

# Putnam 2013

- A1** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

**Solution.** Each face corresponds to exactly 3 vertices, so on average each vertex corresponds to  $20 \times 3 \div 12 = 5$  faces. Since this icosahedron is regular, each vertex corresponds to 5 faces. Suppose that for each vertex, the number written is different. Then the sum of the 5 faces joining a vertex is at least  $0 + 1 + 2 + 3 + 4 = 10$ . Since each vertex corresponds to 3 faces and there are 12 vertices, the total sum of 20 faces is at least  $10 \times 12 \div 3 = 40$ , contradiction.

- A2** Let  $S$  be the set of all positive integers that are not perfect squares. For  $n$  in  $S$ , consider choices of integers  $a_1, a_2, \dots, a_r$  such that  $n < a_1 < a_2 < \dots < a_r$  and  $n \cdot a_1 \cdot a_2 \cdots a_r$  is a perfect square, and let  $f(n)$  be the minimum of  $a_r$  over all such choices. For example,  $2 \cdot 3 \cdot 6$  is a perfect square, while  $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$ , and  $2 \cdot 3 \cdot 4 \cdot 5$  are not, and so  $f(2) = 6$ . Show that the function  $f$  from  $S$  to the integers is one-to-one.

**Solution.** Suppose that  $f(k_1) = f(k_2)$  for some  $k_1 < k_2$ . Let  $k_1 < a_1 < \dots < a_r = f(k_1)$  and  $k_2 < b_1 < \dots < b_s = f(k_2)$  be such choices for  $k_1$  and  $k_2$ . Given that  $k_1 \cdot a_1 \cdots a_r$  and  $k_2 \cdot b_1 \cdots b_s$  are both perfect square, their product  $k_1 \cdot a_1 \cdots a_r \cdot k_2 \cdot b_1 \cdots b_s$  is also a perfect square. Suppose that some number  $g$  appears in both sequence  $\{a_i\}$  and  $\{b_i\}$ , then removing  $g$  from the combined sequence  $k_1 \cdot a_1 \cdots a_r$  and  $k_2 \cdot b_1 \cdots b_s$  yields that  $k_1 \cdot a_1 \cdots a_r \cdot k_2 \cdot b_1 \cdots b_s / g^2$  is still a perfect square. Now, we remove all such repeated elements and sort the numbers, we get  $k_1 < c_1 < \dots < c_t$ , since  $k_1$  only appears once ( $k_1 < k_2$ ) and since  $a_r = b_s$ , this number is removed from both sides and  $c_t < a_r = f(k_1)$ , contradicting the minimality of  $a_r$ .

- A3** Suppose that the real numbers  $a_0, a_1, \dots, a_n$  and  $x$ , with  $0 < x < 1$ , satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \dots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number  $y$  with  $0 < y < 1$  such that

$$a_0 + a_1 y + \dots + a_n y^n = 0.$$

**Solution.** First, since  $0 < x < 1$ , each sequence  $1 - x^n = 1 + x^n + x^{2n} + x^{3n} + \dots$  converges absolutely. Hence we are free to permute the sequence and get the sum in the following sense:

$$0 = \frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \dots + \frac{a_n}{1-x^{n+1}} = \sum_{i=1}^n a_i \left( \sum_{j=0}^{\infty} x^{ij} \right) = \sum_{j=0}^{\infty} \left( \sum_{i=0}^n a_i (x^j)^i \right)$$

Let  $b_j = \sum_{i=0}^n a_i (x^j)^i$  for all  $j \geq 0$ , then it follows that  $\sum_{i=0}^{\infty} b_j$  also converges absolutely to 0. Now let  $k$  to be the minimal index such that  $b_k \neq 0$ . If  $b_j = 0$  for some  $j > k$  then we are done, since we can just pick  $y = x^j$  and since  $j > k \geq 0$ ,  $y \in (0, 1)$ . Otherwise, since  $\sum_{i=k}^{\infty} b_j = \sum_{i=0}^{\infty} b_j = 0$ , there exists a  $j \geq k$  such that  $b_j < 0$  and  $b_{j+1} > 0$ , or vice versa. In either case,  $a_0 + a_1 y + \dots + a_n y^n = 0$  for some  $y \in (x^{j+1}, x^j)$ , which obviously lies in  $(0, 1)$ .

