d^j}{dx^j}p(x)\right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

Answer. $j = 8$.
Solution. Consider $p(x) = x^j$, and we know that $p^{(j)}(x) = j(j-1)\cdots 1 = j!$. The condition implies that $2016|j!$. Since $7! = 5040$ is not divisible by 2016, and $j!|7!$ for $j \leq 7$, we know that $2016 \nmid j!$ for $j \leq 7$. So $j \geq 8$.

Now suppose that $j \geq 8$. Let $p(x) = \sum_{i=0}^{n} a_i x^i$. Notice that differentiating the term x^i j times gives $i(i-1)\cdots(i-j+1)x^{i-j}$; in particular, this term is 0 if $i \leq 7$. Hence multiplying each term by a_i and summing them up we get

$$p^{(j)}(x) = \sum_{i=j}^{n} i(i-1)\cdots(i-j+1)a_i x^{i-j}$$

Notice that we omit all terms with $i < j$ since they contribute 0 to the sum anyway, with reasons explained above. Observe also that the coefficient of x^{i-j} in this derivative is $i(i-1)\cdots(i-j+1) = j!\binom{i}{j}$, hence is divisible by $j!$. For $j \geq 8$, $2016|40320 = 8!|j!$, so each term $i(i-1)\cdots(i-j+1)a_i x^{i-j}$ is divisible by 2016 whenever x is an integer (in particular this holds true for $x = k$). Hence any $j \geq 8$ works, and so the required j is 8.

A2 Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

Answer. $\frac{3+\sqrt{5}}{2}$
Solution. $m = n$ works since the left hand side is n while the right hand side is 0, so $m > n$ and we can then assume that m is positive below. We first try to consider the following inequality:

$$\frac{m!}{(n-1)!(m-n+1)!} > \frac{(m-1)!}{n!(m-n-1)!}$$

Cancelling factors, we are left with $\frac{m}{(m-n)(m-n+1)} > \frac{1}{n}$, so $(m-n)(m-n+1) > mn$. Expanding this, we get

$$m^2 - m(3n-1) + (n^2 - n) < 0$$

Using the formula for quadratic inequality, we get

$$m \in \left(\frac{(3n-1) - \sqrt{(3n-1)^2 - 4(n^2-n)}}{2}, \frac{(3n-1) + \sqrt{(3n-1)^2 + 4(n^2-n)}}{2} \right)$$

This means that $M(n)$ is the unique integer lying in the interval $[\frac{(3n-1)+\sqrt{(3n-1)^2+4(n^2-n)}}{2}-1, \frac{(3n-1)+\sqrt{(3n-1)^2+4(n^2-n)}}{2})$, which also means

$$\frac{M(n)}{n} \in \left[\frac{(3-\frac{1}{n})+\sqrt{(3-\frac{1}{n})^2-4(1-\frac{1}{n})}}{2}-\frac{1}{n}, \frac{(3-\frac{1}{n})+\sqrt{(3-\frac{1}{n})^2-4(1-\frac{1}{n})}}{2}\right)$$

now $\lim_{n\to\infty}\frac{(3-\frac{1}{n})+\sqrt{(3-\frac{1}{n})^2-4(1-\frac{1}{n})}}{2}-\frac{1}{n}=\frac{3+\sqrt{3^2-4}}{2}=\frac{3+\sqrt{5}}{2}$ and $\lim_{n\to\infty}\frac{(3-\frac{1}{n})+\sqrt{(3-\frac{1}{n})^2-4(1-\frac{1}{n})}}{2}=\frac{3+\sqrt{3^2-4}}{2}=\frac{3+\sqrt{5}}{2}$. By Squeeze's theorem, we get $\lim_{n\to\infty}\frac{M(n)}{n}=\frac{3+\sqrt{5}}{2}$, as desired.

A3 Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x)+f\left(1-\frac{1}{x}\right)=\arctan x$$

for all real $x \neq 0$. (As usual, $y=\arctan x$ means $-\pi/2<y<\pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)\,dx.$$

Answer. $\frac{3\pi}{8}$.
Solution. We first focus on the case $x \neq 0, 1$. Plugging $1-\frac{1}{x}$ into the equation above we get $f\left(1-\frac{1}{x}\right)+f\left(-\frac{1}{x-1}\right)=\arctan\left(1-\frac{1}{x}\right)$ and plugging $-\frac{1}{x-1}$ we get $f\left(-\frac{1}{x-1}\right)+f(x)=\arctan(-\frac{1}{x-1})$. Thus adding all these we get:

$$2\left(f(x)+f\left(1-\frac{1}{x}\right)+f\left(-\frac{1}{x-1}\right)\right)=\arctan x+\arctan\left(1-\frac{1}{x}\right)+\arctan\left(-\frac{1}{x-1}\right)$$

