Algebra

A1 (IMO 4) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

Answer. There are three families of functions, with c denoting any integer constant:

- $f(n) = cn^2$.
- $f(n) = 0$ if n even, or c if n odd.
- $f(n) = c$ if n odd, $4c$ if $4 \mid n - 2$, 0 if $4 \mid n$.

We can verify that those work (although in the actual IMO a detail verification is needed...erm screw that).

Solution. Plugging $a = b = c = 0$ gives $3f(0)^2 = 6f(0)^2$ which forces $f(0) = 0$. Plugging $(a, b, c) = (0, n, -n)$ gives $f(n)^2 + f(-n)^2 = 2f(n)f(-n)$, i.e. $(f(n) - f(-n))^2 = 0$. This forces $f(n) = f(-n)$, so f is an even function.

Now we proceed with the following by considering $c = -(a + b)$, bearing in mind that $f(c) = f(-c)$, too. We now have the following:

$$f(a + b)^2 - 2f(a + b)(f(a) + f(b)) + (f(a)^2 + f(b)^2 - 2f(a)f(b)) = 0$$

which is essentially solving the quadratic equation on $f(a+b)$. Using the quadratic formula (and bearing in mind that $f(a)^2 + f(b)^2 - 2f(a)f(b) = (f(a) - f(b))^2$) we have:

$$\begin{aligned} f(a + b) &= \frac{2(f(a) + f(b)) \pm \sqrt{4(f(a) + f(b))^2 - 4(f(a) - f(b))^2}}{2} \\ &= (f(a) + f(b)) \pm \sqrt{4f(a)f(b)} \\ &= (f(a) + f(b)) \pm 2\sqrt{f(a)f(b)} \end{aligned}$$

In particular, if $f(b) = 0$ then $f(a + b) = f(a)$ so f is of period b. Hence $f(1) = 0$ implies $f \equiv 0$ (which can be put into any category above). We thus assume that $f(1) \neq 0$ in the future.

Now we see what happens when we fix $f(1)$. We have $f(2) = 2f(1) \pm 2\sqrt{f(1)^2} = 2f(1) \pm 2f(1)$, i.e. $f(2) = 0$ or $f(2) = 4f(1)$. In the first case, by setting $b = 2$ we have $f(a + 2) = f(a) \pm 2\sqrt{0} = f(a)$, so $f(a + 2) = f(a)$. We now infer that $f(a) = 0$ if a even, and $f(a) = f(1)$ if a odd.

Otherwise, we assume that $f(2) = 4f(1)$. Then $f(3) = f(1) + f(2) \pm \sqrt{f(1)f(2)}$, and setting $f(2) = 4f(1)$ we have $5f(1) \pm \sqrt{2}4f(1)^2 = 5f(1) \pm 4f(1)$, so we have two cases, depending whether $f(3) = f(1)$ or $f(3) = 9f(1)$.

In the first case, $f(3) = f(1)$. We have $f(4) = f(1) + f(3) \pm 2\sqrt{f(1)f(3)} = 2f(1) \pm 2f(1)$, which is equal to either 0 or $4f(1)$. Also $f(4) = f(2) + f(2) \pm 2\sqrt{f(2)^2} = 2f(2) \pm 2f(2)$ which is either 0 or $4f(2) = 16f(1)$. Since both $f(4) \in \{0, 4f(1)\}$ and $f(4) \in \{0, 16f(1)\}$, we have $f(4) = 0$ and therefore f is of period 4, in the form $c, 4c, c, 0$, etc.

Otherwise, $f(3) = 9f(1)$. Now $f(k) = k^2 f(1)$ for $k = 0, 1, 2, 3$. We now show that this is true for all integer k. Since f is even, we only need to prove this for all $k > 0$. We use induction: suppose that $f(n) = n^2 f(1)$ for all $n = 0, 1, \cdots, k$, with $k \geq 3$. We now consider $f(k + 1)$:

- considering $f(1)$ and $f(k)$, we have $f(k + 1) = f(1) + f(k) \pm 2\sqrt{f(1)f(k)} = f(1) + k^2 f(1) \pm 2\sqrt{k^2 f(1)} = f(1)(1 + k^2 \pm 2k) = f(1)(k \pm 1)^2$, i.e. either $f(1)(k + 1)^2$ or $f(1)(k - 1)^2$.

