1. For any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ we have $|\vec{x}\cdot\vec{y}| \le ||\vec{x}||\cdot||\vec{y}||$. (Cauchy-Schawz)

   Proof: Let's present a different proof from class. Denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$. Then the thing we need to prove becomes

   $$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots y_n^2) \ge (x_1y_1 + \cdots + x_ny_n)^2$$

   Subtracting the left hand side by right hand side we have:

   $$\begin{aligned}(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots y_n^2) - (x_1y_1 + \cdots + x_ny_n)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 y_j^2 - \left(\sum_{i=1}^{n} x_i^2 y_i^2 + 2\sum_{1\le i<j\le n} x_iy_ix_jy_j\right)\\ &= \sum_{1\le i<j\le n}(x_iy_j - x_jy_i)^2,\end{aligned}$$

   which is clearly nonnegative. It's also interesting to find when does the equality hold, which happens iff for each $i \ne j$ we have $x_iy_j = x_jy_i$. W.L.O.G. le'ts assume that for each i we have $x_i \ne 0$ or $y_i \ne 0$ (adding the '0' terms will not change the sum on either side). If $x_i = 0$ for some i, then we have $x_jy_i = 0$ for all $j \ne i$, and from $y_i \ne 0$ we have $x_j = 0$ for all j, so $\vec{x}$ is a zero vector. Similarly, $y_i = 0$ for some i implies that $\vec{y}$ is a zero vector. Otherwise, x_i, y_i are nonzero for all i, and thus it's possible to rearrange the equality $x_iy_j = x_jy_i$ to form $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ for all i, j. This means $\frac{x_i}{y_i}$ is the same for all i, and therefore there exists a scalar t such that $\vec{x} = t\vec{y}$. Thus equality holds iff either vector is zero or a scalar mutiple of each other (notice that this invariant won't change even after adding the "zero coordinates" that we assumed didn't exist in the beginning).

   Corollary: $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 + 2\vec{x}\cdot\vec{y} \le ||\vec{x}||^2 + ||\vec{y}||^2 + 2||\vec{x}||\cdot||\vec{y}|| = (||\vec{x}|| + ||\vec{y}||)^2$ is the triangle inequality. This holds for p-norm for any $p \ge 1$, as a consequence of Minkowski's Inequality. The proof is however beyond my current capability.

2. For any set E, E° is open.

   Proof: Recall the definition of $E^\circ = \{x : x \in E, \exists\epsilon > 0 (B_\epsilon(x) \subseteq E)\}$. Now for any $x \in E^\circ$, we have $B_\epsilon(x) \subseteq E$ for some $\epsilon > 0$. This means, for any δ with $0 < \delta < \epsilon$ we have:

   $$y \in B_\delta(x) \to ||y-x|| < \delta \to \forall z \in B_{\epsilon-\delta}(y) : ||z-x|| \le ||z-y|| + ||y-x|| < \epsilon - \delta + \delta = \epsilon,$$

   meaning that $y \in E^\circ$. Therefore $B_\delta(x) \subseteq E^\circ$ and thus E° is open.

3. For any set E, $\overline{E}$ is closed.

   Proof: We need to prove that $\overline{E}^c$ is open. Recall that $\overline{E} = E^\circ \cup \delta E$, and we also have:

   $$E^\circ = \{\vec{x} : \exists\epsilon > 0, B_\epsilon(\vec{x}) \cap E^c = \phi\} \qquad \partial E = \{\vec{x} : \forall\epsilon > 0, B_\epsilon(\vec{x}) \cap E^c, B_\epsilon(\vec{x}) \cap E \ne \phi\}$$

   This means that $\overline{E}^c\{\vec{x} : \exists\epsilon > 0, B_\epsilon(\vec{x}) \cap E = \phi\}$. Clearly $\overline{E}^c$ is disjoint from E, and since for each $\vec{x} \notin \overline{E}$ there is an open ball that is contained in E^c, $\overline{E}^c$ is open.

