Solution to IMO 2015 shortlisted problems.

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Algebra

A1. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Solution. Throughout the solution, we note $f_k(x) = \frac{kx}{x^2+(k-1)}$ for all $x > 0$. Observe that our condition is $a_{k+1} \geq f_k(a_k)$.

We first establish this for a special case where a_1 is arbitrary and equality holds for the recursion (that is, $a_{k+1} = f_k(a_k)$). Notice that $a_3 = \frac{2a_2}{a_2^2+1} \leq 1$, and when $k \geq 2$, $x^2 + (k-1) - kx = (x-1)(x-(k-1))$ which is nonnegative for $x \leq 1$. It then follows that in our case, $a_k \leq 1$ for all $k \geq 3$.

We now also claim that $\sum_{i=1}^n a_i = \frac{n-1}{a_n} + a_n$. Indeed, this holds when $n = 1$. Suppose for some $n \geq 1$ it holds that $\sum_{i=1}^n a_i = \frac{n-1}{a_n} + a_n$, then

$$\sum_{i=1}^{n+1} a_i = \frac{n-1}{a_n} + a_n + \frac{na_n}{a_n^2 + (n-1)} = \frac{n-1+a_n^2}{a_n} + \frac{na_n}{a_n^2+(n-1)} = \frac{n}{a_{n+1}} + a_{n+1}$$

as claimed. Finally, for $n = 2$ the sum is $a_2 + \frac{1}{a_2} \geq 2$ for all $a_2 \in \mathbb{R}_{>0}$; for $n > 2$, since $a_n \leq 1$ and $\frac{(n-1)}{a_n} + a_n$ is decreasing in $0 < a_n \leq \sqrt{n-1}$, we do get $\frac{(n-1)}{a_n} + a_n \leq (n-1)+1 = n$, as desired. Notice that this is also generalizable to $a_3, \cdots, a_n \leq 1$, but where $a_{k+1} \geq f_k(a_k)$ instead of equality, since f_k is increasing in $(0, \sqrt{k-1})$ for all $k \geq 2$.

We now relax the equality constraint. To this end, consider the following claim: suppose for some $m \geq 1$, $a_m \geq 1$ while $a_{m+1}, \cdots, a_n \leq 1$. Then $a_m + \cdots + a_n \geq n - m + 1$. To see why this is true, we see that $a_{m+1} \leq 1$ means that $ma_m \leq a_m^2 + (m-1)$, so $a_m \geq m-1$. Next, the function for all $k \geq 2$, f_k is increasing in $x \in (0, \sqrt{k-1})$ (hence increasing in $(0,1)$). This means with $a_{m+1}, \cdots, a_n \leq 1$ it suffices to consider the case where equality holds (i.e. $a_{k+1} = f_k(a_k)$ for all $k \geq m$). In addition, for all $k \geq 2$, $f_k(x) = f_k(\frac{k-1}{x})$.

Thus we can do induction on m. For $m = 1$ we have exactly the simplified equality case that we established. Now suppose that this is true for $m-1$ for some $m \geq 2$, and consider the case for m (that is, $a_m \geq 1, a_{m+1}, \cdots, n \leq 1$). We note that $a_{m+1} = f_m(a_m) = f_m(\frac{m-1}{a_m})$, Now, consider adding a_{m-1} and substituting a_m with $\frac{m-1}{a_m}$ such that $\frac{m-1}{a_m} = f_{m-1}(a_{m-1})$. By induction hypothesis, $a_{m-1} + \frac{m-1}{a_m} + a_{m+1} + \cdots + a_n \geq n - m$. Now that $a_m = \frac{m-1}{f_{m-1}(a_{m-1})} = a_{m-1} + \frac{m-2}{a_{m-1}}$, we have

$$a_m + 1 - (a_{m-1} + f_{m-1}(a_{m-1})) = \frac{m-2}{a_{m-1}} + 1 - f_{m-1}(a_{m-1}) \geq 0$$

since $f_{m-1}(a_{m-1}) = \frac{m-1}{a_m} \leq 1$. Thus the induction step follows.

We now finish the proof in the following manner. Let $k_0 \geq 3$ be the minimum number such that $a_3, \cdots, a_{k_0} \leq 1$; the initial case yields that $a_1 + \cdots + a_{k_0} \geq k_0$. For a_i's after k_0, we partition them into segments of consecutive numbers where the first element is ≥ 1, and the rest ≤ 1. The second part of our analysis shows that each segment has average ≥ 1, so the conclusion follows.

A2. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Answer. $f \equiv x + 1$, or $x \equiv -1$.

Solution. We first see that these two functions satisfy the equation. The first one gives both sides as $x - y$; the second case just gives -1. It now remains to show that these are the only such functions.

Plugging $y = f(x)$ yields $f(x - f(f(x))) = -1$, meaning that -1 is a value of f. Define y_0 as an integer with $f(y_0) = -1$. Then substituting $y := y_0$ we have

$$f(x + 1) = f(f(x)). \tag{1}$$

For the case where f is injective, we must have $f(x) = x + 1$.

Now, let f not be injective, and let $f(x) = f(z)$ for some $x, z \in \mathbb{Z}$. Then (1) implies $f(x+1) = f(f(x)) = f(f(z)) = f(z+1)$. This means the function is eventually periodic, i.e. there exists N and $d > 0$ such that $f(x) = f(x + d)$ for all $x \geq N$. Now define

$$M = \arg\max\{f(N), \cdots, f(N + d - 1)\} \qquad m = \arg\min\{f(N), \cdots, f(N + d - 1)\}$$

i.e. the arguments that give maximum and minimum numbers in the period, respectively. Now for any positive integer k, plugging $x = M - 1 + kd$ yields

$$f((M-1+kd)-f(y)) = f(f(M-1+kd))-f(y)-1 = f(M+kd)-f(y)-1 = f(M)-f(y)-1$$

Fixing y, for k sufficiently large we have $(M-1+kd)-f(y) \geq N$ so $f((M-1+kd)-f(y)) \leq f(M)$ and therefore $f(y) \leq -1$. Similarly we may substutute $x = m - 1 + kd$ to get

$$f((m - 1 + kd) - f(y)) = f(m) - f(y) - 1$$

and taking k sufficiently large, $f(y) \leq -1$. Combining both yields $f \equiv -1$, a constant.

A3. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n) x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, \cdots, 2n$.

Answer. $n(n - 1)$.

Solution. Let $x = x_1$ and fix other indices, we know that this is a linear function in x (with constant term $\sum\limits_{2 \leq i < j \leq 2n} (j - i - n) x_i x_j$ and coefficient of x as $\sum\limits_{2 \leq i < 2n} (i - 1 - n) x_i$ and therefore the maximum value can be attained when $x = 1$ or -1. Extending this logic to other x_i's, we may assume that $\mid x_i \mid = 1, \forall i \in [1, 2n]$.

Now let p, q be the number of 1's and -1's in the sequence respectively, with $p + q = 2n$. W.L.O.G. let $p \le q$ (replacing x_i by $-x_i$ for all i yields the same sum). We further name $a_1 < a_2 < \cdots < a_p$ and $b_1 < b_2 < \cdots < b_q$ such that $x_{a_i} = 1$ and $x_{b_i} = -1$. Now, the required sum becomes

$$
\begin{aligned}
&\sum_{1\le i<j\le p} (a_j - a_i - n) + \sum_{1\le i<j\le q} (b_j - b_i - n) - \sum(|a_j - b_i| - n) \\
&= \sum_{1\le i<j\le p} (a_j - a_i) + \sum_{1\le i<j\le q} (b_j - b_i) - \sum(|a_j - b_i|) - n\left(\binom{p}{2} + \binom{q}{2} - pq\right) \\
&\overset{(a)}{=} 2\left(\sum_{1\le i<j\le p} (a_j - a_i) + \sum_{1\le i<j\le q} (b_j - b_i)\right) - \sum_{1\le r<s\le 2n} (s-r) - n\left(\binom{2n}{2} - 2pq\right) \\
&\overset{(b)}{=} 2\left(\sum_{i=1}^{p} (2i - p - 1)a_i + \sum_{j=1}^{q} (2j - q - 1)b_j\right) - \sum_{1\le r<s\le 2n} (s-r) - n\left(\binom{2n}{2} - 2pq\right)
\end{aligned}
$$

where (a) is due to that

$$\binom{2n}{2} = \binom{p+q}{2} = \binom{p}{2} + \binom{q}{2} + pq$$

$$\sum_{1\le r<s\le 2n} (s-r) = \sum_{1\le i<j\le p} (a_j - a_i) + \sum_{1\le i<j\le q} (b_j - b_i) + \sum(|a_j - b_i|)$$

and (b) is telescoping sum. Thus our quantity of interest now becomes

$$\sum_{i=1}^{p} (2i - p - 1)a_i + \sum_{j=1}^{q} (2j - q - 1)b_j + pqn$$

Given that $p \equiv q \pmod 2$, and $\{a_i\} \cup \{b_i\} = [1, 2n]$, we see that the coefficient multiset is $\{k : -(q-1) \le k \le (q-1), k \not\equiv q \pmod 2\}$ and such k appears two times if $|k| \le p - 1$, and one time otherwise. It then follows that for fixed p, q, our maximum is attained when the terms $1, 2, \cdots, 2n$ have coefficient $-(q-1), -(q-3), \cdots - (p+1), -(p-1), -(p-1), \cdots + (p-1), +(p-1), +(p+1), \cdots + (q-1)$ in that order.

