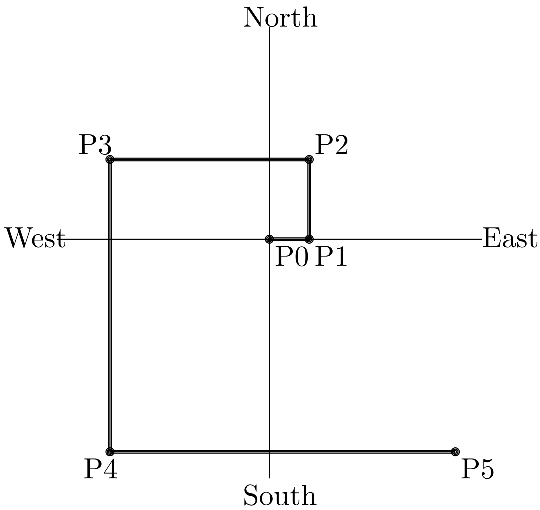
Putnam 2011

A1 Define a growing spiral in the plane to be a sequence of points with integer coordinates $P_0 = (0, 0), P_1, \dots, P_n$ such that $n \geq 2$ and:

- The directed line segments $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ are in successive coordinate directions east (for P_0P_1), north, west, south, east, etc.
- The lengths of these line segments are positive and strictly increasing.



How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ cannot be the last point, P_n, of any growing spiral?

Answer. 10053.

Solution. For $1 \leq x < y$, we can use $|P_0P_1| = x$ and $|P_1P_2| = y$. For (x, y) with $x \geq 3$ and $y \geq 4$ we can also use $|P_iP_{i+1}| = 1, 2, 3, x+1, x+2, x+y-1$ for $1 \leq i \leq 6$, then $x = 1 - 3 + x + 2 = x$ and $y = 2 - (x+1) + (x+y-1) = y$ (we need $x + 1 > 3$ and $y - 1 > 2$, so $x \geq 3$ and $y \geq 4$ at least 4 will work.)

To show that these are the all the possible values, we first show that if $a_1 < a_2 < \cdots a_k$ are increasing sequences of positive numbers, then $\sum_{i=1}^{k}(-1)^{i-1}a_i$ is positive if k is odd, and negative otherwise. If k is odd, then we have $a_k > 0$ and therefore $\sum_{i=1}^{k}(-1)^{i-1}a_i = (a_k - a_{k-1}) + (a_{k-2} - a_{k-3}) + \cdots + (a_3 - a_2) + a_1$ with each of $a_i - a_{i+1} > 0$. Similarly for for k even we have $\sum_{i=1}^{k}(-1)^{i-1}a_i = -(a_k - a_{k-1}) - \cdots - (a_2 - a_1)$ and each term negative. Now going back to the core lemma, each change in the coordinates (for each x- and y-coordinates) are in alternate directions, with magnitude increasing by at least 2 each time. Both start with a positive change, so there must be an odd number of changes for both x and y coordinates. This implies n is congruent to 2 mod 4.

If $n = 2$, then we have the x-coordinate as the length P_0P_1 and the y-coordinate as P_1P_2. In this case we need $x < y$, with $x \geq 1$. If $n \geq 6$, let $x_1, x_2, \cdots, x_{n/2}$ be the lengths of the x-segments, and we have the x-coordinate as $x_1 - x_2 + x_3 - \cdots + x_{n/2} = (x_{n/2} - x_{n/2-1}) + \cdots + x_1$. Since each term in the form $x_i - x_{i-1}$ must be at least 2, so is $(x_{n/2} - x_{n/2-1})$, with $x_1 \geq 1$. This gives $x \geq 3$. Similarly, if $y_1, y_2, \cdots, y_{n/2}$ are the y-segments then each $(y_{n/2} - y_{n/2-1}) \geq 2$ with $y_1 \geq 2$, giving the lower bound for y-coordinate as 4.

Hence for each $0 \le x \le 2011$, if $x = 0$ then all $y \in [0, 2011]$ cannot be one such point (2012 values); if $1 \le x \le 2$ then we need $y \ge x+1$ so $y = 0, 1, \cdots, x$ are impossible, hence $x+1$ values. When $x \ge 3$, each $y \ge 4$ fits. However, those with $y > x$ also have $y \ge 4$, so $y = 0, 1, 2, 3$ are the ones that cannot fit (4 values each). Hence the answer is $2012 + 2 + 3 + \sum_{k=3}^{2011} 4 = 2017 + 4(2009) = 10053$.

