Putnam 2020 (No official contest)

A1 How many positive integers N satisfy all of the following three conditions?

- N is divisible by 2020.
- N has at most 2020 decimal digits.
- The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Answer. 508536.

Solution. Let N to have a 1's and b 0's, so $N = 10^b \times \underbrace{1\cdots 1}_{a}$. Since N is divisible by 20, it must have at least 2 ending zeros. Moreover, $2020 = 101 \times 20$ and $\gcd(10, 101) = 1$, so $101 \mid \underbrace{1\cdots 1}_{a} = \frac{10^a - 1}{9}$. Since $1111 = 101 \times 11$ but $101 \nmid 111, 11, 1$, we see that $\mathrm{ord}_{101}(10) = 4$ so $4 \mid a$. This gives the complete characterization:

$$a + b = 2020, b \geq 2, 4 \mid a, a \geq 1$$

Now for each a we can pick $b = 2, 3, \cdots, 2020 - a$ (thus giving $2019 - a$ choices). The maximum a is 2016. This gives the following:

$$\sum_{k=1}^{504}(2019 - 4k) = 2019 \times 504 - 4 \times 504 \times 505 \div 2 = 1009 \times 504 = 508536$$

A2 Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}.$$

Answer. 4^k.

Solution. Let our sum be S_k and do induction on it. Base case is given by $k = 0$ when we just have 1. Now suppose that our claim holds for some k, that is $S_k = 4^k$ for some $k \geq 0$. We have

$$S_{k+1} - 2S_k = \sum_{j=0}^{k+1} 2^{k+1-j}\binom{k+1+j}{j} - 2\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j} = \binom{2(k+1)}{k+1} + 2\sum_{j=1}^{k} 2^{k-j}\binom{k+j}{j-1}$$

where we used the fact that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$. Notice again that the equivalence

$$2\sum_{j=1}^{k} 2^{k-j}\binom{k+j}{j-1} = 2\sum_{j=0}^{k-1} 2^{k-j-1}\binom{k+1+j}{j} = \frac{1}{2}(S_{k+1} - 2\binom{2k+1}{k} - \binom{2(k+1)}{k+1})$$

which gives

$$\frac{1}{2}S_{k+1} = 2S_k - \binom{2k+1}{k} + \frac{1}{2}\binom{2(k+1)}{k+1}$$

Given that $\binom{2k}{k} = 2\binom{2k-1}{k-1}$, we also have $-\binom{2k+1}{k} + \frac{1}{2}\binom{2(k+1)}{k+1} = 0$ and therefore $S_{k+1} = 4S_k = 4 \cdot 4^k = 4^{k+1}$, as desired.

A3 Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Answer. This sequence diverges.

Solution. We first see that $a_n > 0$ all the while: if $0 < a_n \le \frac{\pi}{2}$ for some n then $0 < a_{n+1} \le 1$, so we have $0 < a_n \le \frac{\pi}{2}$ for all n.

If $a_n \not\to 0$ then the series would diverge, so we may assume $a_n \to 0$. We first analyze the asymptotic behaviour of a_n^2 as a_n small:

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{(a_n - a_{n+1})(a_n + a_{n+1})}{a_n^2 a_{n+1}^2} = \frac{(a_n - \sin a_n)(a_n + \sin a_n)}{a_n^2 \sin^2 a_n} \overset{a_n \to 0}{\sim} \frac{\frac{a_n^3}{6} \cdot 2a_n}{a_n^4} = \frac{1}{3}$$

so as $n \to \infty$, we see that a_n^2 decays in the pace of $\frac{3}{n+c}$ for some constant c. Since $\sum \frac{1}{n}$ diverges, it follows that $\sum a_n^2$ diverges too.

A5 Let a_n be the number of sets S of positive integers for which

$$\sum_{k \in S} F_k = n, \tag{1}$$

where the Fibonacci sequence $(F_k)_{k \ge 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$ and begins $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$. Find the largest number n such that $a_n = 2020$.

Answer. $F_{4040} - 1$.

Solution. We first note the following:

$$\sum_{i=1}^{k} F_i = F_{k+2} - 1 \tag{2}$$

for all $k \ge 1$ (which is clear from $k = 1, 2, 3$ and the rest can be established via induction). If we define $f(n)$ as the index satisfying $F_{f(n)} \le n < F_{f(n)+1}$, then the set S satisfying Equation 1 has $\max\{k : k \in S\}$ either $f(n)$ or $f(n) - 1$. This gives us two scenario whenever $n \ge 2$ (and so $f(n) \ge 3$):

- If the max element in S is $f(n)$, we have $0 \le n - F_{f(n)} < F_{f(n)+1} - F_{f(n)} = F_{f(n)-1}$, which gives us $a_{n-F_{f(n)}}$ choices (all chosen in $\{1, \cdots, f(n) - 2\}$)
- If the max element in S is $f(n) - 1$, then it turned out it's more expedient to consider the elements *not* chosen among $\{1, \cdots, f(n) - 2\}$. By Equation 2, summing over $\{1, \cdots, f(n) - 1\}$ gives $F_{f(n)+1} - 1 \ge n$, so the number of ways here is precisely $a_{F_{f(n)+1}-n-1}$.