- considering $f(2)$ and $f(k-1)$, we have $f(k+1) = f(2)+f(k-1)\pm 2\sqrt{f(2)f(k-1)} = f(1)(2^2+(k-1)^2 \pm 2(2)(k-1)) = f(1)((k-1)\pm 2)^2$, i.e. $f(1)(k+1)^2$ or $f(1)(k-3)^2$.

We thus infer than $f(k+1) = f(1)(k+1)^2$, done.

A3 (IMO 2) Let $n \geq 3$ be an integer, and let $a_2, a_3, \dots, a_n$ be positive real numbers such that $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Solution. As per official solution, we will show that $f_k(x) = \frac{(x+1)^k}{x} \geq \frac{k^k}{(k-1)^{k-1}}$ with equality if and only if $x = \frac{1}{k-1}$. Consider the derivative w.r.t. x:

$$f_k'(x) = \frac{xk(x+1)^{k-1} - (x+1)^k}{x^2} = \frac{(x+1)^{k-1}}{x^2}(xk-(x+1)) = \frac{(x+1)^{k-1}}{x^2}((k-1)x-1)$$

Since $\frac{(x+1)^{k-1}}{x^2} > 0$, we have $f_k'(x) > 0$ iff $(k-1)x - 1 > 0$, i.e. $x > \frac{1}{k-1}$. Also $f_k'(x) < 0$ iff $(k-1)x - 1 < 0$, i.e. $x < \frac{1}{k-1}$. Thus f_k is increasing when $x > \frac{1}{k-1}$ and f_k is decreasing when $x < \frac{1}{k-1}$. We conclude that f_k attains it minimum point at $x = \frac{1}{k-1}$, with $f_k(\frac{1}{k-1}) = \frac{(1+1/(k-1))^k}{1/(k-1)} = \frac{k^k}{(k-1)^{k-1}}$.

Now, having established this, we have

$$\prod_{k=2}^{n}(1+a_k)^k = \prod_{k=2}^{n}\frac{(1+a_k)^k}{a_k} \geq \prod_{k=2}^{n}\frac{k^k}{(k-1)^{k-1}} = \frac{n^n}{1^1} = n^n$$

(since $\prod a_k = 1$). However, if the equality were to hold, we have $a_k = \frac{1}{k-1}$, so $\prod a_k = \frac{1}{(k-1)!} < 1$, which is a contradiction. Hence the equality cannot hold, so the inequality must be strict.

A4 Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Solution. Let $p_1 < p_2 < \cdots$ be a sequence of prime numbers and $\{r_k\}_{k\geq 1}$ be the sequence of rational numbers, such that $(p_nf+g)(r_n) = 0$.

We first show that the sequence $\{r_n\}$ must be bounded. Notice that $\{p_k\}_{k\geq 0} \to \infty$, and $p_n|f(r_n)| = |g(r_n)|$. Let M_1 be a real number such that $f(x), g(x) \neq 0$ for all $x > M_1$, and let M_2 be a real number such that $|f(x)| > |g(x)|$ for all $x > M_2$ (this is true since $\deg(f) > \deg(g)$ and therefore $\frac{|f|}{|g|} \to \infty$). From $p_n|f(r_n)| = |g(r_n)|$ (since $p_n \geq 1$ for primes p_n) we have $|f(r_n)| \leq |g(r_n)|$ since $|\cdot|$ is nonnegative. We therefore know that $r_n \leq \max\{M_1, M_2\}$ for all $n \geq 1$, and therefore $\{r_n\}$ is bounded.

Now, since $\{r_n\}$ is bounded, it has a convergent subsequence. By extracting the convergent subsequence and the corresponding p_n responsible for them, may as well assume that $\{r_n\}$ is itself convergent, and let $\{r_n\} \to r_0$. We show that r_0 is rational and $f(r_0) = 0$. Now that f and g are polynomials, they are continuous functions. Suppose that $f(r_0) > 0$. Since $\{r_n\} \to r_0$, there exists an N such that $f(r_n) > \frac{f(r_0)}{2}$. Since $\{r_n\}$ is also convergent (and bounded), $\{g(r_n)\}$ is also bounded: let N be such that $|g(r_n)| < N$ for all $n \geq 1$. Since $\{p_n\}$ is increasing sequences of primes, it's also unbounded. Thus, there exists p_n that's greater than $\frac{2N}{f(r_0)}$. Now, $p_n|f_n(r_n)| = |g_n(r_n)|$. However,

$$p_n|f_n(r_n)| > p_n|\frac{f(r_0)}{2}| > \frac{2N}{f(r_0)} \cdot \left|\frac{f(r_0)}{2}\right| = N > |g_n(r_n)|$$

which is a contradiction. Hence we cannot have $f(r_0) > 0$. Similarly we cannot have $f(r_0) < 0$. Therefore $f(r_0) = 0$.

