Solution to IMO 2018 shortlisted problems.

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Preface: It's coronavirus lockdown period, and I'm bored. Hence this project.

Algebra

A1 Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying
$$f(x^2 f(y)^2) = f(x)^2 f(y)$$
for all $x, y \in \mathbb{Q}_{>0}$.

Answer. The only function is the constant function $f(x) = 1$.
Solution. It's not hard to see that the aforementioned function works, so let's show that it's the only working function. Substituting $x = \frac{1}{f(y)}$ gives $f(1) = f(\frac{1}{f(y)})^2 f(y)$ and substituting $y = 1$ gives $f(\frac{1}{f(1)})^2 = 1$ (since all function values are positive, hence nonzero), and therefore $f(\frac{1}{f(1)}) = 1$. This means there's a value c_0 such that $f(c_0) = 1$. Therefore plugging $y = c_0$, we get $f(x^2) = f(x)^2$ for all x, and this gives
$$f(x)^2 f(y) = f(x^2 f(y)^2) = f((xf(y))^2) = f(xf(y))^2$$
and since both $f(xf(y))^2$ and $f(x)^2$ are squares of positive rationals, $f(y)$ is also a square of positive rationals for all y.

Now suppose that $f(x) \neq 1$ for some x. Then there is a maximum positive integer n such that for all $x \in \mathbb{Q}^+$, $f(x)$ is a 2^n-th power of a rational. This means for all $y \in \mathbb{Q}^+$,
$$f(y) = \frac{f(xf(y))^2}{f(x)^2}$$
is a square of the 2^n-th power of a rational number. This gives $f(y)$ as a 2^{n+1}-th power of a rational, contradiction.

A2 (IMO 2) Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and
$$a_i a_{i+1} + 1 = a_{i+2},$$
for $i = 1, 2, \dots, n$.

Answer. All n that are multiples of 3.
Solution. (Paraphrased and copy-pasted from my solution on AoPS). For $3 \mid n$ we have the construction that's a repetition of $(2, -1, -1)$. For the rest of the solutions we visualize the numbers on a circle. We'll now eliminate the following cases:

Case $a_i = 0$ for some i; by the cyclic condition above we may assume that $a_1 = 0$. Then we must have $a_2 = a_3 = 1$ and $a_4 = 2$, etc. Now some inductive statement shows that if $a_i, a_{i+1} > 0$ then $a_{i+2} > 0$, and we also have $a_2 > 0$ too. Hence $a_i > 0$ for all $i > 0$, including when $i = n + 1$, contradiction.

Case consecutive positive numbers. That is, $a_i, a_{i+1} > 0$ for some i. Again here we consider indices mod n. Now $a_{i+2} > 1$, and therefore $a_{i+3} > 1$. Subsequently, we have $a_{i+4} = ai + 2ai + 3 + 1 > ai + 2ai + 3 \geq ai + 3 \geq 1$, which shows that this sequence is actually increasing after a_{i+3}. This cannot happen since the sequence of real numbers are on a circle.

This therefore imply that between any two positive numbers there must be at least one negative number; if $a_i, a_{i+1} < 0$ then $a_{i+2} = a_i a_{i+1} + 1 > 1$, so there must be at most two negative numbers between any two consecutive positive numbers. As the goal is to show that there are exactly two negative numbers between any two consecutive positive numbers, it suffices to show that the configuration $+ - +$ cannot happen. To do this we do 'backtracking', i.e. consider a_i, a_{i-1}, a_{i-2}, etc, given the circular structure of our sequence.