Now we will prove that the maximum is achieved iff $\{p, q\} = \{n, n\}$ or $\{n-1, n+1\}$. For $p = q = n$ we have $pqn = n^3$ and the coefficient (or weightage) of $1, 2, \cdots 2n$ as $-(n-1), -(n-1), -(n-3), -(n-3), \cdots + (n-1), +(n-1)$ while for the second case we have pqn as $n(n-1)(n+1)$ (i.e. n less than the first case) and the coefficients as $-n, -(n-2), -(n-2), -(n-4), -(n-4), \cdots (n-2), (n-2), n$. The difference of coefficient between this and the first case will therefore be $-1, +1, -1, +1, \cdots - 1, +1$, from which we know that the resulting difference between the second and the first case is $-1 + 2 - 3 + 4 \cdots - (2n-1) + 2n = n$. (i.e. n more than the first case). Hence, the sum in these two cases are equal.

For other $p < n-1$, we split into two cases. For $p \equiv n \pmod 2$, subtracting the coefficients $-(q-1), -(q-3), \cdots - (p+1), -(p-1), -(p-1), \cdots + (p-1), +(p-1), +(p+1), \cdots + (q-1)$ by $-(n-1), -(n-1), -(n-3), -(n-3), \cdots + (n-1), +(n-1)$ yields $-(q-n), -(q-n-2), -(q-n-2), \cdots 0, 0, 0, \cdots + (q-n-2), +(q-n-2), +(q-n)$ so after multplying by $1, 2, \cdots 2n$ and considering the difference $n(pq - n^2) = -n(q - n)^2$ (since $p + q = 2n$) and the difference is now $(q - n)(2n - 1) + (q - n - 2)(2n - 3) + (q - n - 2)(2n - 5) + \cdots - n(q - n)^2 < 2n((q - n) + 2(q - n - 2) + 2(q - n - 4) + \cdots + 2(2)) - n(q - n)^2 = 2n(q - n + 4 \cdot \frac{q-n-1}{2} \cdot \frac{q-n}{2}) - n(q - n)^2 = 0$. If $p \equiv n - 1 \mod 2$ then use the same weightage to subtract $-n, -(n-2), -(n-2), -(n-4), -(n-4), \cdots (n-2), (n-2), n$ we get $-(q-1-n), -(q-1-n), -(q-3-n), -(q-3-n), \cdots, 0, 0, 0, \cdots + (q-1-n), +(q-1-n)$. The difference is $(q - 1 - n)(2n - 1) + (q - 1 - n)(2n - 3) + \cdots - n((q - n)^2 - 1) < 2(2n-1)(2+4+\cdots+(q-1-n)) - n((q-n)^2-1) = 2(2n-1)\frac{q-n-1}{2} \cdot \frac{q-n+1}{2} - n((q-n)^2-1) < n(q - n - 1)(q + n - 1) - n((q - n)^2 - 1) = 0$. Summing up, we know that for other p, the resulting sum is smaller so we can safely assume that $p = q = n$. If we let $x_1 = x_3 = \cdots = x_{2n-1} = 1$ and $x_2 = x_4 \cdots = x_{2n} = -1$ then obviously, $a_i = 2i - 1$ and $b_i = 2i$, so they have the same weightage $2i - n - 1$. This means the equality case is attained here.

To compute the sum, notice that for each $k \in [1, 2n - 1]$ there are exactly $2n - k$ ordered pairs (r, s) with $s - r = k$ and $1 \leq r, s \leq 2n$. In addition, $r + s \equiv k \pmod 2$ for those pairs satifsying this property, hence $x_r x_s = (-1)^{r+s} = -1^k$. This means our desired maximum sum now becomes $\sum_{k=1}^{2n-1} (-1)^k (k - n)(2n - k)$. Ignoring the case where $k = n$ (which gives $(-1)^k (k - n)(2n - k) = 0$ anyway) and pairing k with $2n - k$ $(1 \leq k \leq 2n - 1)$ gives $(-1)^k(k-n)(2n-k) + (-1)^{2n-k}(n-k)k = (-1)^k(n-k)(2k-2n) = 2(-1)^{k+1}(n-k)^2$. Thus our sum now becomes $2((n-1)^2 - (n-2)^2 + \cdots + (-1)^{n-1}(1^2)) = 2((n-1) + (n-2) + \cdots + 1) = n(n-1)$. An example where this is attained is when $x_1, \cdots, x_{2n}$ are alternating 1's and -1's.

A4. (IMO 5) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Answer. $f(x) = x$ or $f(x) = 2 - x$.

Solution. We first verify that these two functions work. The first one yields both sides as $2x + y + xy$; the second one $2 + y - xy$.

We now show that these are all the functions. Plugging $x = y = 0$ into the original equation gives $f(f(0)) = 0$, so if $f(0) = c$ then $f(c) = 0$. Plugging $x = c, y = 0$ yields $f(f(c)) + f(0) = f(c) + cf(0)$, or $c + c = c^2$, yielding $f(0) \in \{0, 2\}$.

When $y = 1$, $f(x + f(x + 1)) = x + f(x + 1)$, i.e. $x + f(x + 1)$ is a fixed point of f (*). Let $x = 0$ and y be a fixed point, then

$$y + f(0) = f(f(y)) + f(0) = f(y) + yf(0) = y + yf(0) \tag{2}$$

In other words, $(y - 1)f(0) = 0$. If $f(0) = 2$, then $y = 1$, i.e. 1 is the only fixed point of f. This means $x + f(x + 1) \equiv 1$, yielding $f(x) \equiv 2 - x$.

Now consider the case $f(0) = 0$, for which we show that all x is a fixed point of f. First, notice that with $x = -1, y = 1$ gives

$$f(-1) + f(-1) = -1 + f(-1)$$

so -1 is also a fixed point. Reversing the roles $(x = 1, y = -1)$ gives $f(1) + f(-1) = 1 - f(1)$, so $f(1) = 1$. In addition, plugging $y = 0$ gives

$$f(x + f(x)) = x + f(x) \tag{3}$$

so $x + f(x)$ is also a fixed point for all x.

We show that $x + n + f(x)$ is a fixed point of f for all $n \geq 1$. Indeed, if z and $z + 1$ are both fixed point of f, then plugging $x = 1$ and $y = z$, we have

$$f(z + 2) + z = f(1 + f(z + 1)) + f(z) = 1 + f(z + 1) + zf(1) = 2z + 2$$

so $z + 2$ is also a fixed point. Since $x + f(x), x - 1 + f(x)$ are both fixed points of f due to (*) (via change of variables) and (3), the claim now follows by induction on n. In particular, $n = 1$ yields $x + 1 + f(x)$, or $x + f(x - 1)$ fixed points ($\forall x \in \mathbb{R}$), so plugging $y = -1$ yields $f(x + f(x - 1)) + f(-x) = x + f(x - 1) - f(x)$, or $f(x) = -f(x)$. (i.e. f is an odd function).

Finally, with f being odd, replacing x by $-x$ and y by $-y$ in the equation yields

$$-f(x + f(x + y)) + f(xy) = -x - f(x + y) + yf(x)$$

which means we can now compare this with the original equation to get

$$f(x + f(x + y)) = x + f(x + y) \qquad f(xy) = yf(x) \tag{4}$$

In particular, $f(y) = yf(1) = y$ when $x = 1$, so $f(x) \equiv x$.

Combinatorics

C1. In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Solution. We induct on the n, the number of towns. Let T_i, L_i, R_i be town i (counted from the left, so T_1 is leftmost and T_n is rightmost), left bulldozer of town i and right bulldozer of town i, respectively, $\forall i \in [1, n]$. For $n = 1$ there is nothing to prove; for $n = 2$, T_2 is swept $\Leftrightarrow R_1 > L_2 \Leftrightarrow T_1$ isn't swept.

By induction hypothesis we can assume that there is a unique town T_s that is not swept when there are n towns, and add a town T_{n+1} on the right. Obviously, there are no way for R_i and L_j to reach T_s, $\forall i \in [1, s-1], \forall j \in [s+1, n]$. We show that exactly one of T_s and T_{n+1} will be swept. Indeed, denote M such that $R_M = \max\{R_i \mid i \in [s, n]\}$, meaning that R_M can sweep $T_k, \forall k \in [M + 1, n]$. If $R_M > L_{n+1}$, then after sweeping T_n, R_M can sweep T_{n+1} but L_{n+1} can't sweep T_M, so it can't sweep T_s (since $s \leq M$). By our hypothesis no other bulldozer can sweep T_s so here, T_{n+1} is swept but not T_s. Conversely, if $L_{n+1} > R_M$, then L_{n+1} can sweep $T_n, T_{n-1}, \cdots, T_M, \cdots, T_s$. No single town $T_i, \forall i \in [s, n]$ can sweep T_{n+1} and by the second sentence of this paragraph, no town $T_i, \forall i \in [1, s-1]$ can sweep T_{n+1}. Hene T_s can be swept, but not T_{n+1}.

