Putnam 2020 (No official contest)

A1 How many positive integers N satisfy all of the following three conditions?

- N is divisible by 2020.
- N has at most 2020 decimal digits.
- The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Answer. 508536.

Solution. Let N to have a 1's and b 0's, so $N = 10^b \times \underbrace{1\cdots1}_{a}$. Since N is divisible by 20, it must have at least 2 ending zeros. Moreover, $2020 = 101 \times 20$ and $\gcd(10, 101) = 1$, so $101 \mid \underbrace{1\cdots1}_{a} = \frac{10^a-1}{9}$. Since $1111 = 101 \times 11$ but $101 \nmid 111, 11, 1$, we see that $\operatorname{ord}_{101}(10) = 4$ so $4 \mid a$. This gives the complete characterization:

$$a + b = 2020, b \geq 2, 4 \mid a, a \geq 1$$

Now for each a we can pick $b = 2, 3, \cdots, 2020 - a$ (thus giving $2019 - a$ choices). The maximum a is 2016. This gives the following:

$$\sum_{k=1}^{504}(2019 - 4k) = 2019 \times 504 - 4 \times 504 \times 505 \div 2 = 1009 \times 504 = 508536$$

A2 Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}.$$

Answer. 4^k.

Solution. Let our sum be S_k and do induction on it. Base case is given by $k = 0$ when we just have 1. Now suppose that our claim holds for some k, that is $S_k = 4^k$ for some $k \geq 0$. We have

$$S_{k+1}-2S_k = \sum_{j=0}^{k+1} 2^{k+1-j}\binom{k+1+j}{j} - 2\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j} = \binom{2(k+1)}{k+1} + 2\sum_{j=1}^{k} 2^{k-j}\binom{k+j}{j-1}$$

where we used the fact that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$. Notice again that the equivalence

$$2\sum_{j=1}^{k} 2^{k-j}\binom{k+j}{j-1} = 2\sum_{j=0}^{k-1} 2^{k-j-1}\binom{k+1+j}{j} = \frac{1}{2}(S_{k+1} - 2\binom{2k+1}{k} - \binom{2(k+1)}{k+1})$$

which gives

$$\frac{1}{2}S_{k+1} = 2S_k - \binom{2k+1}{k} + \frac{1}{2}\binom{2(k+1)}{k+1}$$

Given that $\binom{2k}{k} = 2\binom{2k-1}{k-1}$, we also have $-\binom{2k+1}{k} + \frac{1}{2}\binom{2(k+1)}{k+1} = 0$ and therefore $S_{k+1} = 4S_k = 4 \cdot 4^k = 4^{k+1}$, as desired.

A3 Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Answer. This sequence diverges.

Solution. We first see that $a_n > 0$ all the while: if $0 < a_n \le \frac{\pi}{2}$ for some n then $0 < a_{n+1} \le 1$, so we have $0 < a_n \le \frac{\pi}{2}$ for all n.

If $a_n \not\to 0$ then the series would diverge, so we may assume $a_n \to 0$. We first analyze the asymptotic behaviour of a_n^2 as a_n small:

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{(a_n - a_{n+1})(a_n + a_{n+1})}{a_n^2 a_{n+1}^2} = \frac{(a_n - \sin a_n)(a_n + \sin a_n)}{a_n^2 \sin^2 a_n} \overset{a_n \to 0}{\sim} \frac{\frac{a_n^3}{6} \cdot 2a_n}{a_n^4} = \frac{1}{3}$$

so as $n \to \infty$, we see that a_n^2 decays in the pace of $\frac{3}{n+c}$ for some constant c. Since $\sum \frac{1}{n}$ diverges, it follows that $\sum a_n^2$ diverges too.

A5 Let a_n be the number of sets S of positive integers for which

$$\sum_{k \in S} F_k = n, \tag{1}$$

where the Fibonacci sequence $(F_k)_{k \ge 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$ and begins $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$. Find the largest number n such that $a_n = 2020$.

Answer. $F_{4040} - 1$.