Now suppose that $a_{i+2} > 0$ and $a_{i+1} < 0$ and $a_i > 0$. We first see that $a_i a_{i+1} < 0$ so $a_{i+2} < 1$. Also $a_{i+3} < 0$ by the "no consecutive positive" lemma we established in the beginning. This means, $a_{i+1}a_{i+2} < -1$ and with $a_{i+2} < 1$ (comparing modulus) we get $a_{i+1} < -1$, actually. Now that $a_i = \frac{a_{i+2}-1}{a_{i+1}}$ with $|a_{i+2} - 1| < 1$ (since $0 < a_{i+2} < 1$) and $|a_{i+1}| > 1$ as proven we have $|a_i| < 1$; combining this with $a_i > 0$ we get $0 < a_i < 1$. Going a bit further, $a_{i-1} = \frac{a_{i+1}-1}{a_i}$; $a_{i+1} < 0$ so $|a_{i+1} - 1| = |a_{i+1}| + 1$; and $0 < a_i < 1$ so $|a_{i-1}| = \frac{|a_{i+1}-1|}{|a_i|} > |a_{i+1}| + 1$, and bear in mind that $a_{i-1} < 0$ here. Therefore $a_{i-1} < a_{i+1} - 1$, i.e. getting more negative. Continuing this backtracking process, we see that the numbers must be alternating in sign, with positive numbers bounded above by 1, and negative numbers given by $a_{i-1} < a_{i+1} - 1$ whenever both numbers are negative. This cannot happen on a circle, reaching our contradiction.

A3 Given any set S of postive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

Solution. Suppose that there exists a set S such that bpth (1) and (2) fail. That is, finite subset sums are pairwise distinct (or injective, if you like), and for each $r < 1$ there's finite F with $\sum_{x\in F} 1/x = r$. This second contrapositive can be extended to include $r = 0$ as all we need is the empty set. This S is infinite because there are at most 2^n subset sums that can be attained by a set of size n and there are infinitely many positive rational numbers $r < 1$. Since $S \subseteq \mathbb{N}$, it's countable so we can enumerate $S\backslash\{1\}$ as $\{a_1 < a_2 < a_3 < \cdots\}$.

Denote $s(F)$ as $\displaystyle\sum_{x\in F}\frac{1}{x}$, and $t(F) = \{s(G) : G \subseteq F\}$ Given that for all $r < \frac{1}{a_1}$ there is F such that $s(F) = r$, such F must not contain a_1 (here, $a_1 > 1$). Thus $\{r \in \mathbb{Q}^+ : r < \frac{1}{a_1}\} \subseteq t(S\backslash\{1, a_1\})$. Now consider the sum $s(S\backslash\{1, a_1\})$. If this diverges, or is $> \frac{1}{a_1}$, then we can choose N (minimal possible) such that

$$s(\{a_2, \cdots, a_N\}) = \sum_{i=2}^{N} \frac{1}{a_i} \geq \frac{1}{a_1}$$

and by the minimality of N, $s(\{a_2, \cdots, a_N\}) < \frac{1}{a_N} + \frac{1}{a_1} < \frac{2}{a_1}$. Therefore $0 \leq s(\{a_2, \cdots, a_N\}) - \frac{1}{a_1} < \frac{1}{a_1}$. By assumption, there's a finite set $F_0 \subseteq \{a_2, a_3, \cdots\}$ with $s(F_0) = s(\{a_2, \cdots, a_N\}) - \frac{1}{a_1}$, so $s(F_0 \cup \{a_1\}) = s(\{a_2, \cdots, a_N\})$. But $F_0 \cup \{a_1\} \neq \{a_2, \cdots, a_N\}$ as one contains a_1 while ther other does not, contradiction. Therefore, $s(S\backslash\{1, a_1\}) \leq \frac{1}{a_1}$ and since $t(S\backslash\{1, a_1\}) \supseteq \{r \in \mathbb{Q} : r < \frac{1}{a_1}\}$, equality must hold, and so $s(S\backslash\{1, a_1\}) = \frac{1}{a_1}$. This means $s(S\backslash\{1\}) = \frac{1}{a_1} + \frac{1}{a_1} = \frac{2}{a_1} \leq \frac{2}{2} = 1$ (as $a_1 \geq 1$). But then we have

$$t(S\backslash\{1\}) \supseteq \{r \in \mathbb{Q} : r < 1\}$$

(a set that includes 1 cannot have reciprocal sum less than 1), we have $a_1 = 2$ and consequently

$$s(S\backslash\{1, a_1\}) = s(S\backslash\{1, 2\}) = \frac{1}{2}$$

Now, we repeat the above for a_2, and the same logic can be applied to get

$$s(S\backslash\{1, a_1, a_2\}) = \frac{1}{a_2} = s(S\backslash\{1, a_1\}) - \frac{1}{a_2} = \frac{1}{2} - \frac{1}{a_2}$$

which means $a_2 = 4$. Continuing this, we get $a_i = 2^i$ for all $i \geq 1$. But then for finite F, the denominator of $s(F)$ is a power of 2 in this case, contradiction.

