Solutions to Tournament of Towns, Fall 2017, Senior

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O-Level

1.

A-Level

2. Six circles of radius 1 with centers in the vertices of a regular hexagon are drawn, so that the center O of the hexagon lies inside all six circles. An angle with angular measure α and vertex O cuts out six arcs in these circles. Prove that the sum of the sizes of these arcs is equal to 6α.

   Solution. We make the assumption that the size of arc is with respect to the center not circumference. The claim is that the sum of the sizes of arcs from the two circles with centers of opposite vertices of the hexagon is always exactly 2α.

   Now let the centers of the two circles ω_1, ω_2 be O_1 and O_2 respectively, which are the opposite vertices of the hexagon. Then O is the midpoint of O_1 and O_2. In addition, let the ray for one of the angles hit the circles ω_1, ω_2 at A_1, A_2, respectively, and the other hit the circles ω_1, ω_2 at B_1, B_2, respectively. Reflect A_2, B_2 in O to get A_3, B_3, respectively. Then we notice:

   - A_3, B_3 both lie on ω_1 since ω_2 is precisely the reflection of ω_1 in O;
   - The arc A_2B_2 and A_3B_3 have the same angle.

   Thus it suffices to show that (A_1B_1) and (A_3B_3) have angle summed to 2α. Here, A_1, O, A_3 are collinear in that order, as are B_1, O, B_3. Since $A_1A_3B_3B_1$ is cyclic, the angle subtended on the circumference satisfies

   $$\angle A_1B_1A_3 + \angle B_1A_1B_3 = \angle A_3B_1A_1 + \angle B_3A_1B_1 = \angle A_3B_1O + \angle B_3A_1O = \angle A_1OA_3 = \alpha$$

   so it follows that the sum of angles subtended in the center is indeed 2α.

3. An analyst made a prediction for the change in the dollar/euro rate for each of the next 12 months: by what percentage the rate would change in October, in November, in December, and so on. It turned out that for every month, he predicted the right percentage but was mistaken if it will go up or down (i.e., if he predicted that the rate will decrease by $x\%$, then the real rate increased by $x\%$, and vice versa). Nevertheless, the dollar/euro rate after 12 months coincided with the prediction. Did the dollar/euro rate go up or down on the whole?

   Answer. It decreases.

   Solution. Let $100 \cdot x_i$ be the actual signed percentage change at the i-th month (that is, positive if it goes up, negative if it goes down). Then the actual proportion change (without percentage) is x_i while what's predicted by the analyst is $-x_i$. The ratio of the dollar/euro rate after k months compared to the beginning is

$$\prod_{i=1}^{k}(1+x_i)$$

and the last sentence suggests that there's a constant R satisfying

$$\prod_{i=1}^{12}(1+x_i) = R = \prod_{i=1}^{12}(1-x_i)$$

Multiplying both sides, we get

$$R^2 = \prod_{i=1}^{12}(1+x_i)(1-x_i) = \prod_{i=1}^{12}(1-x_i^2) \leq 1$$

with equality iff $x_i = 0$ for all i (i.e. the conversion ratio stays constant across the 12 months). Assuming this doesn't happen, we have $R < 1$ and therefore the rate goes down overall.

4. Show that for any infinite sequence $a_0, a_1, \cdots, a_n, \cdots$ of ones and negative ones, we can choose n and k such that

$$|a_0 \cdot a_1 \cdot ... \cdot a_k + a_1 \cdot a_2 \cdot ... \cdot a_{k+1} + ... + a_n \cdot a_{n+1} \cdot ... \cdot a_{n+k}| = 2017.$$

Solution. Let $c = 2 \cdot 2017 = 4034$. Consider the tuples $t_n = (a_{n+1}, \cdots, a_{n+c})$. Since each entry as the tuples are ± 1, there are 2^c possible distinct tuples of such form. Therefore there exists m and n such that $t_m = t_n$, with $m < n$. Now consider k such that $k + 1 = n - m$, and consider

$$s_n = a_0 \cdot a_1 \cdot ... \cdot a_k + a_1 \cdot a_2 \cdot ... \cdot a_{k+1} + ... + a_n \cdot a_{n+1} \cdot ... \cdot a_{n+k}$$

for each n. Then notice that

$$s_n - s_{n-1} = a_n \cdot \cdots \cdot a_{n+k} \qquad s_{n+1} - s_n = a_{n+1} \cdot \cdots \cdot a_{n+k+1} = (s_n - s_{n-1})\frac{a_{n+k+1}}{a_n}$$

but given $t_m = t_n$, $a_{x+k+1} = a_x$ for $x = m+1, m+2, \cdots, m+c$. This gives $s_{x+2} - s_{x+1} = s_{x+1} - s_x$.

