Solution to IMO 2016 shortlisted problems.

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1 Algebra

A1 Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Solution. We start with a preliminary observation: given that $k \geq 2$, and given the set of pairs $K = \{(a,b) : a+b = k, ab \geq 1\}$, then for any $(a_1, b_1), (a_2, b_2) \in K$, $(a_1^2+1)(b_1^2+1) \geq (a_2^2+1)(b_2^2+1)$ iff $|a_1 - b_1| \leq |a_2 - b_2|$. Indeed, for $(a,b) \in K$, $(a^2+1)(b^2+1) = (a+b)^2 + a^2b^2 - 2ab + 1 = k^2 + (ab-1)^2 = k^2 + (\frac{(a+b)^2-(a-b)^2}{2} - 1)^2 = k^2 + (\frac{k^2-(a-b)^2}{2} - 1)^2$, and given that $ab \geq 1$, this function is increasing in ab. In addition, with $a+b$ fixed, this function is also dereasing in $(a-b)^2$, which turns out to also be decreasing in $|a-b|$.

Now let $a+b+c = 3k$, and let $f(a,b,c) = (a^2+1)(b^2+1)(c^2+1)$. W.l.o.g. assume that $a \leq b \leq c$. Let $b \leq k$, then by above, $(b^2+1)(c^2+1) \leq (k^2+1)((b+c-k)^2+1)$ because $b+c \geq 2k$ ($a \leq k$ and $c \geq k$), and $(b+c-k) - k = (b+c) - 2k \leq (b+c) - 2b = c - b$ (since $b \leq k$), which follows that $k(b+c-k) \geq bc \geq 1$. Likewise, if $b \geq k$ then by above, $(a^2+1)(b^2+1) \leq ((a+b-k)^2+1)(k^2+1)$ because $b+a \leq 2k$ ($a \leq k$ and $c \geq k$), and $k - (a+b-k) = 2k - (b+c) \leq 2b - (a-b) = b - a$ (since $b \geq k$), which follows that $k(a+b-k) \geq ab = 1$. Additionally, after the operation, we have $a \leq k \leq b+c-k$ in the first case, and $a+b-k \leq k \leq c$ (a good question to ask might be: what if $a+b-k < 0$ in the second case? This is impossible because we only do this when $b \geq k$). So we only need to verify that the first two has product at least one. In case 1, $ak \geq ab = 1$ and in case 2 we have already verified that $k(a+b-k) \geq ab = 1$. Thus $f(a,b,c) \leq f(a,k,(b+c-k))$ in case 1, or $f(a,b,c) \leq f((a+b-k),k,c)$ in case 2. This means we can focus on the case where $b = k$. Nevertheless, when $b = k$ we have $a+c = 2k$ and by similar procedure we have $(a^2+1)(c^2+1) \leq (k^2+1)^2$, therefore we have: $f(a,b,c) \leq f(a,k,(b+c-k)) \leq f(k,k,k)$ in case 1, $f(a,b,c) \leq f((a+b-k),k,c) \leq f(k,k,k)$ in case 2, and $f(k,k,k) = (k^2+1)^3$ is precisely the cube the right hand side. Q.E.D.

A4 Find all functions $f : (0,\infty) \to (0,\infty)$ such that for any $x, y \in (0,\infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right) \cdots (1).$$

Solution. The only function is $f(x) \equiv \frac{1}{x}$, which works because $xf(x^2)f(f(y)) + f(yf(x)) = x\frac{1}{x^2}y + \frac{1}{y\frac{1}{x}} = \frac{x}{y} + \frac{y}{x} = \frac{x^2}{xy} + \frac{y^2}{xy} = f(xy)\left(f(f(x^2)) + f(f(y^2))\right)$.

First, notice that plugging $x = y = 1$ gives $f(1)f(f(1)) + f(f(1)) = 2f(1)f(f(1))$, and since $f > 0$, we can factorize $f(f(1))$ out to get $f(1) + 1 = 2f(1)$, giving $f(1) = 1$. Next, letting $x = 1$ (and substituting $f(1) \leftarrow 1$ gives $f(f(y)) + f(y) = f(y)(1 + f(f(y^2)))$, giving $f(f(y)) = f(y)f(f(y^2))$. Letting $y = 1$, on the other hand, gives $xf(x^2) + f(f(x)) =$

$f(x)(f(f(x^2))+1)$. Knowing from above that $f(f(x)) = f(x)f(f(x^2))$, we have $xf(x^2) = f(x)\cdots(2)$. In view of this, we can substitute $xf(x^2) \leftarrow f(x)$, $f(f(y)) \leftarrow f(y)f(f(y^2))$, and $yf(x) \leftarrow xyf(x^2)$, giving

$$f(x)f(y)f(f(y^2)) + f(xyf(x^2)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right)\cdots(3).$$