Now since $\{r_n\}$ is a rational sequence, let's write $r_n = \frac{u_n}{v_n}$ with p_n, q_n integers. By the rational root theorem, we have $v_n \mid p_n a_k$ and $u_n \mid p_n a_0 + b_0$ where a_k is the leading coefficient of f (recall that $\deg f > \deg g$) and a_0, b_0 the constant coefficient of f and g, respectively. For clarity, let's focus on all n such that $p_n > a_k$. Since p_n is increasing sequence of primes, there's a N such that $p_n > a_k$ for all $n \geq N$. We may now assume that $N = 1$ (by truncating all the first $N - 1$ terms) and that $p_n > a_k$ for all $n \geq 1$.

If $p_n \nmid v_n$ for infinitely many v_n, then $v_n \mid a_k$ for those all those n's. Since there are infinitely many of those n's, and a_k has only finitely many divisors, there must be infinitely many v_n's that are the same. Let $\{r_{n_k}\}$ be the subsequence such that v_{n_k} are the same, say v_0. Since this subsequence also converges to r_0, u_{n_k} must be the same (say u_0) for sufficiently large k, too. This forces $r_0 = u_0/v_0$, and is thus rational.

Thus $p_n \mid v_n$ for all but finitely many v_n. Again by extracting the appropriate subsequence we may assume $p \mid v_n$ for all n. Let $v_n = p_n w_n$, then from $v_n \mid pa_k$ we have $w_n \mid a_k$. Again by the logic above, there are infinitely w_n that are the same, say, w_0. Since $u_n \mid p_n a_0 + b_0$, we can write $u_n = \frac{p_n a_0 + b_0}{d_n}$, and therefore we have $r_n = \frac{u_n}{v_n} = \frac{p_n a_0 + b_0}{d_n p_n w_n}$, with infinitely many of the w_n equal to w_0. In other words we have yet another subsequence $r_n = \frac{p_n a_0 + b_0}{d_n p_n w_0}$ converging to r_0, i.e. $\frac{p_n a_0 + b_0}{d_n p_n} = \frac{a_0}{d_n} + \frac{b_0}{d_n p_n}$ converging to $r_0 w_0$. If d_n is unbounded, this gives a subsequence of $\frac{p_n a_0 + b_0}{d_n p_n}$ converging to 0, so $r_0 = 0$. Otherwise, there exists infinitely many d_n that's the same, say d_0, so we have $\frac{p_n a_0 + b_0}{d_n p_n} = \frac{p_n a_0 + b_0}{d_0 p_n}$ which converges to $\frac{a_0}{d_0}$ and therefore $r_0 = \frac{a_0}{d_0 w_0}$, establishing the claim.

Combinatorics

C1 Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

Solution. Let M be the maximum of the n numbers. We first notice that at each step, the two numbers $y < x \leq M$ is changed to either $y+1$ and x or $x-1$ and x. Since $y < x$, $y + 1 \leq x \leq M$, so neither $y + 1, x, x - 1$ can exceed M. Thus after any iterations, no number can exceed M, and thus the sum of the n numbers is bounded by Mn. On the other hand, after each iteration, the sum of the n numbers either increases by 1 (in the case of $(y + 1, x)$), or $x - y - 1$ (in the case of $(x - 1, x)$). Since $x - y - 1 \geq 0$ (as $x > y$), the sum of the numbers never decrease. But since the sum of the numbers cannot exceed Mn either, it can only increase finitely many times (each increase adds to the sum by at least 1 since the numbers are all positive integers).

Thus if Alice were to do it infinitely many times, after one point the sum of the numbers remain the same. This falls into the case of $(x, y) \to (x - 1, x)$ and when $x - y - 1 = 0$, i.e. $y = x - 1$. This is basically swapping the two adjacent numbers $(x, y) \to (y, x)$, subject to the constraint $y = x - 1$. We see that the set of the numbers (along with their frequency) do not change, so all those iterations are basically permutations of the numbers.