Therefore we have the iterative formula $a_{n-F_{f(n)}} + a_{F_{f(n)+1}-n-1}$.

Now with this, let's make the following claim:

- For all $k \ge 1$, $a_{F_{2k}-1} = k$.
- For all $k \ge 1$ and $n \ge F_{2k}$, $a_n > k$. In other words, $a_n \ge \lfloor \frac{f(n)}{2} \rfloor + 1$.

These claims would suffice to show that our answer above.

For the first claim, we see that $a_{F_2-1} = a_0 = 1$, and $a_{F_4-1} = a_2 = 2$ (given that 2 can be written 2 and 1+1). Also, $f(F_{2k} - 1) = 2k - 1$, so for $k \ge 3$,

$$a_{F_{2k}-1} = a_{F_{2k}-1-F_{2k-1}} + a_{F_{2k}-F_{2k}} = a_{F_{2k-2}-1} + 1$$

so the inductive hypothesis $a_{F_{2k-2}-1} = k - 1$ would imply $a_{F_{2k}-1} = k$.

Now to prove the second claim, let us again use induction in the following sense: for each k, we consider those n with $F_{2k} \le n < F_{2k+2} - 1$. When $k = 0$ this is just $n = 0$ and $a_n = 1$, and when $k = 1$, $n = 1, 2$ which gives $a_1 = a_2 = 2$.

For $k \geq 2$, for n in the said range we have $f(n)$ either $2k$ or $2k+1$. Now consider the pair of numbers

$$n - F_{f(n)}, F_{f(n)+1} - n - 1$$

which sums up to $F_{f(n)-1} - 1 < F_{2k}$. Thus by induction hypothesis we can deduce that

$$a_n = a_{n-F_{f(n)}} + a_{F_{f(n)+1}-n-1} \geq \lfloor \frac{f(n - F_{f(n)})}{2} \rfloor + \lfloor \frac{f(F_{f(n)+1} - n - 1)}{2} \rfloor + 2$$

Now, let $\lfloor \frac{f(n-F_{f(n)})}{2} \rfloor = x$ and $\lfloor \frac{f(F_{f(n)+1}-n-1)}{2} \rfloor = y$, then

$$f(n - F_{f(n)}) \leq 2x + 1 \Rightarrow n - F_{f(n)} \leq F_{2x+2} - 1$$

and similarly $F_{f(n)+1} - n - 1 \leq F_{2y+2} - 1$. Therefore we have the sum satisfying $F_{f(n)-1} - 1 \leq F_{2x+2} + F_{2y+2} - 2$, or $F_{f(n)-1} \leq F_{2x+2} + F_{2y+2} - 1$. If $x + y < k - 1$, then we have $F_{2k-1} \leq F_{2x+2} + F_{2y+2} - 1$ for some $x + y < k - 1$, and for $x, y \geq 0$. (can substitute $f(n) = 2k$ here since if it holds for $2k + 1$ it will hold for $2k$). We however see that F is convex hence

$$F_{2k-1} \leq F_{2x+2} + F_{2y+2} - 1 \leq F_2 + F_{2(x+y)+2} - 1 = F_{2(x+y)+2} \leq F_{2k-2}$$

which is a contradiction. Therefore $a_n \geq k - 1 + 2 = k + 1$, as claimed.

B3 Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots,$ a point x_{n+1} is chosen uniformly form the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

Answer. $1 + \frac{1}{\delta}$.