Thus for all $x \neq 0, 1$ we have

$$f(x)=\frac{\arctan x-\arctan\left(1-\frac{1}{x}\right)+\arctan\left(-\frac{1}{x-1}\right)}{2}$$

First, notice that arctan is an odd function (well-known), so $-\arctan\left(1-\frac{1}{x}\right)=\arctan\left(\frac{1}{x}-1\right)=\arctan\left(\frac{1-x}{x}\right)$, so we may rewrite $f(x)$ as $\frac{\arctan x+\arctan\left(\frac{1-x}{x}\right)+\arctan\left(\frac{1}{1-x}\right)}{2}$ Second, we consider the following:

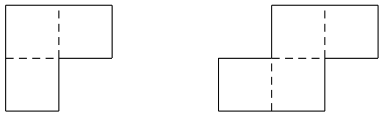
$$f(1-x)=\frac{\arctan(1-x)+\arctan\left(\frac{x}{1-x}\right)+\arctan\left(\frac{1}{x}\right)}{2}$$

Then, we use the fact that if $a>0$, $\arctan a+\arctan\frac{1}{a}=\frac{\pi}{2}$. This gives

$$f(x)+f(1-x)=\frac{\arctan x+\arctan\left(\frac{1-x}{x}\right)+\arctan\left(\frac{1}{1-x}\right)+\arctan(1-x)+\arctan\left(\frac{x}{1-x}\right)+\arctan\left(\frac{1}{x}\right)}{2}$$

$=\frac{3(\frac{\pi}{2})}{2}=\frac{3\pi}{4}$. So $\int_0^1 f(x)+f(1-x)dx=\int_0^1\frac{3\pi}{4}dx=\frac{3\pi}{4}$ (notice that all the computations are not valid when $x=0$ or 1, but the integral is still good even when we remove the two points 0 and 1 from our computation since a finite set of points do not influence the integral, or the integrability of the expression). We also have $\int_0^1 f(x)dx=\int_0^1 f(1-x)dx$, so the required answer is $f(x)=\frac{3\pi}{8}$.

A4 Consider a $(2m-1)\times(2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. The region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into 1×1 squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

Answer. mn.
Solution. We first consider the region as (i, j) with $1 \leq i \leq 2m-1$ and $1 \leq j \leq 2n-1$. We also label each square as types 1, 2, 3, 4 accoridng to the following rule:

(a) Type 1: Both i, j odd.
(b) Type 2: i even, j odd.
(c) Type 3: i odd, j even.
(d) Type 4: Both i, j even.

Now there are mn type 1 cells, $(m-1)n$ type 2 cells, $m(n-1)$ type 3 cells, and $(m-1)(n-1)$ type 4 cells.

Now name the first type of tile as 3-tile and the second type of tile as 4-tile. We first show that each tile covers cells of different type. Consider the 3-tile, and suppose the cell of the tile where its opposite is empty (in our example, it is the top left corner) covers the cell (i, j). The other cells that are being covered are $(i \pm 1, j)$ and $(i, j \pm 1)$. These two cells do not have the same type as (i, j) since for each of them, it cannot happen that both cooridnates have the same parity as that of (i, j). $(i \pm 1, j)$ and $(i, j \pm 1)$ also have different types since i and $i \pm 1$ must have different parity (same for j and $j \pm 1$). For the 4-tile, we argue by considering the Manhattan distance of each cell of the tile. Two cells of the same type must have even distance in both coordinates, and hence an even Manhattan distance. Here in this 4-tile, the only two pairs of cells with even Manhattan distance are diasonally apart, with distance 1 in each of the two coordinates (for both pairs). Hence they cannot have the same type either.

Having established the above, we know that each 4-tile covers each type of cell exactly once, and each 3-tile covers 3 of the types of the cells exactly once. Let a be the number of 4-tiles, and $b_1 + b_2 + b_3 + b_4 = b$ be the number of 3-tiles, with b_i signifying the number of 3-tiles that do not cover any cell of type i. Therefore we have the following: $4a+3b = (2m-1)(2n-1)$, and $a+b-b_1 = mn, a+b-b_2 = (m-1)n, a+b-b_3 = m(n-1)$, and $a+b-b_4 = (m-1)(n-1)$. Thus we know that $b_2 - b_1 = n, b_3 - b_1 = m$ and $b_4 - b_1 = m+n-1$. This forces $b = b_1+b_2+b_3+b_4 \geq n+m+n+m-1 = 2(m+n)-1$, and so $4(a+b) = 4a+3b+b \geq (2m-1)(2n-1)+2(m+n-1) = 4mn$ and we have the number of tiles is $a+b$, which is at least mn.