C2. (IMO 1) We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

Answer for (b). All odd n.

Solution. We first construct a balanced centre-free set when n is odd. Taking $A_1A_2\cdots A_n$ a regular n-gon yields that the perpendicular bisector of A_iA_j passes through $A_{\frac{i+j}{2}}$ for $i+j$ even, or $A_{\frac{i+j+n}{2}}$ for $i+j$ odd (indices taken modulo n). This configuration is therefore balanced, and since P with $PA_i = PA_j = PA_k$ implies P is the centre of the polygon (i.e. $P \neq A_i$, $\forall \in [1,n]$), this configuration is also centre-free.

Now, for even n, consider a circle with centre O and vertices A, B, C such that A, B, C lie on this circle in that order and that triangles AOB and BOC are both equlateral. Obviously, the perpendicular bisector of any point on the circle pass through O, and the perpendicular bisectors of OA, OB, OC pass through B, C, B, respectively. Hence $OABC$ is balanced. Now we add two points X, Y on the circumfence at a time, such that X, Y do not overlap the previous points and XOY equilateral. O is equidistant from any point on the circle, so we only need to consider lines XO, YO. However, their perpendicular bisectors pass through Y, X respectively (so the configuration is balanced too).

Finally, if a configuration of n points $(A_1, A_2, \cdots, A_n)$ is centre-free, then we denote $f(i,j)$ $(i \neq j)$ such that $A_{f(i,j)}$ is equidistant from A_i and A_j (if there are more than one such point we take this f arbitrarily). Now $f(i,j) \notin \{i,j\}$ and $f(i,j) \neq f(i,k)$ if $j \neq k$ (otherwise, $A_{f(i,j)}$ is equidistant from A_i, A_j, A_k, contradiction. Therefore, for fixed i, $\{f(i,j)|j \in [1,n]\backslash\{i\}\} = \{j|j \in [1,n]\backslash\{i\}\}$. Summing this for all i entails that we counted each $f(i,j)$ twice, and for each number k, k is in $n-1$ of the sets $\{j|j \in [1,n]\backslash\{i\}\}$, $i = 1, 2, \cdots n$. Therefore, for each $k \in [1,n]$ there are $\frac{n-1}{2}$ unordered paris of i, j for which $f(i,j)$=A_k. Therefore n is odd.

C3. For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

Answer. $n = 3024$.

Solution. The largest element in A_1 cannot be greater than the smallest element in A_2, so if we sort the numbers in $a_1 < a_2 < \cdots < a_n$ then A_1=$\{a_1, a_2, \cdots, a_i\}$ for some $i \in [1,n]$ and $A_2 = \{a_{i+1}, a_{i+2}, \cdots, a_n\}$. Also, any element in A_1 must divide any element in A_2.

Denote k-partition as the partition of the set in A_1 and A_2 s.t. $A_1 = \{a_1, a_2, \cdots, a_k\}$ and $A_2 = \{a_{k+1}, a_{k+2}, \cdots, a_n\}$. Also denote the LCM and GCD of this k-partition as the LCM of A_1 and GCD of A_2, respectively. We show the following claim:

- for each k with $2 \leq k \leq n-2$ we cannot have $k-1$-, k-, $k+1$-partition to be all good.

- We cannot have both 1− and 2− partition to be good; similarly, $n-2$- and $n-1$-partition cannot be both good.

For the first claim, suppose that $k-1$-partition and k-partition are both good. Then $a_i \mid a_j$ for all $i = 1, \cdots, k$, and $j = k, \cdots, n$. Thus the GCD and LCM of both the $k-1$-partition and k-partition are both a_k. In particular, $\gcd(\{a_{k+1}, \cdots, a_n\}) = a_k < a_{k+1}$ so $a_{k+1} \nmid a_i$ for some $i > k+1$. Now consider the $k+1$-partition into $A_1 = \{a_1, \cdots, a_{k+1}\}$ and $A_2 = \{a_{k+2}, \cdots, a_n\}$. Then LCM of A_1 is a_{k+1}. But since $a_{k+1} \nmid a_i$ for some $i > k+1$, the GCD of A_2 cannot be a_{k+1}, a contradiction.

For the second claim it suffices to show that in our example above $k \neq 2$, and $k \neq n-1$. A 1-partition that's good means $\gcd\{a_2, \cdots, a_n\} = a_1$, so $a_2 \nmid a_k$ for some $k > 2$. But $\mathrm{lcm}\{a_1, a_2\} = a_2 \neq \gcd\{a_3, \cdots, a_n\}$ in this case. Similarly, an $n-1$-partition that's good means $\mathrm{lcm}\{a_1, \cdots, a_{n-1}\} = a_n$, so $a_i \nmid a_{n-1}$ for some $i < n-1$. But then $\mathrm{lcm}\{a_1, \cdots, a_{n-2}\} \neq a_{n-1} = \gcd\{a_{n-1}, a_n\}$.

Finally, denote $1 \leq x_1, x_2, \cdots x_m \leq n-1$ be all indices such that x_i-partition is not good. From above, $x_1 \leq 2$ and $x_m \geq n-2$, while $x_{i+1} - x_i \leq 3$, $\forall i \in [1, m]$. Notice, also, that 2015=$n-1-m$. This means $n-2 \leq x_m \leq x_1 + 3(m-1) \leq 2 + 3(m-1)$, so $n \leq 3m+1 = 3(n-1-2015)+1 = 3n-6047$. We have $2n \geq 6047$, or $n \geq 3024$ since $n \in \mathbb{N}$. This can be achieved by taking $a_{3i+1} = 2^{i+1} \cdot 3^i$, $a_{3i+2} = 2^i \cdot 3^{i+1}$, $a_{3i+3} = 2^{i+1} \cdot 3^{i+1}$, $\forall i \in [0, 1007]$. where a $3i+2$-partition will give both LCM and GCD of $2^{i+1} \cdot 3^{i+1}$ ($\forall i \in [0, 1007]$) and a $3i+3$-partition will give both LCM and GCD of $2^{i+1} \cdot 3^{i+1}$.

C5. (IMO 6) The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

(i) $1 \leq a_j \leq 2015$ for all $j \geq 1$,

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

for all integers m and n such that $n > m \geq N$.

Solution. Denote b_n as $a_n + n$, and we know that b_i is distinct for every b. Let $x_1 < x_2 < \cdots < x_k$ be first k numbers such that $b_i \neq a_j$, $\forall i \geq 1$, $\forall 1 \leq j \leq k$. It follows that there are exactly $x_k - k$ such i's such that $b_i < x_k$ (1). If $k > 2015$, then from the fact $n < b_n \leq n + 2015$ we know that for $i \leq x_k - k$, $b_i \leq i + 2015 \leq x_k - k + 2015 < x_k$. Therefore, $b_1, b_2, \cdots, b_{x_k - k} \leq x_k$ so from point 1, $\{b_1, b_2, \cdots b_{x_k-k}\} = \mathbb{N} \cap [1, x_k] \backslash \{x_i \mid 1 \leq i \leq k\}$. Now $b_{x_k-k+1} \geq x_k + 1$, and $a_{x_k-k+1} = b_{x_k-k+1} - (x_k - k + 1) \geq x_k + 1 - (x_k - k + 1) = k > 2015$, contradiction. Therefore k is finite (at least 1, since $b_n > 1$ for all n.

We show that $b = k$ and $N = x_k$ satisfies the conclusion we want to prove. Now, we can further assert that for any $N > x_k$, there are exactly $N - k$ indices i such that $b_i \leq N$. (2) Denote by S_n the sum $\sum_{i=1}^n b_i$, the sum of elements in the set $B_n = \{b_i \mid 1 \leq k \leq n\}$. Now, $\sum_{j=m+1}^n (a_j - b)$ is precisely $S_n - S_m - \sum_{j=m+1}^n j - (n-m)b$. We consider S_n, in general, and prove that $T_n \leq S_n \leq T_n + (k-1)(2015-k)$, where T_n is taken as $\sum_{i=1}^{n+k} i - \sum_{j=1}^k x_k$, the elements in the set $C_n = \{1, 2, \cdots, n+k\} \backslash \{x_i \mid 1 \leq i \leq k\}$. This is the n smallest possible sequence that appears in set $\{b_1, b_2, \cdots\}$ so $S_n \geq T_n$ follows from here.