Solution. Let's consider the function $F_n(\delta) = \mathbb{P}[x_n < \delta]$, and $f_n(\delta) = \frac{d}{d\delta} F_n$. This has the following recursive formula:

$$F_1(\delta) = \delta \qquad F_n(\delta) = F_{n-1}(\delta) + \int_\delta^1 \frac{\delta}{x} f_{n-1}(x) dx$$

Let's now show that

$$F_n(\delta) = \delta \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{\delta})^k}{k!} \right) \tag{3}$$

Using induction, we see that this holds for $n = 1$, and if this holds for some n, i.e.

$$F_n(x) = x \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{x})^k}{k!} \right) \qquad F_n(x) = \sum_{k=0}^{n-1} \frac{\ln(\frac{1}{x})^k}{k!} - \sum_{k=1}^{n-1} \frac{\ln(\frac{1}{x})^{k-1}}{(k-1)!} = \frac{\ln(\frac{1}{x})^{n-1}}{(n-1)!}$$

and therefore

$$F_{n+1}(\delta) = \delta \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{\delta})^k}{k!} \right) + \int_\delta^1 \frac{\delta}{x} \frac{\ln(\frac{1}{x})^{n-1}}{(n-1)!} dx = \delta \left(\sum_{k=0}^{n} \frac{\ln(\frac{1}{\delta})^k}{k!} \right)$$

establishing 3.

To finish the solution, we have $\mathbb{P}[Z = k] = \mathbb{P}[x_n < \delta \wedge x_{n-1} \geq \delta] = F_k(\delta) - F_{k-1}(\delta) =$

$\delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!}$. Therefore,

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{k=1}^{\infty} k\delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty} \delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} + (k-1)\delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty} \delta\frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} + \sum_{k=2}^{\infty} \delta\frac{\ln(\frac{1}{\delta})^{k-2}}{(k-2)!} \\
&= \delta e^{\ln(\frac{1}{\delta})} + \delta\ln(\frac{1}{\delta})e^{\ln(\frac{1}{\delta})} \\
&= 1 + \ln\frac{1}{\delta}
\end{aligned}$$

as claimed.

B4 Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

Answer. $\frac{1}{4040}$.

Solution. In general, we show that the average of $1 + \sum_{j=1}^{2n-1} \alpha^{s_j}$ over V_n is $\frac{1}{2n}$ for any $\alpha > 0$.

Consider $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ and $t(\mathbf{v}) := (t_0, \cdots, t_{2n-1})$ be such that $t_k = s_{k+1} - s_k$. We say $\mathbf{v} \sim \mathbf{v}'$ if $t(\mathbf{v}')$ can be obtained from $t(\mathbf{v})$ via cyclic shift (hence we could also say $t(\mathbf{v}) \sim t(\mathbf{v})'$). Now t maps V_n to $T_n := \{(t_0, \cdots, t_{n-1}) \subseteq \{-1, 1\}^{2n}, \sum t_i = 0\}$, and is a bijection. Moreover, relation defined via cyclic shift is both symmetric (just reverse cycle) and transitive, and also $\mathbf{v} \sim \mathbf{v}$ holds for all $\mathbf{v}$. Thus $\sim$ is an equivalence relation.

Denote, now, the equivalence class of each $\mathbf{v}$:

$$E_{\mathbf{v}} = \{\mathbf{v}' : \mathbf{v}' \sim \mathbf{v}\}$$

We'll show that the average of $\frac{1}{q}$ in $E_{\mathbf{v}}$ is $\frac{1}{2n}$. Let $t(\mathbf{v}) = (t_0, t_1, \cdots, t_{2n-1})$ and for each $\mathbf{v}' \sim \mathbf{v}$ can be written as $t(\mathbf{v})' = (t_j, t_{j+1}, \cdots, t_{2n+j-1})$ for some $j \geq 0$ (indices taken modulo $2n$). Thus if $\mathbf{v} = (0, s_1, \cdots, s_{n-1}, 0)$ we have $\mathbf{v}' = (0, s_{j+1} - s_j, \cdots, s_{2n} - s_j, s_1 - s_j, \cdots, s_{j-1} - s_j, 0)$, and $\frac{1}{q(\mathbf{v}')} = \frac{\alpha^{s_j}}{q(\mathbf{v})}$. Now considering $j = 0, \cdots, 2n-1$ we see that the average of $\frac{1}{q}$ is now

$$\frac{1}{2n}\sum_{j=0}^{2n-1}\frac{\alpha^{s_j}}{q(\mathbf{v})} = \frac{1}{2n}$$

since $\frac{\alpha^{s_j}}{q(\mathbf{v})}$ is just 1. But we're not done yet – we need to show that averaging over all $j = 0, \cdots, 2n-1$ is the *true* average of this equivalence class. This is to show that as we loop over $j = 0, \cdots, 2n-1$ each $\mathbf{v}' \in E_{\mathbf{v}}$ shows up equally many times. If we extend $t_0, \cdots, t_{2n-1}$ infinitely (hence having $2n$ as period) and let g as its *minimal* period, we see that each $\mathbf{v}' \in E_{\mathbf{v}}$ shows up $\frac{2n}{g}$ times, proving our claim.