Solutions to APMO 2016 Problems

Anzo Teh

1. We say that a triangle ABC is great if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90°$ and $AB = AC$.

   Solution. Consider D the internal angle bisector from A to BC. This means $AP = AQ$ and the reflection of D with line PQ (say E) will also be on the bisector line AD. If ABC were to be great, then either $E = A$ or E is the midpoint M of arc BC not containing A of the circumcircle of ABC. Given that M and D are both on the same side of PQ, E' cannot be M. Therefore $E = A$ must hold. This gives $\angle PE'Q = \angle PAQ = \angle BAC = \angle PDQ = 180° - \angle BAC$, and therefore $\angle A = 90°$.

   Now we restrict our attention to $\angle A = 90°$. This means that $PDQA$ will be a rectangle, so $E' = A$ only when $PDQA$ is a square, i.e. when AD bisects $\angle A$. When this is not the case, E must be somewhere else on the circumcircle. But since $\angle PEQ = \angle PAQ = 90°$. E is the second intersection of circles APQ and ABC. This holds iff triangles EPB and EQC are similar. In particular, $\frac{EP}{PB} = \frac{EQ}{QC}$. Now since $EP = PD$ and $EQ = DQ$, we have $\frac{DP}{PB} = \frac{DQ}{QC}$ but with $\angle DPB = \angle DQC = 90°$, triangles DPB and DQC are similar. This means $\angle ABC = \angle ACB$ so $AB = AC$.

   Conversely, if $AB = AC$ and $\angle A = 90°$, then $EP = PD = PB$ and $EQ = DQ = QC$ will always hold, thereby giving the condition for spiral similarity and ensuring E is on the circumcircle ABC.

2. A positive integer is called fancy if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

   where $a_1, a_2, \cdots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

   Answer. $n = 2^{101} - 1$.

   Solution. We first show that if a number $k \geq 100$ has at most 100 1's in its binary representation, then it's fancy. Suppose it has $\ell \leq 100$ 1's in this representation. If $\ell = 100$ we're good (just let $a_1, \cdots, a_{100}$ be the places where the 1's occur). Otherwise, let $a_1, \cdots, a_\ell$ be the places where the 1's occur. Since $k \geq 100$, at least 1 number, say, $a_1 > 0$. We may then decompose 2^{a_1} into $2^{a_1-1} + 2^{a_1-1}$, increasing the number of summands from ℓ to $\ell + 1$. We can always repeat this until there are $100 \leq k$ summands, thereby proving this a fancy number.

   Now, we have shown that any $k \geq 100$ and $k < 2^{101} - 1$ are fancy: $2^{101} - 1$ is the smallest number with more than 100 1's in its binary representation. For $k = 1, 2, \cdots, 99$, the number $100k \geq 100$ but $100k \leq 9900 < 2^{101} - 1$, so these k's have fancy multiple.

   To show that $n = 2^{101} - 1$ works, we show that any fancy number cannot be devisible by n. Now, $2^{101} \equiv 1 \pmod{n}$, so if $a_1 \equiv b_1 \pmod{101}$ then $2^{a_1} \equiv 2^{b_1} \pmod{n}$. In addition, we have $2^a + 2^a = 2^{a+1}$. Therefore we can transform $a_1, \cdots, a_{100}$ into the following $b_1, \cdots, b_k$ using the following procedures:

- When $a_i \geq 101$ for some i, take the remainder of a_i modulo 101 (resulting in $0 \leq a_i < 101$).
- When $a_i = a_j$ for some $(i \neq j$, remove a_i and a_j and replace with $a_i + 1$.

Both procedures never increases the number of terms in the sequence $b_1, \cdots, b_k$; in fact, the second one decreases (hence can only be applied finitely many times). Between two consecutive instances of second procedure, the first one can only be applied at most $k \leq 100$ times. Therefore this process must terminate. In addition, from previous points, the sum $2^{b_1} + \cdots 2^{b_k}$ does not change modulo n. Therefore, $2^{a_1} + \cdots + 2^{a_{100}} \equiv 2^{b_1} + \cdots + 2^{b_k} \pmod{n}$.

Finally, the final instance of $b_1, b_2, \cdots, b_k$ are numbers at most 100 and are distinct. Since $k \geq 1$ must hold, $2^{b_1} + \cdots + 2^{b_k} > 0$ must hold. Therefore we have (bearing in mind that $k \leq 100$)

$$0 < 2^{b_1} + \cdots + 2^{b_k} \leq 2^{100} + \cdots + 2^{101-k} \leq 2^{100} + \cdots + 2^1 = n - 1 < n$$

and therefore $n \nmid 2^{b_1} + \cdots + 2^{b_k}$. Consequently, $n \nmid 2^{a_1} + \cdots + 2^{a_{100}}$.

3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

   Solution. We consider the transformation into pole-and-polar w.r.t. to ω, and for each object ℓ (point or line) let ℓ' be the image after this transformation (we have $\ell'' = \ell$). This means, $(AB)' = F$, $(AC)' = E$, $(EF)' = AB \cap AC = A$ so R' is a line that passes through A. In addition, P' is the line through F perpendicular to EF. Consequently, $(PR)'$ is a point Q on P', so that $QF \perp EF$. N' is the line joining $(PR)'$ and $(AC)' = E$, so $N' = QE$. Since this transformation maps parallel lines to two points collinear with the center O, the line through R parallel to AC has image X that's intersection of line $R' = AQ$ and the line $O(AC)' = OE$. Thus M' is the line through $(AB)' = F$ and X. The goal is to show that $N' \equiv QE$ and $M' \equiv FX$ intersect on ω. Thus the problem can be reformulated into the following:

   Let E and F be points on ω with AE, AF tangents to ω. Let Q be any point satisfying $QF \perp EF$, and X be intersection of lines OE and AQ. Prove that FX and QE intersect on ω.