A2 Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S.

Answer. We necessarily have $S = \frac{3}{2}$.

Solution. Now we write $a_n = \dfrac{b_n + 2}{b_{n-1}}$ for $n \ge 2$ so $\dfrac{1}{a_1 \cdots a_n} = \dfrac{1}{\prod_{k=1}^{n} a_k} = \dfrac{1}{\prod_{k=2}^{n} \dfrac{b_k + 2}{b_{k-1}}} = \dfrac{1}{(b_n + 2)\prod_{k=2}^{n-1}\left(1 + \frac{2}{b_k}\right)}$ This motivates us to do the following telescoping sum: we consider the difference $\dfrac{3}{2} - \sum_{k=1}^{n} \dfrac{1}{a_1 \cdots a_k}$ for each n. When $n = 1$ we have $\dfrac{3}{2} - \dfrac{1}{a_1} = \dfrac{3}{2} - 1 = \dfrac{1}{2}$ and when $n = 2$ we have $\dfrac{1}{2} - \dfrac{1}{b_2 + 2} = \dfrac{b_2}{2(b_2 + 2)} = \dfrac{1}{2(1 + \frac{2}{b_2})}$. We claim from here that $\dfrac{3}{2} - \sum_{k=1}^{n} \dfrac{1}{a_1 \cdots a_k} = \dfrac{1}{2\prod_{k=2}^{n}(1 + \frac{2}{b_k})}$. Suppose that this is true for some n (we have done base case $n = 2$ above), then for $n+1$ we have

$$\begin{aligned}
\frac{3}{2} - \sum_{k=1}^{n} \frac{1}{a_1 \cdots a_k} &= \frac{1}{2\prod_{k=2}^{n}(1 + \frac{2}{b_k})} - \frac{1}{(b_{n+1} + 2)\prod_{k=2}^{n}\left(1 + \frac{2}{b_k}\right)} \\
&= \frac{1}{\prod_{k=2}^{n}(1 + \frac{2}{b_k})}\left(\frac{1}{2} - \frac{1}{b_{n+1} + 2}\right) \\
&= \frac{1}{\prod_{k=2}^{n}(1 + \frac{2}{b_k})}\left(\frac{b_{n+1}}{2(b_{n+1} + 2)}\right) \\
&= \frac{1}{\prod_{k=2}^{n+1}(1 + \frac{2}{b_k})}
\end{aligned}$$

and therefore we have $S = \dfrac{3}{2} - \lim_{n\to\infty} \dfrac{1}{\prod_{k=2}^{n}(1 + \frac{2}{b_k})}$. Since (b_k) is bounded, there is M positive such that $b_k \le M$ for each k. This means $\dfrac{1}{\prod_{k=2}^{n}(1 + \frac{2}{b_k})} \le \dfrac{1}{\prod_{k=2}^{n}(1 + \frac{2}{M})} = \dfrac{1}{(1 + \frac{2}{M})^{n-1}}$ and so $\lim_{n\to\infty} \dfrac{1}{\prod_{k=2}^{n}(1 + \frac{2}{b_k})} \le \lim_{n\to\infty} \dfrac{1}{(1 + \frac{2}{M})^{n-1}} \to 0$. So $S = \dfrac{3}{2}$.

A3 Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x\, dx}{\int_0^{\pi/2} x^r \cos x\, dx} = L.$$

Answer. $c = -1$ and $L = \frac{2}{\pi}$.

Solution. Denote $S_r = \int_0^{\pi/2} x^r \sin x\, dx$ and $C_r = \int_0^{\pi/2} x^r \cos x\, dx$. We first find the relation between S_{r+1} and S_r for each r. In fact, we will prove that $\lim_{r\to\infty} \frac{S_{r+1}}{S_r} = \frac{\pi}{2}$. First, for each r we have $S_{r+1} = \int_0^{\pi/2} x^{r+1} \sin x \, dx = \int_0^{\pi/2} x \cdot x^r \sin x \, dx \le \int_0^{\pi/2} \frac{\pi}{2} x^r \sin x \, dx = \frac{\pi}{2} S_r$. On the other hand, we show that for each $\epsilon > 0$, there exists r_0 such that $\frac{S_{r+1}}{s_r} > \frac{\pi}{2} - \epsilon$ for all $r \ge r_0$. Now let $0 < \delta < \epsilon$. We split S_r into two parts: $\int_0^{\pi/2-\delta} x^r \sin x \, dx$ and $\int_{\pi/2-\delta}^{\pi/2} x^r \sin x \, dx$. Since $\sin x \le 1$ for all x, we have