Finally, by convention $s_{-1} = 0$ and $s_n - s_{n-1} = \pm 1$ for all n. If $|s_{m+1}| \geq 2017$ then there must be an $x \leq m + 1$ such that $|s_{m+1}| = 2017$ and we're done. Otherwise, we have $s_{m+1}, s_{m+2}, \cdots, s_{m+c}$ all one more than the term before or one less than the term before. This means $s_{m+c} = s_{m+1} + (c - 1)$ or $s_{m+c} = s_{m+1} - (c - 1)$. Since $-2016 \leq s_{m+1} \leq 2016$, s_{m+c} must lie outside the $[-2016, 2016]$ interval and so there's an $x \leq c$ with $|s_{m+x}| = 2017$, done.

5. You must cut a piece of cheese into parts following the rules:

   (1) The first cut must divide the cheese into two pieces, every next cut divides one of the existing pieces into two;

   (2) after every cut, the ratio of the weight of any piece to the weight to any other one must be greater than a given number R.

   (a) Prove that for $R = 0.5$ we can cut the cheese so that the process will never stop (i.e., after any number of cuts, we will still be able to make one more cut).

   (b) Prove that if $R > 0.5$, then at some point we will have to stop cutting.

(c) What is the greatest number of parts we can achieve if $R = 0.6$?

Answer to (c). 6 parts.

Solution. For (a) the algorithm is simply "take the largest piece and cut that into even pieces". Then at any time, there's a nonnegative integer k such that the weight of the piece is either $\frac{1}{2^k}$ or $\frac{1}{2^{k+1}}$ of the original.

For the next two parts, let's consider the following lemma:

Lemma. Let $R > 0.5$, such that $(2R)^k > \frac{1}{R}$. Then we cannot have more than $2k$ parts.

Proof: let a be the weight of piece we're cutting. Then for any other piece with weight b we have $\frac{1}{2}a \geq Rb$ so $a \geq 2Rb > b$. This means, we must choose the heaviest piece to cut, and this piece must have weight at least $2R\times$ the next one.

Consider a configuration where the pieces have weights $a_1 \geq \cdots a_\ell$, such that there exists $m < \ell$ such that $\frac{a_m}{a_{m+1}} < 2R$ (if multiple such m's exist, choose the smallest such m). For $i = 1, \cdots, m-1$, denote d_i as the maximum integer such that $(2R)^{d_i} \leq \frac{a_i}{a_{i+1}}$, denote the score of the score s of the configuration as $\sum_{i=1}^{m-1} d_i$.

We claim that we cannot do more than s cuts from here. Indeed, if $d_1 = 0$ then $\frac{a_1}{a_2} < 2R$ then we cannot make anymore cut. Otherwise, we consider cutting a_1 into $b_1 \geq b_2$, with $\frac{b_2}{b_1} \geq R$. It then follows that $\frac{b_1}{a_1} \leq \frac{1}{1+R} \leq \frac{1}{2R}$ since $R \leq 1$. Let's now consider one of the two cases:

Case 1. $b_1 \leq a_2$. Now, given that $\frac{a_m}{a_{m+1}} < 2R$, if b_2 is the smallest piece then its ratio with the next smallest piece will not contribute to the score of this new configuration. Suppose that $a_i \geq b_1 \geq a_{i+1}$ for some i, Defining $e_{i1} = \lfloor \frac{\log(a_i/b_1)}{\log(2R)} \rfloor$ and $e_{i2} = \lfloor \frac{\log(b_1/a_{i+1})}{\log(2R)} \rfloor$ we get $e_{i1} + e_{i2} \leq \lfloor \frac{\log(a_i/a_{i+1})}{\log(2R)} \rfloor = d_i$ (basically, $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor$ for all reals x, y). We may also do the same for b_2, and then the score is now no more than $\sum_{j=2}^{m-1} d_j + (e_{i1} + e_{i2} - d_i) \leq \sum_{j=2}^{m-1} d_j = s - d_1 < s$.

Case 2. $b_1 > a_2$. With $\frac{b_1}{a_1} \leq \frac{1}{2R}$, $\lfloor \frac{\log(b_1/a_2)}{\log(2R)} \rfloor \leq \lfloor \frac{\log(a_1/a_2)}{\log(2R)} - 1 \rfloor = d_1 - 1$. A similar proof idea would yield that the score now is at most $s - 1$.

Therefore in each of the cases we have the score decreasing by at least 1, showing that we cannot perform more than s cuts.

Finally, consider the first moment where we have this m with $\frac{a_m}{a_{m+1}} < 2R$. Since $(2R)^k > \frac{1}{R}$, m cannot exceed k and the score cannot exceed $k-1$, so this must happen when we have done k cuts. It then follows that no more than $2k-1$ cuts can be made.