- A4** A finite collection of digits 0 and 1 is written around a circle. An arc of length  $L \geq 0$  consists of  $L$  consecutive digits around the circle. For each arc  $w$ , let  $Z(w)$  and  $N(w)$  denote the number of 0's in  $w$  and the number of 1's in  $w$ , respectively. Assume that

$|Z(w) - Z(w')| \leq 1$  for any two arcs  $w, w'$  of the same length. Suppose that some arcs  $w_1, \dots, w_k$  have the property that

$$Z = \frac{1}{k} \sum_{j=1}^k Z(w_j) \text{ and } N = \frac{1}{k} \sum_{j=1}^k N(w_j)$$

are both integers. Prove that there exists an arc  $w$  with  $Z(w) = Z$  and  $N(w) = N$ .

**Solution.** Let  $n$  be the number of bits written around the circle, and  $m$  the number of 1's written. We first prove the following lemma: for each  $0 \leq k \leq n$ , consider  $S_k = \{Z(w) : |w| = k\}$  where  $|w|$  is the length of  $w$ . Then  $S_k = \{\lfloor \frac{km}{n} \rfloor, \lceil \frac{km}{n} \rceil\}$ . To prove this, we notice that from the problem statement,  $\max(S_k) - \min(S_k) \leq 1$ . We consider two possible cases:

- The quantity  $\frac{km}{n}$  is not an integer. Then  $\lceil \frac{km}{n} \rceil - \lfloor \frac{km}{n} \rfloor = 1$ . Now consider the  $n$  arcs  $W_1, W_2, \dots, W_n$  with length  $k$ ; each point belongs to exactly  $k$  of the arcs. Thus  $\sum_{i=1}^n Z(W_i) = km$  since there are  $m$  one's written, and the average of  $Z(w_i)$  is  $\frac{km}{n}$ . By piegonholw principle, there is at least one  $W_i$  with  $Z(W_i) \geq \lceil \frac{km}{n} \rceil$  and one  $W_i$  with  $Z(W_i) \leq \lfloor \frac{km}{n} \rfloor$ . Since  $\max(W_i) - \min(W_i) \leq 1$  and  $\lceil \frac{km}{n} \rceil - \lfloor \frac{km}{n} \rfloor = 1$ , the conclusion follows.
- Now that the quantity  $\frac{km}{n}$  is an integer, meaning that  $\lceil \frac{km}{n} \rceil = \frac{km}{n} = \lfloor \frac{km}{n} \rfloor$ . Denote the  $n$  arcs by  $W_1, \dots, W_n$ , and by the logic above, the average of  $Z(w_i)$  is  $\frac{km}{n}$ . If there is  $w_i$  with  $Z(w_i) < \frac{km}{n}$ , i.e.  $Z(w_i) \leq \frac{km}{n} - 1$ , then there must be  $w_i$  with  $Z(w_i) > \frac{km}{n}$ , i.e.  $Z(w_i) \geq \frac{km}{n} + 1$ . This is a contradiction that  $\max(S_k) - \min(S_k) \leq 1$ , so we have  $Z(W_i) = \frac{km}{n}$  for each  $i$ .

Now going back to the problem. For each arc  $w$ , we have  $|w| = Z(w) + N(w)$ . Thus considering the  $w_1, \dots, w_k$  given in the problem we have

$$Z + N = \frac{1}{k} \sum_{j=1}^k Z(w_j) + \frac{1}{k} \sum_{j=1}^k N(w_j) = \frac{1}{k} \sum_{j=1}^k (Z(w_j) + N(w_j)) = \frac{1}{k} \sum_{j=1}^k |w_j|$$