Combinatorics

C1 Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

Solution. We use the following:

$$\{2 \cdot 3^{n-1}, 3^{n-1}, 2 \cdot 3^{n-2}, 3^{n-2}, \cdots, 2 \cdot 3, 3, a, b\}$$

where a and b are simply any pair of numbers such that the sum of the set is $2 \cdot 3^n$. This is possible because the first $2(n-1)$ elements have sum

$$3^n + 3^{n-1} + \cdots + 3^2 = 3^2(\frac{3^{n-1}-1}{3-1}) = \frac{3 \cdot 3^n - 9}{2} < 2 \cdot 3^n$$

(To avoid a, b being the same as any other elements in the set we could just take a, b both not divisible by 3). A set with m elements can be taken as:

$$\{2 \cdot 3^{n-1}, 2 \cdot 3^{n-2}, \cdots, 2 \cdot 3^{n-m+1}, 3^{n-m+1}\}$$

which all has sum 3^n.

C3 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Solution. Label the stones as $1, 2, \cdots, n$. At each turn, all it matters is the square where a stone is chosen to be thrown (and not a particular stone from that square). Therefore we can assume that at each turn when a particular square is chosen, the stone with the largest label in the square is chosen to be moved. If the square has k stones, then the stone with largest label has at label at least k but it can only be moved at most k steps. We then conclude that for each stone with label i, it can only be moved by at most i steps to the right.

Thus we conclude that $\lceil \frac{n}{i} \rceil$ steps is required to move stone labelled i from 0 to n, and summing up gives the inequality.

Geometry

G1 (IMO 1) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Solution. (Paraphrased from my post on AoPS). Skipping angle-chasing algebra here, it suffices to show that $\angle AFD = \angle AGE$. We first show that these two angles must be at most $90°$. To begin with, the fact that D lies on line segment AB means that $\angle AFD \leq \angle AFB = 180° - \angle ACB$, so we may assume that $\angle ACB < 90°$. Next, we have $BF = FD$ (by the definition of perpendicular bisector), and $\angle FBA \leq angle ACB$ (think of the angle subtended by AB and angle subtended by AF). This gives $\angle BFD \geq 180^\circ - 2\angle ACB$ and thus $\angle AFD \leq \angle ACB \leq 90^\circ$. Similarly $\angle AGE \leq 90^\circ$. Now that the sin function is injective in $[0, 90^\circ]$, it's enough to prove that $\sin\angle AFD = \sin\angle AGE$. This isn't hard either: by sine rule we have

$$\frac{\sin\angle AFD}{AD} = \frac{\sin\angle FAD}{FD} = \frac{\sin\angle FAB}{FB} = \frac{\sin\angle GAC}{GC} = \frac{\sin\angle GAE}{GE} = \frac{\sin\angle AGE}{AE}$$

where the desired equality follows from that $AD = AE$. (Notice the implicit use of the facts $FD = FB, GC = GE$, and $\frac{\sin\angle FAB}{FB} = \frac{\sin\angle GAC}{GC}$ follows from that FB and GC are chords of the same circle.

G2 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Solution. Consider the circumcircle Γ of PXY, and consider the antipode D of P w.r.t. Γ. Consider A_1 as the second intersection of DM and Γ. We'll show that $A = A_1$.

Let's first see how would A be defined if only P, X, Y, M are given: we have $BM = MC$ so PM is the median line to BC. With $AP \parallel BC$, the lines $(PA, PM; PB, PC)$ is a harmonic bundle. Therefore the line PA can be constructed this way, and also $PA \perp AM$ since $PA \parallel BC$ and $AM \perp BC$. Therefore all it remains to show is that $(PA_1, PM; PB, PC)$ is harmonic bundle, and that $PA_1 \perp A_1M$, The second relation $PA_1 \perp A_1M$, or equivalently $\angle PA_1M = 90^\circ$, follows from that PM is the antipode of Γ so we're left with showing $(PA_1, PM; PB, PC)$ is harmonic bundle.