In the special case where $xy = 1$ we have $f(x)f(y)f(f(y^2))+f(f(x^2)) = 1\left(f(f(x^2)) + f(f(y^2))\right)$, so $f(x)f(y) = 1$ whenever $xy = 1$. In other words, $f(\frac{1}{x}) = \frac{1}{f(x)}$. Having this in mind, we substitute $\frac{1}{x}$ and $\frac{1}{x}$ in place of x and y into (3) to turn $f(\frac{1}{x})f(\frac{1}{x})f(f(\frac{1}{x^2})) + f(\frac{1}{x^2}f(\frac{1}{x^2}))$ $= f(\frac{1}{x^2})\left(f(f(\frac{1}{x^2})) + f(f(\frac{1}{x^2}))\right)$ into $\frac{1}{f(x)f(x)f(f(x^2))} + \frac{1}{f(x^2f(x^2))} = \frac{2}{f(x^2)f(f(x^2))}$. Notice, however, we also have (by substituting $x = y$ into (3)) to get $f(x)f(x)f(f(x^2))+f(x^2f(x^2)) = 2f(x^2)f(f(x^2))\cdots(3)$. Thus we have
$\frac{1}{f(x)f(x)f(f(x^2))} + \frac{1}{f(x^2f(x^2))} = \frac{2}{f(x^2)f(f(x^2))} = \frac{4}{f(x)f(x)f(f(x^2))+f(x^2f(x^2))}$, and cross multiplying gives $(f(x)f(x)f(f(x^2)) + f(x^2f(x^2)))^2 = 4(f(x)f(x)f(f(x^2)) \cdot f(x^2f(x^2)))$, a.k.a. $(f(x)f(x)f(f(x^2)) - f(x^2f(x^2)))^2 = 0$. Therefore $f(x)f(x)f(f(x^2)) = f(x^2f(x^2)) = f(x^2)f(f(x^2))$. Factorizing out $f(f(x^2))$ gives $f(x)^2 = f(x^2)$, and from (2), $f(x) = xf(x^2) = xf(x)^2$, or $xf(x) = 1$, or $f(x) = \frac{1}{x}$.

A5 Consider fractions $\frac{a}{b}$ where a and b are positive integers.
(a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}+1$.
(b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}$.

Solution. For part (a), we partion the set of positive integers according to their integer square roots, that is, the sets $S_1 = \{1,2,3\}$, $S_2 = \{4,5,6,7,8\}$, $S_3 = \{9,10,11,12,13,14,15\}$, etc. Consider $S_k = \{k^2, k^2+1, \cdots k^2+2k\}$, and we claim that $b = k$ and $b = k+1$ alone will jointly work for the sets. Indeed, considering $c \in [0,k]$ we have $(k+\frac{a}{k})^2 = k^2+2a+(\frac{a}{k})^2$. With $(\frac{a}{k})^2 \le 1$, we have $\sqrt{k^2+2a} \le k+\frac{a}{k} \le \sqrt{k^2+2a+1}$, so $b = k$ works for $k^2, k^2+2, \cdots, k^2+2k$. Meanwhile for $c \in [0,k+1]$ we have $(k+\frac{a}{k+1})^2 = k^2+\frac{2ak}{k+1}+(\frac{a}{k+1})^2$ $= k^2+2a-\frac{2a}{k+1}+(\frac{a}{k+1})^2$. Notice that $-\frac{2a}{k+1}+(\frac{a}{k+1})^2 = \frac{a^2-2a(k+1)}{(k+1)^2} = \frac{(a-(k+1))^2-(k+1)^2}{(k+1)^2}$ $= \frac{(a-(k+1))^2}{(k+1)^2} - 1$, and with $0 \le a \le k+1$ we have $-1 \le \frac{(a-(k+1))^2}{(k+1)^2} - 1 \le 0$. Therefore $\sqrt{k^2+2a-1} \le k+\frac{a}{k+1} \le \sqrt{k^2+2a+1}$, and this works for $n = k^2+1, k^2+3, \cdots, k^2+2k-1$. Therefore all elements in S_k are covered. As for part (b) we show that there's no fraction $\frac{a}{b}$ (with $b \le k$) lying in the interval $[\sqrt{k^2+1}, \sqrt{k^2+2}]$. Notice that, $k < \sqrt{k^2+1} < \sqrt{k^2+2} < \sqrt{k^2+2k+1} = k+1$. Assume that $\frac{a}{b}$ satisfies this property, then from $\frac{a}{b} > k$ and $b \le k$ we have $(\frac{a}{b})^2 \ge (k+\frac{1}{k})^2 = k^2+2+\frac{1}{k^2} > k^2+2$, contradiction.

A6/IMO 5 The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

Solution. The answer is 2016. Anything fewer doesn't work, because for some i, the factor $x-i$ appears on both sides, so i is itself a root.

It remains to show that 2016 is good to go. We claim that the equation $\prod_{i=1}^{504}(x-(4i-3))(x-(4i)) = \prod_{i=1}^{504}(x-(4i-2))(x-(4i-1))$ has no real solution by showing that the left-hand side is always strictly smaller than the right hand side. We first eliminate the obvious cases where $LHS > 0$ while $RHS < 0$. Observe that whenever $x \in (4i+1, 4i+2)$ for some $i \in [0, 503]$, there are $2i+1$ negative factors (and the rest $1007-2i$ positive) on the left (hence negative) while $2i$ negative factors (and the rest $1008-2i$ positive) on the right (hence positive). Also whenever $x \in (4i-1, 4i)$ for some $i \in [1, 504]$, there are $2i-1$ negative factors (and the rest $1009-2i$ positive) on the left (hence negative) while $2i$ negative factors (and the rest $1008-2i$ positive) on the right (hence positive). Thus in both of the cases the left is less than 0 while the right is more than 0. As for the endpoints $x \in \{1, 2, \cdots, 2016\}$, if $x = 4i$ or $x = 4i+1$ then LHS=0 while RHS has $2i$ negative factors (while the rest positive) hence positive. If $x = 4i-1$ or $x = 4i-2$ then the right is 0 while the left has $2i-1$ negative factors (while the rest positive) hence negative.