And in addition, for each  $w_j$  we have  $Z(w_j) \in \{\lfloor \frac{|w_j|m}{n} \rfloor, \lceil \frac{|w_j|m}{n} \rceil\}$  so  $\frac{|w_j|m}{n} - 1 < Z(w_j) < \frac{|w_j|m}{n} + 1$ . This means,  $\frac{1}{k} \sum_{j=1}^k (\frac{|w_j|m}{n} - 1) < Z < \frac{1}{k} \sum_{j=1}^k (\frac{|w_j|m}{n} + 1)$ , i.e.  $(\frac{m}{kn} \sum_{j=1}^k |w_j|) - 1 < Z < (\frac{m}{kn} \sum_{j=1}^k |w_j|) + 1$ , and since  $\frac{1}{k} \sum_{j=1}^k |w_j| = Z + N$ , we have

$$(\frac{m}{n}(Z + N)) - 1 < Z < (\frac{m}{n}(Z + N)) + 1$$

Since  $Z$  is an integer, this also implies that  $Z \in \{\lfloor \frac{m}{n}(Z + N) \rfloor, \lceil \frac{m}{n}(Z + N) \rceil\}$ . This means we can find an arc  $w$  of length  $Z + N$  that has  $Z(w) = Z$ , and therefore  $N(w) = N$ .

**A5** For  $m \geq 3$ , a list of  $\binom{m}{3}$  real numbers  $a_{ijk}$  ( $1 \leq i < j < k \leq m$ ) is said to be area definite for  $\mathbb{R}^n$  if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of  $m$  points  $A_1, \dots, A_m$  in  $\mathbb{R}^n$ . For example, the list of four numbers  $a_{123} = a_{124} = a_{134} = 1, a_{234} = -1$  is area definite for  $\mathbb{R}^2$ . Prove that if a list of  $\binom{m}{3}$  numbers is area definite for  $\mathbb{R}^2$ , then it is area definite for  $\mathbb{R}^3$ .

**Solution.** Let's first claim the following:

*Lemma.* Consider a plane  $p$ , and let  $A_1 A_2 A_3$  be on a plane  $q$ . Let  $B_i$  be the projection from  $A_i$  to  $p$ . Then

$$|\triangle B_1 B_2 B_3| = |\cos \angle(p, q)| \cdot |\triangle A_1 A_2 A_3|$$

where  $\angle(p, q)$  is the angle between the two planes.

Proof: if  $A_1A_2A_3$  is degenerate (i.e. on the same line), then so would  $B_1B_2B_3$ . Now assume  $A_1A_2A_3$  is not degenerate. By doing parallel shift on  $p$  (which wouldn't change the area of  $B_1B_2B_3$ ), we may shift  $p$  such that  $A_1$  on  $p$ , and therefore  $B_1 = A_1$ .

Now, let the line  $A_2A_3$  to intersect  $p$  at  $C$  (the case when  $A_2A_3$  is parallel to  $C$  can be dealt separately). Then  $AC$  is the intersection line of  $p$  and  $q$  (either that, or  $p$  and  $q$  coincide). In addition, we have  $C, B_2, B_3$  collinear in this case. If  $h_A, h_B$  are the perpendiculars from  $A_2, B_2$  to line  $AC$ , then  $h_A$  and  $h_B$  intersect on  $AC$  and  $|h_B| = \cos \angle(p, q)|h_A|$ . Therefore,

$$|\triangle ACB_2| = \frac{1}{2}AC|h_B| = \frac{1}{2}AC \cos \angle(p, q)|h_A| = \cos \angle(p, q)|\triangle ACA_2|$$

and similarly  $|\triangle ACB_3| = \cos \angle(p, q)|\triangle ACA_3|$ . If  $A_2, A_3$  are on the same side of  $C$ , then so will  $B_2, B_3$  and in this case,

$$\begin{aligned} |\triangle AB_3B_2| &= ||\triangle ACB_2| - |\triangle ACB_3|| \\ &= |\cos \angle(p, q)|\triangle ACA_2| - \cos \angle(p, q)|\triangle ACA_3|| = |\cos \angle(p, q)||\triangle AA_3A_2| \end{aligned}$$

and if  $A_2, A_3$  are on different side of  $C$ , then so will  $B_2, B_3$  and in this case,

$$\begin{aligned} |\triangle AB_3B_2| &= ||\triangle ACB_2| + |\triangle ACB_3|| \\ &= |\cos \angle(p, q)|\triangle ACA_2| + \cos \angle(p, q)|\triangle ACA_3|| = |\cos \angle(p, q)||\triangle AA_3A_2| \end{aligned}$$

as desired.

