Solutions to Tournament of Towns, Spring 2020, Senior

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O-Level

1.

A-Level

2. Alice had picked positive integers a, b, c and then tried to find positive integers x, y, z such that $a = l.c.m.(x, y)$, $b = l.c.m.(x, z), c = l.c.m.(y, z)$. It so happened that such x, y, z existed and were unique. Alice told this fact to Bob and also told him the numbers a and b. Prove that Bob can find c. (Note: l.c.m = least common multiple.)

   Solution. Consider, now, arbitrary a, b, c, and for each prime p dividing abc consider $v_p(a), v_p(b), v_p(c)$. We need to find x, y, z such that

   $$v_p(a) = \max(v_p(x), v_p(y)) \qquad v_p(b) = \max(v_p(x), v_p(z)) \qquad v_p(c) = \max(v_p(y), v_p(z))$$

   Since x, y, z are unique, $v_p(x), v_p(y), v_p(z)$ must be uniquely determined. Since each of $v_p(x), v_p(y), v_p(z)$ are considered twice in $v_p(a), v_p(b), v_p(c)$, considering largest among $v_p(x), v_p(y), v_p(z)$ yield that the two largest quantity among $v_p(a), v_p(b), v_p(c)$ must both be equal to $\max\{v_p(x), v_p(y), v_p(z)\}$ and nonzero because $p \mid abc$. W.l.o.g. suppose $v_p(z)$ turns out to be the largest, then $v_p(b) = v_p(c) = v_p(z)$ and given $v_p(a) = \max(v_p(x), v_p(y)) \leq v_p(z)$, $v_p(x)$ and $v_p(y)$ can be constructed by taking one being equal to $v_p(a)$ and the others any number $\leq v_p(a)$. In particular, all $(v_p(a), v_p(a)), (0, v_p(a))$ and $(v_p(a), 0)$ are valid numbers. By the uniqueness of x, y, z we must have $v_p(a) = 0$. It then follows that for each p dividing abc, two of $v_p(a), v_p(b), v_p(c)$ are equal and positive, while the third one 0.

   Knowing this, if $p \mid abc$ then p divides exactly two of a, b, c, so it divides at least one of a, b. Hence, when constructing c, we only need to consider all p dividing a or b:

   - If, for such p, $v_p(a) = v_p(b) > 0$, then $v_p(c) = 0$.
   - Otherwise, one of $v_p(a), v_p(b)$ is positive while the other zero, match $v_p(c)$ with the positive one.

   This determines c uniquely.

4. Henry invited $2N$ guests to his birthday party. He has N white hats and N black hats. He wants to place hats on his guests and split his guests into one or several dancing circles so that in each circle there would be at least two people and the colors of hats of any two neighbours would be different. Prove that Henry can do this in exactly $(2N)!$ different ways. (All the hats with the same color are identical, all the guests are obviously distinct; $(2N)! = 1 \cdot 2 \cdot \ldots \cdot (2N)$.)

   Solution. We do induction on N. If $N = 1$ the two kids must be in the same circle, so the 2 ways depending who gets white hat (i.e. the other gets black hat).

Induction: suppose the hypothesis holds for $1, \cdots, N-1$ for some $N \geq 2$. We now focus on the circle containing the kid numbered 1. Notice that the alternating colour condition says that the circle size must be even.

Let $2k$ be the size. Fixing the position of kid 1, there are $\frac{(2N-1)!}{(2N-2k)!}$ ways to choose $2k-1$ other people to be in the same circle as this kid. Given this, there are two ways to distribute the hats (kid 1 gets black or white, the rest is then determined uniquely). Now, the unselected $2N-2k$ kids form other circles, which have $(2N-2k)!$ ways by induction hypothesis. This gives a total of

$$\frac{(2N-1)!}{(2N-2k)!} \cdot 2 \cdot (2N-2k)! = 2(2N-1)!$$

and combining $k = 1, 2, \cdots, N$ we get the total number of ways as $2N(2N-1)! = (2N)!$, completing the induction step.