$$\int_0^{\pi/2-\delta} x^r \sin x \, dx \le \int_0^{\pi/2-\delta} x^r \, dx = \frac{(\pi/2-\delta)^{r+1}}{r+1}$$

and

$$\int_{\pi/2-\delta}^{\pi/2} x^r \sin x \, dx \ge \int_{\pi/2-\delta}^{\pi/2} (\pi/2-\delta)^r \sin(\pi/2-\delta) \, dx = \delta(\pi/2-\delta)^r \sin(\pi/2-\delta)$$

which means

$$\frac{\int_{\pi/2-\delta}^{\pi/2} x^r \sin x \, dx}{S_r} \ge \frac{\delta(\pi/2-\delta)^r \sin(\pi/2-\delta)}{\delta(\pi/2-\delta)^r \sin(\pi/2-\delta) + \frac{(\pi/2-\delta)^{r+1}}{r+1}} = \frac{\delta \sin(\pi/2-\delta)}{\delta \sin(\pi/2-\delta) + \frac{\pi/2-\delta}{r+1}}$$

We see that this ratio converges to 1 as $r \to \infty$, and since $\delta < \epsilon$, the ratio $\dfrac{\int_{\pi/2-\delta}^{\pi/2} x^r \sin x \, dx}{S_r} > \frac{\pi/2-\epsilon}{\pi/2-\delta}$ for sufficiently large r. Now we also have

$$\begin{aligned} S_{r+1} &= \int_0^{\pi/2} x^{r+1} \sin x \, dx \\ &> \int_{\pi/2-\delta}^{\pi/2} x^{r+1} \sin x \, dx \\ &\ge \int_{\pi/2-\delta}^{\pi/2} (\pi/2-\delta) x^r \sin x \, dx \\ &> (\pi/2-\delta)(\frac{\pi/2-\epsilon}{\pi/2-\delta}) S_r \\ &= (\pi/2-\epsilon) S_r \end{aligned}$$

with the last inequality holds true for sufficiently large r. This concludes the claim that $\frac{S_{r+1}}{S_r} > \frac{\pi}{2} - \epsilon$ for all sufficiently large r. Considering the fact that this holds for each $\epsilon > 0$, we have $\lim_{r\to\infty} \frac{S_{r+1}}{S_r} = \frac{\pi}{2}$.

Now going back to the problem, by virtue of integration by parts we get $C_r = \int_0^{\pi/2} x^r \cos x \, dx = [\frac{x^{r+1}}{r+1} \cos x]_0^{\pi/2} + \int_0^{\pi/2} \frac{x^{r+1}}{r+1} \sin x \, dx = 0 + \frac{1}{r+1} S_{r+1} = \frac{S_{r+1}}{r+1}$ and by the claim above we have $\frac{2}{\pi} = \lim_{r\to\infty} \frac{S_r}{S_{r+1}} = \lim_{r\to\infty} \frac{S_r}{(r+1)C_r} = \lim_{r\to\infty} \frac{S_r}{rC_r} \lim_{r\to\infty} \frac{r^{-1}S_r}{C_r}$ since $\frac{r+1}{r}$ as $r \to \infty$. Thus $c = -1$ and $L = \frac{2}{\pi}$.

A4 For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Answer. When n is odd. For this example we can use A as the matrix full with ones, and return the answer $A - I$. (Basically, the ij-entry is 1 iff $i \ne j$).

Solution. It suffices to produce a contradiction when n is even. Now, consider the matrix A of $n \times n$ with the desired property, and it will be more useful to consider it in the $\mathbb{Z}_2$ space. Let v be the $n \times 1$ matrix with all entries 1 (i.e. $\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$). Then Av is contains the sum of entries of each row, which is essentially also the dot product of each row with itself in $\mathbb{Z}_2$. Hence, $Av = 0$, and thus v is in the null space of A (also v is nonempty). On the other hand, the ij-th entry of AA^t is the dot product of the i-th and j-th row of A, and is therefore odd if $i \neq j$, and even otherwise. This gives $AA^t = B - I$ where B is the $n \times n$ matrix with all ones.

