Solution to IMO 2017 shortlisted problems.

Anzo Teh

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1 Algebra

A1 Let $a_1, a_2, \dots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

Solution. We will actually prove that $(x+a_1)(x+a_2)\cdots(x+a_n) > M(x+1)^k$ for all $x > 0$. Now, dividing by M on both sides (i.e. dividing by $a_1 a_2 \cdots a_n$ on the left hand side) (bearing in mind that M is positive) we get that the desired inequality is equivalent to $(1+\frac{x}{a_1})(1+\frac{x}{a_2})\cdots(1+\frac{x}{a_n}) = \frac{x+a_1}{a_1} \cdot \frac{x+a_2}{a_2} \cdots \frac{x+a_n}{a_n} > (x+1)^k$.

Before we proceed, we prove a key fact: for all $x > 0$ and all i we have $(1+\frac{x}{a_i})^{a_i} \geq 1+x$, with equality happening if and only if $a_i = 1$. Here I will show two proofs to it:

- Expanding the left hand side (thankfully a_i is a positive integer) gives

$$(1+\frac{x}{a_i})^{a_i} = \sum_{j=0}^{a_i} \binom{a_i}{j} x^j = 1 + x + \sum_{j=2}^{a_i} \binom{a_i}{j} x^j$$

  Clearly the last term is positive if $x > 0$ and $a_i > 1$.

- Consider the expression $f(x) = (1+\frac{x}{a_i})^{a_i} - (1+x)$, where $f(0) = 0$ and $f'(x) = (1+\frac{x}{a_i})^{a_i-1} - 1$. This derivative is positive if $x > 0$ and $a_i - 1 > 0$ (i.e. $a_i = 1$), which will follow that $f(x) > 0$ for all $x > 0$ if $a_i > 1$.

Thus we have $(1+\frac{x}{a_i}) \geq (1+x)^{\frac{1}{a_i}}$ for all $x > 0$ with equality iff $a_i = 1$. This means, $(1+\frac{x}{a_1})(1+\frac{x}{a_2})\cdots(1+\frac{x}{a_n}) \geq (x+1)^{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} = (x+1)^k$, with equality iff $a_i = 1$ for all i. This cannot happen, otherwise $M = 1$. So the strict inequality always hold.

2 Combinatorics

C1

3 Geometry

G1

4 Number Theory

N1 (IMO #1) For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \dots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

Answer. All n divisible by 3.
Solution. For the first part we show the that if $3 \nmid a_0$, then $a_i \equiv 2 \pmod 3$ for some i. Notice first that if $3 \nmid a_i$ then $3 \nmid a_{i+1}$ (regardless whether a_i is a perfect square). If a_0 has remainder 2 modulo 3 we are done. Otherwise, $a_0 \equiv 1 \pmod 3$ so $a_0 + 3k$ is a perfect square for some k. Find the minimal such k, and we have $a_k = a_0 + 3k = c^2$ for some c, and $a_{k+1} = c$. If $c \equiv 2 \pmod 3$ we are done. Otherwise, we have $c \geq 4$ and $c - 2 \equiv 2 \pmod 3$ so $(c-2)^2 \equiv 1 \pmod 3$, showing that $a_0 \geq (c-2)^2 + 3$. With $c \geq 4$ we have $(c-2)^2 + 3 > c$, so $a_0 > a_{k+1}$. Letting $0 = b_0$ and $b_1, b_2, \cdots$ be indices such that a_{b_i} is a perfect square yields that $a_{b_0} > a_{b_1} > \cdots$, so this sequence must terminate, meaning that we have $a_i \equiv 2 \pmod 3$ for some i. Now it's easy to prove that this a_i cannot be a perfect square, so for all $j > 0$ we have $a_{i+j} = a_i + 3j$, showing that all numbers appear a finite number of times.

For the case where $3|a_0$ we will do something similar: keep looking for the next square. Again let k be the least index with a_k a perfect square, say, c^2. Then $a_0 \geq (c-3)^2 + 3$ because $3|c$. Now if $c > 3$ then $c = a_{k+1} > a_0$, so again constructing the sequence $b_0, b_1, \cdots$ gives $a_{b_0} > a_{b_1} > \cdots$, hence it must terminate. The only way to terminate is when $c \leq 3$, in which the equality must hold since $a_i > 0$ for all i. Hence the sequence goes $3 \to 6 \to 9 \to 3 \to 6 \to 9$, so each of $3, 6, 9$ appears infinitely many times.

N2 Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=1}^{p-1} a_i 10^i$$

The goal of Eduardo is to make M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

Solution. If $p = 2$ or $p = 5$, Eduardo just have to choose $a_0 = 0$ and the games is his, forever. Hence from now on we assume that $\gcd(p, 10) = 1$. Eduardo first lets $a_{p-1} = 0$. Then he considers $10^j \pmod p$ for $j = 0, \cdots, p-2$, and consider the minimum k such that $10^k = 1 \pmod p$. Now two cases arise:

- If k is even, then it must happen that $10^{k/2} \equiv -1 \pmod p$. Now pair the indices $0, 1, \cdots, p-2$ in the following manner: if $j = ak + b$ with $0 \leq b < k$ then pair j with $j + k/2$ if $b < k/2$, and with $j - k/2$ otherwise. Now notice that if j, ℓ are a pair then 10^j and 10^ℓ are negatives of each other, and these pairs form a partition of the numbers $0, 1, \cdots, p-2$. Now at each turn, Fernando chooses j and a_j, and is j is paired with ℓ then Eduardo chooses ℓ with $a_\ell = a_j$, so that $a_j 10^j + a_\ell 10^\ell \equiv 0 \pmod p$. This will allow $p|M$ in the end.
- Otherwise, let $b_j = 10^j$ for all $0, 1, \cdots, k-1$. Notice that $(p-1)/k$ must be even, so for each j there are an even number of indices ℓ with $10^\ell \equiv b_j \pmod p$. Now for each j and all such $(p-1)/k$ ℓ's, we pair the indices arbitrarily (so that there are $(p-1)/2k$ pairs). Each time when when Fernando chooses j and a_j, and suppose that j is paired with some ℓ, Eduardo chooses $a_\ell = 9 - a_j$, so that the contribution to M mod p is $9b_j$. Therefore, the resulting M has congruence $\sum 9 * (p-1) * b_j/(2k) = 9 * (p-1)/2k \sum b_j$. If $p = 3$ the factor 9 already implies $3|M$. Otherwise, notice that $(p-1)/k \sum b_j = \sum_{i=0}^{p-2} 10^j = \frac{10^{p-1}-1}{10-1}$, which is divisible by p since $p|10^{p-1} - 1$ by Fermat's little theorem, and $p \nmid 9 = 10 - 1$ for $p \neq 3$. Since $(p-1)/k$ is not a multiple of p, $\sum b_j$ is a multiple of p, and so is $9 * (p-1)/2k \sum b_j$ and M.