Now we turn back to the problem. Consider any list of  $\{a_{ijk}\}_{1 \leq i < j < k \leq m}$  that's area definite w.r.t.  $\mathbb{R}^2$ . Consider also, any  $m$  points  $A_1, \dots, A_m$  in  $\mathbb{R}^3$ . Define  $\Pi$  as the set of all planes passing through  $(0, 0, 0)$ , and let  $B_{i,p}$  be the projection  $A_i$  to plane  $p \in \Pi$ . Let's consider the quantity

$$S = \int_{p \in \Pi} \sum_{1 \leq i < j < k \leq m} a_{i,j,k} |\triangle B_{i,p}B_{j,p}B_{k,p}| dp$$

Since all points  $B_{i,p}$  are on plane  $p$ ,  $\sum_{1 \leq i < j < k \leq m} a_{i,j,k} |\triangle B_{i,p}B_{j,p}B_{k,p}| \geq 0$  since it's our numbers are area definite w.r.t.  $\mathbb{R}^2$ . Hence  $S \geq 0$ .

Let's now compute  $S$  in another way based on our lemma. Denote  $p_{i,j,k}$  as the plane passing through  $A_i, A_j, A_k$ , then by our lemma we can rewrite  $S$  as

$$\begin{aligned} S &= \int_{p \in \Pi} \sum_{1 \leq i < j < k \leq m} a_{i,j,k} |\cos \angle(p, p_{i,j,k})| \cdot |\triangle A_iA_jA_k| dp \\ &= \sum_{1 \leq i < j < k \leq m} a_{i,j,k} |\triangle A_iA_jA_k| \int_{p \in \Pi} |\cos \angle(p, p_{i,j,k})| dp \end{aligned}$$

But then  $\int_{p \in \Pi} |\cos \angle(p, q)| dp$  is the same for all planes  $q$ , and it's positive. To see why, we can consider the unit vectors  $p_1$  and  $q_1$  from  $(0, 0, 0)$  perpendicular to  $p, q$  respectively, so that  $\cos \angle(p, q) = p_1 \cdot q_1 = \cos \angle(p_1, q_1)$ . For each  $\theta \in [0, \pi]$ , we have

$$\int_{p: \cos(p, q) = \theta} 1 = 2\pi \sin \theta$$

since such set  $p$  is just the great circle on the unit sphere through the origin that's angle  $\theta$  away from  $q$ , hence having radius  $\sin \theta$ . Therefore we have

$$\int_{p \in \Pi} |\cos \angle(p, q)| = \int_0^\pi 2\pi |\sin \theta \cos \theta| = \pi \int_0^\pi |\sin 2\theta| = 2\pi$$

Thus there's constant  $C = 2\pi$  where  $\int_{p \in \Pi} |\cos \angle(p, q)| dp = C$  and

$$\sum_{1 \leq i < j < k \leq m} a_{i,j,k} |\triangle A_i A_j A_k| = \frac{S}{C}$$

and the conclusion thus follows when  $S \geq 0$  and  $C > 0$ .

**B1** For positive integers  $n$ , let the numbers  $c(n)$  be determined by the rules  $c(1) = 1$ ,  $c(2n) = c(n)$ , and  $c(2n+1) = (-1)^n c(n)$ . Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

**Answer.**  $-1$ .

**Solution.**

$$\begin{aligned} \sum_{n=1}^{2013} c(n)c(n+2) &= c(1)c(3) + \sum_{n=1}^{1006} c(2n)c(2n+2) + \sum_{n=1}^{1006} c(2n+1)c(2n+3) \\ &= c(1)c(3) + \sum_{n=1}^{1006} c(n)c(n+1) + \sum_{n=1}^{1006} (-1)^n c(n)(-1)^{n+1} c(n+1) \\ &= c(1)c(3) + \sum_{n=1}^{1006} c(n)c(n+1) + \sum_{n=1}^{1006} (-1)^{2n+1} c(n)c(n+1) \\ &= c(1)c(3) + \sum_{n=1}^{1006} c(n)c(n+1) - \sum_{n=1}^{1006} c(n)c(n+1) \\ &= c(1)c(3) \\ &= c(1)(-1)^1 c(1) \\ &= -1 \end{aligned}$$

**B2** Let  $C = \bigcup_{N=1}^{\infty} C_N$ , where  $C_N$  denotes the set of 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^N a_n \cos(2\pi n x)$$

for which:

- (i)  $f(x) \geq 0$  for all real  $x$ , and
- (ii)  $a_n = 0$  whenever  $n$  is a multiple of 3.