If $x > 2016$ then we have LHS and RHS both greater than 0 (since all remaining 2016 factors are positive). Nevertheless, in light of the relation $(x-(4i-2))(x-(4i-1))-(x-(4i-3))(x-(4i)) = (4i-1)(4i-2)-(4i-3)(4i) = 2$ we have $|(x-(4i-2))(x-(4i-1))| > |(x-(4i-3))(x-(4i))|$, and thus $\prod_{i=1}^{504}|(x-(4i-3))(x-(4i))| < \prod_{i=1}^{504}|(x-(4i-2))(x-(4i-1))|$. Since each side is positive, $\prod_{i=1}^{504}(x-(4i-3))(x-(4i)) < \prod_{i=1}^{504}(x-(4i-2))(x-(4i-1))$. The case $x < 1$ is symmetrical and hence analogous.

We are thus left with the trickiest case: $x \in (4i-2, 4i-1)$ for some $i \in [1, 504]$, whereby both sides are negative. The goal is therefore to show that $|LHS| > |RHS|$. We still want to keep in mind that $(x-(4i-2))(x-(4i-1))-(x-(4i-3))(x-(4i)) = (4i-1)(4i-2)-(4i-3)(4i) = 2$, and that both $(x-(4i-2))(x-(4i-1))$ and $(x-(4i-3))(x-(4i))$ are positive for $x \notin [4i-3, 4i]$. Now, let $x \in (4i-2, 4i-1)$ for some $i \in [1, 504]$, then from $(x-(4i-2))(x-(4i-1)) = (x-(4i-1.5))-\frac{1}{4} \geq -\frac{1}{4}$ we get $\frac{|(x-(4i-2))(x-(4i-1))|}{|(x-(4i))(x-(4i-3))|} = \frac{c}{c+2} = 1-\frac{2}{c+2} \leq 1-\frac{2}{2+\frac{1}{4}} = \frac{1}{9}$ where $c = |(x-(4i-2))(x-(4i-1))|$. Next, let's investigate $\frac{|(x-(4j-2))(x-(4j-1))|}{|(x-(4j))(x-(4j-3))|}$ for some $j < i$. We know that $x > 4i+1$, so $(x-(4j-2))(x-(4j-1)) > (4i-4j-1)(4i-4j)$ and therefore $\frac{|(x-(4j-2))(x-(4j-1))|}{|(x-(4j))(x-(4j-3))|}$ $\frac{(x-(4j-2))(x-(4j-1))}{(x-(4j-2))(x-(4j-1))-2} = 1+\frac{2}{(x-(4j-2))(x-(4j-1))-2} < 1+\frac{2}{(4(i-j)-1)(4i-4j)-2} = 1+\frac{1}{2(4(i-j)-1)(i-j)-1}$. It's also not hard to verify that $2(4(i-j)-1)(i-j)-1 < (i-j+1)^2-1$ for $j \leq i-1$, so we in turn have $1+\frac{1}{2(4(i-j)-1)(i-j)-1} < 1+\frac{1}{(i-j+1)^2-1}$ Thus $\frac{\prod_{j=1}^{i-1}(x-(4j-2))(x-(4j-1))}{\prod_{j=1}^{i-1}(x-(4j-3))(x-(4j))} < \prod_{j=1}^{i-1}(1+\frac{2}{(4i-4j-1)(4i-4j)-2})$

$< \prod_{j=-\infty}^{i-1}(1+\frac{2}{(4i-4j-1)(4i-4j)-2}) < \prod_{j=-\infty}^{i-1}(1+\frac{1}{(i-j+1)^2-1}) = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdots$

$= \lim_{x\to\infty} 2\cdot\frac{x-1}{x} = 2$. Likewise, $\frac{\prod_{j=i+1}^{504}(x-(4j-2))(x-(4j-1))}{\prod_{j=i+1}^{504}(x-(4j-3))(x-(4j))} < 2$, (we can drop the modulus since they are all greater than 0. Thus $\frac{\prod_{i=1}^{504}|(x-(4i-2))(x-(4i-1))|}{\prod_{i=1}^{504}|(x-(4i-3))(x-(4i))|}$ $< 2\times\frac{1}{9}\times 2 < \frac{4}{9}$, and we are done. (OMG the long proof...)

2 Combinatorics

C2 Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:
• each cell contains a distinct divisor;
• the sums of all rows are equal; and
• the sums of all columns are equal.