Using directed angles, $\angle(A_1P, PX) = \angle(A_1D, DX)$ and $\angle(A_1P, PY) = \angle(A_1D, DY)$. Using the relation $\angle PXM = \angle PYM$, and $\angle PXD = \angle PYD = 90^\circ$, we have $\angle MXD = \angle MYD$. We also have the relation (consider the concurrent lines DM, MX, MY and the triangle DXY)

$$\begin{aligned} 1 &= \frac{\sin\angle XDM}{\sin\angle YDM} \cdot \frac{\sin\angle DYM}{\sin\angle XYM} \cdot \frac{\sin\angle MXY}{\sin\angle MXD} \\ &= \frac{\sin\angle XDM}{\sin\angle YDM} \cdot \frac{\sin\angle MYD}{\sin\angle MXD} \cdot \frac{\sin\angle MXY}{\sin\angle XYM} \\ &= \frac{\sin\angle XDM}{\sin\angle YDM} \cdot \frac{\sin\angle MXY}{\sin\angle MYX} \end{aligned}$$

due to Ceva because $\angle MYD = \angle MXD$. By looking at point M and triangle PXY we also have

$$\begin{aligned} 1 &= \frac{\sin\angle MXY}{\sin\angle MYX} \cdot \frac{\sin\angle MYP}{\sin\angle YPM} \cdot \frac{\sin\angle XPM}{\sin\angle PXM} \\ &= \frac{\sin\angle MXY}{\sin\angle MYX} \cdot \frac{\sin\angle XPM}{\sin\angle YPM} \end{aligned}$$

because $\angle MYP = \angle PXM$. Thus considering the two equations give

$$\frac{\sin\angle XDM}{\sin\angle YDM} = \frac{\sin\angle XPM}{\sin\angle YPM}$$

and by the earlier directed angle identity

$$\frac{\sin\angle XPA_1}{\sin\angle YPA_1} = \frac{\sin\angle XDM}{\sin\angle YDM} = \frac{\sin\angle XPM}{\sin\angle YPM}$$

which then show that $(PA_1, PM; PX, PY)$ is a harmonic bundle.

G6 (IMO 6) A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

Solution. Now consider the second intersection (namely Y) of the circles ABX and CDX, we get (using oriented angles to prevent case distinction) $\angle(YB, YX) = \angle(AB, AX) = \angle(CD, CX) = \angle(YD, YX)$ (the first and the last inequality follows from that A, B, Y, X and C, D, Y, X are each concyclic; the middle one follows from the problem condition). Thus we get that YD, YB are actually the same line (so Y, D, B collinear). Also from we have $\angle XBC = \angle XDA$ which translates into $\angle(XB, BC) = \angle XD, AD)$ so we get $\angle(AY, YX) = \angle(AB, BX) = \angle(AB, BC) - \angle(XB, BC)$ and $\angle(CD, AD) - \angle(XD, AD) = \angle(CD, DX) = \angle(CY, YX)$ This gives the important property:

$$\angle(AY, CY) = \angle(AY, YX) - \angle(CY, YX) = \angle(AB, BC) - \angle(CD, AD)$$

Now the condition $AB \cdot CD = \angle BC \cdot DA$ means that the tangents to triangle ABD and the tangents to triangle BCD will intersect at a point Z on line BD (think of Appolonius circle) and by the properties of tangents we have $\angle(BD, AD) = \angle(AB, BZ)$ so $\angle(AB, BD) - \angle(BD, AD) = \angle(AB, BD) - \angle(AB, AZ) = \angle(AZ, BD)$ and similarly we have $\angle(BD, BC) - \angle(CD, BD) = \angle(BD, CZ)$ so adding these two up we get

$$\angle(AY, CY) = \angle(AB, BC) - \angle(CD, AD) = \angle(AB, BD) - \angle(BD, AD) + \angle(BD, BC) - \angle(CD, BD)$$

$$= \angle(AZ, BD) + \angle(BD, CZ) = \angle(AZ, CZ)$$

Thus Y lies on the circle ACZ, and the fact that Z is the centre of Appolonius circle means that $AZ = ZC$. WLOG assume that Z is closer to B than D. Now $\angle BXA = \angle BYA = \angle AYZ = \angle ZCA = \angle ZAC = \angle ZYC = 180° - \angle DYC = \angle DXC$, the second last inequality follows from our earlier claim that B, Y, D are collinear. So $\angle BXA + \angle DXC = 180°$, Q.E.D.

