Solutions to Tournament of Towns, Fall 2018, Senior

Anzo Teh

O-Level

1.

A-Level

3. Prove that

   (a) any integer of the form $3k - 2$, where k is an integer, is the sum of a square and two cubes of some integers

   (b) any integer is the sum of a square and three cubes of some integers.

   Solution. (a) is taken care by the following:

   $$(3k + 5)^2 + k^3 + (-(k + 3))^3 = 9k^2 + 30k + 25 - 9k^2 - 27k - 27 = 3k - 2$$

   (b) uses (a) in that for any integer n, either n, $n - 1$ or $n - 8$ is in the form of $3k - 2$ (corresponding to the case where n has remainder $1, 2, 0$ modulo 3). If $w \in \{n, n-1, n-8\}$ is in the form of $3k - 2$ then one of $w + 0^3, w + 1^3, w + 2^3$ is equal to n, and each of these expression is then sum of a square and 3 cubes.