Denote by X' the set containing all elements not in set X. To prove the right inequality, from point 1.1 we know that if $i \in B_n \cap C'_n$, then $n + k + 1 \leq i \leq n + 2015$ and $\mid B_n \cap C'_n \mid \leq 2015 - k$; if $j \in C_n \cap B'_n$ then $n+2 \leq i \leq n+k$. and $\mid C_n \cap B'_n \mid \leq k-1$. But since $\mid B_n \cap C'_n \mid = \mid C_n \cap B'_n \mid$ we must have this number at most $p = \min(k-1, 2015-k)$. Now, we have $S_n - T_n$=sum of elements in $B_n \cap C'_n$-sum of elements in $C_n \cap B'_n \leq (n+2015) + (n+2014) \cdots + (n+2016-p) - ((n+2)+(n+3)+\cdots+(n+p+1)) = 2013 + 2011 + \cdots + (2013 - 2(p-1)) = \frac{p}{2} \cdot (4028 - 2p)$=$p(2014-p)$, which is precisely $(k-1)(2015-k)$.

Finally, $T_n - T_m$=$(m+k+1) + (m+k+2) + \cdots + (n+k)$=$k(n-m) + \sum_{j=m+1}^{n} j$, so $\sum_{j=m+1}^{n}(a_j - k) = S_n - S_m - \sum_{j=m+1}^{n} j - (n-m)k = (S_n - T_n) - (S_m - T_m)$. From above, $0 \leq S_n - T_n, S_m - T_m \leq (k-1)(2015-k)$, so $-(k-1)(2015-k) \leq (S_n - T_n) - (S_m - T_m) \leq (k-1)(2015-k)$. Now $(k-1)(2015-k) \leq (\frac{k-1+2015-k}{2})^2 = 1007^2$ by AM-GM inequality. Q.E.D.

Geometry

G1. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

Solution. Denote P as $AC \cap HG$. Now $IP = PH$, from H beign the orthocentre $\angle ACH = \angle HBA$, from $HG \parallel AB$ we have $\angle HBA = \angle BHG$, from $CH \perp AB, HG$ and $BC \perp AH, BG$ we have $\angle CHG = \angle CBG = 90^\circ$, so $CHBG$ is cyclic and $\angle BHG = \angle BCG$. Moreover $\triangle IPJ \sim \triangle CPG$. So $IJ = CG \cdot \frac{IP}{CP} = CG \cdot \frac{PH}{CP} = CG \cdot \sin \angle PHC = CG \cdot \sin \angle ACH = CG \cdot \sin \angle HBA = CG \cdot \sin \angle BHG = CG \cdot \sin \angle BCG = BG = AH$.

G2. (IMO 4) Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Solution. Since AO is the perpendicular bisector of FG (why? $OF = OG$ and $AF = AG$), we only need the fact $\angle AFK = \angle AGL$. First, we show that $\angle AFD - \angle AGE = \angle ABC - \angle ACB$. Indeed, let FG intersect BC at R, and for sake of simplicity assume that R lies on ray CB beyond B. Taking the triangle RGB and the exterior angle at B yields $\angle GBC = \angle FGB + \angle GRB$; taking the triangle RGD and exterior angle at D yields $\angle GDE = \angle FGD + \angle GRD$. Now $\angle ABC - \angle ACB = (\angle ABG + \angle GBC) - (\angle ACF + \angle FCB) = \angle GRB$ (bearing in mind that $\angle ABG = \angle ACF$ since $AG = AF$)= $\angle GRD = \angle GDE - \angle FGD$= $\frac{1}{2}\angle GAE - \frac{1}{2}\angle FAD = (90^\circ - \angle AGE) - (90^\circ - \angle AFD) = \angle AFD - \angle AGE$ (since $AF = AD$ and $AG = AE$). Now, $\angle AFK - \angle AGL = (\angle AFD - \angle KFD) - (\angle AGE - \angle LGE) = (\angle ABC - \angle ACB)$ - $(\angle ABC - \angle ACB)$=0, since $BFKD$ and $CGLE$ are both cyclic and $\angle KFD = \angle KBD = \angle ABC$, $\angle LGE = \angle LCE = \angle ACB$.

G3. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Solution. By Menelaus' theorem applied on triangle DHB and line CH we have $\frac{AH}{HB} = \frac{PD}{PB}$. Now, let AD intersect ω at T, and let CT intersect ω again at Q'. We are then left to prove that PQ' is tangent to ω, so $Q' \equiv Q$. Since $\angle DTB = \angle ATB = \angle ACB = 90°$, $ACTB$ is cyclic and so $\angle CBA = \angle CTA = \angle Q'TD = \angle Q'BD$. With $\angle DQ'B = 90°$ we have $\triangle ABC \sim \triangle DBQ'$. Finally, if the tangent to Q' intersects BD at P', then $\frac{P'D}{P'B} = (\frac{DQ'}{Q'B})^2 = (\frac{AC}{AB})^2 = \frac{AH}{HB} = \frac{PD}{PB}$, yielding $P \equiv P'$ and thus PQ' is tangent to ω.

G4. Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Answer. $\sqrt{2}$.

Solution. Let S be the common midpoint of BT and PQ. We claim that, if $BMQP$ is cyclic (regardless of whether T, A, B, C are concyclic) then S lies on a fixed line. Indeed, consider any P, Q, P', Q' with $BMQP$ and $BMQ'P'$ cyclic, and P, P' on AB, Q, Q' on BC, then it is not hard to see that $\triangle MPP' \sim \triangle MQQ'$. Denoting S' as midpoint of $P'Q'$ we know that there exists a spiral similarity centred at M that brings P to P', Q to Q' and S to S'. Moreover, $\angle(PP', SS') = \angle(MP, MS)$, i.e. if there is another point S'' with this property then S, S', S'' are collinear. So S lies on a fixed line. Denote P_0, Q_0, S_0 as P, Q, S in the case when $PQ \parallel AC$. We know that S_0 is on BM, so $\angle PMS = \angle P_0M_0S_0$ $= \angle P_0Q_0B = \angle ACB$. Similarly $\angle QMS = \angle BAC$. In degenerate case where Q coincides with B, let P_1 be the midpoint of BP and we have $\angle P_1MB = \angle QMS = \angle BAC$, meaning $BM^2 = BP_1 \cdot AB$. Similarly, $BM^2 = BQ_1 \cdot BC$, where Q_1 is defined as S when $P \equiv B$. This means that if O is the circumcentre of triangle ABC, then $BO \perp P_1Q_1$ since P_1Q_1 is antiparallel to BC.

Now T, A, B, C concyclic iff $\angle BSO = 90°$. From $S \in P_1Q_1$ and $P_1Q_1 \perp BO$, if h_1 is the disance from B to P_1Q_1 we have $BS^2 = h_1 \cdot R$ (with R the circumradius of $\triangle ABC$). Notice also that $\triangle BP_1Q_1$ and $\triangle BCA$ are similar with similitude $\frac{BM^2}{BA \cdot BC}$. Therefore if h is the perpendicular distance from B to AC then $BS^2 = h \cdot R(\frac{BM^2}{BA \cdot BC})$. Since $\frac{hR}{BA \cdot BC}$ is $\left(\frac{BS}{BM}\right)^2$, this ratio is what we sought for.

It is not hard to notice that $\frac{hR}{BA \cdot BC} = \frac{2R|\triangle ABC|}{BA \cdot BC \cdot AC}$, where $|\triangle ABC|$ is the area of triangle ABC. Indeed, it is well-known that $|\triangle ABC| = \frac{BA \cdot BC \cdot AC}{4R}$. Therefore $\frac{2R|\triangle ABC|}{BA \cdot BC \cdot AC} = \frac{1}{2}$. Finally, $BT = 2BS = 2\left(\sqrt{\frac{1}{2}}\right) BM = \sqrt{2}BM$.

G5. Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Solution. Denote the centre of Γ as O, and W.L.O.G. we have $CA < CB$. Denote also the angle α and $\angle CDA = \angle CID = \frac{1}{2}\angle COD$. We know that $AC^2 = AD^2 + DC^2 - 2 \cdot AD \cdot CD \cdot \cos\alpha$ and $BC^2 = BD^2 + DC^2 - 2 \cdot BD \cdot CD \cdot \cos(180° - \alpha)$. With $\cos(180° - \alpha) = -\cos\alpha$ and $AD = BD$, we have $\cos\alpha = \frac{BC^2 - AC^2}{4 \cdot AD \cdot CD}$ (1), and $AC^2 + BC^2 = 2AD^2 + 2DC^2$. (2)

Notice that $CQ = QP$ iff $OQ \perp CM$. Also name the point E as the midpoint of FG, and X the intersection of FG and the tangent to Γ at C. Since $FG \parallel AB$ and E lies on CD, it's not hard to notice that $XC = XE$. Moreover, $OD \perp FG$, $OE \perp CD$, and $\angle CEX = \alpha$. What we need now reduces to the fact $\angle MCE = \angle QOE$ (or $\sin\angle MCE = \sin\angle QOE$) and equivalently $\angle CME = 180° - \angle QOD$ (also $\sin\angle CME = \sin\angle QOD$.) However, with $\angle MCE + \angle CME = 180° - \angle MEC = \angle EOD = \angle QOD - \angle QOE$ we only need the fact $\frac{ME}{CE} = \frac{\sin\angle MCE}{\sin\angle CME} = \frac{\sin\angle QOE}{\sin\angle QOD} = \frac{QE}{QD} \div \frac{OE}{OD}$. The first equality follows by sine rule, and the last equality is well-known in trigonometry. The second equality left to be proven.