G7 Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

Solution. The solution will be elaborated, but the end goal is to prove that the point of tangency will be the point P.

First, denote ℓ as the perpendicular bisector of OP, then O_A, O_B and O_C. We'll use the fact that the orthocenter and circumcenter are harmonic conjugates of each other w.r.t. a triangle. Let's first do some angle chasing: given that $OA = OP = OB$, OO_A, OO_B is internal angle bisector of $\angle OAP$ and $\angle OPB$, respectively. If m_C is the perpendicular bisector of AB then m_C is also the angle bisector of $\angle AOB$ and some angle chasing yields that m_C and OP are conjugates w.r.t. $\angle O_AOO_B$. Since $OP \perp O_AO_B$, the circumcenter of OO_AO_B lies on m_C, which is perpendicular to AB and therefore parallel to ℓ_C.

We now consider the point O, the line ℓ with points O_A, O_B, O_C on it, and the lines ℓ_A, ℓ_B, ℓ_C. Consider the point C_1 as the intersection of ℓ_A and ℓ_B; we first show that O_A, O_B, C_1, P are concyclic. Since O and P are symmetric w.r.t. ℓ, $\angle(O_AP, O_BP) = -\angle(O_AO, O_BO)$. If we name m_A, m_B as how we name m_C before (that is, these lines pass through O, and contain the circumcenters of triangles OO_BO_C, OO_CO_A and OO_AO_B, respectively), then $\ell_A \parallel m_A$ and similarly for the others. Therefore:

$$\angle(O_AC_1, O_BC_1) = \angle(\ell_A, \ell_B) = \angle(m_A, m_B)$$

The last angle might be hard to gauge, but given m_A contain the circumcenter of triangle OO_BO_C, it also contains the antipode of O w.r.t. this circumcircle, namely, X_A. This means $90^\circ = \angle(OO_B, O_BX_A) = \angle(OO_C, O_CX_A)$. This gives

$$\angle(m_A, OO_C) = \angle(OX_A, OO_C) = \angle(O_BX_A, O_BO_C)$$

$$= \angle(O_BX_A, OO_B) + \angle(OO_B, O_BO_C) = 90^\circ + \angle(OO_B, O_BO_C) = 90^\circ + \angle(OO_B, \ell)$$

and similarly $\angle(m_B, OO_C) = 90^\circ + \angle(OO_A, \ell)$. Therefore

$$\angle(m_A, m_B) = \angle(m_A, OO_C) - \angle(m_B, OO_C) = \angle(OO_B, \ell) - \angle(OO_A, \ell) = \angle(OO_B, OO_A) = \angle(O_AP, O_BP)$$

which then gives $\angle(O_AP, O_BP) = \angle(m_A, m_B) = \angle(O_AC_1, O_BC_1)$ so C_1 lies on the circle O_AO_BP, as claimed.

Define A_1, B_1 similarly above and we get identities like above; we are now in a position to show that P lies on the circumcircle of $A_1B_1C_1$. To see why:

$$\angle(C_1P, A_1P) = \angle(C_1P, PO_B) - \angle(A_1P, PO_B) = \angle(C_1O_A, O_AO_B) - \angle(A_1P_C, O_BO_C)$$

$$= \angle(C_1O_A, \ell) - \angle(A_1P_C, \ell) = \angle(C_1O_A, A_1P_C) = \angle(C_1B_1, A_1B_1)$$

which shows C_1, B_1, A_1, P are indeed concyclic. To show that OP is tangent to this circle, we need to show that $\angle(A_1P, OP) = \angle(A_1C_1, C_1P)$. We first have

$$\angle(A_1P, OP) = \angle(A_1P, A_1O_B) + \angle(A_1O_B, OP) = \angle(O_CP, O_CO_B) + \angle(\ell_B, OP)$$