A5 Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r}$$

with $1 \leq r \leq |G|$ and $m_n, n_1, m_2, n_2, \ldots, m_r, n_r \in \{1, -1\}$. (Here $|G|$ is the number of elements of G.)

Solution. We first let S to be the set of elements in G that can be written in the desired form, but with the condition $1 \leq r \leq G$ relaxed. To start with, $gh, g^{-1}h, gh^{-1}, g^{-1}h^{-1} \in$

S. We also have S closed in multiplication. Now if $x = g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r} \in S$ then $x^{-1} = (g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r})^{|G|-1} \in S$ too, so we have $(g^{-1}h)^{-1} = h^{-1}g \in S$, and thus $ghh^{-1}g = g^2 \in S$. Since the order of G is odd, we also have $g = (g^2)^{\frac{ord(g)+1}{2}} \in S$, and $gg^{-1}h = h \in S$. Since g and h generate G, all elements in G are in S.

It remains to show that the restriction $1 \le r \le |G|$ can be imposed. For the identity element e we observe that $(gh)^|G| = e$ (just to consider the possibility of $r = 0$). Otherwise, if $x = g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r}$ with $r > |G|$, and suppose that this r is the minimal number of index needed to represent x in our desired form, then considering the element $x_k = g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_k}h^{n_k}$, and by the fact that $r > |G|$, we have $x_i = x_j$ for some $1 \le i \ne j \le r$. This way, we also have $x = g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_i}h^{n_i}g^{m_{j+1}}h^{n_{j+1}}\cdots g^{m_r}h^{n_r}$, contradicting the minimality of r.

B2 Define a positive integer n to be squarish if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

Answer.$\alpha = \frac{3}{4}, \beta = \frac{4}{3}$
Solution. We first consider the number of squarish numbers between 1 and n^2. Consider the number between $(k-1)^2 + 1$ to k^2 for some k, inclusive. For each of the numbers $m \in [(k-1)^2 + 1, k(k-1)]$, the closest square is $(k-1)^2$. For each of the numbers $m \in [k(k-1)+1, k^2]$, the closest square is k^2. In the first category, the distance from $(k-1)^2$ is $1, 2, \cdots, k-1$, so there are $\lfloor\sqrt{k-1}\rfloor$ squarish numbers. In the second category, the distance to k^2 is $k-1, k-2, \cdots, 0$, so the number of squarish numbers is then $\lfloor\sqrt{k-1}\rfloor + 1$. Therefore we have the sum as

$$S(n^2) = \sum_{k=1}^{n} 2\lfloor\sqrt{k-1}\rfloor + 1$$

The next thing is to evaluate the expression $\sum_{k=1}^{n}\lfloor\sqrt{k-1}\rfloor$. Again we use the following inequality: $\sum_{k=1}^{n}(\sqrt{k-1} - 1) < \sum_{k=1}^{n}\lfloor\sqrt{k-1}\rfloor \le \sum_{k=1}^{n}\sqrt{k-1}$. Since $\sqrt{x}$ is also an increasing function, we have the following:

$$\int_0^{n-1}\sqrt{x}dx \le \sum_{k=1}^{n}\sqrt{k-1} = \sum_{k=0}^{n-1}\sqrt{k} \le \int_0^{n}\sqrt{x}dx$$

(we are using the fact that $\int_{k-1}^{k} f(x)dx \le f(k) \le \int_k^{k+1} f(x)dx$ for f increasing, so $\int_0^{n-1} f(x)dx \le \sum_{k=0}^{n-1} f(k) \le \int_0^n f(x)dx$, given that $f(0) = 0$ for $f(x) := \sqrt{x}$). By evaluating the intergral $\int_0^n \sqrt{x}dx = \frac{2}{3}n^{3/2}$, we have