Let's now order the permutations of n numbers in the following way: if σ and γ are two permutations, let k_0 be the leftmost index such that $\sigma(k_0) \neq \gamma(k_0)$. We say that $\sigma < \gamma$ iff $\sigma(k_0) < \gamma(k_0)$. Since $y < x$, each iteration changes the arrangement of the n numbers to a smaller permutation. But since there are only $n!$ permutation, there's only finitely many possible such iterations too. This proves the problem.

C2 Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Answer. $\lfloor \frac{2n-1}{5} \rfloor$

Solution. Suppose that we have k such pairs of numbers, then the $2k$ numbers must have sum at least $1 + 2 + \cdots + 2k = k(2k+1)$. Since these k pairs have sum that are different integers not exceeding n, the sum is at most $n + (n-1) + \cdots + (n-k+1) = \frac{k}{2} \cdot (2n-k+1)$. Thus, $k(2k+1) \leq \frac{k}{2} \cdot (2n-k+1)$, which is the same as $2(2k+1) \leq 2n-k+1$. Rearranging yields $5k \leq 2n-1$, i.e. $k \leq \frac{2n-1}{5}$. Since k must be an integers, $k \leq \frac{2n-1}{5}$.

Now we show that this is an attainable bound, which we split into two cases:

- For $n = 5m+1$ and $5m+2$, we have (in both cases) $\lfloor \frac{2n-1}{5} \rfloor = 2m$. Consider the first $4m$ numbers and we pair them in the following fashion:

  $$(1, 4m-1), (2, 4m-3), \cdots, (m, 2m+1), (m+1, 4m), (m+2, 4m-2), \cdots, (2m, 2m+2)$$

  these give the sum as following:

  $$4m, 4m-1, \cdots, 3m+1, 5m+1, 5m, \cdots, 4m+2$$

  so these are $2m$ pairs of sum $3m+1, 3m+2, \cdots, 4m, 4m+2, 4m+3, \cdots, 5m, 5m+1$ which are all distinct and at most $5m+1 \leq n$.

- Now for $n = 5m+3, 5m+4, 5m+5$, $\lfloor \frac{2n-1}{5} \rfloor = 2m+1$. We now need the first $4m+2$ numbers and consider the following pairing:

  $$(1, 3m+2), (2, 3m+3), \cdots, (m+1, 4m+2), (m+2, 2m+2), (m+3, 2m+3), \cdots, (2m+1, 3m+1)$$

  these give the sum as the following:

  $$(3m+3), (4m+5), \cdots, (5m+1), (5m+3), (3m+4), (3m+6), \cdots, (5m+2)$$

  which are the distinct $2m+1$ sums of all integers in the range $3m+3, 3m+4, \cdots, 5m+3$.

C4 Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:
(a) On every move of his B passes 1 coin from every box to an adjacent box.
(b) On every move of hers A chooses several coins that were not involved in B's previous move and are in different boxes. She passes every coin to and adjacent box. Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Geomtery

G1 (IMO 1) Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Solution. We first show that A, F, L, J lie on a circle. Now that BK and BM are both tangent to the excircle ω of ABC, we have $\angle BKJ = BMJ = 90^\circ$, and moreover

$MJ = MK$, so M and K are symmetric w.r.t. BJ. This means, $BJ \perp MK$. By some angle chasing we have

$$\begin{aligned}
\angle JFL &= \angle JFM \\
&= 90^\circ - \angle KMF \\
&= 90^\circ - (180^\circ - \angle KML) \\
&= 90^\circ - 180^\circ + (180^\circ - \frac{1}{2}\angle KJL) \\
&= 90^\circ - \frac{1}{2}(180^\circ - \angle KAL) \\
&= \frac{1}{2}\angle KAL \\
&= \angle LAJ
\end{aligned}$$

so A, F, L, J are indeed concyclic. Since $\angle ALJ = 90^\circ$, $\angle AFJ = 90^\circ$, too, and thus $AS \perp BJ$. Combined with $MK \perp BJ$ we have $AS \parallel MK$. But then $BK = BM$ so $BA = BS$, too. This gives $MS = MB + BS = BK + BA = AK$.

Similarly, $MT = AL$. Since AK and AL are both tangents to ω, we have $AK = AL$, so $MS = MT$. Since M, S, T are all on BC, they are collinear, so M is indeed the midpoint of ST.

G2 Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Solution. Since G and H lie on the different side of DF, we are proving that $\angle DHF + \angle DGF = 180^\circ$, which is the same as proving $\angle DEF + \angle DGF = 180^\circ$.

