Solutions to Tournament of Towns, Fall 2011, Senior

Anzo Teh

O-Level

1. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

   Solution. Suppose there's a guest who've eaten an odd number of berries. Then it cannot eat twice as many berries as the guest before it, so it must 6 fewer berries than the guest before, who therefore also eats and odd number of berries. Thus by iterating this around the table, each guest eats 6 fewer berries than the guest before. But this is impossible (by considering the guest who eats the most berries). Therefore each guest eats an even number of berries, and therefore the total number of berries eaten is also even, and hence cannot be 2011.

3. In a convex quadrilateral $ABCD, AB = 10, BC = 14, CD = 11$ and $DA = 5$. Determine the angle between its diagonals.

   Answer. 90°.

   Solution. Let the diagonals meet at P and denote $\theta = \angle PAB$. This gives $\angle PCD = \theta$ and $\angle PAD = \angle PBC = 180^\circ - \theta$. Then:

$$AB^2 + CD^2 = (PA^2 + PB^2 - 2PA \cdot PB \cos\theta) + (PC^2 + PD^2 - 2PC \cdot PD \cos\theta)$$

$$= PA^2 + PB^2 + PC^2 + PD^2 - 2\cos\theta(PA \cdot PB + PC \cdot PD)$$

   and similarly $AD^2 + BC^2 = PA^2 + PB^2 + PC^2 + PD^2 + 2\cos\theta(PA \cdot PD + PC \cdot PB)$. Here, $AD^2 + BC^2 = 5^2 + 14^2 = 221 = AB^2 + CD^2$ so subtracting the two equations give

$$0 = 2\cos\theta(PA \cdot PD + PC \cdot PB + PA \cdot PB + PC \cdot PD)$$

   but since $PA, PB, PC, PD > 0$ by the convexity of $ABCD$, $\cos\theta = 0$ must hold. Therefore $\theta = 90^\circ$.

4. Positive integers $a < b < c$ are such that $b + a$ is a multiple of $b - a$ and $c + b$ is a multiple of $c - b$. If a is a 2011-digit number and b is a 2012-digit number, exactly how many digits does c have?

   Answer. 2012.

   Solution. c must have at least 2012 digits as per the inequality condition. Now if $(b + a) = k(b - a)$ for some integer k then $\frac{b}{a} = \frac{k+1}{k-1} = 1 + \frac{2}{k-1} \leq 1 + 2 = 3$ since $k - 1 \geq 1$. Therefore $b \leq 3a$ and similarly $c \leq 3b$. This gives $c \leq 9a < 10a$ and since a has 2011 digits, c cannot have more than 2012 digits.

5. In the plane are 10 lines in general position, which means that no 2 are parallel and no 3 are concurrent. Where 2 lines intersect, we measure the smaller of the two angles formed between them. What is the maximum value of the sum of the measures of these 45 angles?

   Answer. 2250°.

Solution. Label the 10 angles in clockwise rotational order as $\ell_1, \cdots, \ell_{10}$. The angles (ℓ_i, ℓ_{i+1}) (indices taken modulo 10) form a rotational order of $180°$. and therefore the sum of smaller angles cannot be more than $180°$. In a similar way, using clockwise order, the sum of angles travelled between (ℓ_i, ℓ_{i+k}) for each $k \le 5$ and summing across all $i = 1, 2, \cdots, 10$ is at most $k \cdot 180°$. Therefore the for $k = 1, 2, 3, 4$ the total angle is at most $(1+2+3+4) \cdot 180° = 1800°$. Finally, the last five angles (ℓ_i, ℓ_{i+5}) for $i = 1, 2, 3, 4, 5$ gives at most $90°$ each, giving the bound of $2250°$ in total.

This can be achived by, for example, making ℓ_i to form an angle of $i \cdot 18°$ clockwise with the x-axis.

A-Level

2. Given that $0 < a, b, c, d < 1$ and $abcd = (1-a)(1-b)(1-c)(1-d)$, prove that $(a+b+c+d) - (a+c)(b+d) \ge 1$.

   Solution. Denote $u = a + c$ and $v = b + d$. Then we need to show that

   $$(u-1)(v-1) = uv - (u+v) + 1 = (a+c)(b+d) - (a+b+c+d) + 1 \le 0$$

   Thus it suffices to show that we cannot have both $u, v > 1$ and similarly we cannot have both $u, v < 1$. If $u, v > 1$, then $(1-a)(1-c) = ac + 1 - u < ac$ and similarly $(1-b)(1-d) < bd$. Therefore $(1-a)(1-b)(1-c)(1-d) < abcd$ now. Similarly $u, v < 1$ gives $(1-a)(1-c) > ac$ and $(1-b)(1-d) < bd$ and therefore $(1-a)(1-b)(1-c)(1-d) > abcd$. These are contradictions.