Now $\frac{OE}{OD} = \cos\alpha = \frac{BC^2-AC^2}{4\cdot AD\cdot CD}$ by (1). We need the equivalence $CH' = HA = \frac{AD^2}{AC}$ and $CI' = BI = \frac{AD^2}{BC}$ (bearing in mind that $AD = BD$). If we let R to be $H'I' \cap AB$ then the relation $\frac{RA}{RB} = \frac{AH'}{H'C} \cdot \frac{CI'}{I'B}$ holds by Menelaus' theorem. Changing $H'C$ into $AC - CH'$ and BI' into $BC - CI'$ yields $\frac{AH'}{H'C} \cdot \frac{CI'}{I'B} = \frac{AC^2-AD^2}{BC^2-AD^2}$. This, in turn, means $\frac{RA}{RD} = \frac{2(AC^2-AD^2)}{AC^2-AD^2+BC^2-AD^2}$ (since D is the midpoint of AB), so by Menelaus' theorem on $\triangle ACD$ and line $H'I'$ we have $\frac{2(AC^2-AD^2)}{AC^2-AD^2+BC^2-AD^2} = \frac{AH'}{H'C} \cdot \frac{CQ}{QD}$. With $\frac{AH'}{H'C} = \frac{AC^2-AD^2}{AD^2}$ we have $\frac{CQ}{QD} = \frac{2AD^2}{AC^2+BC^2-2AD^2}$. But $\frac{CQ}{CD} = \frac{CQ}{CQ+QD} = \frac{2AD^2}{AC^2+BC^2}$ and $CE = \frac{1}{2}CD$, $\frac{CQ}{CE} = \frac{4AD^2}{AC^2+BC^2}$, $\frac{CQ}{QE} = \frac{CQ}{CE-CQ} = \frac{4AD^2}{AC^2+BC^2-4AD^2}$ and $\frac{QE}{QD} = \frac{AC^2+BC^2-4AD^2}{2AC^2+2BC^2-4AD^2}$, and therefore $\frac{QE}{QD} \div \frac{OE}{OD} = \frac{AC^2+BC^2-4AD^2}{2AC^2+2BC^2-4AD^2} \cdot \frac{4\cdot AD\cdot CD}{BC^2-AC^2}$. (3)

Now $H'F = CF - CH' = \frac{1}{2}AC - \frac{AD^2}{AC} = \frac{AC^2-2AD^2}{2AC}$. Similarly $I'G = \frac{BC^2-2AD^2}{2BC}$. Therefore by Menelaus' theorem again $\frac{MF}{MG} = \frac{H'F}{H'C} \cdot \frac{CI'}{I'G} = \frac{AC}{BC} \cdot \left(\frac{AC^2-2AD^2}{2AC} / \frac{BC^2-2AD^2}{2BC}\right) = \frac{AC^2-2AD^2}{BC^2-2AD^2}$. (It is easy to prove that $\frac{CI'}{CH'} = \frac{AC}{BC}$.) This means $\frac{MF}{FG} = \frac{AC^2-2AD^2}{BC^2-AC^2}$ $(FG = MG - MF)$, so $ME = MF + \frac{1}{2}FG = FG\left(\frac{AC^2-2AD^2}{BC^2-AC^2} + \frac{1}{2}\right) = FG\left(\frac{BC^2+AC^2-4AD^2}{2(BC^2-AC^2)}\right)$. Therefore, $\frac{ME}{CE} = \frac{FG}{CE} \cdot \frac{BC^2+AC^2-4AD^2}{2(BC^2-AC^2)} = \frac{AD}{CD} \cdot \frac{BC^2+AC^2-4AD^2}{(BC^2-AC^2)}$, since $AB = 2AD = 2FG$ and $CD = 2CE$. (4)

Finally, combining (2), (3) and (4), we have the relation $2CD^2 = AC^2 + BC^2 - 2AD^2$ that implies $\frac{AC^2+BC^2-4AD^2}{2AC^2+2BC^2-4AD^2} \cdot \frac{4\cdot AD\cdot CD}{BC^2-AC^2} = \frac{AD}{CD} \cdot \frac{BC^2+AC^2-4AD^2}{(BC^2-AC^2)}$ This proves $\frac{ME}{CE} = \frac{QE}{QD} \div \frac{OE}{OD}$.

G6. (IMO 3) Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Solution. Let QH intersect Γ again at U, we know that $\angle AQU = \angle ABU = \angle ACU = 90°$, so from $BU, CH \perp AB$ and $CU, BH \perp AC$ we have $BHCU$ a parallelogram, so HU bisects BC and we have Q, H, M collinear.

Now denote by X the midpoint of QH. First, the circumcircle of MFH is tangent to the circumcircle of QKH, because they intersect at H and from $\angle HFM = \angle QKH = 90°$, the centres of the circles must lie on midpoints of HM and QH, respectively, and H lies on the line joining the centres. Next, denote by R the radical centre of circumcircles of HFM, QKH, MKF, The radical axis of HFM and MKF is FM (i.e. BC) and the radical axis of HFM and QKH is HR, with $HR \perp QH$. It suffices to prove that KR is tangent to the circumcrcle of QKH since KR is the radical axis of the circles that we want to prove them tangent.

Now, $OX^2 = R^2 - QX \cdot XU$ (power of point)$=R^2 - XH \cdot (XH + 2HM) = R^2 - XH^2 -$

$2XH \cdot HM$ ($HM = MU$ and $XH = XQ$), $XR^2 = XH^2 + HR^2$ and $OR^2 = OM^2 + MR^2 = (R^2 - QM \cdot MU) + HM^2 + HR^2 = (R^2 - (2XH + HM) \cdot HM) + HM^2 + HR^2 = R^2 - 2XH \cdot HM + HR^2$. Combining the three yields $OX^2 + XR^2 = OR^2$, so $\angle OXR = 90°$. Finally, since OX is the perpendicular bisector of KQ, we know that XO is an angle bisector of lines XH and XK, With $\angle OXR = 90°$, XR is another angle bisector of these two lines. Therefore, $\angle HXR = \angle KXR$. With $XK = XH$, $\triangle XHR \cong \triangle XKR$, so $\angle XKR = \angle XHR = 90°$, and KR is indeed tangent to circumcircle of QKH.

G7. Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Solution. We first start with the following lemma.

Lemma 1. *Let ℓ_0, ℓ_1, ℓ_2 be lines that are either concurrent or parallel. Let A, C be on ℓ_0, A_1, C_1 on ℓ_1 and A_2, C_2 on ℓ_2. Let C_2A_1 intersect AA_2 and CC_1 at S and Q, respectively. Let A_2C_1 intersect AA_1 and CC_2 at P and R, respectively. Then PQ, RS, AC will also be either concurrent and parallel.*

Proof. The problem condition tells us that the triangles AA_1A_2 and CC_1C_2 are Desargues' perspective of each other. This means, if we denote $AA_1 \cap CC_1$ as X_1 and $AA_2 \cap CC_2$ as X_2 then the intersection of A_1A_2 and C_1C_2 will lie on X_1X_2 too (or possibly, X_1X_2, A_1A_2, C_1C_2 are all parallel).

By Menelaus' theorem we have $\frac{A_1A}{A_1X_1} \cdot \frac{A_2X_2}{A_2A} = \frac{C_1C}{C_1X_1} \cdot \frac{C_2X_2}{C_2C}$. Denoting the intersection of A_1C_2 as T (possibly point of infinity) and considering triangles AX_1X_2 and CX_1X_2 gives $\frac{A_1A}{A_1X_1} \cdot \frac{SX_2}{SA} \cdot \frac{X_1T}{X_2T} = -1 = \frac{C_2X_2}{C_2C} \cdot \frac{QC}{QX_1} \cdot \frac{X_1T}{X_2T}$, which gives $\frac{A_1A}{A_1X_1} \cdot \frac{SX_2}{SA} = \frac{C_2X_2}{C_2C} \cdot \frac{QC}{QX_1}$. Similarly, $\frac{A_2X_2}{A_2A} \cdot \frac{PA}{PX_1} = \frac{C_1C}{C_1X_1} \cdot \frac{RX_2}{RC}$. Combining everything above gives $\frac{PA}{PX_1} \cdot \frac{SX_2}{SA} = \frac{RX_2}{RC} \cdot \frac{QC}{QX_1}$. By Menelaus' theorem again PS and QR either intersect on X_1X_2 or both parallel to X_1X_2, so APS and CQR are also perspective of each other. Thus AC, PQ, RS are concurrent or parallel. □

Now denote the incircles of $APOS$, $BQOP$, $CROQ$, and $DSOR$ by ω_A, ω_B, ω_C, ω_D, respectively. Denote $T(W, XY)$ by the point of tangency of circle ω_W to line XY too.

