# Solution to APMO 2019 Problems

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Problem 1. The fact that $f(a)+b \mid a^2+f(a)f(b)$ implies that

$$f(a)+b \mid a^2+f(a)f(b)-f(b)(f(a)+b)=a^2-bf(b)\dots(*)$$

Plugging $a=b=1$ gives $f(1)+1 \mid 1-f(1)$, which also means $f(1)+1 \mid 2$. Given that $f(1)+1 \leq 2$ and $f(1) \geq 1$ (since it's a positive integer), we have $f(1)=1$.

Now we show by induction on $n$ that $f(n)=n$ for all $n \in \mathbb{N}$, with the base case done above. Assume for some $n$ that $f(n)=n$. Now, plugging $b=n+1$ and $a=1$ into (*) gives $n+2=f(1)+(n+1) \mid 1-(n+1)f(n+1)$ and since $n+1 \equiv -1 \pmod{n+2}$, we have $1-(n+1)f(n+1) \equiv 1+f(n+1) \pmod{n+2}$, meaning that $n+2 \pmod 1+f(n+1)$. Since $f(n+1)>0$, we have $1+f(n+1) \geq n+2$, i.e. $f(n+1) \geq n+1$. On the other hand, plugging $a=n+1$ and $b=n$ gives the following:

$$f(n+1)+n \mid (n+1)^2-nf(n)=(n+1)^2-n^2=2n+1$$

so $f(n+1)+n \leq 2n+1$, and therefore $f(n+1) \leq n+1$. Combining the two inequalities we have that $f(n+1)=n+1$. This finishes the inductive step.

Finally, to show that the identity function works, we have $f(a)+b=a+b$ and $a^2+f(a)f(b)=a^2+ab=a(a+b)$.

Problem 2. Assume that $\{a_n\}$ is a sequence of integers. We first show that for each $k$ there exists $\ell \geq k$ such that $2^{m-1} \leq a_\ell < 2^m$. If $a_k$ satisfies this property we are done. Otherwise we have the following cases:

- $a_k \geq 2^m$. Let $m_1 \geq m$ be such that $2^{m_1} \leq a_k < 2^{m_1+1}$. Iterating $a_{\ell+1}:=a_\ell/2$ for $m_1-m+1$ times gives $2^{m-1} \leq a_{k+m_1-m+1} < 2^m$.
- $a_k < 2^{m-1}$. Then $a_{k+1}=a_k^2+2^m > 2^m$ and the rest is taken care as of the previous case.

This means we can create a subsequence $\{a_{x_n}\}$ from the sequence $\{a_n\}$ such that $x_n$ are all the numbers satisfying $2^{m-1} \leq a_{x_n} < 2^m$.

Now, consider the sequence $\{b_n\}$ with $b_n=v_2(a_{x_n})$, the highest power of 2 dividing $a_{x_n}$. This sequence $b_n$ cannot decrease forever (since it has to be nonnegative for $a_{x_n}$ to be a positive integer). Let $k$ to be such that $b_{k+1} \geq b_k$. We notice the following: $a_{x_{n+1}}=(a_{x_n}^2+2^m)/2^x$ where $x$ is the minimum index such that $(a_{x_n}^2+2^m)/2^x < 2^m$. However, since $a_{x_n} \geq 2^{m-1}$, we have $a_{x_n}^2+2^m \geq 2^{2m-2}+2^m=2^m(2^{m-2}+1)$ so $2^x > 2^{m-2}+1$, and therefore $x \geq m-1$. Consequently, we have

$$v_2(a_{x_{n+1}})=v_2((a_{x_n}^2+2^m)/2^x)=v_2(a_{x_n}^2+2^m)-x \leq v_2(a_{x_n}^2+2^m)-(m-1)$$

and in the context of $k$, $v_2(a_{x_k}) \leq v_2 a_{x_{k+1}} \leq v_2(a_{x_n}^2+2^m)-(m-1)$.