5. Let $ABCD$ be an inscribed quadrilateral. Let the circles with diameters AB and CD intersect at two points X_1 and Y_1, the circles with diameters BC and AD intersect at two points X_2 and Y_2, the circles with diameters AC and BD intersect at two points X_3 and Y_3. Prove that the lines X_1Y_1, X_2Y_2 and X_3Y_3 are concurrent.

   Solution. Denote the intersection of AB and CD as Z_1, BC and AD as Z_2, and AC and BD as Z_3. Notice that Z_1 might be point of infinity. If both Z_1 and Z_2 are point of infinity, then $ABCD$ is a rectangle and all the three target lines will meet at the center O of the circle surrounding $ABCD$. If Z_1 is but Z_2 is not, then $ABCD$ is an isoceles trapezoid and X_2Y_2 will be the shared perpendicular bisector of AB and CD (the circles of diameters BC and AD are literally mirror images of each other under this perpendicular bisector), and so is X_3Y_3. Thus X_2Y_2 and X_3Y_3 coincide in this case.

   Hence we may assume that Z_1, Z_2, Z_3 are not point of infinity. We now consider the following: taking the two circles with diameters AB and CD, and also the circumcircle of $ABCD$, the radical axes from the 2 out of 3 circles are AB, CD, and X_1Y_1. Hence X_1Y_1 passes through Z_1. Moreover, if M_ℓ is the midpoint of segment ℓ then X_1Y_1 is perpendicular to M_{AB} and M_{CD} (radical axis is always perpendicular to the line joining the centers of two circles). We also have $\angle Z_1M_{AB}O = \angle Z_1M_{CD}O = 90°$ so $Z_1M_{AB}OM_{CD}$ is cyclic. This gives the following relation (bearing in mind thar X_1Y_1 and $M_{AB}M_{CD}$ are perpendicular)

$$\angle(Z_1M_{AB}, X_1Y_1) = 90° - \angle(M_{AB}M_{CD}, Z_1M_{AB}) = 90° - \angle(OM_{CD}, OZ_1) = \angle(OZ_1, Z_1M_{CD})$$

   or rather, $\angle(AB, X_1Y_1) = \angle(OZ_1, CD)$. i.e. X_1Y_1 is the reflection of Z_1O in the angle bisector of $\angle M_{AB}Z_1M_{CD}$. In a similar way we have $\angle(BC, X_2Y_2) = \angle(OZ_2, AD)$ and $\angle(AC, X_3Y_3) = \angle(OZ_3, BD)$.

   Now, let T_i be the second intersection of X_iY_i with the circumcircle of $Z_1Z_2Z_3$ ($T_i \neq Z_i$ unless the X_iY_i is tangent to the circumcircle). Next, Brokard's theorem says that O is the orthocenter of $Z_1Z_2Z_3$. In the context of directed angle, this means

$$\angle(Z_1Z_3, Z_2Z_3) = \angle(Z_2O, Z_1O)$$

   Given that $ABCD$ is cyclic, we also have $\angle(AB, AD) = \angle(CB, CD)$. This then means

$$\angle(X_1Y_1, X_2Y_2) = \angle(X_1Y_1, AB) + \angle(AB, AD) + \angle(AD, X_2Y_2)$$

$$= \angle(CD, OZ_1) + \angle(CB, CD) + \angle(OZ_2, BC) = \angle(OZ_2, OZ_1) = \angle(Z_1Z_3, Z_2Z_3)$$

   and since X_1Y_1 passes through Z_1 and X_2Y_2 passes through Z_2, we have X_1Y_1 and X_2Y_2 intersecting on the circle $Z_1Z_2Z_3$, and therefore $T_1 = T_2$. Similarly $T_2 = T_3$. Hence X_iY_is concur on the same point $T_1 = T_2 = T_3$.