We first prove that $PR = QS$. take E, the exsimilicenter of ω_B and ω_C and consider the triangle formed by $e(BC), O, Q$. Now, ω_B and ω_C are incircle and excircle of this triangle, so $OT(C, OQ) = QT(B, OQ)$ (*). In the case where $PR \parallel BC$, (*) follows by symmetry, We similarly have $PT(B, PR) = OT(A, PR) = OT(A, SQ)$, $OT(B.PR) = OT(B, SQ)$, $OT(C, PR) = OT(C, SQ) = QT(B, SQ)$ and $RT(C, PR) = OT(D, PR) = OT(D, QS) = ST(A, QS)$ Therefore, $PR = PO + OR = PT(B, PR) + OT(B.PR) + OT(C, PR) + RT(C, PR) = OT(A, SQ) + OT(B, SQ) + QT(B, SQ) + ST(A, QS) = SO + OQ = QS$. It then follows that 0=$SQ - PR = SO - OR + OQ - OP = SD - DR + BQ - BP$. Similarly $0 = SO - OP + OQ - OR = SA - AP + QC - RC$. Adding the two equations we have $0 = SD + SA + BQ + QC - DR - RC - BP - AP = AD + BC - DC - AB$, and this last relation yields $ABCD$ circumscribed.

Now, let AD and BC intersect PR, at A_2 and C_1, respectively, AB, CD intersect QS at A_1 and C_2 respectively. Denote T as the exsimilicenter of ω_A and ω_C (possible point at infinity). Denote also ω_O as the incircle of $ABCD$ (which we proved exist). Then A is the exsimilicenter of ω_O and ω_A; B, of ω_O and ω_B; C, of ω_O and ω_C; and D, of ω_O and

ω_D. Thus by Monge's theorem, A, C, T are collinear. Moreover, A_1 is the exsimilicenter of ω_A and ω_B; and C_1, of ω_B and ω_C. Again by Monge's theorem, A_1, C_1, T collinear. Similarly, A_2, C_2, T collinear. So AC, A_1C_1, A_2C_2 parallel or concurrent. Keeping in mind that $P = C_1A_2 \cap AA_1$, $Q = CC_1 \cap C_2A_1$, $R = C_1A_2 \cap CC_2$, $S = A_1C_2 \cap AA_2$, we can use the lemma aforementioned to finish our proof. Q.E.D.

Number Theory

N1. Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Answer. All positive integers except 1.

Solution. When $M = 1$, we have $a_n = 1.5$ for all $n \geq 0$. It suffices to show that all other M's work.

Denote $\nu_2(x)$ as the highest power of 2 dividing a positive integer x. We claim the following: if x is such that $x - 1.5$ is an integer, then

- $x\lfloor x \rfloor$ is an integer if $\nu_2(x - 1.5) = 0$;
- $\nu_2(x\lfloor x \rfloor - 1.5) = \nu_2(x - 1.5) - 1$ if $\nu_2(x - 1.5) > 0$ (i.e. if $\lfloor x \rfloor$ is odd).

Indeed, for both of the cases, $x\lfloor x \rfloor = x(x - 0.5) = \frac{(2x-1)x}{2}$. If $\nu_2(x - 1.5) = 0$, then $2x - 1$ is divisible by 4 and therefore $(2x - 1)x$ is even (since $2x$ is an integer). Otherwise, let $x - 1.5 = c \cdot 2^k$ for some c odd and $k \geq 1$, then $(2x - 1)x = (c \cdot 2^k + \frac{3}{2})(c \cdot 2^{k+1} + 2) = (c \cdot 2^{k+1} + 3)(c \cdot 2^k + 1) \equiv 3c \cdot 2^k + 3 \equiv 2^k + 3 \pmod{2^{k+1}}$, and therefore $x\lfloor x \rfloor - \frac{3}{2} \equiv 2^{k-1} \pmod{2^k}$, as desired.

Thus from the claim, for each $M \geq 2$, we have $a_0 - \frac{3}{2} = M - 1 > 0$ so there is a nonnegative integer c with $\nu_2(a_0 - \frac{3}{2}) = c$. From our analysis above, for each $k = 0, \cdots, c$ we have $\nu_2(a_k - \frac{3}{2}) = c - k$, and finally a_{c+1} is an integer.

N2. Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Solution. If $a = 1$ then $b! + 1 \mid b!$, absurd. So we can assume that $a, b \geq 2$ and $3a < 2b + 2$ implies $b > a$, which we can safely assume.

Now change the problem to be $a!(1 + (a+1)(a+2)\cdots b) \mid a!b!$, or $1 + (a+1)(a+2)\cdots b \mid b!$. If $b > \frac{3a}{2} - 1$, then $\underline{}a \geq \frac{a}{2}$ for a even and $\geq \frac{a-1}{2}$ otherwise. Let $p \mid b!$ for some prime p and $p \leq \frac{a-1}{2}$. Since $(a+1)(a+2)\cdots b$ consists of $b - a \geq \frac{a-1}{2}$ consecutive integer, p divides this number and therefore $p \nmid 1 + (a+1)(a+2)\cdots b$. Consequently, if prime p s.t. $p \mid \gcd(1 + (a+1)(a+2)\cdots b, b!)$ then $\frac{a+1}{2} \leq p \leq a$ and $p \nmid a+1, a+2, \cdots b$. Therefore, since $2p > a$ we must have $2p > b$ as well, yielding $p \parallel b!$. Considering all those primes yield the gcd of the two numbers is at most $(\frac{a}{2} + 1)(\frac{a}{2} + 2)\cdots a$, for a even, or $a(a-2)\cdot \cdots \cdot \frac{a+1}{2}$ (or $\frac{a+3}{2}$) (notice that we eliminated all even factors since they cannot be prime). In both cases they are less than $1 + (a+1)(a+2)\cdots b$, so the problem condition cannot hold.

N3. Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1x_2\ldots x_{n+1} - 1$ is divisible by an odd prime.

Solution. Let $a_k = x_k - 1 = \frac{m-n}{n+k}$, we know that $m-n$ is divisible by $n+1, n+2, \cdots 2n+1$ so it is divisible by $l = lcm(n+1, n+2, \cdots 2n+1)$. Let $g(k) = \frac{x}{n+k}$ and we show that eactly one of $g(k)$ among $g(1), g(2), \cdots g(n+1)$ is odd. Indeed, if $2^c \le n < 2^{c+1}$ then $2^{c+1} \le 2n < 2^{c+2}$ and we know that exactly one power of 2, which is 2^{c+1}, is in $n+1, n+2, \cdots 2n+1$. Conversely, only one number is divisible by 2^{c+1} and $g(2^{c+1}-n)$ is odd, but $g(k)$ is even for other k.

Now, let $x = \frac{m-n}{l}$ and $a_k = g(k)\cdot x$. Now $P(x) = x_1 x_2 \cdots x_{n+1} - 1 = (\prod_{k=1}^{n+1}(g(k)\cdot x + 1)) - 1 = x(\prod_{i=1}^{n+1} c_i x^{i-1})$, where c_i is $x(\prod_{b_1<b_2<\cdots b_i} g(b_1)g(b_2)\cdots g(b_i))$. If $P(x)$ is a power of 2, then x must be a power of 2, and from the fact that c_1 is odd but $c_2, c_3, \cdots c_{n+1}$ all even we conclude that $(\prod_{i=1}^{n+1} c_i x^{i-1})$ must be 1, which is impossible as $g(1), g(2), \cdots g(n+1)$ are all at least one and $(\prod_{i=1}^{n+1} c_i x^{i-1}) > c_i \ge n+1$. Contradiction is achieved and $P(x)$ is divisible by an odd prime.

N4. Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \ge 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \ge 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \ge N$.

Solution. We partition the sequence (a_n) into *groups* of adjacent numbers such that a_{i+1} and a_i are in the same group if and only if $a_{i+1} > a_i$. Obviously, $a_{i+1} - 1 \mid a_i$, so $a_{i+1} > a_i \Leftrightarrow a_{i+1} = a_i + 1 \Leftrightarrow a_i \mid b_i \Leftrightarrow b_{i+1} = b_i - 1$. The tail of a group is defined as the last (and therefore the largest) number in that group; we claim that the sequence of numbers containing all tails of groups must be non-increasing. Indeed, let a_n be a tail, then $a_{n+1} = \gcd(a_n, b_n) + 1$, and for sake of simplicity denote this number by $g+1$. We then have $b_{n+1} = \dfrac{a_n b_n}{g} - 1$. Now, suppose that the next tail is at least a_n, then $(a_{n+k}, b_{n+k}) = (g+k, \dfrac{a_n b_n}{g} - k)$, $\forall k \le a_n - g$. We therefore have $a_{n+a_n-g} = a_n$ and $b_{n+a_n-g} = \dfrac{a_n b_n}{g} - a_n + g$. Notice that $g = \gcd(a_n, b_n)$ so $g \mid b_n$, and a_n divides both $\dfrac{a_n b_n}{g}$ and a_n, so $\gcd(a_{n+a_n-g}, b_{n+a_n-g}) = g < a_n$, and this a_{n+a_n-g} is a tail now (hence cannot exceed a_n).