We now have three cases:

- Case 1: $v_2(a_{x_k}) < m/2$. This means that $v_2(a_{x_k}^2) < m$ and therefore $v_2(a_{x_k}^2 + 2^m) = v_2(a_{x_k}^2) = 2v_2(a_{x_k})$. Consequently $v_2(a_{x_n}^2 + 2^m) - (m-1) = 2v_2(a_{x_n}) - (m-1) \geq v_2(a_{x_n})$, or $v_2(a_{x_n}) \geq m-1$. But we have assumed that $v_2(a_{x_k}) < m/2$ so $m - 1 < m/2$, i.e. $m < 2$.
- Case 2: $v_2(a_{x_k}) > m/2$. This means that $v_2(a_{x_k}^2) > m$ and therefore $v_2(a_{x_k}^2 + 2^m) = m$. This means that $v_2(a_{x_k}^2 + 2^m) - (m-1) = m - (m-1) = 1 \geq v_2(a_{x_k}) > m/2$, so $m < 2$, too.
- Case 3: $v_2(a_{x_k}) = m/2$. Write $a_{x_k} = 2^{m/2}y$ with $y$ odd, then $a_{x_k}^2 + 2^m = 2^m(y^2+1)$, with $y^2 + 1$ even. However, since $-1$ is not a quadratic residue mod 4, we have $v_2(y^2+1) = 1$ and therefore $v_2(a_{x_k}^2 + 2^m) = m + 1$. Now $(m+1) - (m-2) \geq m/2$, so $m \leq 4$.

The case $m = 1$ means that the only possible $a_{x_n}$ at all times is 1 (since it's the only positive integer smaller than 2). However, $a_{x_n+1} = 1^2 + 2 = 3$ and $a_{x_n+2} = 3/2$ is not an integer. Hence the case $m = 1$ is out. This leaves with $m \geq 2$, and only case 3 is valid here, which also means only $m$ even needs to be considered: $m = 2$ and $m = 4$. If $m = 4$, the only $a_{x_k}$ with $2^3 \leq a_{x_k} < 2^4$ and $v_2(a_{x_k}) = m/2 = 2$ is 12, which follows the following progression: $a_{x_k+1} = 12^2 + 16 = 160$, then $80, 40, 20, 10, 10^2 + 16 = 116, 29, 29/2$, showing that 12 fails here. Hence $m = 4$ doesn't work either.

Hence only $m = 2$ works, with the only possible $a_{x_k} = 2$, which will then follow the $2, 8, 4, 2, 8, 4$ cycle and thus works. Now, to find such suitable $a_1$, let's consider the following:

- If $a_1 < 4$, we already know $a_1 = 2$ works. Now $a_1 = 1$ means $a_2 = 1+4 = 5, a_3 = 5/2$; and $a_1 = 3$ means $a_2 = 9 + 4 = 13$, $a_3 = 13/2$. So only $a_1 = 2$ works.
- If $a_1 \geq 4$ then if $a_k$ is the smallest $k$ with $a_k = 2$ then $a_1 = 2 \cdot 2^{k-1} = 2^k$, so any power of 2.

This gives $m = 2$ and $a_1 = 2^k$ $(k \geq 1)$ as the only possible solutions.

Problem 3. Let the line passing through $B$ and parallel to $AM$ intersect $\Gamma$ again at $V$, and line passing through $C$ and parallel to $AM$ intersect $\Gamma$ again at $U$. Let $UV$ intersect $BC$ again at $W$ and let $AW$ intersect $\Gamma$ again at $T$. $U$ and $V$ do not depend on $P$ (given that $A, B, C$ are fixed), and neither do the points $W$ and $T$. We show that the circle $AXY$ passes through $T$, thus solving the problem.

First, notice that $D, P, U$ are collinear. Since $AP \parallel CU$, we have $\angle(BM, AP) = \angle(BC, AP) = \angle(BC, CU)$ and since $B, C, D, U$ is concyclic and so are points $B, D, P, M$, $\angle(BC, CU) = \angle(BD, DU)$ and $\angle(BD, DP) = \angle(BM, MP) = \angle(BM, AP)$. Thus $\angle(BD, DU) = \angle(BM, AP) = \angle(BD, DP)$, showing that $\angle(DU, DP) = 0$ and so $D, U, P$ must be on the same line, and similarly for points $C, P, V$. Next, since $\angle(BD, DP) = \angle(BM, MP) = \angle(CM, MP) = \angle(CX, XP) = \angle(CX, DP)$ (since $D, P, X$ collinear), we have $BD \parallel CX$ and similarly $CE \parallel BY$.

