Solutions to Tournament of Towns, Fall 2020, Senior

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O-Level

1.

A-Level

1. There were n positive integers. For each pair of those integers Boris wrote their arithmetic mean onto a blackboard and their geometric mean onto a whiteboard. It so happened that for each pair at least one of those means was integer. Prove that on at least one of the boards all the numbers are integer.

   Solution. Suppose that this is false. Choose a number on the blackboard that's not integer. This corresponds to a pair with different parity, which follows that there are both even and odd numbers among the n positive integers.

   Now choose a number on the whiteboard that's not integer. Then it corresponds to number a, b such that ab is not perfect square. Then $a+b$ has to be even, hence a and b are of the same parity. Now choose c that's different parity with a and b, then ac and bc both have to be perfect square, so abc^2 is a perfect square, and so is ab, contradiction.

2. Baron Munchausen presented a new theorem: if a polynomial $x^n - ax^{n-1} + b^x n - 2 + \dots$ has n positive integer roots then there exist a lines in the plane such that they have exactly b intersection points. Is the baron's theorem true?

   Answer. Yes.

   Solution. Let $a_1, \cdots, a_n$ be the n roots, then

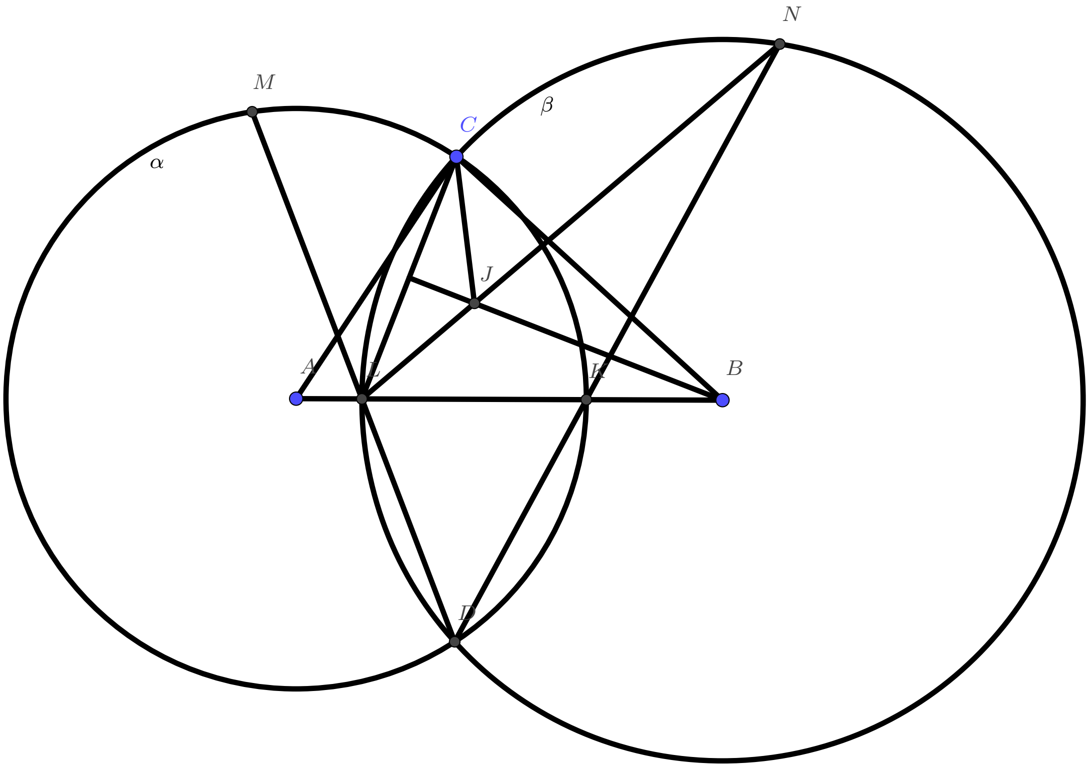
   $$a = a_1 + \cdots + a_n \qquad b = \sum_{1 \le i < j \le n} a_i a_j$$

   Now create n classes $L_1, \cdots, L_n$, each L_i containing a_i parallel lines but each pair of lines in L_i and L_j are never parallel to each other when $L_i \neq L_j$. It follows that the number of intersection is

   $$\sum_{1 \le i < j \le n} |L_i| \cdot |L_j| = \sum_{1 \le i < j \le n} a_i a_j = b$$

3. Two circles α and β with centers A and B respectively intersect at points C and D. The segment AB intersects α and β at points K and L respectively. The ray DK intersects the circle β for the second time at the point N, and the ray DL intersects the circle α for the second time at the point M. Prove that the intersection point of the diagonals of the quadrangle $KLMN$ coincides with the incenter of the triangle ABC.