There are a few ways to prove this (one of which is the trigonometric bash I submitted as my homework), but one way is to explore the similarities arising from the fact that $ABCD$ is cyclic. Now, by some angle chasing we have (note the use of equal angle at parallelograms, and the use of exterior angle $\angle EDC + \angle ECD = \angle AED$ and $\angle FDE + \angle AED = \angle FAE$)

$$\angle FDG = \angle FDE + \angle EDC + \angle CDG = \angle FDE + \angle EDC + \angle ECD = \angle FDE + \angle AED = \angle FAE = \angle FBE$$

with the last equality following from the $ABCD$ is cyclic. In addition,

$$\begin{aligned}
\frac{FD}{DG} &= \frac{FD}{CE} \\
&= \frac{FB \sin \angle FBD / \sin \angle FDB}{BE \sin \angle EBC / \sin \angle BCE} \text{ (sin rule on triangle} FBD) \\
&= \frac{FB \sin \angle EBC / \sin \angle BCE}{BE \sin \angle EBC / \sin \angle BCE} (\angle BCE = \angle FDB) \\
&= \frac{FB}{BE}
\end{aligned}$$

so together with $\angle FDG = \angle FBE$, we can conclude that triangles FDG and FBE are similar. In particular, $\angle DGF = \angle BEF$. But since B, E, D are on a straight line in that order, we have $\angle BEF + \angle FED = 180^\circ$, and therefore $\angle DGF + \angle FED = 180^\circ$.

G3 In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Solution. There are two things we need to prove:

- Let I be the incenter of triangle ABC. Then $CI \perp I_1I_2$.
- CI is also the radical axis of the circumcircles of ACI_1 and BCI_2.

Once we establish the two, it will follow that $CI \perp O_1O_2$, and therefore with $CO \perp I_1I_2$ we have $O_1O_2 \parallel I_1I_2$.

We will in fact show an overarching claim: ABI_2I_1 is cyclic. We have learned so many times that $DFCB$ is cyclic because $\angle CFB = \angle CDB = 90°$, therefore $\angle ADF = \angle ACB$, meaning that triangles ADF and ACB are similar. Moreover, their similitude is $\frac{AF}{AB} = \cos\angle CAB$. Thus, $\frac{AI_1}{AI} = \cos\angle CAB$, too. But then I and I_1 both lie on the internal angle bisector of $\angle CAB$, so we have

$$II_1 = IA - I_1A = IA(1 - \cos\angle CAB) = IA(2\sin^2\frac{\angle CAB}{2}) = 2IA\sin^2\angle IAB$$

(notice the use of double angle formula: $\cos 2x = 1 - 2\sin^2 x$) and similarly $II_2 = 2IB\sin^2\angle IBA$. Thus we now have

$$\frac{II_1 \cdot IA}{II_2 \cdot IB} = \frac{2IA\sin^2\angle IAB \cdot \angle IA}{2IB\sin^2\angle IBA \cdot IB} = \frac{IA^2}{IB^2} \cdot \frac{\sin^2\angle IAB}{\sin^2\angle IBA} = \frac{\sin^2\angle IBA}{\sin^2\angle IAB} \cdot \frac{\sin^2\angle IAB}{\sin^2\angle IBA} = 1$$

(notice the use of sine rule on triangle IAB in the second last equality). Thus $II_1 \cdot IA = II_2 \cdot IB$, and so by power of point theorem I_1ABI_2 is cyclic. Moreover, the power of point from I to the circumcircle of ACI_1 and BCI_2 are $II_1 \cdot IA$ and $II_2 \cdot IB$, respectively. Since these two are equal, I has equal power from the two circles, and hence lie on the radical axis of the two circles. Since C lies on both the circles, CI is the radical axis of these two circles. Finally, if G is the intersection from CI to I_1I_2 then (the second equality is the exterior angle)

$$\angle I_1IG + \angle II_1G = (\angle ICA + \angle IAC) + \angle IBA = \frac{\angle BCA + \angle BAC + \angle CBA}{2} = \frac{180°}{2} = 90°$$

which shows that CI and I_1I_2 are indeed perpendicular.

G4 Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Solution. W.L.O.G. assume $AB < AC$. We first notice that, if M is the second intersection of the angle bisector AD of $\angle BAC$ and the circumcircle of ABC, then $BM = MC$, and therefore the perpendicular from M to BC (say, M_1) will be the midpoint of BC. Since $DM_1 = M_1E$, we also have $DM = MY$.