Notice that the last equality is the same as $\angle(O_CP, O_CO_B) + \angle(\ell_B, OP) = \angle(O_CP, \ell) + \angle(\ell_B, OP) = -\angle(OO_C, \ell) + \angle(\ell_B, OP)$ as O and P are symmetric w.r.t ℓ. But by before, $\angle(m_B, OO_C) = 90^\circ + \angle(OO_A, \ell)$ so by similar logic $90^\circ + \angle(OO_C, \ell) = \angle(m_B, OO_A) = \angle(\ell_B, OO_A)$. Thus

$$\angle(A_1P, OP) = -\angle(OO_C, \ell) + \angle(\ell_B, OP) 90^\circ - \angle(\ell_B, OO_A) + \angle(\ell_B, OP) = 90^\circ + \angle(OO_A, OP) = \angle(OO_A, \ell)$$

the last equality because OP perpendicular ℓ. Meanwhile,

$$\angle(A_1C_1, C_1P) = \angle(\ell_B, C_1P) = \angle(\ell_B, O_AC_1) + \angle(O_AC_1, C_1P)$$

$$= \angle(\ell_B, \ell_A) + \angle(\ell, O_BP) = \angle(OO_A, OO_B) + \angle(OO_B, \ell) = \angle(OO_A, \ell)$$

where $\angle(\ell_B, \ell_A) = \angle(O_BP, \angle O_AP) = \angle(OO_A, OO_B)$ and $\angle(\ell, O_BP) = \angle(OO_B, \ell)$ are because O and P are symmetric w.r.t. ℓ. Thus $\angle(A_1P, OP) = \angle(OO_A, \ell) = \angle(A_1C_1, C_1P)$, Q.E.D.

Number Theory

N1 Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

Answer. All pairs (n, k) such that neither of them divides each other.
Solution. If $k \mid n$ then sk is a proper divisor of sn, so sn always has strictly more divisors than sk.

N2 Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

- Each number in the table is congruent to 1 modulo n.
- The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the number in the j^{th} column. Prove that the sums $R_1 + \cdots R_n$ and $C_1 + \cdots C_n$ are congruent modulo n^4.

Solution. A routine algebra exercise. The ij-th entry can be written as $na_{ij} + 1$ for some positive integer a_{ij}, which satisfies that for all i and j,

$$n \mid \sum_{k=1}^{n} a_{ik} \qquad n \mid \sum_{k=1}^{n} a_{ij}$$

We also have

$$R_i = \prod_{k=1}^{n} (na_{ik} + 1) \equiv 1 + n\sum_{k=1}^{n} a_{ik} + n^2 \sum_{1 \le k < \ell \le n} a_{ik}a_{i\ell} + n^3 \sum_{1 \le k < \ell < m \le n} a_{ik}a_{i\ell}a_{im}$$

$$C_j = \prod_{k=1}^{n} (na_{kj} + 1) \equiv 1 + n\sum_{k=1}^{n} a_{kj} + n^2 \sum_{1 \le k < \ell \le n} a_{kj}a_{\ell j} + n^3 \sum_{1 \le k < \ell < m \le n} a_{kj}a_{\ell j}a_{mj}$$

both taken modulo n^4. We now compare the coefficients of $1, n, n^2, n^3$ in $R_1 + \cdots R_n$ and $C_1 + \cdots C_n$. We will, in fact, show that the difference of each term is divisible by n^4. The coefficient of 1 in both expressions are n, and the coefficient of n in $R_1 + \cdots R_n$ is:

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}$$

which is the also the coefficeint of n in $R_1 + \cdots R_n$. Next, we notice that

$$\sum_{1 \le k < \ell \le n} a_{ik}a_{i\ell} = \frac{1}{2}\left((\sum_{k=1}^{n} a_{ik})^2 - \sum_{k=1}^{n} a_{ik}^2 \right)$$

so summing this across all i gives the coefficient of n^2 in $R_1 + \cdots R_n$ as

$$\frac{1}{2}\left(\sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik})^2 - \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^2 \right)$$

and similarly for $C_1 + \cdots C_n$:

$$\frac{1}{2}\left(\sum_{j=1}^{n}(\sum_{k=1}^{n} a_{kj})^2 - \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}^2 \right)$$