Determine the maximum value of  $f(0)$  as  $f$  ranges through  $C$ , and prove that this maximum is attained.

**Answer.** 3.

**Solution.** Consider the following:

$$\begin{aligned} f(x) &= 1 + \frac{4}{3} \cos(2\pi x) + \frac{2}{3} \cos(4\pi x) \\ &= 1 + \frac{4}{3} \cos(2\pi x) + \frac{2}{3} (2 \cos^2(2\pi x) - 1) \\ &= \frac{1}{3} (1 + 4 \cos(2\pi x) + 4 \cos^2(2\pi x)) \\ &= \frac{1}{3} (1 + 2 \cos(2\pi x))^2 \end{aligned}$$

which is clearly nonnegative all the time. We also have  $f(0) = 1 + \frac{4}{3} + \frac{2}{3} = 3$ , establishing the equality. To show that 3 is indeed the maximum, it suffices to show that  $\sum_{n=1}^N a_n \leq 2$  at all times. But plugging  $x = \frac{1}{3}$  gives  $\cos(\frac{2}{3}n\pi) = -\frac{1}{2}$  if  $n$  is not divisible by 3, and 1 otherwise. Considering that  $a_n = 0$  whenever  $n$  is a multiple of 3, we have  $f(\frac{1}{3}) = 1 - \frac{1}{2} \sum_{n=1}^N a_n \geq 0$ . Thus  $\sum_{n=1}^N a_n \leq 2$  must hold. Finally note the motivation to get the example  $f(x)$  as shown in the beginning: we simply find a suitable  $a$  such that  $2ax^2 + (2-a)x + (1-a)$  is always nonnegative, which is essentially asking for the discriminant  $(2-a)^2 - 4(2a)(1-a) \leq 0$ , and we get  $a = \frac{2}{3}$  as the sole answer.

**B3** Let  $P$  be a nonempty collection of subsets of  $\{1, \dots, n\}$  such that:

- (i) if  $S, S' \in P$ , then  $S \cup S' \in P$  and  $S \cap S' \in P$ , and
- (ii) if  $S \in P$  and  $S \neq \emptyset$ , then there is a subset  $T \subset S$  such that  $T \in P$  and  $T$  contains exactly one fewer element than  $S$ .

Suppose that  $f : P \rightarrow \mathbb{R}$  is a function such that  $f(\emptyset) = 0$  and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in P.$$

Must there exist real numbers  $f_1, \dots, f_n$  such that

$$f(S) = \sum_{i \in S} f_i$$

for every  $S \in P$ ?

**Answer.** Yes.

**Solution.** Let  $S_0$  be the subset such that  $|S_0| \geq |S|$  for all  $S \in P$ . We first show that  $S \subseteq S_0$  for all  $S \in P$ . Indeed, for an arbitrary set  $S \in P$ , we have  $S \cup S_0 \in P$  and  $|S \cup S_0| \geq |S_0|$ . By the maximality of  $S_0$  we must have  $|S \cup S_0| = |S_0|$ , which can only happen when  $S \subseteq S_0$ , as desired.

Now, w.l.o.g. let  $S_0 = \{1, 2, \dots, k\}$ . By (ii) there exists  $S_1 \subseteq S_0$  with one element fewer than  $S_0$ ; w.l.o.g. let it be  $\{1, 2, \dots, k-1\}$ . Continuing this trend we can assume that for all  $0 \leq i \leq k$ ,  $S_{k-i} = \{1, 2, \dots, i\} \in P$ . Consider, now,  $f_1, f_2, \dots, f_n$  such that  $f_i = f(S_{k-i}) - f(S_{k-i+1})$  for  $i = 1, 2, \dots, k$ , and arbitrary for  $i = k+1, \dots, n$ . The identity  $f(S) = \sum_{i \in S} f_i$  holds when  $S = S_0, S_1, \dots, S_k$  with  $S_k$  being the empty set (because  $f(\emptyset) = 0$ ).

To show that this identity holds for all  $S \in P$ , we first notice that  $S \subseteq S_0 = \{1, 2, \dots, k\}$ , so only  $f_1, \dots, f_k$  matter. We will proceed using the following premise with parameter  $p = 0, 1, \dots, k$ : the  $f(S) = \sum_{i \in S} f_i$  identity holds for all  $p$ -element subsets  $S$ . We are to prove this statement for all  $p = 0, 1, \dots, k$ , and we will proceed by induction.