We now show that $AD \cdot DM = XD \cdot EY$. Now, $EY = DY\cos\angle DYE = 2DM\cos\angle DYE$. Moreover, we can use the well-known fact that the perpendicular from A to BC and AO are the isogonal conjugate (i.e. the reflection in AD) and since $DX \perp BC$, we have $\angle XAD = \angle XDA$. Since $XD \perp EY$, too, $\angle XDA = \angle EYD$. This also means $DX = XA$, and that $AD = 2DX\cos\angle EYD$, and thus $DX = \frac{AD}{2\cos\angle EYD}$. Therefore we have

$$XD \cdot EY = \frac{AD}{2\cos\angle EYD} \cdot 2DM\cos\angle DYE = AD \cdot DM$$

but by the power of point theorem (since $ABMC$ is cyclic) we have $AD \cdot DM = BD \cdot DC$. It follows that $XD \cdot EY = BD \cdot DC$ too.

Now reflect Y in the perpendicular bisector of BC to get Y' and we have $XD \cdot DY' = BD \cdot DC$ since X, D, Y' would be collinear. It follows that $BXCY'$ is cyclic. But then $BCYY'$ is also cyclic (isoceles trapezoid), so the conclusion follows.

G5 (IMO 5) Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

Solution. Let Y to be the orthocenter of triangle AXB. Since YX is perpendicular to AB, Y, X, C, D are collinear. Now, consider the circle ω_A centered at A with radius AC, and circle ω_B centered at B with radius BC. We now consider the pole P' of BX with respect to ω_A. This pole P' lies on the polar of B. Since $\angle BCA = 90^\circ$, BC is tangent to ω_A and since CD is perpendicular to AB, CD is indeed to polar of B and P' is on CD. Moreover, by the definition of pole, P' also lies on perpendicular line from A to BX. Thus $P'A \perp BX$ and $P'X \perp AB$, showing that P' is the orthocenter of AXB. Thus $P' = Y$, too, i.e. Y is the pole of BX w.r.t. ω_A. Similarly Y is also the pole of AX w.r.t. ω_B. But since L is on both BX and ω_A, YL is tangent to ω_A, and similarly YK is tangent to ω_B. But since Y is on CD, the radical axis of ω_A and ω_B, we have $YK = YL$. Finally, $MK^2 = YM^2 - YK^2 = YM^2 - YL^2 = ML^2$ so $MK = ML$ (we use the fact that $\angle YKM = \angle YLM = 90^\circ$).

G6 Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Number Theory

N1 Call admissible a set A of integers that has the following property: If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k. Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Answer. All (m, n) satisfying $\gcd(m, n) = 1$.
Solution. First, let $d = \gcd(m, n)$. The set $d\mathbb{Z} = \{dn : n \in \mathbb{Z}\}$ contains m and n, and is admissible since $d \mid x$ and $d \mid y$ implies that for all $k \in \mathbb{Z}$ we have x^2, kxy, y^2 are all divisible by d, so $d \mid x^2 + kxy + y^2$. So we need $d = 1$ for $d\mathbb{Z} = \mathbb{Z}$.

On the other hand, suppose $\gcd(m, n) = 1$. Letting $x = y = m$ we have $m^2 + km^2 + m^2 = (k+2)m^2 \in A$, so $m^2\mathbb{Z} \subseteq A$, and similarly $n^2\mathbb{Z} \subseteq A$. Since $\gcd(m^2, n^2) = 1$ too, we can find a and b such that $am^2 + bn^2 = 1$, by Euclidean theorem. Also am^2, bn^2 both $\in A$. Now let $x = am^2$ and $y = bn^2$ and $k = 2$ we have $(x+y)^2 = x^2 + 2xy + y^2 \in A$ but since $x + y = 1$ we have $1 \in A$. Finally, letting $x = y = 1$ we have $1 + k + 1 = k + 2 \in A$ for every integer k, so $A = \mathbb{Z}$.