Notice that $\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}^2$ are simply sum of squares of all a_{ik} so it suffices to show that $2n^2$ divides

$$\sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik})^2 - \sum_{j=1}^{n}(\sum_{k=1}^{n} a_{kj})^2$$

By before, $n \mid \sum_{k=1}^{n} a_{ik}$ and $n \mid \sum_{k=1}^{n} a_{ij}$ so we can name $\sum_{k=1}^{n} a_{ik} = nr_i$ and $\sum_{k=1}^{n} a_{ij} = nc_j$. Substituting this and factoring n^2 term, all it needs is to show that

$$2 \mid \sum_{i=1}^{n} r_i^2 - \sum_{j=1}^{n} c_j^2$$

However, r_i and r_i^2 have the same parity, and $\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j$ is the sum of all the a_{ij}s. Therefore

$$\sum_{i=1}^{n} r_i^2 \equiv \sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j \equiv \sum_{j=1}^{n} c_j^2 \pmod 2$$

as desired.

Finally we need to tackle the coefficient of n^3. Again we have that for a set of variables $x_1, \cdots, x_n$ then

$$\sum_{i<j<k} x_i x_j x_k = \frac{1}{6}\left(\left(\sum_{i=1}^{n} x_i\right)^3 - \sum_{i=1}^{n} x_i^3 - 3\sum_{i \neq j} x_i^2 x_j\right) = \frac{1}{6}\left(S_1^3 - S_3 - 3\sum_{i=1}^{n} x_i^2(S_1 - x_i)\right)$$
$$= \frac{1}{6}(S_1^3 + 2S_3 - 3S_1S_2)$$

where S_k is the sum of k-th power $\sum_{i=1}^{n} x_i^k$. Using the r_i and c_j notations before we now have the coefficient of n^3 in R_i as

$$\sum_{i \leq k < \ell < m \leq n} a_{ik} a_{il} a_{im} = \frac{1}{6}((nr_i)^3 + 2\sum_{k=1}^{n} a_{ik}^3 - 3nr_i S_i^{r(2)})$$

where $S_i^{r(2)}$ is simply the sum of square of squares of row i. Similarly the coefficeint of n^3 in C_j is

$$\frac{1}{6}((nc_j)^3 + 2\sum_{k=1}^{n} a_{kj}^3 - 3nc_j S_j^{c(2)})$$

with the c in $S_j^{c(2)}$ denoting the notion of column. Thus we are essentially comparing

$$\frac{1}{6}(n^3 \sum_{i=1}^{n} r_i^3 + 2\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^3 - 3n\sum_{i=1}^{n} r_i S_i^{r(2)})$$

vs

$$\frac{1}{6}(n^3 \sum_{j=1}^{n} c_j^3 + 2\sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}^3 - 3n\sum_{j=1}^{n} c_j S_j^{c(2)})$$

now $\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^3 = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}^3$ because they are just the sum of cubes of all a_{ij}s. The other two terms on each side has factor n so it suffices to show that $\sum_{i=1}^{n} r_i^3 \equiv \sum_{j=1}^{n} c_j^3 \pmod 6$, and $\sum_{i=1}^{n} r_i S_i^{r(2)} \equiv \sum_{j=1}^{n} c_j S_j^{c(2)} \pmod 2$. The first one is due to the fact that for all integers k, $k^3 \equiv k \pmod 6$ and therefore

$$\sum_{i=1}^{n} r_i^3 \equiv \sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j \equiv \sum_{j=1}^{n} c_j^3 \pmod 6$$

where the middle equality is because they are the same as the overall sum of squares on grid. The second one is due to the fact that $S_i^{r(2)}$ is the sum of squares of row which has same parity of sum of row r_i so in mod 2 (and also $x \equiv x^3 \pmod 2$),

$$\sum_{i=1}^{n} r_i S_i^{r(2)} \equiv \sum_{i=1}^{n} r_i r_i (r_i^2) = \sum_{i=1}^{n} r_i^3 \equiv \sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j \equiv \sum_{j=1}^{n} c_j^3 \equiv \sum_{j=1}^{n} c_j S_j^{c(2)}$$

as desired.