$$\begin{aligned}\frac{4}{3}(n-1)^{3/2} - n &\le (\sum_{k=1}^{n} 2\sqrt{k-1}) - n \\ &= \sum_{k=1}^{n}(2(\sqrt{k-1}-1)+1) \\ &< \sum_{k=1}^{n}(2\lfloor\sqrt{k-1}\rfloor+1) \\ &\le \sum_{k=1}^{n}(2\sqrt{k-1}+1) \\ &= n + 2\sum_{k=1}^{n}\sqrt{k-1} \\ &= \frac{4}{3}n^{3/2}+n\end{aligned}$$

meaning that $S(n^2) \in [\frac{4}{3}(n-1)^{3/2} - n, \frac{4}{3}n^{3/2}+n]$. Also notice that $S(n)$ is increasing with n (we need it in the rest of the proof). If we consider N in general, then $\lfloor\sqrt{N}\rfloor^2 \le N \le \lceil\sqrt{N}\rceil$. Thus

$$\frac{4}{3}(\lfloor\sqrt{N}\rfloor-1)^{3/2} - \lfloor\sqrt{N}\rfloor \le S(\lfloor\sqrt{N}\rfloor) \le S(n) \le S(\lceil\sqrt{N}\rceil) \le \frac{4}{3}(\lceil\sqrt{N}\rceil)^{3/2} + \lceil\sqrt{N}\rceil$$

again we note that $\lfloor\sqrt{N}\rfloor > \sqrt{N}-1$ and $\lceil\sqrt{N}\rceil < \sqrt{N}+1$. Thus we get $\frac{4}{3}(\sqrt{N}-2)^{3/2} - \sqrt{N} \le S(N) \le \frac{4}{3}(\sqrt{N}+1)^{3/2} + \sqrt{N}+1$. Notice also that

$$\lim_{N\to\infty}\frac{\frac{4}{3}(\sqrt{N}-2)^{3/2}-\sqrt{N}}{N^{3/4}} = \frac{4}{3} = \lim_{N\to\infty}\frac{\frac{4}{3}(\sqrt{N}+1)^{3/2}+\sqrt{N}+1}{N^{3/4}}$$

, so the limit must exist by Squeeze's theorem and equal to $\frac{4}{3}$, as desired.

B4 Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

Answer.

Solution. Denote a_{ij} as the (i,j)-th entry of A, and b_{ij} as the (i,j)-th entry of $A - A^t$. Notice that $b_{ij} = a_{ij} - a_{ji}$, which has $\frac{1}{2}$ chance of being zero, and $\frac{1}{4}$ chance of being -1, and $\frac{1}{4}$ chance of being 1, if $i \ne j$. If $i = j$ then $b_{ij} = 0$ at all times.

Now consider $\det(A - A^t) = \sum_{\sigma\in S}\operatorname{sgn}(\sigma)\prod_{i=1}^{2n} b_{i\sigma(i)}$ where S is the set of permutations of $\{1, 2, \cdots, n\}$, σ is its permutation, and sgn $= 1$ if σ is even, and -1 otherwise. The task is to find $E(\sum_{\sigma\in S}\operatorname{sgn}(\sigma)\prod_{i=1}^{2n} b_{i\sigma(i)})$, whixh is equal to $\sum_{\sigma\in S}\operatorname{sgn}(\sigma)E(\prod_{i=1}^{2n} b_{i\sigma(i)})$ by the linearity of expectation. Hence we can go ahead and investigate $E(\prod_{i=1}^{2n} b_{i\sigma(i)})$ individually for each of the $(2n)!$ permutations σ. We note the following:

(a) If $\sigma(i) = i$ for some i, then $b_{i\sigma(i)} = 0$ at all times, so $\prod_{i=1}^{2n} b_{i\sigma(i)} = 0$, and the expected value of this term is 0.

(b) If $\sigma(\sigma(k)) \ne k$ for some k, then the factor $b_{\sigma(k)k}$ is not present in the term $\prod_{i=1}^{2n} b_{i\sigma(i)}$. Recall that b_{ij} is dependent with b_{kl} if and only if $\{i,j\} \ne \{k,l\}$, so $b_{k\sigma(k)}$ is indep-dendent from $b_{i\sigma(i)}$ for all $i \ne k$, and is thus independent to $\prod_{i=1, i\ne k}^{2n} b_{i\sigma(i)}$. Now by the independence of $b_{k\sigma(k)}$ from the rest of the terms we get

$$E(\prod_{i=1}^{2n} b_{i\sigma(i)}) = E(b_{k\sigma(k)})E(\prod_{i=1,i\ne k}^{2n} b_{i\sigma(i)}) = 0E(\prod_{i=1,i\ne k}^{2n} b_{i\sigma(i)}) = 0$$

since $E(b_{ij}) = 0(\frac{1}{2}) + 1(\frac{1}{4}) - 1(\frac{1}{4}) = 0$ for all i, j. Hence this gives a 0 expectation too.