In particular for (c), $1.2^3 = 1.728$ and $\frac{1}{0.6} \simeq 1.666$, i.e. we cannot do more than 6 cuts. An example of 6 cuts is as follows (normalizing the weights we may assume the total was $M = 1.2^3 + 1.2^2 + 1.2 + 1.2$ to start with)

$$\begin{aligned} M &\to (1.2^3 + 1.2^2, 1.2 + 1.2) \to (1.2 + 1.2, 1.2^3, 1.2^2) \to (1.2^3, 1.2^2, 1.2, 1.2) \\ &\to (1.2^2, 1.2, 1.2, 1.2^3/2, 1.2^3/2) \to (1.2, 1.2, 1.2^3/2, 1.2^3/2, 1.2^2/2, 1.2^2/2) \end{aligned} \tag{1}$$

6. A triangle ABC is given. Let I be the center of its excircle tangent to the segment AB, and let A_1 and B_1 be the points where the segments BC and AC touch the corresponding excircles. Let M be the midpoint of the segment IC, and let the segments AA_1 and BB_1 intersect at point N. Prove that the points N, B_1, A, and M are concyclic.

Solution. In fact, we'll show that M is the second intersection of circles AB_1N and BA_1N. First, consider the second intersection of line IC with circumcircle of CAA_1, namely M'. $M'A = M'A_1$ and therefore by Ptolemy's theorem,

$$M'C \cdot AA_1 = M'A \cdot (CA + CA_1)$$

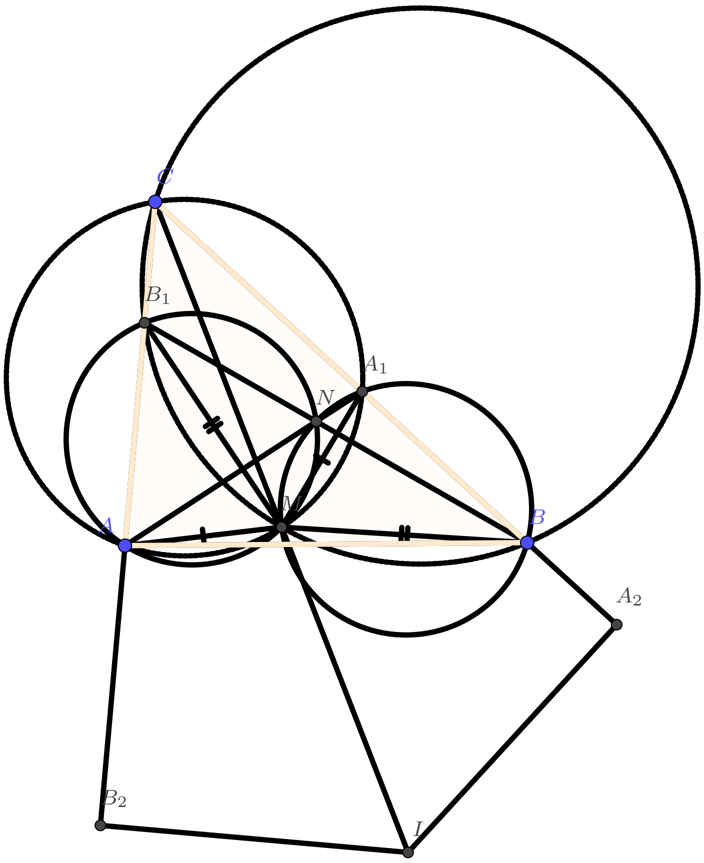
but if the excircle opposite C touches CA and CB at B_2 and A_2 respectively, we have B_2, C, A_2, I concyclic and therefore

$$IC \cdot A_2B_2 = IA_2 \cdot (CA_2 + CB_2)$$

and given that $M'AA_1$ and IA_2B_2 are similar, the two equations above give:

$$\frac{M'C}{CA + CA_1} = \frac{IC}{CA_2 + CB_2}$$

but by definition of A_2 and B_2, CA_2 and CB_2 are each equal to s, the semiperimeter of triangle ABC, same goes to the sum $CA+CA_1$ given the definition of A_1. We $IC = 2CM'$, and so $M' = M$, i.e. M, A, C, A_1 are concyclic and therefore $MA = MA_1$. Similarly $MB = MB_1$.



Notice also that $AB_1 = BA_1$: each of them are equal to the length of tangent from C to the incircle of ABC. Therefore by spiral similarity, if O is the intersection of the circles AB_1N and BA_1N we have $OA = OA_1$ and $OB = OB_1$ but then O and M are both on the intersection of perpendicular bisectors of AA_1 and BB_1 (which are not parallel as the lines themselves intersect at N), we have $O = M$, so M is indeed on the intersection of the two circles.