Let $CX$ and $BY$ intersect at $R$, and $BD$ and $CE$ intersect at $Q$. Since $BD$ is the radical axis of $\Gamma$ and circle $BPM$, and $CE$ the radical axis of $\Gamma$ and circle $CPM$, $Q$ is the radical center of these circles, hence on $PM$ the radical axis of $BMP$ and $CPM$. Since $BQCR$ is a parallelogram, $R$ is also on $PM$. Now consider the following:

$$\angle(YP, PX) = \angle(EP, PD) = \angle(EP, PM) + \angle(PM, PD) = \angle(EC, CM) + \angle(BM, BD)$$

$$= \angle(EC, CD) = \angle(EQ, QB) = \angle(BR, RC) = \angle(RY, RX)$$

which shows that $R, Y, P, X$ are concyclic. Furthermore, we also have

$$\angle(BY, YX) = \angle(RY, YX) = \angle(RP, PX) = \angle(MP, PX) = \angle(MC, CX) = \angle(BC, CX)$$

showing that $B, Y, X, C$ are also concyclic. Finally, with $BD \parallel CX$ and $CE \parallel BY$ we also have

$$\frac{DP}{PE} = \frac{DP}{PQ} \div \frac{PE}{PQ} = \frac{PX}{PR} \div \frac{PY}{PR} = \frac{PX}{PY}$$

and therefore there exists a constant $k$ such that $PX = kPD$ and $PY = kPE$. Since $D, E, U, V$ are concyclic on $\Gamma$ with the intersection $P$, we have $PX \cdot PU = kPD \cdot PU = kPE \cdot PV = PY \cdot PE$, showing that $Y, X, U, V$ are also concyclic.

We can now complete our solution. $UV$ is the radical axis of circle $YXUV$ and $\Gamma$, $XY$ is the radical axis of circle $YXUV$ and $YXBC$, and $BC$ is the radical axis of circle $YXBC$ and $\Gamma$. Thus, $UV, BC, XY$ concur at pre-defined point $W$. Moreover, $XY$ is the radical axis of the circles $AXY$ and $YXBC$ and again $BC$ is the radical axis of $YXBC$ and $\Gamma$. Again, the radical axis of $AXY$ and $\Gamma$, $XY$ and $BC$ will concur at the same point $W$. Since $T$ is the second intersection of $\Gamma$ and $AW$, we have $T$ as the second intersection of $\Gamma$ and $AXY$, so $T$ is on triangle $AXY$. Q.E.D.

Problem 5. Now plug $x = y = 0$ we have $f(f(0)) = f(f(0)) + 3f(0)$ so $f(0) = 0$. Plugging only $y = 0$ gives $f(x^2) = f(f(x)) + 2f(0) = f(f(x))$ so $f(f(x)) = f(x^2)$ for all $x$. Also, we have the following:

$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy) = f(x^2) + (y^2) + 2f(xy)$$

$$= f(y^2) + f(x^2) + 2f(yx) = f(f(y)) + (x^2) + 2f(yx) = f(y^2 + f(x))$$

so $f(x^2+f(y)) = f(y^2+f(x))$. In addition, we have $f(x^2)+(y^2)+2f(xy) = f(x^2+f(y)) = f((-x)^2 + f(y)) = f((-x)^2) + (y^2) + 2f(-xy)$ so $f(-xy) = f(xy)$ for all $x$ and $y$. Letting $y = 1$ yields $f(x) = f(-x)$, showing that $f$ is an even function. Combining this with $f(0) = 0$, we only need to consider those $x$ with $x > 0$.