(c) Finally, assume that the two scenarios above do not happen, so for all i we have $\sigma(i) \neq i$ but $\sigma(\sigma(i)) = i$. This means that for each i, there are $\frac{1}{2}$ chance where $b_{i\sigma(i)} = b_{\sigma(i)i} = 0$, $\frac{1}{4}$ chance when $b_{i\sigma(i)} = -b_{\sigma(i)i} = 1$, and $\frac{1}{4}$ chance when $b_{i\sigma(i)} = -b_{\sigma(i)i} = -1$. Now in the first case the product $b_{i\sigma(i)}b_{\sigma(i)i} = 0$ while in the second and third case this product is -1, so the expected value of this product is $-\frac{1}{2}$. Since each such product is independent of all other $b_{j\sigma j}$ and $b_{\sigma jj}$, the expectation is then

$$E(\prod_{i=1}^{2n} b_{i\sigma(i)}) = \prod E(b_{i\sigma(i)}b_{\sigma(i)i}) = \left(-\frac{1}{2}\right)^n$$

The task now is to consider all permutations falling into the third category. First, this gives rise to n distinct orbits, so the parity of permutation is congruent to $2n - n = n \pmod 2$, and thus $\mathrm{sgn}(\sigma) = (-1)^n$ for all such σ. Second, the number of such permutations depends on the pairing of the $2n$ numbers, so this is exactly the ways to split these numbers into pairs. In general we have the number of pairs as:

$$\frac{\prod_{i=1}^{n} \binom{2i}{2}}{n!} = \frac{(2n)!}{2^n n!}$$

Hence the final expected value is $(\frac{1}{2})^n \cdot \frac{(2n)!}{2^n n!} = \frac{(2n)!}{4^n n!}$.

B5 Find all functions f from the interval $(1, \infty)$ to $(1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

Answer. $f(x) = x^c$ for any constant $c > 0$.

Solution. We first show that the aforementioned functions satisfy the problem condition. Indeed, given that $x^2 \leq y \leq x^3$ for any pairs of $x, y \in (1, \infty)$, from $x^2 \leq y$ and $c > 0$ we have $(x^2)^c \leq y^c$, and from $y \leq x^3$ we have $y^c \leq (x^3) \leq c$, as desired.

To show that there's no other suitable functions, denote $g : (0, \infty) \to \infty$ by $g(x) = \log f(e^x)$. Then the condition now becomes: if $e^{2x} \leq 3^y \leq e^{3x}$ then $2\log(f(e^x)) \leq \log(f(e^y)) \leq 3\log(f(e^x))$. In other words, if $2x \leq y \leq 3x$ then $2g(x) \leq g(y) \leq 3g(x)$. Notice that if a function g is such a solution, then for any constant $c > 0$, cg is also a solution. Hence we can assume that $g(1) = 1$.

We now show that $g(x) = x$ for all $x > 0$. Before that, let's show a lemma:

Lemma (a). Let $4 \leq x \leq 3^n$. Then $f(x) \leq 3^n f(1)$.
Proof: Let k be the integer satisfying $3^{k-1} < x \leq 3^k$ with $k \leq n$. Then $3^{\frac{k-1}{k}} < \sqrt[k]{x} \leq 3$. If $k \geq 3$ then $3^{\frac{k-1}{k}} > 3^{\frac{2}{3}} = \sqrt[3]{9} > 2$ and if $k = 2$ then $4 \leq x \leq 9$ and $2 \leq \sqrt{x} \leq 3$. In either case, $2 \leq \sqrt[k]{x} \leq 3$. This gives

$$f(x) = f(x^{\frac{1}{k}} x^{\frac{k-1}{k}}) \leq 3f(x^{\frac{k-1}{k}}) = 3f(x^{\frac{1}{k}} x^{\frac{k-2}{k}}) \leq 3^2 f(x^{\frac{k-2}{k}}) \leq \cdots \leq 3^k f(x) \leq 3^n f(x)$$

as desired (since $k \leq n$).