Solution. The answer is $n = 1$, which works with 1 being placed in a 1×1 table. To show that this fails for other n, first prime factorize it into $\prod_{i=1}^{k} p_i^{a_i}$. If r is the number of rows and c is the number of columns then $rc = \prod_{i=1}^{k}(a_i + 1)$, the number of divisors of n. W.l.o.g. $r \geq c$ and therefore $r \geq \sqrt{\prod_{i=1}^{k}(a_i + 1)} = \prod_{i=1}^{k} \sqrt{(a_i + 1)}$. We have also known that the sum of divisors is $\prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}$. Knowing that one of the cells contains n, the sum of each row must be greater than n, (n cannot be the only cell in that row, otherwise all cells would have to contain the same number which is absurd for $n > 1$). This means that the sum of each column is greater than rn, giving the following inequality:

$$\prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1} > rn \geq \prod_{i=1}^{k} \sqrt{(a_i + 1)} p_i^{a_i}$$

or equivalently, $\prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1} > \prod_{i=1}^{k} \sqrt{(a_i + 1)}$

Now for each prime p, we are interested to investigate the ratio $\frac{1}{p-1}(p - \frac{1}{p^{a_i}}) : \sqrt{(a_i + 1)}$. For $p = 2$ we have $(2 - \frac{1}{2^{a_i}}) : \sqrt{(a_i + 1)}$. Notice that for $a_i \geq 3$, $(2 - \frac{1}{2^{a_i}}) < 2$ while $\sqrt{(a_i + 1)} \geq 2$. so the ratio is smaller than 1. for $a_i = 1$, $(2 - \frac{1}{2^{a_i}}) = \frac{3}{2}$ and $\sqrt{(a_i + 1)} = \sqrt{2}$ so the ratio is $\frac{3}{2\sqrt{2}}$, for $a_i = 2$ we have $\frac{7}{4} \div \sqrt{3} = \frac{7}{4\sqrt{3}}$. Knowing that $\frac{3}{2\sqrt{2}} = \sqrt{\frac{9}{8}} > \sqrt{\frac{49}{16}} = \frac{7}{4\sqrt{3}}$ the maximum ratio is $\sqrt{\frac{9}{8}}$. For $p \geq 3$ we have $\frac{1}{p-1}(p - \frac{1}{p^{a_i}})$ decreasing with p with a_i fixed because $\frac{1}{p-1}(p - \frac{1}{p^{a_i}}) = 1 + \frac{1}{p} + \cdots + \frac{1}{p^{a_i}} \leq 1 + \frac{1}{3} + \cdots + \frac{1}{3^{a_i}} = \frac{1}{2}(3 - \frac{1}{3^{a_i}})$. When $a_i = 1$ the ratio is $\frac{4}{3\sqrt{2}} = \sqrt{\frac{8}{9}}$, when $a_i \geq 2$ the ratio is at most $\frac{1}{2}(3 - \frac{1}{3^{a_i}}) \div \sqrt{a_i + 1}$ $\leq \frac{1}{2}(3) \div \sqrt{2 + 1} = \frac{3}{2\sqrt{3}} = \sqrt{\frac{3}{4}} < \sqrt{\frac{8}{9}}$. Thus the maximum possible ratio is $\sqrt{\frac{8}{9}}$.

Summing up, for n consisting at least two distinct prime factors the ratio $\prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1} \div$ $\prod_{i=1}^{k} \sqrt{(a_i + 1)}$ cannot exceed $\sqrt{\frac{9}{8}} \times \sqrt{\frac{8}{9}}^{i-1} \leq 1$, contradicting that $\prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1} > \prod_{i=1}^{k} \sqrt{(a_i + 1)}$. Hence $i = 1$ and from the previous paragraph, $p < 3$ and thus $p = 2$. However, this implies n is a power of 2 and from $a_i \geq 1$, at least two rows must be used (we assumed $r \geq c$). The row containing n must therefore have sum at least $2n$, but for n a power of two the sum of divisors is $2n - 1$, contradiction.

C4/IMO 2 Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

• in each row and each column, one third of the entries are I, one third are M and one third are O; and
• in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Solution. The answer is all n divisible by 9. We start by showing an example for $n = 9$, given below:

I	M	O	M	O	I	O	I	M
M	M	M	O	O	O	I	I	I
I	M	O	M	O	I	O	I	M
O	I	M	I	M	O	M	O	I
I	I	I	M	M	M	O	O	O
O	I	M	I	M	O	M	O	I
M	O	I	O	I	M	I	M	O
O	O	O	I	I	I	M	M	M
M	O	I	O	I	M	I	M	O

For $n = 9k$ for some k we just have to split the grid into k^2 9×9 grids, and fill each one with the letters above. For sake of verification, observe that there are exactly 3 I's, 3 M's and 3 O's in each column or each row of a single 9×9 grid. Also, each diagonal is in the form of either $R_m = \{(i, j) : i + j = m\}$, or $L_m = \{(i, j) : i - j = m\}$, for some m satisfying $1 \le (i, j) \le n$. Now for R_m, the size $|R_m|$ is $m - 1$ for $m \le n + 1$, and $2n + 1 - m$ for $m \ge n + 1$. Notice that 3 divides $|R_m|$ iff $m \equiv 1 \pmod{n}$ (first case), or iff $m \equiv 1 \pmod{n}$ (second case). Thus it is not hard to see that the diagonals are in the form of $(1, m - 1), (2, m - 2), \cdots (m - 1, 1)$ in the first case, and $(m - n, n), (m - n + 1, n - 1), \cdots (n, m - n)$ in the second case. In each of the cases we can group them into groups of three, such that, if we further split each 9×9 grids into 3×3 grids, each group contains three cells along the main diagonal. Nevertheless, from the construction above we see that each main diagonal in the 3×3 grids have one I, one M and one O. Thus this set of diagonal works too. A similar conclusion can be yielded for diagonals in the form of L_m.