N2 Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Answer. The only such triple is $(2, 251, 252)$.
Solution. The left hand side is divisible by x while the right hand side is congruent to $2012(2) = 4024$ modulo x. We therefore have $x \mid 4024 = 503 \times 2^3$, with 503 being a prime. Next, we show that x cannot be divisible by 4. Suppose it is, then the left-hand-side is divisible by $4^3 = 64$, and $4 \mid x$ implies $503 \times 2^3 \times 2^2 = 2012(4) \mid 2012xyz$ so $2012xyz$ is divisible by $2^5 = 32$. It then follows that 4024 is divisible by 32 (since $32 \mid 64$) too, which is a contradiction. Finally, we show that x cannot be greater than $\sqrt[3]{2012}$. For all $x \geq 2$ we have $xyz + 2 \leq z^3 + 2 < z^3 + y^3$ since $x \leq y \leq z$, so either $x = 1$ or $x^3 < 2012$, as claimed. Thus the maximal possible x is 11, but since it also has to be a divisor of 8×503 and cannot be divisible by 4, we have either $x = 1$ or $x = 2$.

If $x=1$, we essentially have $y^3+z^3=2012(yz+2)$, and notice that y^3+z^3 is divisible by 2012. Since 503 is a prime that's $\equiv 2 \pmod 3$, we have $y^3 \equiv z^3 \pmod{503} \to y \equiv z \pmod{503}$, so here $y^3 \equiv (-z)^3 \to y \equiv -z$, meaning $503 \mid y+z$. In addition, y^3+z^3 is even, so $y+z$ must also be even (y and z must be of the same parity). It then follows that $y+z$ is divisible by 1006. Let $y+z=1006k$, and $y^3+z^3=(y+z)(y^2-yz+z^2)=1006k(y^2-y(1006k-y)+(1006k-y)^2)$ while $2012(yz+2)=2012(y(1006k-y)+2)$. This gives $k(y^2-y(1006k-y)+(1006k-y)^2)=2(y(1006k-y)+2)$.

- If $k=1$, we have $(y^2-y(1006-y)+(1006-y)^2)=2(y(1006-y)+2)$, i.e. $3y^2-3018y+1006^2=-2y^2+2012y+4$, i.e. $5y^2-5030y+(1006^2-4)=0$. This reduces to a quadratic equation in y with discriminant $5030^2-4(5)(1006^2-4)=5\times1006^2+80=4(5)(503^2+4)$. We see that $503^2+4\equiv 4+4\equiv 3$ modulo 5, so this discriminant is divisible by 5 but not 5^2, and thus not a perfect square. It follows that this quadratic equation has no integer solution.
- If $k=2$ we have $y^2-y(2012-y)+(2012-y)^2=y(2012-y)+2$, so $3y^2-3(2012)y+2012^2=2012y-y^2+2$, i.e. $4y^2-4(2012)y+(2012^2-2)=0$. Again we treat it as a quadratic equation with discriminant $4^2(2012^2)-4(2012^2-2)=4(4(2012^2)-(2012^2-2))$. Since $4\mid 2012$, we see the term $4(2012^2)-(2012^2-2)$ is divisible by 2 but not 4, hence cannot be a perfect square. This also means that $4(4(2012^2)-(2012^2-2))=2^2(4(2012^2)-(2012^2-2))$ cannot be a perfect square, showing that once again this qudratic equation cannot have solution.
- Let's see what happens as $k\geq 3$. We have $y+z=1006k$ and therefore by the power-mean inequality, $y^3+z^3\geq 2(\frac{y+z}{2})^3=\frac{(1006k)^3}{4}=2\times503^3\times k^3$. On the other hand, on the right hand side we have $2012(yz+2)\leq 2012(\frac{(y+z)^2}{4}+2)=2012(\frac{(1006k)^2}{4}+2)=2012(503^2k^2+2)$. Combining these two inequalities give

  $$2\times503^3\times k^3\leq 2012(503^2k^2+2)\overset{\div1006}{\to}503^2\times k^3\leq 2(503^2k^2+2)$$

  This means, $503^2k^2(k-2)\leq 4$, and we see that this inequality fails when $k\geq 3$, so no solution for $k\geq 3$.

