55th International Mathematical Olympiad

IMO 2014

Cape Town - South Africa

Contestant: MAS1

Problem: 1

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We prove a claim below:

Let $n$ such that the condition $\frac{a_0+a_1+\cdots+a_n}{n} \le a_{n+1}$ holds. Then, for $m>n$, $\frac{a_0+a_1+\cdots+a_m}{m} \le a_m$ holds.

We induct on $m$. The base case $m=n+1$ is clear, as:

$$\cancel{\frac{a_0+a_1+\cdots+a_n}{n} \le a_{n+1} \Rightarrow \frac{a_0+a_1+\cdots+a_n+a_{n+1}}{n+1} < \frac{\sum_{i=0}^{n+1} a_{n+1}-i}{n+1}}$$

$$\cancel{= \frac{(n+2)a_{n+1} - \frac{(n+1)(n+2)}{2}}{(n+1)} =}$$

$$\frac{a_0+a_1+\cdots+a_n}{n} \le a_{n+1} \Rightarrow \frac{a_0+\cdots+a_n+a_{n+1}}{n+1} \le \frac{n a_{n+1}+a_{n+1}}{n+1} = a_{n+1}.$$

Inductive step: if $\frac{a_0+a_1+\cdots+a_m}{m} \le a_m \le a_{m+1}$, then $\frac{a_0+a_1+\cdots+a_m+a_{m+1}}{m+1} < \frac{m a_m + a_{m+1}}{m+1} < \frac{m a_{m+1}+a_{m+1}}{m+1} = a_{m+1}$, completing the claim.

Similarly, if $a_n < \frac{a_0+a_1+\cdots+a_n}{n}$ for some $n$, then we can prove that $a_{m+1} < \frac{a_0+a_1+\cdots+a_m}{m}$ for $m<n$. Again, take base case $m=n-1$ we have $a_n < \frac{a_0+a_1+\cdots+a_n}{n} \Rightarrow \frac{a_0+a_1+\cdots+a_{n-1}}{n-1} > \frac{na_n-a_n}{n-1} = a_n.$

Inductive step: Let $a_{m+1} < \frac{a_0+a_1+\cdots+a_m}{m}$ for some $m \in [1, n-1]$, with $a_m < a_{m+1}$ we have

~~$a_m < \frac{a_0+a_1+a_m+\cdots+}{}$~~ $\frac{a_0+a_1+\cdots+a_{m-1}}{m-1} \geq \frac{(a_0+\cdots+a_m)-a_m}{m-1} > \frac{m a_m - a_m}{m-1} = a_m$.

This completes the inductive step.

Hence, if $a_n < \frac{a_0+a_1+\cdots+a_n}{n} \leq a_{n+1}$ for some $n$, then,

$\rightarrow$ for $m>n$, $\frac{a_0+a_1+\cdots+a_m}{m} \leq a_m$

$\rightarrow$ for $m<n$, $\frac{a_0+a_1+\cdots+a_m}{m} > a_{m+1}$

which shows at most one integer $n$ has this property stated in problem.

This leaves us to prove that such $n$ does exist.

Suppose that for every $n \in \mathbb{N}$, either $\frac{a_0+a_1+\cdots+a_n}{n} \leq a_n$ (1) or $\frac{a_0+a_1+\cdots+a_n}{n} > a_{n+1}$ (2). For $n=1$, $a_0+a_1 > a_1$ as $a_0>0$, so (1) doesn't hold and (2) holds. Also, if (1) holds for some $n$, then (1) holds for all $m>n$, and if (2) holds for some $n$, then (2) holds for all $m<n$.

Suppose there exists an $n$ s.t. (2) holds for $n$ and (1) holds for $n+1$. Then $\frac{a_0+a_1+\cdots+a_n}{n} > a_{n+1}$ and $\frac{a_0+a_1+\cdots+a_{n+1}}{n+1} \leq a_{n+1}$.

However, we have $a_0+a_1+\cdots+a_n > n a_{n+1}$, $a_0+a_1+\cdots+a_{n+1} \leq (n+1)a_{n+1} \Rightarrow$ $a_{n+1} < (n+1)a_{n+1} - n a_{n+1} = a_{n+1}$, contradiction.

Hence, if (1) holds for some $n$, then (2) doesn't hold for $n-1$, and by our assumption (1) holds for ~~n-2~~ $n-1$. Repeating that it yields (1) holds for ~~2~~ $n=1$, which is absurd. ~~It~~

It remains the case for $\frac{a_0+a_1+\dots+a_n}{n} > a_{n+1}$ for all $n$. This immediately implies the fact $\frac{a_0+a_1+\dots+a_n}{n} > a_n$, or

$a_0+a_1+\dots+a_n - na_n > 0$.

We show that $a_0+a_1+\dots+a_{n+1}-(n+1)a_{n+1} < a_0+a_1+\dots+a_n - na_n$, which shows that the function $f(n)=a_0+a_1+\dots+a_n-na_n$ is strictly decreasing, so $a_0+a_1+\dots+a_n-na_n \leq 0$ at some point.

However, $a_0+a_1+\dots+a_{n-1}+a_n+a_{n+1}-(n+1)a_{n+1} < a_0+a_1+\dots+a_n-na_n$ is equivalent to $a_{n+1}-(n+1)a_{n+1} < -na_n$, or $-na_{n+1} < -na_n$. This is true as $a_{n+1} > a_n$.

The solution is then complete, showing that $\frac{a_0+a_1+\dots+a_n}{n} > a_{n+1}$ cannot hold for all $n$, and there is a $n$ with

$$a_n < \frac{a_0+a_1+\dots+a_n}{n} \leq a_{n+1}.$$

$a_n = n+1$

$1+2+3+4\cdots$

$$n+1 < \left(\frac{\frac{(n+1)(n+2)}{2}}{n}\right) \leq n+2$$

$$2 < \frac{1+2}{1} \leq 3$$

$$3 < \frac{1+2+3}{2} \leq 4 \ ?$$

$$4 < \frac{1+2+3+4}{3} \leq 5$$

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}$$

$3.33\cdots$