To show that $9|n$ is necessary, observe from the first condition that $3|n$. Let $n = 3k$ and let's split the table into k^2 3×3 cells. Notice from the logic (of diagonals characterization) as of above, the center of each 3×3 cell ((i, j) where $i, j \equiv 2 \pmod{3}$) lie on both R_m and L_m with both size divisible by 3; the four corners (((i, j) where $i, j \not\equiv 2 \pmod{3}$) lie on exactly one of the sets satifying the properties; the four sides (((i, j) where exactly one of i and j is congruent to 2 mod 3) lie on none of them. Thus, when we mark the cells in each column, each row, and each diagonal with size divisible by 3, the center cells are marked 4 times, the corners thrice, and the sides twice (as illustrated below).

3	2	3
2	4	2
3	2	3

Let c be the number of M's on the center cells. Considering just the $3i - 1$-th column for $i \in [1, k]$ and the $3j - 1$-th row for $j \in [1, k]$ yields $2k^2$ M's being counted. Each cell on the "side" is being counted once, each cell on the "center" twice, an each cell on the "corner" none. This gives the number of M's on the side as $2k^2 - c$, which follows that there must be $k^2 + c$ M's at the corner. Now let's see what happens as we consider all such markings (all columns, all rows, and all diagonals of size divisible by 3). Observe that for each 3×3 cells we have $3 + 2 + 3 + 2 + 4 + 2 + 3 + 2 + 3 = 24$ markings, so each letter (M, in particular) has $8k^2$ markings. This means $8k^2 = 4c + 2(2k^2 - c) + 3(k^2 + c) = 3c + 7k^2$, or $c = \frac{k^2}{3}$. Hence $3|k^2$, or $3|k$, or $9|n$.

C7/IMO 6 There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Feridun has to choose an endpoint of each segment and place a goose on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each goose will immediately jump forward to the next intersection point on its segment. Geese never change the direction of their jumps. Feridun wishes to place the geese in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Feridun can always fulfill his wish if n is odd.

Solution. (a) Let the segments be $\ell_1, \ell_2, \cdots, \ell_n$. Let P_{ij} be the intersection of line ij. For each segment ℓ_i we aim to investigate the number of points on each side of P_{ij} (other than P_{ij}). Since there are $n-2$ such points (which is odd), one side has even number of points and the other side odd. We call this odd side of ℓ_i w.r.t. point P_{ij}.

Now place the first goose arbitrarily on ℓ_1. For $i \in [2, n]$ we do the following: if the goose corresponding to ℓ_1 is place on the odd side of ℓ_1 w.r.t. P_{1i}, Feridun places one goose at the even side of ℓ_i w.r.t. P_{1i} (and vice versa). We now proceed to the following claim: using the procedure detailed above, for each two distinct integers $i, j \in [1, n]$, the geese corresponding to l_i and l_j lie on different parity of l_i and l_j, respectively, both w.r.t. P_{ij}. Indeed, consider the triangle formed by lines ℓ_1, ℓ_i and ℓ_j. Menelaus' theorem says that any line either intersects none or two of the segments $P_{ij}P_{1i}$, $P_{1j}P_{1i}$, $P_{ij}P_{1j}$. Thus considering lines l_k with $k \notin \{1, i, j\}$ we know that it has even number of total intersection points with segments $P_{ij}P_{1i}$, $P_{1j}P_{1i}$, $P_{ij}P_{1j}$. If this number is even on $P_{1j}P_{1i}$, then each endpoint is on the odd side of l_1 w.r.t. one of P_{1j} and P_{1i}, and even on the other. Thus according of our choice of placing the geese , either one goose come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}, or vice versa. The intersection with $P_{ij}P_{1i}$ and $P_{ij}P_{1j}$ will be both odd or both even. If it's both odd and in the first case (one goose come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}), then the goose corresponding to i come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}, which works for this pair of (i, j). The other three subcases can be treated equally. If this number is odd on $P_{ij}P_{1i}$, then each endpoint is on the odd side of l_i w.r.t. both P_{1i} and P_{1j}, or vice versa (both even). According to our choice again, both geese come from the odd side of l_i w.r.t. P_{1i} and the and of l_j w.r.t. P_{1j}, or both from the even side of their respective lines. The intersection with $P_{ij}P_{1i}$ and $P_{ij}P_{1j}$ will be one odd and one even, for the same endpoint w.r.t the lines l_i and l_j, exactly one of them will change sign when switching from P_{1i} to P_{ij} and from P_{1j} to P_{ij}. Again this (i, j) works.

Finally, to see why the geese won't intersect at the same time, observe that if this happens for some of (i, j), then the geese must have encountered the same number of points before. This implies that they have to come both from the odd side or the even side of the line, contradiction.