Base case: when  $p = 0$  we have emptyset (proven above), and when  $p = 1$  we have  $S = \{j\}$  for some  $1 \leq j \leq k$ . We have  $S_{k-j} = \{1, 2, \dots, j\}$  and  $S_{k-j+1} = \{1, 2, \dots, j-1\}$ .

Moreover,  $f(S_{k-j+1}) = \sum_{i=1}^{j-1} f_i$  and  $f(S_{k-j}) = \sum_{i=1}^j f_i$  by how  $f_i$ 's are defined. Now by the definition of  $f$ ,

$$f(S_{k-j}) = f(S \cup S_{k-j+1}) = f(S) + f(S_{k-j+1}) - f(S \cup S_{k-j+1})$$

since  $S = \{j\}$ . Considering that  $S \cup S_{k-j+1} = \emptyset$ , we have  $f(S \cup S_{k-j+1}) = 0$  and therefore

$$f(S) = f(S_{k-j}) - f(S_{k-j+1}) = \sum_{i=1}^j f_i - \sum_{i=1}^{j-1} f_i = f_j$$

as desired.

Now let  $2 \leq q \leq p$  be such that the preimise is true for all  $p = 1, 2, \dots, q-1$ . Consider, now, any  $q$ -element subset  $S = \{a_1, a_2, \dots, a_q\}$ . By condition (ii), there exists a subset of  $S$  in  $P$  with one fewer element; w.l.o.g. let it be  $\{a_1, \dots, a_{q-1}\}$ . Consider, now, the set  $S_{k-a_q} = \{1, 2, \dots, a_q\} \in P$ . Consider now the two equations:

$$f(S_{k-a_q} \cup \{a_1, \dots, a_{q-1}\}) = f(S_{k-a_q}) + f(\{a_1, \dots, a_{q-1}\}) - f(S_{k-a_q} \cap \{a_1, \dots, a_{q-1}\})$$

$$f(S_{k-a_q} \cup \{a_1, \dots, a_q\}) = f(S_{k-a_q}) + f(\{a_1, \dots, a_q\}) - f(S_{k-a_q} \cap \{a_1, \dots, a_q\})$$

First, notice that  $\{a_1, \dots, a_{q-1}\}$  and  $\{a_1, \dots, a_q\}$  differ only by an element  $a_q$ , and since  $a_q \in S_{k-a_q}$ , we have  $\{1, 2, \dots, a_q\} \cup \{a_1, \dots, a_{q-1}\} = \{1, 2, \dots, a_q\} \cup \{a_1, \dots, a_q\}$ . Comparing the two equations now give

$$f(\{a_1, \dots, a_q\}) - f(\{a_1, \dots, a_{q-1}\}) = f(S_{k-a_q} \cap \{a_1, \dots, a_q\}) - f(S_{k-a_q} \cap \{a_1, \dots, a_{q-1}\})$$

Since  $a_q \in S_{k-a_q}$ , we have  $S_{k-a_q} \cap \{a_1, \dots, a_{q-1}\} \subset S_{k-a_q} \cap \{a_1, \dots, a_q\}$ , differing only by an element  $a_q$ . If  $\{a_1, \dots, a_q\} = S_{k-a_q}$  then the condition  $f(S) = \sum_{f_i \in S} f_i$  holds for

this  $S = \{a_1, \dots, a_q\}$ . Otherwise,  $S_{k-a_q} \cap \{a_1, \dots, a_q\}$  will have less than  $q$  elements. By the induction hypothesis,  $f(S_{k-a_q} \cap \{a_1, \dots, a_q\}) - f(S_{k-a_q} \cap \{a_1, \dots, a_{q-1}\}) = f_{a_q}$ , and therefore  $f(\{a_1, \dots, a_q\}) = f(\{a_1, \dots, a_{q-1}\}) + f_{a_q}$ . But by induction hypothesis again  $f(\{a_1, \dots, a_{q-1}\}) = f_{a_1} + \dots + f_{a_{q-1}}$ , and from here the conclusion follows.