Solution. We first note the following:

$$\sum_{i=1}^{k} F_i = F_{k+2} - 1 \tag{2}$$

for all $k \ge 1$ (which is clear from $k = 1, 2, 3$ and the rest can be established via induction). If we define $f(n)$ as the index satisfying $F_{f(n)} \le n < F_{f(n)+1}$, then the set S satisfying Equation 1 has $\max\{k : k \in S\}$ either $f(n)$ or $f(n) - 1$. This gives us two scenario whenever $n \ge 2$ (and so $f(n) \ge 3$):

- If the max element in S is $f(n)$, we have $0 \le n - F_{f(n)} < F_{f(n)+1} - F_{f(n)} = F_{f(n)-1}$, which gives us $a_{n - F_{f(n)}}$ choices (all chosen in $\{1, \cdots, f(n) - 2\}$)
- If the max element in S is $f(n) - 1$, then it turned out it's more expedient to consider the elements *not* chosen among $\{1, \cdots, f(n) - 2\}$. By Equation 2, summing over $\{1, \cdots, f(n) - 1\}$ gives $F_{f(n)+1} - 1 \ge n$, so the number of ways here is precisely $a_{F_{f(n)+1} - n - 1}$.

Therefore we have the iterative formula $a_{n - F_{f(n)}} + a_{F_{f(n)+1} - n - 1}$.

Now with this, let's make the following claim:

- For all $k \ge 1$, $a_{F_{2k} - 1} = k$.
- For all $k \ge 1$ and $n \ge F_{2k}$, $a_n > k$. In other words, $a_n \ge \lfloor \frac{f(n)}{2} \rfloor + 1$.

These claims would suffice to show that our answer above.

For the first claim, we see that $a_{F_2 - 1} = a_0 = 1$, and $a_{F_4 - 1} = a_2 = 2$ (given that 2 can be written 2 and 1+1). Also, $f(F_{2k} - 1) = 2k - 1$, so for $k \ge 3$,

$$a_{F_{2k} - 1} = a_{F_{2k} - 1 - F_{2k-1}} + a_{F_{2k} - F_{2k}} = a_{F_{2k-2} - 1} + 1$$

so the inductive hypothesis $a_{F_{2k-2} - 1} = k - 1$ would imply $a_{F_{2k} - 1} = k$.

Now to prove the second claim, let us again use induction in the following sense: for each k, we consider those n with $F_{2k} \le n < F_{2k+2} - 1$. When $k = 0$ this is just $n = 0$ and $a_n = 1$, and when $k = 1$, $n = 1, 2$ which gives $a_1 = a_2 = 2$.

For $k \geq 2$, for n in the said range we have $f(n)$ either $2k$ or $2k+1$. Now consider the pair of numbers

$$n - F_{f(n)}, F_{f(n)+1} - n - 1$$

which sums up to $F_{f(n)-1} - 1 < F_{2k}$. Thus by induction hypothesis we can deduce that

$$a_n = a_{n-F_{f(n)}} + a_{F_{f(n)+1}-n-1} \geq \lfloor \frac{f(n - F_{f(n)})}{2} \rfloor + \lfloor \frac{f(F_{f(n)+1} - n - 1)}{2} \rfloor + 2$$

Now, let $\lfloor \frac{f(n-F_{f(n)})}{2} \rfloor = x$ and $\lfloor \frac{f(F_{f(n)+1}-n-1)}{2} \rfloor = y$, then

$$f(n - F_{f(n)}) \leq 2x + 1 \Rightarrow n - F_{f(n)} \leq F_{2x+2} - 1$$

and similarly $F_{f(n)+1} - n - 1 \leq F_{2y+2} - 1$. Therefore we have the sum satisfying $F_{f(n)-1} - 1 \leq F_{2x+2} + F_{2y+2} - 2$, or $F_{f(n)-1} \leq F_{2x+2} + F_{2y+2} - 1$. If $x + y < k - 1$, then we have $F_{2k-1} \leq F_{2x+2} + F_{2y+2} - 1$ for some $x + y < k - 1$, and for $x, y \geq 0$. (can substitute $f(n) = 2k$ here since if it holds for $2k+1$ it will hold for $2k$). We however see that F is convex hence

$$F_{2k-1} \leq F_{2x+2} + F_{2y+2} - 1 \leq F_2 + F_{2(x+y)+2} - 1 = F_{2(x+y)+2} \leq F_{2k-2}$$

which is a contradiction. Therefore $a_n \geq k - 1 + 2 = k + 1$, as claimed.

B1 For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

Answer. 1990.