3 Geometry

G1/IMO 1 Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Solution. The fact that $\angle CBF = 90°$ and M being the midpoint of CF should very well suggest us to draw the circumcircle of $\triangle BCF$. As it turns out, D and X seems to lie on this circle (and that's almost everything we need). Now $\angle DCF = \angle DCA =$

$\angle DAC = \angle BAF = \angle ABF = 90^\circ - \frac{1}{2}\angle BFC$. $DC = DA = \frac{CA}{2\cos\angle DCA} = \frac{CF-AF}{2\cos(90^\circ - \frac{1}{2}\angle BFC)}$ $= \frac{CF-BF}{2\sin\frac{1}{2}\angle BFC} = \frac{CF-CF\cos\angle BFC}{2\sin\frac{1}{2}\angle BFC} = CF \cdot \frac{1-(1-2\sin^2\frac{1}{2}\angle BFC)}{2\sin\frac{1}{2}\angle BFC} = CF\cdot\sin\frac{1}{2}\angle BFC = CF\cdot\cos(90^\circ - \frac{1}{2}\angle BFC) = CF\cdot\cos\angle DCF$. If D' is on ray CD satisfying $\angle CD'F = 90^\circ$ we have $CD' = CF\cos\angle D'CF = CF\cos\angle DCF = CD$, so $D = D'$ and D lies on the circumcircle of BCF. Moreover, $\angle DFC = 90^\circ - \angle DCF = \frac{1}{2}\angle BFC = \angle BFD$, so BD and DC subtend the same angle and $BD = DC$.

Now $EA = ED$ and $\angle CAD = \angle EAD$ so $\angle EDA = \angle EAD = \angle CAD$, so $ED \parallel CM$. With $EX \parallel AM$ we have E, X, D collinear, too. Moreover, $DE = \frac{DA}{2\cos\angle EDA} = \frac{DA}{2\cos\angle CAD}$ $= \frac{DC}{2\cos\angle DCA} = \frac{CF\cos\angle DCF}{2\cos\angle DCF} = \frac{CF}{2} = CM$, so $CMED$ is a parallelogram. Thus $\angle DEM = \angle DCM = \angle DCA = \angle CAD$, meaning that $DEMA$ is an isoceles trapezoid (so $\angle EAM = \angle EDM$). With $AMXE$ parallelogram we also have $\angle DXM = \angle EXM = angle EAM = \angle EDM = \angle XDM$, so $MX = MD$ and X lies on the circumcircle of BCF, too. With $DX \parallel CF$ we conclude that $DXFC$ is an isoceles trapezoid so $CD = FX$, from previous identity we have $BD = DC$ so $FX = BD$.

Finally, we already had F, E, X collinear and $\angle DBA = \angle DBC + \angle CBA = \angle DFC + (90^\circ - \angle FBA) = \angle DFC + (\frac{1}{2}\angle BFC) = \angle DFC + (\angle DFC) = \angle DMC$, so B lies on the circle containing D, E, A, M too. This means $\angle BED = \angle BMD$, and since BD and FX subtend the same angle on circle BCF we have $\angle BMD = \angle FMX = \angle FEX$ (the last equality follows from that $EXFM$ is isoceles trapezoid, hence cyclic). Therefore, B, E, F are in fact collinear, and E is the intersection of DX and BF. Hence ME is the common perpendicular bisector of segments BX and DF (since $BXFD$ is an isoceles trapezoid), and the intersection of BD and FX will lie on this perpendicular bisector too.

G2 Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Solution. W.L.O.G. let $AB < AC$. First, well-known spiral similarity property should dictate the similarity of triangles BXF an CXE, so $\frac{CX}{CE} = \frac{BX}{BF}$. Also, let's also invoke an identity for triangles (feel free to verify it; I'm not gonna do this):

$$\frac{BX}{XC}\cdot\frac{\sin\angle BXD}{\sin\angle CXD} = \frac{BD}{DC}.$$

Denoting N_1 as the other intersection of XD and Γ gives $\frac{\sin\angle BXD}{\sin\angle CXD} = \frac{BN_1}{CN_1}$. Similarly we have $\frac{AB}{AC}\cdot\frac{\sin\angle ABM}{\sin\angle ACM} = \frac{BM}{CM} = 1$. ALso let N_2 as the other intersection of AM and Γ and we have $\frac{\sin\angle ABM}{\sin\angle ACM} = \frac{BN_2}{CN_2}$. Therefore all we need is $\frac{\sin\angle ABM}{\sin\angle ACM} = \frac{\sin\angle BXD}{\sin\angle CXD}$, and it's not hard to see that $\frac{\sin\angle ABM}{\sin\angle ACM} = \frac{AC}{AB}$, so we are left with proving the fact $\frac{BF}{EC}\cdot\frac{AC}{AB} = \frac{BD}{DC}$.

Now, $\frac{BD}{DC} = \frac{\tan\frac{1}{2}\angle C}{\tan\frac{1}{2}\angle B}$, $\frac{AC}{AB} = \frac{\sin\angle B}{\sin\angle C} = \frac{2\sin\frac{1}{2}\angle B\cos\frac{1}{2}\angle B}{2\sin\frac{1}{2}\angle C\cos\frac{1}{2}\angle C}$. Also $IE = IF$, and by angle chasing we have $\angle FIB = \angle ICE = \frac{1}{2}\angle C$, $\angle EIC = \angle IBF = \frac{1}{2}\angle B$. Therefore BIF and ICE similar, yielding $\frac{BF}{EC} = (\frac{BF}{FI})^2 = (\frac{\sin\frac{1}{2}\angle C}{\sin\frac{1}{2}\angle B})^2$, now it's no longer difficult to prove that $(\frac{\sin\frac{1}{2}\angle C}{\sin\frac{1}{2}\angle B})^2\cdot\frac{2\sin\frac{1}{2}\angle B\cos\frac{1}{2}\angle B}{2\sin\frac{1}{2}\angle C\cos\frac{1}{2}\angle C} = \frac{\tan\frac{1}{2}\angle C}{\tan\frac{1}{2}\angle B}$.