Lemma (b). Let $x \geq 2^n$ with $n \geq 2$. Then $f(x) \geq 2^n f(1)$.
Proof: the trick is essentially the same: let k be the integer satisfying $2^k \leq x < 2^{k+1}$, then $2 \leq \sqrt[k]{x} < 2^{1+1/k} \leq 2^{\frac{3}{2}} = \sqrt{8} < 3$. This means

$$f(x) = f(x^{\frac{1}{k}} x^{\frac{k-1}{k}}) \geq 2f(x^{\frac{k-1}{k}}) = 2f(x^{\frac{1}{k}} x^{\frac{k-2}{k}}) \geq 2^2 f(x^{\frac{k-2}{k}}) \leq \cdots \leq 2^k f(x) \geq 2^n f(x)$$

since $k \geq n$ in this case.

To solve this problem, we also need another lemma:

Lemma 2. Let x be an irrational number, then $\{\{nx\} : n = 1, 2, \cdots\}$ is dense in $(0, 1)$ (i.e., the fractional part of the number).
Proof: Let $(a, b) \subseteq (0, 1)$ be an interval of length $\varepsilon > 0$, and let $N > \frac{1}{\varepsilon}$ be an integer. Consider the $\{\{nx\} : n = 1, 2, \cdots, N+1\}$. Since x is irrational, all the $\{nx\}$ are different. By pigeonhole principle, there exists $m, n \leq N+1$ such that $0 < |\{mx\} - \{nx\}| < \varepsilon$. From here, we can conclude that $0 < \{(m-n)x\} < \varepsilon$. Consider, now, any $k \geq 1$ and the quantity $\{kx\}$. If $\{kx\} \in (a, b)$ then we are done. Otherwise, we first consider the case where $m > n$. If $\{kx\} \leq a$ let ℓ be the minimum positive integer such that $\{kx\} + \ell\{(m-n)x\} > a$. Then since $\{(m-n)x\} < \varepsilon$, the minimality of ℓ suggests that $\{kx\} + \ell\{(m-n)x\} < a + varepsilon = b$, so $\{(k + \ell(m-n))x\} \in (a, b)$. Otherwise, $\{kx\} \geq b$, in which case we have to consider ℓ as the minimum positive integer such that $\{kx\} + \ell\{(m-n)x\} > 1 + a$. Since a similar analysis shows that $1 + a < \{kx\} + \ell\{(m-n)x\} < 1 + b$, we have $\{(k + \ell(m-n))x\} \in (a, b)$. This is valid since $m > n$ and therefore $k + \ell(m-n) > 0$. In the case where $m < n$, if $\{kx\} \leq b$ then let ℓ be the minimum positive integer such that $\{kx\} - \ell\{(m-n)x\} < b$. By the similar proof before we would have $\{kx\} - \ell\{(m-n)x\} > a$ too, so $\{(k + \ell(n-m))x\} \in (a, b)$. Otherwise, $\{kx\} \leq a$ and let ℓ be the minimum positive integer such that $\{kx\} - \ell\{(m-n)x\} < -(1-b)$. By the similar proof before we would have $\{kx\} - \ell\{(m-n)x\} > -(1-a)$ so $-(1-a) < \{kx\} - \ell\{(m-n)x\} < -(1-b)$ and hence $\{(k + \ell(n-m))x\} \in (a, b)$, as desired. Notice we actually showed something slightly stronger (well not quite) – for each open interval $(a, b) \in (0, 1)$ and $k \geq 1$ there exists $\ell \in k$ such that $\{\ell x\} \in (a, b)$.

We are ready to go back to the proof now, which we assumed $g(1) = 1$. Suppose there exists x_0 with $g(x_0) > x_0$. Let $g(x_0) = cx_0$ with $c > 1$. If $n, m \geq 2$ are such that $2^n x_0 \leq 3^m$, then choosing $y \in [2^n x_0, 3^m]$ and by lemma 1 we have $g(y) \geq 2^n g(x_0) = 2^n c x_0$ and $g(y) \leq 3^m g(1) = 3^m$. Therefore, $2^n c x_0 \leq g(y) \leq 3^m$, or $2^n c x_0 \leq 3^m$. This means that whenever $n, m \geq 2$ are integers such that $2^n x_0 \leq 3^m$ (or $n \log 2 + \log x_0 \leq m \log 3$) we have $2^n x_0 c \leq 3^m$ (or $n \log 2 + \log x_0 + \log c \leq m \log 3$). This is to mean that there cannot exist $m, n \geq 2$ such that $m \log 3 - n \log 2 \in [\log x_0, \log x_0 + \log c]$. Let q be an integer that is greater than 2, $2 \log 3 / log 2$ and $2 \log 2 / log 3$. Given that $\log 2 / \log 3$ is irrational,