Since the tail of the sequence cannot decrease forever, it must remain constant at one point. Now, for sufficiently large indices when the tail remains constant, from above we know that if a_n and a_m are both tails then $\gcd(a_n, b_n) = \gcd(a_m, b_m)$. It folows that the number succeeding each tail must be the same too, and that's the smallest number of the group thereafter. Summing up, the smallest and the biggest number (i.e. first and last number) in a period becomes constant, and $a_{i+1} = a_i + 1$ for a_i, a_{i+1} in the same group. We therefore conclude that the groups must be identical at one point, hence eventually periodic.

N5. (IMO 2) Find all positive integers (a, b, c) such that

$$ab - c, \quad bc - a, \quad ca - b$$

are all powers of 2.

Answer. $(2,2,2), (2,2,3), (2,6,11), (3,5,7)$ and their permutations.

Solution. We first verify that these triples work. The first one yields $ab - c = 2$; the second one gives 1 and 2; the third one, 1, 16, 64; the last, 8, 16, 32.

We now show that there are no other triples satisfying the condition, and it suffices to consider triples up to permutation. Notice that all numbers are strictly greater than 1, otherwise if $a = 1$ then $b - c$ and $c - b$ cannot be both positive. Throughout the solution we denote $\nu_2(a)$ as the highest exponent of 2 dividing a. Now we have these few cases.

Case 1: two of the three numbers are the same (i.e. we have (a, a, b) instead) then $a, b - 1, a^2 - b$ are all powers of 2. In particular, a is even, and b is either odd or 2. If b is odd, then $a^2 - b = 1$, i.e. $b - 1, b + 1$ both powers of 2 (since a^2 is also a power of 2). The only possibility is $b - 1 = 2, b + 1 = a^2 = 4$, giving the triple $(2, 2, 3)$. If $b = 2$, with a^2 divisible by 4, $a^2 - 2 = a^2 - b \equiv 2 \pmod 4$, so $a^2 - 2 = 2$. This gives $a = 2$, giving the triple $(2, 2, 2)$.

Case 2: all three numbers are distinct. Notice that $(ab - c) - (bc - a) = (b + 1)(a - c) \neq 0$ so the three target numbers $ab - c, bc - a, ca - b$ are also pairwise distinct. In particular, they cannot be all odd. This would have been the case if among a, b, c, exactly two of them are odd, so this case can be eliminated. We now claim that they a, b, c cannot be all even, either. W.l.o.g. suppose that $\nu_2(a) = x, \nu_2(b) = y$ and $\nu_2(c) = z$, with $1 \leq x \leq y \leq z$. Then since $\nu_2(ac) = x + z > y = \nu_2(b)$ and $\nu_2(bc) = y + z > x = \nu_2(a)$, we have $\nu_2(ac - b) = y$ and $\nu_2(bc - a) = x$, hence $ac - b = 2^y$ and $bc - a = 2^x$. Adding them, we get $(c - 1)(a + b) = 2^x + 2^y$. With $2^x \mid a, 2^y \mid b$, however, we are now forced with $c = 2, a = 2^x, b = 2^y$, so $z = 1$. With $x \leq y \leq z$ we get $x = y = z$. But now $a = b = c = 2$, contradicting that they are pairwise distinct.

Subcase 2a: two of them (say, a, b) even, and c odd. W.l.o.g. set $a < b$, then $ac - b < bc - a$. This means $ab - c = 1$, so $c = ab - 1$. Note that $\nu_2(ac) = a$ and $\nu_2(bc) = b$, so if $\nu_2(a) \neq \nu_2(b)$ then $\nu_2(ac - b) = \nu_2(bc - a) = \min(\nu_2(a), \nu_2(b))$, forcing $ac - b = bc - a$, which contradicts $a \neq b$. Thus $\nu_2(a) = \nu_2(b)$, let this quantity be ℓ. Expanding, we get $ac - b = a^2 b - (a + b)$ and $bc - a = ab^2 - (a + b)$. Notice also that $\nu_2(a^2 b) = \nu_2(ab^2) = 3\ell$ so by the same logic as above, we need $\nu_2(a + b) = 3\nu_2(a)$ too (otherwise $\nu_2(ac - b) = \nu_2(bc - a) = \min(\nu_2(a + b), \nu_2(a^2 b)))$. We also have $ac - b + bc - a = (a + b)(c - 1) = (a + b)(ab - 2)$. Since a, b are both even, $\nu_2(ab - 2) = 1$ and therefore $\nu_2(ac - b + bc - a) = 1 + \nu_2(a + b) = 1 + 3\ell$, which yields the smaller of them $ac - b$ has

$$ac - b = a^2 b - (a + b) = 2^{3\ell + 1}$$

Let $a' = \frac{a}{2^\ell}$ and $b' = \frac{b'}{2^\ell}$, which are both odd integers. In addition, $a \neq b$ means $a' \geq 1, b' \geq 3$. Thus the equation above can be further analyzed as

$$2 = (a')^2 b' - \frac{a' + b'}{2^{2\ell}} \geq a'b' - \frac{a' + b'}{2^{2\ell}} \geq (a' - \frac{1}{2^{2\ell}})(b' - \frac{1}{2^{2\ell}}) - \frac{1}{2^{4\ell}} \geq (1 - \frac{1}{2^{2\ell}})(3 - \frac{1}{2^{2\ell}}) - \frac{1}{2^{4\ell}}$$
$$= 3 - \frac{4}{2^{2\ell}} \geq 3 - \frac{4}{4} = 2$$

so all inequalities above are, in fact, equality. This means we have $a' = 1, b' = 3, \ell = 1$, which gives the solution $a = 2, b = 6, c = 11$.

Subcase 2b: a, b, c are all odd, so all the three key terms are even, meaning at least two of them are divisible by 4. W.l.o.g. we assume that $a < b < c$. Then $ac - b$ and $bc - a$ have sum $(c - 1)(a + b)$ and difference $(c + 1)(b - a)$. In addition, since $ac - b < bc - a$ and are distinct powers of 2, we have $\nu_2(ac - b) = \nu_2(c - 1) + \nu_2(a + b) = \nu_2(c + 1) + \nu_2(b - a)$.

Notice that $4 \nmid 2c$, so $c - 1, c + 1$ cannot both be divisible by 4, i.e. $\min\{\nu_2(c - 1), \nu_2(c + 1)\} = 1$. In a same way, $a + b + b - a = 2b \equiv 2 \pmod 4$, so $\min\{\nu_2(a + b), \nu_2(b - a)\} = 1$. Along with $\nu_2(c - 1) + \nu_2(a + b) = \nu_2(c + 1) + \nu_2(b - a)$, we have

$$\nu_2(c + 1) = \nu_2(a + b) \qquad \nu_2(c - 1) = \nu_2(b - a)$$

Suppose that $\nu_2(c+1) = 1 < \ell = \nu_2(c-1)$ for some ℓ, writing $c = 2^\ell c' + 1$, we have

$$2^{\ell+1} = ac - b > ac - c = (a-1)(2^\ell c' + 1) \geq 2(2^\ell c' + 1) > 2^{\ell+1}$$

(given that $a > 1$ and a is odd). This is a contradiction.

Thus we need $\nu_2(c-1) = 1 < \ell = \nu_2(c+1)$. Now we write $c = 2^\ell c' - 1$. Similar to above, we get

$$2^{\ell+1} = ac - b > (a-1)c = (a-1)(2^\ell c' - 1) \tag{5}$$

If $a \geq 5$, then (5) gives $2^{\ell+1} > 4(2^\ell c' - 1) \geq 4(2^\ell - 1) = 2^{\ell+2} - 4$, or $4 > 2^{\ell+1}$. But we have $\ell \geq 2$, contradiction. Hence $a = 3$. Now dividing 2 on both sides of (5) gives $2^\ell > 2^\ell c' - 1$, so $c' = 1$, which also gives $b = 2^\ell - 3$.

Finally, we now have $ab - c = 3(2^\ell - 3) - (2^\ell - 1) = 2^{\ell+1} - 8$. The only possibility here is $\ell = 3$ (to make it a power of 2). This gives the solution $(3, 5, 7)$.

N6. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$;
(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

Solution. If $f(m) = f(k)$, then $f^n(m) = f^n(k), \forall n \in \mathbb{N}$. By (i), n divides both $f^n(m) - m$ and $f^n(m) - k$, so n divides $m - k$ for all positive integers n. This means $m - k = 0$ and f is injective. From (i) again we have $f(m) - m > 0, \forall m \in \mathbb{N}$. This allows us to partition the set of positive integers into groups such that for one group $A = \{a_i \mid i \geq 0\}$, $a_{k+1} = f(a_k)$, $\forall k \geq 1$ and we name a_0 as the element such that there exists no integer i such that $f(i) = a_0$. By (ii), the number of set A is finite, so name them as $A_1, A_2, \cdots A_p$.

We proceed with this lemma.