G6 Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Solution. Let ω_1 be the circumcircle of ABC and ω_2 the circumcircle of ADC, then these two circles are symmetric w.r.t. AC. Also notice that BP passes through M_1, the midpoint of arc AC of ω_1 not containing B, and DP passes through M_2, the midpoint of arc AC of ω_2 not containing D.

We first start with a preliminary observation: X lies on ω_2 and Y lies on ω_1. W.L.O.G. for this section we assume that $AB \leq AC$. Indeed, let X' be on BM satisfying $MX' \cdot MB = MA^2 = MC^2$. Then $\angle X'AC = \angle MBA$ and $\angle X'CA = \angle MBC$. Thus $\angle ADC = \angle ABC = \angle MBA + \angle MBC = \angle X'AC + \angle X'CA = \pi - \angle AX'C$, so X' lie on ω_2. In addition, let BM intersect ω_1 again at X'', then X' and X'' are symmetrical w.r.t. AC. Combining with the fact that M_1 and M_2 are also symmetrical w.r.t. AC (being the midpoint of arc) we have $X'M_2 = X''M_1$. Knowing that the two circles have the same radius further allows us to assert $\angle X'BP = \angle X''BM_1 = \angle X'DM_2 = \angle X'DP$, showing that D, B, P, X' cyclic hence $X' = X$. Similarly, Y lies on ω_1.

Next, let N_1 be diametrically opposite M_1 w.r.t. ω_1 and define similarly for N_2. We claim that XE passes through N_2 by claiming that XE is the internal angle bisector of $\angle AXC$. Indeed, by angle bisector theorem we have $\frac{AE}{EC} = \frac{AB}{BC}$. Invoking our X'' from the previous section (i.e. the other intersection of BM and ω_1) gives $AXCX''$ parallelogram. Now invoking a little bit more trigonometric bashing we have $1 = \frac{AM}{CM} = \frac{AB}{BC} \cdot \frac{\sin\angle ABM}{\sin\angle CBM} = \frac{AB}{BC} \cdot \frac{AX''}{CX''} = \frac{AB}{BC} \cdot \frac{CX}{AX}$, so $\frac{AX}{CX} = \frac{AB}{BC} = \frac{AE}{EC}$, and the conclusion follows by the angle bisector theorem. Analogously, YF passes through N_1.

Finally, considering triangle PEF, and letting the perpendicular from P to reach AC at P_1 we have (considering signed length) $\frac{EP_1}{FP_1} = \frac{\cot\angle FEP}{\cot\angle EFP}$. Similarly if letting perpendcular from Q to reach AC at Q_1 we have $\frac{EQ_1}{FQ_1} = \frac{\cot\angle FEQ}{\cot\angle EFQ}$. Now $\cot\angle FEP = \cot\angle MEM_1 = \frac{MM_1}{EM}$, $\cot\angle EFP = \cot\angle MFM_2 = \frac{MM_2}{FM}$. Considering $MM_2 = MM_1$ we have $\frac{\cot\angle FEP}{\cot\angle EFP} = \frac{FM}{EM}$. Analogously, $\cot\angle FEQ = \cot\angle FEQ = \cot\angle MEN_2 = \frac{MN_2}{EM}$, and $\cot\angle EFQ = \cot\angle N_1FM = \frac{MN_1}{FM}$. Therefore we have $\frac{\cot\angle FEQ}{\cot\angle EFQ} = \frac{FM}{EM}$ since again it is not hard to verify that $MN_2 = MN_1$. (For signed convention we can say that $ME < 0$ if it's nearer to A than B, and > 0 otherwise). Therefore, $\frac{EP_1}{FP_1} = \frac{EQ_1}{FQ_1}$, so $P_1 \equiv Q_1$ and the two perpendicular lines coincide.

4 Number Theory

N1 For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Solution. The answer is the constant polynomial $P(x) = c$ where $c \in \{1, 2, \cdots, 9\}$, or the identity polynomial $P(x) = x$. In the first case we have $S(P(n)) = S(c) = c = P(\text{anything}) = P(S(n))$, and in the second case $S(P(n)) = S(n) = P(S(n))$.

Now let $P(x) = \sum_{i=0}^{k} a_i x^i$, then for sufficiently large n (in particular, $10^n > max\{a_i(9^i) : i \in [0,k]\}$) and for each $c \in \{1, 2, \cdots, 9\}$ we have $P(c \cdot 10^n) = \sum_{i=0}^{k} a_i(c^i)(10^{ni})$. Since $a_i(c^i)(10^{ni}) < 10^{(n+1)i}$ (because $a_i(c^i) < 10^n$ by our choice of n), the number $P(c \cdot 10^n)$ are in the form of $(a_k c^k)(0 \cdots 0)(a_{k-1}c^{k-1})(0 \cdots 0) \cdots (0 \cdots 0)(a_0 c^0)$ when laid in decimal form. Therefore $S(P(c \cdot 10^n)) = \sum_{i=0}^{k} S(a_i(c^i))$, and $P(S(c \cdot 10^n)) = P(c) = \sum_{i=0}^{k} a_i(c^i)$. Knowing that $S(x) \leq x$ with equality holds if and only if $0 \leq x \leq 9$ (indeed, if $k = \sum_{i=0}^{k} b_i(10^i)$ then