3. In triangle ABC, points A_1, B_1, C_1 are bases of altitudes from vertices A, B, C, and points C_A, C_B are the projections of C_1 to AC and BC respectively. Prove that line C_AC_B bisects the segments C_1A_1 and C_1B_1.

   Solution. Let H be the orthocenter, C_0 the reflection of H in C_1, and the projection of C_0 to AC and BC as C_A^0, C_B^0. It's well-known that C_0 is on the circumcircle of ABC, so line $C_A^0C_B^0$ is a Simpson line that passes through the projection of C_0 to AB, which is actually C_1.

   Now, the projection of H to AC and BC are B_1 and A_1, respectively. Given that C_1 is the midpoint of C_0H, C_AC_B lies exactly halfway between the lines $C_A^0C_B^0$ and C_1B_1. (That is, all three lines are parallel and C_AC_B is equidistant from $C_A^0C_B^0$ and C_1B_1). But given that $C_A^0C_B^0$ passes through C_1, we conclude that C_AC_B bisects the segments C_1A_1 and C_1B_1.

6. Prove that the integer $1^1 + 3^3 + 5^5 + .. + (2^n-1)^{2^n-1}$ is a multiple of 2^n but not a multiple of 2^{n+1}.

   Solution. We'll go by induction on n. Base case $n = 2$ yields the sum 28 which is a multiple of 4 but not 8.

   For inductive step, we need the following intermediate steps:

   Lemma 1: for all $k \ge 1$, and a odd, we have $a^{2^k} \equiv 1 \pmod{2^{k+2}}$.

   Proof: base case we have $k = 1$ and $1^2, 3^2, 5^2, 7^2 \equiv 1 \pmod 8$. Inductive step: suppose $a^{2^k} \equiv 1 \pmod{2^{k+2}}$, write $a^{2^k} = 2^{k+2}c + 1$. Then

   $$a^{2^{k+1}} = (a^{2^k})^2 \equiv (2^{k+2}c+1)^2 = 2^{2k+4}c^2 + 2^{k+3}c + 1 \equiv 1 \pmod{2^{k+3}}$$

   as desired.

   Lemma 2: for all $n \ge 2$, the following identities hold:

   $$(2^{n+1}-1)^{2^{n+1}-1} + 2^{n+1}-3)^{2^{n+1}-3} + \cdots + (2^n+1)^{2^n+1} - (1^1+3^3+5^5+..+(2^n-1)^{2^n-1}) \equiv 0 \pmod{2^{n+2}}$$

Proof: we play around with $(a+2^n)^{a+2^n} - a^a$ for each $1 \le a \le 2^n - 1$ and a odd. We know from lemma 1 that $a^{2^n} \equiv 1 \pmod{2^{n+2}}$ and therefore $(a+2^n)^{a+2^n} - a^a \equiv (a+2^n)^a - a^a$. Now

$$(a+2^n)^a - a^a = \sum_{i=1}^{a} \binom{a}{i} (2^n)^i a^{a-i} \equiv a(2^n)a^{a-1} = a^a(2^n) \pmod{2^{n+2}}$$

using the fact that $2n \ge n+2$ for $n \ge 2$ (and therefore all term with $i \ge 2$ above are divisible by 2^{n+2}). Thus we have

$$(2^{n+1}-1)^{2^{n+1}-1} + 2^{n+1} - 3)^{2^{n+1}-3} + \cdots + (2^n+1)^{2^n+1} - (1^1 + 3^3 + 5^5 + .. + (2^n-1)^{2^n-1})$$

$$\equiv 2^n(1^1 + 3^3 + \cdots + (2^n-1)^{(2^n-1)}) \pmod{2^{n+2}}$$

and by induction hypothesis, $1^1 + 3^3 + \cdots + (2^n-1)^{(2^n-1)}$ is divisible by 2^n and therefore $2^n(1^1 + 3^3 + \cdots + (2^n-1)^{(2^n-1)})$ is divisible by 2^{2n}, hence divisible by 2^{n+2}, as claimed.

Now we can complete the proof. Let the sum $1^1 + 3^3 + 5^5 + .. + (2^n-1)^{2^n-1} = S_n$; we've shown that $2^{n+2} \mid S_{n+1} - 2S_n$ so $S_{n+1} \equiv 2S_n \pmod{2^{n+2}}$. By induction hypothesis there's a number a odd such that $S_n = a \cdot 2^n$ so $S_{n+1} \equiv a \cdot 2^{n+1} \pmod{2^{n+2}}$ is divisible by 2^{n+1} but not 2^{n+2}, as desired.