   Solution. Let LN intersect the internal angle bisector of $\angle B$ at J. We show that J is the incenter of triangle ABC, which is the same as showing that CJ bisects $\angle ACB$.



Now, $BC = BL$ is the radius of β, so with BJ bisects $\angle ABC$, BJ is the perpendicular bisector of CL. Therefore, $\angle JCB = \angle JLB$. In addition, C and D are symmetric w.r.t. AB, so we can do angle chasing (henceforth $\angle A, B, C$ correspond to the angle at triangle ABC)

$$\angle JLB = \angle JLD - \angle KLD = 180^\circ - \angle LDK - \angle LNK - \angle KLD = \angle LKD - \angle LCD = \angle LKC - \angle LCD$$

where we used $\angle LND = \angle LNK = \angle LCD$ since C, D, L, N are on the same circle β. Now, $CD \perp AB$ so $\angle LCD = 90^\circ - \angle CLB = \angle JBL = \frac{\angle B}{2}$, while $\angle LKC = \angle AKC = 90^\circ - \frac{\angle A}{2}$. It therefore follows that

$$\angle JCB = \angle JLB = 90^\circ - \frac{\angle A}{2} - \frac{\angle B}{2} = \frac{\angle C}{2}$$

and so CJ bisects angle ACB. Thus LN passes through the incenter of I, and similarly we can show that so that KM.

4. There are two round tables with n dwarves sitting at each table. Each dwarf has only two friends: his neighbours to the left and to the right. A good wizard wants to seat the dwarves at one round table so that each two neighbours are friends. His magic allows him to make any $2n$ pairs of dwarves into pairs of friends (the dwarves in a pair may be from the same or from different tables). However, he knows that an evil sorcerer will break n of those new friendships. For which n is the good wizard able to achieve his goal no matter what the evil sorcerer does?

Answer. All n odd.

Solution. Label the people on the first round table as $a_1, \cdots, a_n$ in that order, and second as $b_1, \cdots, b_n$ in that order. If n is odd, then the wizard can add the two classes of friendship links:

$$\text{Class 1} \ : (a_i, b_i), \forall i = 1, \cdots, n$$

$$\text{Class 2} \ : (a_i, b_{n-i+1}), \forall i = 1, \cdots, n$$

Since n is odd, after the removal of n links by sorcerer, there's one class with at least $\frac{n+1}{2}$ links left. If this class is Class 1, then there's k such that (a_k, b_k) and (a_{k+1}, b_{k+1}) both remain (notice we take indices modulo n since things are in a circle, so (a_n, b_n) and (a_1, b_1) would still count). Then the wizard can form the following cycle of length $2n$ as follows (again indices modulo n)

$$a_k \to b_k \to b_{k-1} \cdots \to b_{k+1} \to a_{k+1} \to a_{k+2} \to \cdots \to a_{k+1} \to a_k$$

The case for Class 2 is similar:

$$a_k \to b_{n-k+1} \to b_{n-k+2} \to \cdots \to b_{n-k} \to a_{k+1} \to a_{k+2} \to \cdots \to a_{k+1} \to a_k$$

Now when n is even, it makes sense to distinguish between a_{odd}, a_{even}, b_{odd}, b_{even}, each containing $\frac{n}{2}$ of them. Consider the $2n$ new links (x, y), which contains $4n$ entries in total. Hence, one of a_{odd}, a_{even}, b_{odd}, b_{even} appears at most n times. Let's say, w.l.o.g., that a_{even} appears $\leq n$ times in the $2n$ links created by the wizard, counted with multiciplicity: in particular, a_{even} appears in at most n of the $2n$ new friendship links. Then the evil sorcerer can remove all links containing a_{even}.

We now show that the wizard cannot achieve their goal given the action of sorcerer. Suppose such a seating of $2n$ dwarfs is possible. Consider, now, partitioning the circle into a few segments as possible (arcs) where each segment contains only the old friendship not added by the wizard. Then each segment either have all a's or all b's. Now consider the a-segment. By the action of the sorcerer, the endpoints of the segments must be a_{odd}, and each element in a segment alternates between a_{odd} and a_{even}. It follows that each segment has exactly one more a_{odd} than a_{even}, so in total there are more a_{odd} than a_{even}, which is a conatradiction.