$S(k) = \sum_{i=0}^{k} b_i$, so $k - S(k) = \sum_{i=0}^{k} b_i(10^i - 1) \geq 0$, with equality holds iff $b_i = 0$ for $i \geq 1$,) we have $a_i(c^i) \leq 9$ for all $c \in \{0, 1, \cdots 9\}$. This means $k \leq 1$ (if we assume that $a_k > 0$). If $k = 0$ then we get $a_0 \leq 9$, yielding the constant solution. If $k = 1$, then $9a_1 \leq 9$ (when $c = 9$) and $a_1 = 1$, yielding $P(x) = x + c$ for some constant c (and since $c = a_0$ we have $c = a_0 \leq 9$ too). This entails $S(P(n)) = S(n+c)$ and $P(S(n)) = S(n)+c$ for all $n \geq 2016$, and letting $n = 10^d - 1$ we have $S(n) = 9d$, and for $c \geq 1$, $S(n + c) = S(10^k - 1 + c) = c$, which doesn't hold for $d = 5$. Therefore $c = 0$ and we get the identity polynomial.

N3/IMO 4. A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a + 1), P(a + 2), \ldots, P(a + b)\}$$

is fragrant?

Solution. The answer is $b = 6$. Observe that this solution works because the set $\{P(197), P(198), P(199), P(200), P(201), P(202)\}$ has $P(199) \equiv P(202) \equiv P(1) = 3 \equiv 0 \pmod 3$, $P(198) \equiv P(2) = 7 \equiv 0 \equiv 21 = P(4) \equiv P(200) \pmod 7$, $P(197) \equiv P(7) = 57 \equiv 0 \equiv 133 = P(11) \equiv P(201) \pmod{19}$.

First, notice that $P(n) - P(n - 1) = n^2 + n + 1 - (n^2 - n - 1) = 2n$, and knowing that $n^2 + n + 1 \equiv n + n + 1 = 2n + 1 \equiv 1 \pmod 2$, we know that if $p|P(n)$ and $p|2n$ then $p|n$ (since $P(n)$ is relatively prime to 2), and consequently $p|n^2 + n$ and $p|1$, showing that $P(n)$ and $P(n - 1)$ are relatively prime. This means, $b = 2$ fails, and $b = 3$ fails too sine $P(a+1)$ and $P(a+3)$ are both relatively prime to $P(a+2)$. (We will use profusely the fact that $P(a)$ and $P(a + 1)$ cannot have any common prime factor throughout the solution).

Now, for $b = 4$ and $b = 5$ our strategy is to determine an upper bound for $\gcd(P(n), P(n+c))$ for $c = 2, 3$. Observe that $P(n + c) - P(n) = 2cn + c^2 + c = c(2n + c + 1)$. For $c = 2$ this is the same as $2(2n + 3)$. If $p|P(n + 2)$ and $p|P(n)$ then $p|2(2n + 3)$, and therefore $p|2n + 3$ with P being odd at all times. This entails $2n \equiv -3 \pmod p$, and $0 \equiv 4P(n) = 4n^+4n + 1 = (2n)^2 + 2(2n) + 1 \equiv (-3)^2 - 3 + 1 = 7 \pmod 7$. Hence $p = 7$ and $n \equiv 2 \pmod 7$. Now for $b = 4$, knowing that $P(a+2)$ is relatively prime with $P(a+1)$ and $P(a + 3)$ it must have a common prime factor with $P(a + 4)$, and by the previous step this prime factor has to be 7. Similarly $P(a + 1)$ and $P(a + 3)$ must both be divisible by 7. This means $P(a + 1), P(a + 2), P(a + 3), P(a + 4)$ are all divisible by 7 for some a, contradicting that any two neighbouring elements are coprime.

Finally for $b = 5$ we investigate $c = 3$ as in the previous paragraph. Now $3(2n + 3 + 1) = 3(2n + 4) = 3(2)(n + 2)$. If a prime p satisfies $p|P(n)$ and $p|P(n + 3)$ simultaneously then either $p = 3$ or $p|n+2$ (again p must be relatively prime to 2 so this can be easily factored out). In the second case we have $n \equiv= 2 \pmod p$, so $P(n) \equiv P(-2) = 4 - 2 + 1 = 3 \equiv \pmod p$, forcing $p = 3$ (no choice!) Thus viewing the set $\{P(a + 1), \cdots, P(a + 5)\}$ we know that $P(a+3)$ must have a common factor with $P(a+1)$ or $P(a+5)$, and by previous paragraph this common factor has to be 7. Thus neither of $P(a + 2)$ nor $P(a + 4)$ can be divisible by 7, and they cannot have common prime factor (again by previous paragraph). This entails $P(a+1)$ and $P(a+4)$ must have common factor, and by what we established earlier this factor must be 3. Similarly, $P(a + 2)$ and $P(a + 5)$ must both be divisible by 3. However, $P(a + 1)$ and $P(a + 2)$ are both divisible by 3, contradiction.