Now $\det(AA^t) = \det(A)\det(A^t) = \det(A^t)\det(A) = \det(A^tA)$ and since $Av = 0$, we have $A^tAv = 0$ too, so A^tA and AA^t cannot be invertible in $\mathbb{Z}_2$. On the other hand, consider the matrix $B - I = AA^t$, and we claim that the determinant is odd by induction on n. Base case when $n = 2$ and we have $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with determinant -1 (and hence odd). Now suppose that for some even n, B_{n-2} has odd determinant. We consider B_n: $\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \ddots & & & \\ 1 & 1 & \cdots & 0 \end{pmatrix}$. Consider, now, B_{1k} for $k > 1$ where B_{ij} is the matrix obtained by deleting i-th row and j-th column from B, and we have $\det(B) = \sum_{k=1}^{n}(-1)^{k-1}b_{1k}\det(B_{1k}) = \sum_{k=2}^{n}\det(B_{1k})$ since b_{1k} is 1 except for $b_{11} = 0$, and also removing all the $(-1)^k$'s since we are doing $\mathbb{Z}_2$. Now each $C = B_{1k}$ for $k \geq 2$, the matrix has the following form: $c_{1j} = 1$ for all j's, and $c_{j\ell} = 1$ with the exception when $j \geq 2$ and $\ell = j - 1$ for $j < k$, and $\ell = j$, otherwise. Since row reduction preserves the determinant, we subtract every row by the first row. Since the first row is all ones, we essentially flipped all rows 2 to $n - 1$. Thus we now have $c_{j\ell} = 0$ unless $j \geq 2$, and $\ell = j - 1$ for $j < k$ and $\ell = j$, otherwise. This means, there's exactly 1 nontrivial entries in each row $c_{j(j-1)}$ $(j < k)$ or c_{jj} $(j \geq k)$, and each of them are in different rows and columns. Multiplying them with $c_{1(k-1)} = 1$ gives the only possible contribution to the determinant of C, i.e. $\pm 1 = 1$ in $\mathbb{Z}_2$. Thus $\det(B) = \sum_{k=2}^{n}\det(B_{1k}) = \sum_{k=2}^{n} 1 = n - 1 = 1$ since n is even. Thus now B is invertible, which is a contradiction.

B1 Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < \left| h\sqrt{m} - k\sqrt{n} \right| < 2\varepsilon.$$

Solution. We first show that for all $\varepsilon > 0$, there exists m and n such that $0 < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon$.

Let $d > 0$ be the greatest common divisor of h^2 and k^2. By Euclid's algorithm, there exists m_0 and n_0 such that $h^2m_0 - k^2n_0 = d$. And if m_0 and n_0 are such solutions, other solutions can be obtained by changing (m_0, n_0) with $(m_0 + xk^2/d, n_0 + xh^2/d)$ for all $x \geq 0$.

We now proceed to another crucial observation: $\lim_{N\to\infty} \sqrt{N+d} - \sqrt{N} = 0$. To this end, notice that for each $\varepsilon > 0$, we have $(\sqrt{N} + \varepsilon)^2 = N + \varepsilon^2 + 2\varepsilon\sqrt{N} > N + 2\varepsilon\sqrt{N}$, so choosing N such that $d < 2\varepsilon\sqrt{N}$ (i.e. $N > (\frac{d}{4\varepsilon^2})$) we get $(\sqrt{N} + \varepsilon)^2 > N + d$ and therefore $\sqrt{N+d} - \sqrt{N} < \varepsilon$ for all such N. This means, fixing N_0 such that $0 < \sqrt{N+d} - \sqrt{N} < \varepsilon$ for all $N > N_0$ and choosing x such that $n_0 + xh^2/d > N$ we have $0 < h\sqrt{m_0 + xk^2/d} - k\sqrt{n_0 + xh^2/d} < \varepsilon$. In other words, there exists m_1 and n_1 such that $0 < h\sqrt{m_1} - k\sqrt{n_1} < \varepsilon$ (by assigning $m_1 = m_0 + xk^2/d$ and $n_1 = n_0 + xh^2/d$).

Finally, since $0 < h\sqrt{m_1} - k\sqrt{n_1}$, let $c = \varepsilon / h\sqrt{m_1} - k\sqrt{n_1}$. Consider the number $g = \lfloor c \rfloor + 1$. From the choices of m_1 and n_1, we also have $c > 1$, and from $c < g = \lfloor c \rfloor + 1 \leq c + 1$ we have $1 < g/c < 2$. Thus, making $m = g^2m_1$ and $n = g^2n_1$ we get

$$h\sqrt{m} - k\sqrt{n} = g(h\sqrt{m_1} - k\sqrt{n_1}) = \varepsilon \cdot (g/c)$$

and with $g/c \in (1, 2)$ we gave $h\sqrt{m} - k\sqrt{n} \in (\varepsilon, 2\varepsilon)$.