Solution. We use the fact that $d(2k) = d(k)$ and $d(2k+1) = d(k) + 1$ for all $k \geq 0$ (note that $d(0) = 0$). Then we have

$$S = (-1)^{d(2020)} 2020^3 + \sum_{k=0}^{1009} (-1)^{d(k)} ((2k)^3 - (2k+1)^3) \equiv -\sum_{k=0}^{1009} (-1)^{d(k)} (12k^2 + 6k + 1) \pmod{2020}$$

which then becomes

$$-\sum_{k=0}^{504} (-1)^{d(k)} (12(2k)^2 + 6(2k) + 1) - (12(2k+1)^2 + 6(2k+1) + 1) = \sum_{k=0}^{504} (-1)^{d(k)} (48k + 18)$$

$$= (-1)^{d(504)} (48 \cdot 504 + 18) + \sum_{k=0}^{251} (-1)^{d(k)} (48(2k) + 18) - (48(2k+1) + 18)$$

Notice that the last summation is

$$\sum_{k=0}^{251} (-1)^{d(k)} (-48) = \sum_{k=0}^{125} (-1)^{d(k)} (-48 + 48) = 0$$

and therefore we just need to compute $d(504) = d(63) = 6$, and $48 \cdot 504 + 18 = 12 \cdot 2016 + 18 \equiv 12(-4) + 18 = -30 \equiv 1990 \pmod{2020}$.

B2 Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

Answer. Alice wins if and only if either n or k (or both) is odd.

Solution. We first design a winning strategy for Bob when k and n are both even. Split the holes into $\frac{n}{2}$ pairs of consecutive holes. We show that Bob can maintain the following invariant after his turns: each pair either has both pegs or none of the pegs. This invariant holds in the beginning, with the leftmost $\frac{k}{2}$ pairs having both pegs; the rest having no pegs. Starting from this condition, if Alice has no legal move she loses; otherwise, she moves a peg from a pair (namely p_1) of holes with both pegs into a pair of holes (namely p_2) with no peg, so p_2 is to the right of p_1. Thus Ben can move another peg from p_1 to another hole in p_2, so now p_1 has no peg while p_2 has both pegs, maintaining the invariant (and hence guaranteeing his victory).

On the other hand, suppose we do not have both k and n even; we show that Alice can make one move and reverse Ben's role from previous argument. Consider, now, the case where k is even while n odd. Alice can move the leftmost peg to the immediate right of the rightmost peg (i.e. the holes at positions $2, 3, \cdots, k+1$ are occupied). Since the leftmost hole is never going to be used again, we can assume we only have the rightmost $n-1$ holes with the k pegs, returning our case to the k, n even again except Bob is moving next, so Alice wins.

Finally, consider the case where k is odd. If n is odd, we move the rightmost peg to the rightmost hole; and if n is even, we move the leftmost peg to the rightmost hole. In this case after this initial Alice's move, the rightmost hole is occupied and therefore will not be emptied again. In the former case we're left with considering holes $1, 2, \cdots, n-1$ (from the left) with the first $k-1$ occupied; in the latter, since the leftmost hole is not going to be occupied, this left us with $2, 3, \cdots, n-1$ from theleft with the first $k-1$ occupied. In either case, we have even number of holes to consider with even number of leftmost holes occupied with pegs, so we can repeat the strategy of k, n even except with Bob starting next, so Alice wins.

B3 Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$, a point x_{n+1} is chosen uniformly form the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

Answer. $1 + \frac{1}{\delta}$.

Solution. Let's consider the function $F_n(\delta) = \mathbb{P}[x_n < \delta]$, and $f_n(\delta) = \frac{d}{d\delta} F_n$. This has the following recursive formula:

$$F_1(\delta) = \delta \qquad F_n(\delta) = F_{n-1}(\delta) + \int_\delta^1 \frac{\delta}{x} f_{n-1}(x) dx$$

Let's now show that

$$F_n(\delta) = \delta \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{\delta})^k}{k!} \right) \tag{3}$$

Using induction, we see that this holds for $n = 1$, and if this holds for some n, i.e.

$$F_n(x) = x \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{x})^k}{k!} \right) \qquad F_n(x) = \sum_{k=0}^{n-1} \frac{\ln(\frac{1}{x})^k}{k!} - \sum_{k=1}^{n-1} \frac{\ln(\frac{1}{x})^{k-1}}{(k-1)!} = \frac{\ln(\frac{1}{x})^{n-1}}{(n-1)!}$$

and therefore

$$F_{n+1}(\delta) = \delta\left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{\delta})^k}{k!}\right) + \int_\delta^1 \frac{\delta}{x}\frac{\ln(\frac{1}{x})^{n-1}}{(n-1)!}dx = \delta\left(\sum_{k=0}^{n} \frac{\ln(\frac{1}{\delta})^k}{k!}\right)$$

establishing 3.