   Now, consider the intersection $Z \neq E$ of QE and ω, $Y \neq F$ the intersection of QF and ω. This gives EY the diameter of ω. Let EY and FZ intersect at X', and we'll show that $X' = X$. Since OE and EY are the same line, it suffices to show that X', Q, A are collinear.

   Back to the polar transformation again, by Brokard's theorem on the quadrilateral $FZEY$, if H is the intersection of EF and ZY then QX' has polar H which is on EF. The pole of A is EF which passes through H. Therefore La Hire's theorem gives the consequence that the poles of X', Q, A concur at H, so these three points are collinear. Thus $X' = X$, and XF and QF meet at Z which is on ω.

4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

5. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(z+1)f(x+y) = f(xf(z)+y) + f(yf(z)+x),$$

for all positive real numbers x, y, z.

Answer. The only function is the identity function $f(x) \equiv x$, where both sides are equal to $(z+1)(x+y)$.

Solution. We start with the following lemma:
Lemma. If $x_1, x_2, y_1, y_2 > 0$ satisfy $x_1 + y_1 = x_2 + y_2$, then $f(x_1) + f(y_1) = f(x_2) + f(y_2)$.

Proof: we first show that f is unbounded. Suppose otherwise, then $f \leq M$ for all f, and therefore $(z+1)f(x+y) \leq 2M$ for all x, y, z. Let $x = y = 1$ and we have $(z+1)f(2) \leq 2M$ for all $z > 0$. This can only happen when $f(2) \leq 0$, contrasicting that f only takes positive values.

Now we consider the set of pairs $A_{z,w} : \{(xf(z)+y, yf(z)+x) : x+y = w\}$, focusing only on z with $f(z) > 1$ (which exists by the argument above). We see that if $(a, b) \in A_{z,w}$ then $a + b = (x+y)(f(z)+1) = w(f(z)+1)$, hence staying the same across the set. We also see that, as x varies in $(0, w)$, we have

$$a = xf(z) + y = xf(z) + (w-x) = x(f(z)-1) + w$$

so a takes all values in the range $(w, wf(z))$ and similarly for b. In addition, if $a, b \in A_{z,w}$ then $f(a) + f(b) = (z+1)f(w)$ which is again the same for all $A_{z,w}$. Hence it suffices to show that there exists z, w such that (x_1, y_1) and $(x_2, y_2) \in A_{z,w}$.

Choose z such that $f(z) > \max\{\frac{x_1}{y_1}, \frac{y_1}{x_1}, \frac{x_2}{y_2}, \frac{y_2}{x_2}\}$, and $w = \frac{x_1+y_1}{f(z)+1} = \frac{x_2+y_2}{f(z)+1}$. Then $A_{z,w}$ have pairs (a, b) satisfying $a+b = x_1+y_1$ and $w < a < wf(z)$. Since $f(z) > \frac{x_1}{y_1}$, we have $w < y_1$ and $x_1 < wf(z)$ and therefore $(x_1, y_1) \in A_{w,z}$. Similarly $(x_2, y_2) \in A_{w,z}$. This proves the lemma. Consequently, there's a function g satisfying $f(x) - f(y) = g(x-y)$ for all $x > y$. It can be proven that this g is additive.

Now, consider the difference when (x, y, z) is replaced with $(x, y, z + \Delta z)$. For the left hand side we have $(z + \Delta z + 1)f(x+y) - (z+1)f(x+y) = \Delta z f(x+y)$, and for right hand side we have

$$f(xf(z+\Delta z)+y) - f(xf(z)+y) = f(x(f(z)+g(\Delta z))+y) - f(xf(z)+y) = g(xg(\Delta z))$$

and similarly $f(yf(z+\Delta z)+x) - f(yf(z)+x) = g(yg(\Delta z))$.Therefore, we have for all $x, y, \Delta z > 0$:

$$\Delta z f(x+y) = g(xg(\Delta z)) + g(yg(\Delta z)) = g((x+y)g(\Delta z))$$

or, simply, with $x + y = w$ we have $\Delta z f(w) = g(wg(\Delta z))$ for all $w, \Delta z > 0$. Plugging w_1, w_2 into w and fixing $\Delta z > 0$ we have

$$\Delta z f(w_1) + \Delta z f(w_2) = g(w_1 g(\Delta z)) + g(w_2 g(\Delta z)) = g((w_1+w_2)g(\Delta z)) = \Delta x f(w_1+w_2)$$

and after dividing by Δz we get $f(w_1) + f(w_2) = f(w_1 + w_2)$. This means f is also additive.

Finally, notice also that if $f(z_1) = f(z_2)$ then for any x, y we have $(z_1+1)f(x+y) = (z_2+1)f(x+y)$ so $z_1 = z_2$, showing that f is injective. Combined with the additivity of the functions we have

$$(z+1)f(x+y) = f(xf(z)+y) + f(yf(z)+x)$$

$$= f(xf(z)) + f(yf(z)) + f(x) + f(y) = f(xf(z)+yf(z)) + f(x+y)$$

and again substituting $w = x + y$ gives $zf(w) = f(wf(z))$. Setting $z = 1$ gives $f(w) = f(wf(1))$ and by injectivity of f, $w = wf(1)$ so $f(1) = 1$. Setting $w = 1$ gives $z = zf(1) = f(f(z))$. Given that f is additive and is from positive reals to positive reals, it's also strictly increasing. This means, if $f(z) < z$ then $z = f(f(z)) < f(z) < z$ which is a contradiction. Similarly $f(z) > z$ means $z = f(f(z)) > f(z) > z$, also contradiction. Therefore, $f(z) = z$ must hold for all z.