B2 Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

Answer. 2 and 5

Solution. We will use without proof that a rational solution exists to $px^2 + qx + r = 0$ if and only if the discriminant $q^2 - 4pr$ is a perfect square. In other words, we want to solve for $q^2 - 4pr = s^2$ with s being an integer. Rearranging gives $(q - s)(q + s) = 4pr$, with the prime factorization of $4pr$ being $2 \times p \times r$.

If both p and r are 2, we have $(q - s)(q + s)$ is 16, so $(q - s, q + s)$ is either $(1, 16), (2, 8)$ or $(4, 4)$. The first one will force q and s to be non-integer; the second one gives (q, s) as $(5, 3)$. The third example gives $(4, 0)$, neither of which is a prime. Thus the only possibility is $(p, q, r) = (2, 5, 2)$.

If one of them, say p is 2 while r prime, then $(q - s)(q + s) = 8r$. Bearing in mind that $q - s \equiv q + s \mod 2$, both factors have to be even and therefore in the category of $(2, 4r), (4, 2r)$. Since $r > 2$, we have $2r > 4$. This forces q, s to be $(2r + 1, 2r - 1)$ in the first case, and $(r + 2, r - 2)$ in the second case. Thus we have $(p, q, r) = (2, 2r + 1, r)$, $(r, 2r + 1, 2)$, $(2, r + 2, r)$ or $(r, r + 2, 2)$, condition on that $2r + 1$ or $r + 2$ actually being a prime.

If both p and r are odd primes, we have $(q - s)(q + s) = 4pr = 2p \times 2r$. Again both $q - s$ and $q + s$ are even, so $(q - s, q + s)$ are $(2, 2pr)$ or $(2p, 2r)$, assuming $p \leq r$. The first case gives $(q, s) = (pr + 1, pr - 1)$ and the second case gives $(p + r, p - r)$. Notice, however, that this is hardly possible: since p and r are odd, $q = pr + 1$ and $q = p + r$ are both odd, and greater than 2, hence cannot be even.

Thus a prime $r \notin \{2, 5\}$ will appear two times when $2r + 1$ is prime, when $r + 2$ being a prime, when $\frac{r-1}{2}$ is a prime, when $r - 2$ is a prime. If r were to appear at least 7 times, then all conditions must hold. If $r \geq 7$, then one of $r - 2, r, r + 2$ must be divisible by 3, contradiction. Hence $r \geq 7$ is impossible. When $r = 3$, $r - 2 = 1$ is not prime. Now we claim that the primes 2 and 5 are possible: we have an example $(2, 5, 2)$ as above and since $2r + 1 = 11, 5 + 2 = 7, 5 - 2 = 3$ are primes, we can do $(2, 11, 5), (5, 11, 2), (2, 7, 5), (5, 7, 2), (2, 5, 3), (3, 5, 2)$. These give the 7 occurences of 2 and 5.

B3 Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

Answer. Yes.

Solution. We need to see if $\lim_{x \to 0} \frac{f(x) - f(0)}{x}$ is defined. By the rules of limits we have

$$\lim_{x \to 0} \frac{f(x)g(x) - f(0)g(0)}{x} = (fg)'(0)$$

$$\lim_{x \to 0} \frac{f(x)g(0) - f(0)g(x)}{x} = \lim_{x \to 0} \frac{f(x)/g(x) - f(0)/g(0)}{x} \cdot \lim_{x \to 0} g(0)g(x) = (f/g)'(0) \cdot g(0)^2$$

Adding the two limits up give

$$\begin{aligned}(fg)'(0) + (f/g)'(0) \cdot g(0)^2 &= \lim_{x \to 0} \frac{f(x)g(x) - f(0)g(0)}{x} + \lim_{x \to 0} \frac{f(x)g(0) - f(0)g(x)}{x} \\ &= \lim_{x \to 0} \frac{(f(x) - f(0))(g(x) + g(0))}{x}\end{aligned}$$

and since $\lim_{x\to 0} g(x) + g(0) = 2g(0) \neq 0$ (before f is continuous at 0), we have

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x\to 0} \frac{(f(x) - f(0))(g(x) + g(0))}{x} \div \lim_{x\to 0}(g(x) + g(0)) \\ &= (fg)'(0) + (f/g)'(0) \cdot g(0)^2 \div 2g(0) \end{aligned}$$

as desired.