Lemma 2. *Let $A = \{a_i \mid i \geq 0\}$ be an increasing sequence such that $j - i \mid a_j - a_i$, and that there exists a positive integer s with $a_{i+1} - a_i < s$ for infinitely many i. Then $a_0, a_1, \cdots$ necessarily forms an arithmetic progression.*

Proof. For every i and every M, we can find infinitely many $j > i + M$ s.t. j satisfies this property $a_{j+1} - a_j < s$. Now, take $M > |a_{i+1} - a_i| + s$ and we have $j - i$ divides both $a_j - a_i$ and $a_{j+1} - a_{i+1}$. It therefore divides $(a_{j+1} - a_{i+1}) - (a_j - a_i) = (a_{j+1} - a_j) - (a_{i+1} - a_i)$ and $|(a_{j+1} - a_j) - (a_{i+1} - a_i)| \leq |(a_{j+1} - a_j)| + |(a_{i+1} - a_i)| \leq |(a_{j+1} - a_j)| + s < j - i$, meaning that $(a_{j+1} - a_j) - (a_{i+1} - a_i) = 0$. Taking this for infinitely many j yields $a_{j+1} - a_j = x$, a constant for infinitely many j. This, in turn, allows us to take any i and j with $a_{j+1} - a_j = x$ and $j - i > |(a_{j+1} - a_j)| + x$. Repeating the above yields the same thing, whereby $a_{i+1} - a_i = a_{j+1} - a_j = x$. Thus, $a_{i+1} - a_i$ is indeed a constant for all i, i.e. the elements form an arithmetic progression. $\square$

We prove, inductively, that elements in all sets satisfy the lemma condition, hence form arithmetic progressions. First we prove that this is true for at least one set. Now, among any interval $[k(p+1)+1, (k+1)(p+1)]$, there must exist two integers i and j in this interval such that $i, j \in A_c$ for some integer c. We record such c, and consider different

$c \in [1, p]$ for all integers k. This means, there exist a number c that is recorded infinitely many times. W.l.o.g. let $c = 1$, so this is fulfilled for A_1 since we can substitute $p + 1$ into s in the lemma. Suppose that this is true for groups $A_1, A_2, \cdots A_i$, and let $D=$ lcm $(d_1, d_2, \cdots, d_i)$. Also name $B = A_{i+1} \cup A_{i+1} \cup \cdots \cup A_p$ (the unselected). This means that, if $D \mid a - b$ for $a, b \in \mathbb{N}$, we have $a \in A_k \Leftrightarrow b \in A_k$, $\forall k \in [1, i]$. More importantly, this means $a \in B \Leftrightarrow b \in B$. With this, we conclude that B contains at least an element in the interval $[a + 1, a + D]$ for each a (otherwise $A_1, A_2, \cdots A_i$ jointly contains all positive integers), and at least $p - i + 1$ elements among $[a + 1, a + (p - i + 1)D]$. In other words, there are at least two elements in the interval belong to the same set, say A_c, and record this c. Repeat this for infinitely many disjoint intervals of length $(p - i + 1)D$, and since the choice of c is finite (in the set $[i + 1, p]$), at least one such c recorded infinitely many times, hence such c (say, $i + 1$) fulfills the condition in the lemma (this time we have $s = (p - i + 1)D$).

Finally, since every element in each set form an arithmetic progression, we denote D=lcm $(d_1, d_2, \dots d_p)$ (whereby d_i is the common difference of A_i) and prove that $f(i) - i$ has period D. Indeed, for every $i \in \mathbb{N}$. i and $i + D$ are in the same group, say, A_c. Hence $f(i + D) - (i + D) = f(i) - i = d_c$. Q.E.D.

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called k-good if $\gcd(f(m) + n, f(n) + m) \leq k$ for all $m \neq n$. Find all k such that there exists a k-good function.

Answer. All $k \geq 2$.

Solution. We first eliminate $k = 1$. Indeed, suppose that $\gcd(f(m)+n, f(n)+m) = 1$ for all $m \neq n$. Considering $m = 1, n = 3$ means one of $f(1), f(3)$ must be even. Considering $m = 2, n = 4$ means one of $f(2), f(4)$ must be odd. This means there is m, n such that $m, f(n)$ are both odd and $n, f(m)$ are both even, so both $f(m)+n$ and $f(n)+m$ are even, which is a contradiction.

We now construct a 2-good function, which will then imply this it's also k-good for all $k \geq 2$. Let P be the set of all odd primes and 4. A condition that we will be checking is that for all $p \in P$ $m \neq n$, $f(m) + n$ and $f(n) + m$ cannot be simultaneously divisible by p. For each number $p \in P$, we also keep track of an auxillary (modular) function $g_p : \{1, 2, \cdots, p\} \to \{0, 1, \cdots, p - 1\}$, with the goal that $g_p(m) + n$ and $g_p(n) + m$ are not simulatenously divisible by p for any pairs (m, n). Since this is the only goal for this g_p, we may also extend g_p periodically modulo p, i.e. $g_p(n + p) = f_p(n)$.

Now we construct f and g_p iteratively. We set $f(1) = 1$, and $g_p(1) = 1$ for all p odd prime or 4. Thereafter, for each $n \geq 2$ we construct $f(n)$ and $g_p(n)$ for each $p > n$ such that the following invariants are maintained:

- For each $p \in P$ such that either $p < n$ or $p \nmid n + f(n)$, we have $g_p(n) \equiv f(n) \pmod{p}$;
- For each $m < n$ and $p \in P$, either $p \nmid n + g_p(m)$ or $p \nmid m + f(n)$;
- For each $m \leq n$ and $p \in P$, either $p \nmid n + g_p(m)$ or $p \nmid m + g_p(n)$ (in particular, $p \nmid n + g_p(n)$).

We show that such a construction of $f(n)$ and $g_p(n)$ (for all $p > n$) is possible. Notice that when constructing $f(1) = g_p(1) = 1$, such an invariant is maintained since $1 + f(1) = 2$ is divisible by no member in P. Next, suppose for some $n \geq 2$ this invariant is maintained for $f(1), \cdots, f(n-1)$ and all $g_p(k)$ for all $k \leq \min(n-1, p)$. We now do the construction in the following manner: constructing $g_p(n)$ for just a subset of $p > n$, then $f(n)$ consistent with these $g_p(n)$'s, and finally $g_p(n)$ for the rest of p. Notice for the final invariant we only need to care about those p with $p \geq n$.

We claim that the set $\{p \in P : \exists m < n : p \mid n + g_p(m)\}$ is finite. Indeed, suppose that $p \mid n + g_p(m)$ for some $m < n$. Then by the invariant maintained, either $p \mid m + f(m)$ or $g_p(m) \equiv f(m) \pmod p$, i.e. the second case gives $p \mid n + f(m) \pmod p$. Thus p must divide one of $\{n + f(m) : m < n\} \cup \{m + f(m) : m < n\}$, and therefore finite.

Let $P_n \subseteq P \cap \{n, n+1, \cdots\}$ be such numbers. We now show that for each $p \in P_n$, $g_p(n)$ can be constructed in such a way that for each $m \leq n$, p does not simultaneously divide $g_p(m) + n$ and $g_p(n) + m$. If $n < p$, we can easily take, say, $g_p(n) \equiv -(n+1) \pmod p$. If $n = p$, from $g_p(1) = 1$ we get $p \nmid g_p(1) + p$, and therefore we may then choose $g_p(n) \equiv -1 \pmod p$.

Denote, now, $Q_n = P_n \cap (P \cup \{1, \cdots, n-1\})$, which are precisely the primes p where we have the values $g_p(n)$ for. Now, since the elements in P (and therefore Q_n) are pairwise coprime, The equivalence $f(n) \equiv g_p(n \pmod p)$ has a unique solution modulo $\prod_{p \in P : p < n} p$ by Chinese Remainder Theorem. Choose one such solution $f(n)$, the second invariant is automatically maintained given that for all $m < n$ and $p \in P$, we have $p \nmid n + g_p(m)$. and now for any $p \notin Q_n$, we may now define $g_p(n)$ such that $g_p(n) = f(n) \pmod p$ if $p \nmid n + f(n)$. If $p \mid n + f(n)$, we may use the previous recipe (e.g. $g_p(n) \equiv -(n+1) \pmod p$ if $n < p$, and $g_p(n) \equiv -1 \pmod p$ if $n = p$).

We are now ready to show that our function f is now 2-good. Let $p \in P$ be arbitrary and consider $m \neq n$. If $f(n) \equiv g_p(n \pmod p)$ and $f(m) \equiv g_p(m \pmod p)$ both hold, then by our construction on g_p, p cannot simultaneously divide $n + f(m), m + f(n)$. Otherwise, suppose $f(n) \not\equiv g_p(n)$, which then implies both $p \geq n$ and $p \mid n + f(n)$. If $p \mid m + f(n)$, then $p \mid m - n$ (so $m > p$), but then $f(m) \equiv g_p(m \pmod p) \equiv g_p(n \pmod p) \not\equiv f(n) \pmod p$, so $p \nmid n + f(m)$.