To finish the solution, we have $\mathbb{P}[Z=k] = \mathbb{P}[x_n < \delta \wedge x_{n-1} \geq \delta] = F_k(\delta) - F_{k-1}(\delta) = \delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!}$. Therefore,

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{k=1}^{\infty} k\delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty} \delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} + (k-1)\delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty} \delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} + \sum_{k=2}^{\infty} \delta\frac{\ln(\frac{1}{\delta})^{k-2}}{(k-2)!} \\
&= \delta e^{\ln(\frac{1}{\delta})} + \delta\ln(\frac{1}{\delta})e^{\ln(\frac{1}{\delta})} \\
&= 1 + \ln\frac{1}{\delta}
\end{aligned}$$

as claimed.

B4 Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

Answer. $\frac{1}{4040}$.

Solution. In general, we show that the average of $1 + \sum_{j=1}^{2n-1} \alpha^{s_j}$ over V_n is $\frac{1}{2n}$ for any $\alpha > 0$.

Consider $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ and $t(\mathbf{v}) := (t_0, \cdots, t_{2n-1})$ be such that $t_k = s_{k+1} - s_k$. We say $\mathbf{v} \sim \mathbf{v}'$ if $t(\mathbf{v}')$ can be obtained from $t(\mathbf{v})$ via cyclic shift (hence we could also say $t(\mathbf{v}) \sim t(\mathbf{v})'$). Now t maps V_n to $T_n := \{(t_0, \cdots, t_{n-1}) \subseteq \{-1, 1\}^{2n}, \sum t_i = 0\}$, and is a bijection. Moreover, relation defined via cyclic shift is both symmetric (just reverse cycle) and transitive, and also $\mathbf{v} \sim \mathbf{v}$ holds for all $\mathbf{v}$. Thus $\sim$ is an equivalence relation.

Denote, now, the equivalence class of each $\mathbf{v}$:

$$E_{\mathbf{v}} = \{\mathbf{v}' : \mathbf{v}' \sim \mathbf{v}\}$$

We'll show that the average of $\frac{1}{q}$ in $E_{\mathbf{v}}$ is $\frac{1}{2n}$. Let $t(\mathbf{v}) = (t_0, t_1, \cdots, t_{2n-1})$ and for each $\mathbf{v}' \sim \mathbf{v}$ can be written as $t(\mathbf{v})' = (t_j, t_{j+1}, \cdots, t_{2n+j-1})$ for some $j \geq 0$ (indices taken modulo $2n$). Thus if $\mathbf{v} = (0, s_1, \cdots, s_{n-1}, 0)$ we have $\mathbf{v}' = (0, s_{j+1} - s_j, \cdots, s_{2n} - s_j, s_1 - s_j, \cdots, s_{j-1} - s_j, 0)$, and $\frac{1}{q(\mathbf{v}')} = \frac{\alpha^{s_j}}{q(\mathbf{v})}$. Now considering $j = 0, \cdots, 2n-1$ we see that the average of $\frac{1}{q}$ is now

$$\frac{1}{2n}\sum_{j=0}^{2n-1}\frac{\alpha^{s_j}}{q(\mathbf{v})} = \frac{1}{2n}$$

since $\frac{\alpha^{s_j}}{q(\mathbf{v})}$ is just 1. But we're not done yet – we need to show that averaging over all $j = 0, \cdots, 2n-1$ is the *true* average of this equivalence class. This is to show that as we loop over $j = 0, \cdots, 2n-1$ each $\mathbf{v}' \in E_{\mathbf{v}}$ shows up equally many times. If we extend $t_0, \cdots, t_{2n-1}$ infinitely (hence having $2n$ as period) and let g as its *minimal* period, we see that each $\mathbf{v}' \in E_{\mathbf{v}}$ shows up $\frac{2n}{g}$ times, proving our claim.