N5 Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t$$

Is it possible that both xy and zt are perfect squares?

Answer. No.

Solution. Suppose otherwise, write $xy = a^2$ and $zt = b^2$. Then $(x-y)^2 = (x+y)^2 - 4xy = (a^2 - b^2)^2 - 4a^2$ is a perfect square, say c^2. Similarly, $(a^2 - b^2) - 4b^2$ is a perfect square, say d^2.

The relation above can be best summarized into:

$$X^2 = p^2 + q^2 = r^2 + s^2$$

where $X = a^2 - b^2$, and $|p^2 - r^2| = 4(a^2 - b^2) = 4X$. It's also important to note that since $x, y, z, t > 0$, $a^2 - b^2, a, b$ are all positive. We further write $p = p_1 2^{p_0}$ and $q = q_1 2^{q_0}$ where p_1, q_1 both odd. Then

$$X^2 = (p_1 2^{p_0})^2 + (q_1 2^{q_0})^2$$

If $p_0 = q_0$ then $X^2 = 2^{2p_0}(p_1^2 + q_1^2)$ and given that $2^{2p_0} = (2^{p_0})^2$ is a perfect square, so is $p_1^2 + q_1^2 \equiv 1 + 1 = 2 \pmod 4$ (as both p_1, q_1 odd). This contradicts that no square can be congruent to 2 mod 4. Hence $p_0 \neq q_0$ and w.l.o.g. let $p_0 > q_0$, which will imply that the highest power dividing X^2 is 2^{2q_0}. In other words, if k is the highest power of 2 dividing X then each p, q, r, s are divisible by k, so they can be written as $2^k p_0, 2^k q_0, 2^k r_0, 2^k s_0$.

Write $X = 2^k X_0$ with X_0 odd. Dividing 2^k from both sides give

$$X_0^2 = p_0^2 + q_0^2 = r_0^2 + s_0^2$$

with $|p_0^2 - r_0^2| = \frac{4X}{2^{2k}} = \frac{4X_0}{2^k}$ and given X_0 odd, we have $k \leq 2$. We'll use the fact that, since X_0 odd, one of p_0 and q_0 must be odd, and the other even. Similarly, one of r_0 and s_0 odd and the other even.

- If $k = 0$, we have $|p_0^2 - r_0^2| = |q_0^2 - s_0^2| = 4X_0 \equiv 4 \pmod 8$. To satisfy $|p_0^2 - r_0^2| = 4X_0$ we will need p_0, r_0 both even or both odd, so w.l.o.g. let them be both odd because $|q_0^2 - s_0^2| = 4X_0$ must be satisfied too. However, $p_0^2 \equiv r_0^2 \equiv 1 \pmod 8$ whenever p_0, r_0 odd, contradicting $|p_0^2 - r_0^2| \equiv 4 \pmod 8$.
- $k = 1$ means $2X_0 \equiv 2 \pmod 4$ is impossible: difference of squares cannot have $\equiv 2 \pmod 4$.
- $k = 0$ means $|p_0^2 - r_0^2| = |q_0^2 - s_0^2| = X_0$, which is odd. Now consider $p_0^2 + q_0^2 = X^2$ and w.l.o.g. let $p_0 > q_0$, which also means $p_0 > X_0\sqrt{\frac{1}{2}}$. Now, assuming $r_0 \neq p_0$

  $$|r_0^2 - p_0^2| \geq \min\{|(p_0+1)^2 - p_0^2|, |(p_0-1)^2 - p_0^2|\} = \min\{|2p_0-1|, |2p_0+1|\} = 2p_0 - 1 > 2X_0\sqrt{\frac{1}{2}} - 1$$

  which gives $X_0 > 2X_0\sqrt{\frac{1}{2}} - 1 = \sqrt{2}X_0 - 1$ which means $X_0 \leq 2$. But since X_0 is odd, $X_0 = 1$ and the only solution is $1^2 + 0^2$. This gives $X = 4$, and the only possible combinations of (p, q) and (r, s) are $(4, 0, 0, 4)$ which we can solve as $a = 2, b = 0$. Now $b = 0$ means $zt = 0$, contradiction.

Therefore the contradiction above means a solution does not exist.