**B5** Let  $X = \{1, 2, \dots, n\}$ , and let  $k \in X$ . Show that there are exactly  $k \cdot n^{n-1}$  functions  $f : X \rightarrow X$  such that for every  $x \in X$  there is a  $j \geq 0$  such that  $f^{(j)}(x) \leq k$ .

[Here  $f^{(j)}$  denotes the  $j$ th iterate of  $f$ , so that  $f^{(0)}(x) = x$  and  $f^{(j+1)}(x) = f(f^{(j)}(x))$ .]

**Solution.** We perform induction on  $n$  and  $n - k$ . When  $k = n$  (when  $n - k = 0$ ) then any function  $f : X \rightarrow X$  works, so there are  $n^n$  such functions; when  $k = n - 1$ , the only requirement is that  $f(n) \neq n$  so there are  $(n - 1)n^{n-1}$  such functions.

Thus now consider any  $k \leq n - 2$ . Observe that since  $f^0(x) = x$ , there is no restriction on  $f(1), f(2), \dots, f(k)$ , giving  $n^k$  choices to each of them. We first make a following detour to a lemma: there exists  $x > k$  with  $f(x) \leq k$ . Suppose not, then we have  $f : \{k + 1, \dots, n\} \rightarrow \{k + 1, \dots, n\}$  and for each  $x > k$  we have  $f^{(j)}(x) > k$  for any  $j \geq 0$ , contradiction.

Now fix  $m \in [1, n - k]$  such that exactly  $m$  of the numbers  $k < x \leq n$  have  $f(x) \leq k$ . This gives rise of  $\binom{n - k}{m}$  ways to choose those  $x$ , and each of them takes values  $\{1, 2, \dots, k\}$ , giving rise to  $k^m$  of them. Now w.l.o.g. assume that those  $m$  elements are  $k + 1, \dots, k + m$ . To see how would  $f(x)$  looks like for the other  $x > k + m$ 's, consider the problem of  $g : \{1, 2, \dots, n - k\} \rightarrow \{1, 2, \dots, n - k\}$  where for each  $x$ ,  $g^{(j)}(x) \leq m$  for some  $j \geq 0$ . Here,  $g(1), \dots, g(m)$  can be arbitrary, (i.e.  $(n - k)^m$  choices), and by the induction hypothesis  $n - k < n$  since  $k \geq 1$ , the number of such  $g$ 's is  $m \cdot (n - k)^{n - k - 1}$ , meaning that there are  $(n - k)^{n - k - m - 1}$  choices for  $g(m + 1), \dots, g(n - k)$ . Thus going back to our original problem here, we consider  $f|_{k+m+1, \dots, n}$  such that  $f^{(j)}(x) \leq k + m$  for some  $j$  (this is because, given that  $f(x) > k$  for all  $x > k + m$ , if  $j_0$  is the minimum  $j$  with  $f^{(j_0)}(x) \leq k$  then  $f^{(j_0-1)}(x) \in \{k + 1, \dots, k + m\}$ ). This gives  $m \cdot (n - k)^{(n - k - m - 1)}$  choices on  $f|_{k+m+1, \dots, n}$ , giving rise of  $\binom{n - k}{m} \cdot k^m \cdot m \cdot (n - k)^{(n - k - m - 1)}$  of them in total.

Hence considering all such  $m \in [1, n - k]$  gives

$$\begin{aligned}
\sum_{m=1}^{n-k} \binom{n-k}{m} \cdot k^m \cdot m \cdot (n-k)^{(n-k-m-1)} &= \sum_{m=1}^{n-k} \frac{(n-k)!}{m!(n-k-m)!} \cdot k^m \cdot m \cdot (n-k)^{(n-k-m-1)} \\
&= \sum_{m=1}^{n-k} \frac{(n-k)(n-k-1)!}{(m-1)!(n-k-m)!} \cdot k^{m-1} \cdot k \cdot (n-k)^{(n-k-m-1)} \\
&= k \sum_{m=1}^{n-k} \binom{n-k-1}{m-1} \cdot k^{m-1} (n-k)^{(n-k-m)} \\
&= k(k+n-k)^{m-k-1} \\
&= kn^{n-k-1}
\end{aligned}$$

so combining with the arbitrary choice of  $f|_{1,2,\dots,k}$  we have  $kn^{n-k-1}n^k = kn^{n-1}$ , as desired.