Solutions to Tournament of Towns, Fall 2017, Senior

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O-Level

1.

A-Level

3. An analyst made a prediction for the change in the dollar/euro rate for each of the next 12 months: by what percentage the rate would change in October, in November, in December, and so on. It turned out that for every month, he predicted the right percentage but was mistaken if it will go up or down (i.e., if he predicted that the rate will decrease by $x\%$, then the real rate increased by $x\%$, and vice versa). Nevertheless, the dollar/euro rate after 12 months coincided with the prediction. Did the dollar/euro rate go up or down on the whole?

   Answer. It decreases.

   Solution. Let $100 \cdot x_i$ be the actual signed percentage change at the i-th month (that is, positive if it goes up, negative if it goes down). Then the actual proportion change (without percentage) is x_i while what's predicted by the analyst is $-x_i$. The ratio of the dollar/euro rate after k months compared to the beginning is

$$\prod_{i=1}^{k}(1+x_i)$$

   and the last sentence suggests that there's a constant R satisfying

$$\prod_{i=1}^{12}(1+x_i) = R = \prod_{i=1}^{12}(1-x_i)$$

   Multiplying both sides, we get

$$R^2 = \prod_{i=1}^{12}(1+x_i)(1-x_i) = \prod_{i=1}^{12}(1-x_i^2) \leq 1$$

   with equality iff $x_i = 0$ for all i (i.e. the conversion ratio stays constant across the 12 months). Assuming this doesn't happen, we have $R < 1$ and therefore the rate goes down overall.

4. Show that for any infinite sequence $a_0, a_1, \cdots, a_n, \cdots$ of ones and negative ones, we can choose n and k such that

$$|a_0 \cdot a_1 \cdot \ldots \cdot a_k + a_1 \cdot a_2 \cdot \ldots \cdot a_{k+1} + \ldots + a_n \cdot a_{n+1} \cdot \ldots \cdot a_{n+k}| = 2017.$$

   Solution. Let $c = 2 \cdot 2017 = 4034$. Consider the tuples $t_n = (a_{n+1}, \cdots, a_{n+c})$. Since each entry as the tuples are ± 1, there are 2^c possible distinct tuples of such form. Therefore there exists m and n such that $t_m = t_n$, with $m < n$. Now consider k such that $k + 1 = n - m$, and consider

$$s_n = a_0 \cdot a_1 \cdot ... \cdot a_k + a_1 \cdot a_2 \cdot ... \cdot a_{k+1} + ... + a_n \cdot a_{n+1} \cdot ... \cdot a_{n+k}$$

for each n. Then notice that

$$s_n - s_{n-1} = a_n \cdot \dots \cdot a_{n+k} \qquad s_{n+1} - s_n = a_{n+1} \cdot \dots \cdot a_{n+k+1} = (s_n - s_{n-1})\frac{a_{n+k+1}}{a_n}$$

but given $t_m = t_n$, $a_{x+k+1} = a_x$ for $x = m+1, m+2, \cdots, m+c$. This gives $s_{x+2} - s_{x+1} = s_{x+1} - s_x$.

Finally, by convention $s_{-1} = 0$ and $s_n - s_{n-1} = \pm 1$ for all n. If $|s_{m+1}| \geq 2017$ then there must be an $x \leq m+1$ such that $|s_{m+1}| = 2017$ and we're done. Otherwise, we have $s_{m+1}, s_{m+2}, \cdots, s_{m+c}$ all one more than the term before or one less than the term before. This means $s_{m+c} = s_{m+1} + (c-1)$ or $s_{m+c} = s_{m+1} - (c-1)$. Since $-2016 \leq s_{m+1} \leq 2016$, s_{m+c} must lie outside the $[-2016, 2016]$ interval and so there's an $x \leq c$ with $|s_{m+x}| = 2017$, done.

6. A triangle ABC is given. Let I be the center of its excircle tangent to the segment AB, and let A_1 and B_1 be the points where the segments BC and AC touch the corresponding excircles. Let M be the midpoint of the segment IC, and let the segments AA_1 and BB_1 intersect at point N. Prove that the points N, B_1, A, and M are concyclic.

   Solution. In fact, we'll show that M is the second intersection of circles AB_1N and BA_1N. First, consider the second intersection of line IC with circumcircle of CAA_1, namely M'. $M'A = M'A_1$ and therefore by Ptolemy's theorem,

$$M'C \cdot AA_1 = M'A \cdot (CA + CA_1)$$

   but if the excircle touches CA and CB at B_2 and A_2 respectively, we have B_2, C, A_2, I concyclic and therefore

$$IC \cdot A_2B_2 = IA_2 \cdot (CA_2 + CB_2)$$

   and given that $M'AA_1$ and IA_2B_2 are similar, the two equations above give:

$$\frac{M'C}{CA + CA_1} = \frac{IC}{CA_2 + CB_2}$$

   but by definition of A_2 and B_2, CA_2 and CB_2 are each equal to s, the semiperimeter of triangle ABC, same goes to the sum $CA+CA_1$ given the definition of A_1. We $IC = 2CM'$, and so $M' = M$, i.e. M, A, C, A_1 are concyclic and therefore $MA = MA_1$. Similarly $MB = MB_1$.

   Notice also that $AB_1 = BA_1$: each of them are equal to the length of tangent from C to the incircle of ABC. Therefore by spiral similarity, if O is the intersection of the circles AB_1N and BA_1N we have $OA = OA_1$ and $OB = OB_1$ but then O and M are both on the intersection of perpendicular bisectors of AA_1 and BB_1 (which are not parallel as the lines themselves intersect at N), we have $O = M$, so M is indeed on the intersection of the two circles.