4. For any set E, $(\overline{E})^c = (E^c)^\circ$.

   Proof: For any $\vec{x}$ we have

   $$\vec{x} \in \overline{E} \leftrightarrow (\vec{x} \in E^\circ \vee \vec{x} \in \partial E) \leftrightarrow (\exists\epsilon > 0(B_\epsilon(\vec{x}) \subseteq E)) \vee (\forall\epsilon > 0(B_\epsilon(\vec{x}) \nsubseteq E) \wedge (B_\epsilon(\vec{x}) \nsubseteq E^c)) \leftrightarrow (\forall\epsilon > 0(B_\epsilon(\vec{x}) \nsubseteq E^c))$$

   the last relation is indeed equivalent to $\vec{x} \notin (E^c)^\circ$, and hence concludes the proof.

5. For any set E we have $E^\circ \subseteq E \subseteq \overline{E}$.

   Proof: For any $\vec{x} \in E^\circ$ there exists an $\epsilon > 0$ such that $B_\epsilon(\vec{x}) \subseteq E$, so $\vec{x} \in B_\epsilon(\vec{x}) \subseteq E$ and $\vec{x} \in E$. For the second part we use the previous identity: $(\overline{E})^c = (E^c)^\circ \subseteq E^c$, and therefore $E \subseteq \overline{E}$.

6. Any union of open sets is open.

   Proof: Let $\vec{x}$ be in the union U of open sets, meaning that $\vec{x} \in U_i$ for some open set U_i (where U_i is an open set that is 'part of' U). Then there exists an open ball $B_\epsilon(\vec{x}) \subseteq U_i \subseteq U$; the former follows from the definition of openness and the latter follows from the definition of union.

   Corollary: Any intersection of closed sets is closed.

7. Any *finite* intersection of open sets is open.

   Proof: Consider the sets $U_1, \cdots, U_n$ and let $\vec{x}$ be in their intersection. This means for each $i \in [1, n]$ there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(\vec{x}) \subseteq U_i$. Letting $m = \min\{\epsilon_i : i \in [1, n]\} > 0$ and we have $B_m(\vec{x}) \subseteq U_i$ for all i, thus $B_m \subseteq \cap U_i$.

   Corollary. Any *finite* union of closed sets is closed.

   Note on finiteness. The condition of finiteness is necessary. For example, the set $\cap B_{\frac{1}{n}}(\vec{O}) = \vec{O}$ cannot be open although each $B_{\frac{1}{n}}(\vec{O})$ is open.

8. For each set E we have $E^\circ = \cup V : V \subseteq E$ and V open.

   Proof: We have already proven that E° is open, and that $E^\circ \subseteq E$. This shows that E° is in fact one of the V's into the consideration and therefore $E^\circ \subseteq \cup V$.

   Conversely, let $V : V \subseteq E$ and V open. Then for each $\vec{x} \in V$ there exists $\epsilon > 0$ such that $B_\epsilon(\vec{x}) \subseteq V \subseteq E^\circ$ so $\vec{x} \in E^\circ$ too. Thus $V \subseteq E^\circ$ and therefore $\cup V : V \subseteq E$ and V open $\subseteq E^\circ$.

   Corollary. The following three are equivalent:

   - E is open.
   - $E^\circ = E$.
   - $\partial E \cap E = \phi$.

9. For each set E we have $\overline{E} = \cap V : E \subseteq V$ and V closed.

   Proof: By the previous proofs we have: $(\overline{E})^\circ = (E^c)^\circ = \cup W : W \subseteq E^c$ and W open. Thus we have $\overline{E} = (\cup W)^c = \cap W^c$ where $W \subseteq E^c$ and W open. The last condition is the same as $E \subseteq W^c$ and W^c closed, so substituting W^c with V we get $\overline{E} = \cap V$ with $E \subseteq V$ and V closed.

   Corollary. The following three are equivalent:

   - E is closed.
   - $\overline{E} = E$.
   - $\partial E \subseteq E$.