Now suppose that for some $x_1, x_2 \geq 0$ we have $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. This means that $f(x_1^2) = f(f(x_1)) = f(f(x_2)) = f(x_2^2)$. Fix $y$, and consider the following separately:

$$f(y^2 + f(x_1)) = f(y^2) + f(x_1^2) + 2f(yx_1)$$

$$f(y^2 + f(x_2)) = f(y^2) + f(x_2^2) + 2f(yx_2)$$

and comparing both sides, we have $f(x_1y) = f(x_2y)$ for all $y$. If $x_1 = 0$ then we have $f(0) = f(x_2y)$ for all $y$ and combining this with $x_2 > 0$ we have $f(x) = 0$ for all $x$. Thus we can assume $0 < x_1 < x_2$. We now consider the following equivalence:

$$f((x_1z)^2 + f(y)) = f(y^2 + f(x_1z)) = f(y^2 + f(x_2z)) = f((x_2z)^2 + f(y))$$

where $z$ is arbitrary real number (we have shown before that $f(x_1z) = f(x_2z)$ for any real $z$. Suppose that $f(y) < 0$ for this $y$. Let $z$ be such that $(x_1z)^2+f(y) = 0$, or $x_1z = \sqrt{-f(y)}$ (we can find such $z$ since $x_1 > 0$). Then $f((x_1z)^2 + f(y)) = f((x_2z)^2 + f(y)) = 0$. Since $(x_2z)^2 + f(y) \neq 0$ as $x_1 \neq x_2$ and both $x_1, x_2 > 0$, we have $f(c) = 0$ for some $c \neq 0$ and by above this applies $f \equiv 0$. Thus we can assume that $f(y) \geq 0$. If $f$ is not identically 0, choose $y$ such that $f(y) > 0$. Suppose also that $x_1 > x_2$ and consider the following ratio: $r = \dfrac{x_1^2z^2 + f(y)}{x_2^2z^2 + f(y)}$. By the fact that $f(x_1) = f(x_2) \to f(x_1z) = f(x_2z)$ for all $z$, we have $f(x) = f(kx)$ for all $x$ where $k = x_1/x_2$. We also know that, as $z$ varies in $[0, \infty)$, the ratio varies continuously in $[1, x_1^2/x_2^2)$. For any $r \in [1, x_1^2/x_2^2)$ we have $f(x) = f(rx)$ for all $x$; this also holds for $r = x_1^2/x_2^2$ since $f(x_1^2) = f(x_2^2)$. Hence for all $x$, $f$ is constant in the interval $[x, Rx]$ with $R = x_1^2/x_2^2$ and considering this for all $x > 0$ we conclude that $f$ takes constant value on $\mathbb{R}^+$. If this constant $c = 0$ then $f \equiv 0$; otherwise, by before we can assume $c > 0$ ($c < 0$ leads to the $f \equiv 0$ conclusion too). If $c > 0$, $f$ is defined by $f(x) = 0$ for $x = 0$ and $c$ otherwise, giving the following for any pairs of $x, y > 0$

$$f(x^2 + f(y)) = f(x^2 + c) = c; f(f(x)) + f(y^2) + 2f(xy) = f(c) + c + 2c = 4c$$

so $c = 4c$ forcing $c = 0$.

Now assumt that there's no $x_1 \neq x_2$, both nonnegative, such that $f(x_1) = f(x_2)$. This means that $f$ is injective on $\mathbb{R}_{\geq 0}$. Going back to the function $f(f(x)) = f(x^2)$ and the fact that $f$ is even, we consider $x > 0$. If $f(x) > 0$, then from the injectivity of $f$ on $\mathbb{R}^+$ we have $f(x) = x^2$. Otherwise, $f(x) < 0$ and by the even function property $f$ is injective on nonpositive reals too. Given also that $f(x^2) = f(-x^2) = f(f(x))$ we have $f(x) = -x^2$. This limits our function to, for each $x$, $f(x) = \pm x^2$. If $x > 0$ is a number such that $f(x) = -x^2$, letting $x = y$ gives $0 = f(0) = f(x^2 - x^2) = f(f(x)) + 3f(x^2) = 4f(x^2)$ (recall that $f(f(x)) = f(x^2)$) so $4f(x^2) = 0$, which is a contradiction since $x^2 \neq 0$ and $f(x^2) = 0$ means $f \equiv 0$. Hence $f$ must be $x^2$.

It turns out that 0 and $x^2$ both works: the first gives 0 on both sides; the second gives $(x^2 + y^2)^2$ on both sides.