We now proceed to the second case: $x=2$, i.e. $8(y^3+z^3)=2012(2yz+2)$, or $y^3+z^3=503(yz+1)$. By the same logic as above, we need $503\mid y+z$. If $y+z=503k$ then by the power-mean inequality (again) we have $y^3+z^3\geq\frac{503^3k^3}{4}$ and $503(yz+1)\leq 503(\frac{503^2k^2}{4}+1)$, so

$$\frac{503^3k^3}{4}\leq 503(\frac{503^2k^2}{4}+1)\overset{\times4\div503}{\to}503^2k^3\leq 503^2k^2+4$$

which implies $503^2k^2(k-1)\leq 4$, as well. Again from here we can see $k\leq 1$ so we only need to care about this case. Knowing this, we have $x+y=503$ and therefore $y^2-yz+z^2=yz+1$ becomes $(y-z)^2=1$. Since $z\geq y$, we have $z-y=1$ and $z+y=503$. This implies $y=251$ and $z=252$. We can check that this triple $(x,y,z)=(2,251,252)$ fulfills the problem condition, and is thus the only triple of our interest.

N3 Determine all integers $m\geq 2$ such that every n with $\frac{m}{3}\leq n\leq\frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Answer. All m's that are prime.

Solution. We use the following formula to calculate the highest power of a prime p dividing $n!$:

$$v_p(n!)=\sum_{k=1}^{\infty}\left\lfloor\frac{n}{p^k}\right\rfloor$$

and notice that $\binom{n}{m-2n}=\frac{n!}{(m-2n)!(3n-m)!}$.

We first see what happens when m is prime. Then for all n within the stipulated condition we have $n < m$ and so $\gcd(n, m) = 1$. Choose any n arbitrary, subject to the inequality constraint. Let p to be any prime dividing n and let $k_0 = v_p(n)$. We now have

$$v_p\left(\frac{n!}{(m-2n)!(3n-m)!}\right) = v_p(n!) - v_p((m-2n)!) - v_p((3n-m)!) = \sum_{k=1}^{\infty}\left\lfloor\frac{n}{p^k}\right\rfloor - \left\lfloor\frac{m-2n}{p^k}\right\rfloor - \left\lfloor\frac{3n-m}{p^k}\right\rfloor$$

We first notice that for each k, $\left\lfloor\frac{n}{p^k}\right\rfloor - \left\lfloor\frac{m-2n}{p^k}\right\rfloor - \left\lfloor\frac{3n-m}{p^k}\right\rfloor \geq 0$ since $\lfloor a+b\rfloor \geq \lfloor a\rfloor + \lfloor b\rfloor$ for all real numbers a and b. It now remains to show that for all $k \leq k_0$ we have $\left\lfloor\frac{n}{p^k}\right\rfloor - \left\lfloor\frac{m-2n}{p^k}\right\rfloor - \left\lfloor\frac{3n-m}{p^k}\right\rfloor \geq 1$. Since $p^{k_0} \mid v_p(n)$, $\frac{n}{p^k}$ is an integer and therefore $\left\lfloor\frac{n}{p^k}\right\rfloor = \frac{n}{p^k}$. However, as $\gcd(n, m) = 1$, m is not divisible by p and so neither is $m - 2n$ nor $3n - m$. We therefore have $\left\lfloor\frac{m-2n}{p^k}\right\rfloor < \frac{m-2n}{p^k}$ and $\left\lfloor\frac{3n-m}{p^k}\right\rfloor < \frac{3n-m}{p^k}$. Therefore,

$$\begin{aligned}\left\lfloor\frac{n}{p^k}\right\rfloor - \left\lfloor\frac{m-2n}{p^k}\right\rfloor - \left\lfloor\frac{3n-m}{p^k}\right\rfloor &= \frac{n}{p^k} - \left\lfloor\frac{m-2n}{p^k}\right\rfloor - \left\lfloor\frac{3n-m}{p^k}\right\rfloor \\ &> \frac{n}{p^k} - \frac{m-2n}{p^k} - \frac{3n-m}{p^k} \\ &> 0\end{aligned}$$

and since the floor functions are integers, so is the differences and sums among them. Therefore $\left\lfloor\frac{n}{p^k}\right\rfloor - \left\lfloor\frac{m-2n}{p^k}\right\rfloor - \left\lfloor\frac{3n-m}{p^k}\right\rfloor \geq 1$ for all $k \leq k_0$, establishing the claim. Therefore this identity holds when m is prime.

Next, let's see what happens if m is composite. If m is divisible by 2, choosing $n = \frac{m}{2}$ gives $\binom{n}{m-2n} = \binom{n}{0} = 1$ so we have $n \nmid 1$. If m is divisible by 3, choosing $m = \frac{m}{3}$ gives $\binom{n}{m-2n} = \binom{n}{n} = 1$ so we have $n \nmid 1$, too. We can therefore assume that the smallest prime dividing m is at least 5.

N6 Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.