Putnam 2010

A1 Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same?

[When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is at least 3.]

Answer. $\lfloor \frac{n+1}{2} \rfloor$.
Solution. The sum in each box is at least n since this must be the case for the box containing n. Since the total sum of the n numbers is $\frac{n(n+1)}{2}$, the number of boxes cannot exceed $\frac{n+1}{2}$, so the answer is at most $\lfloor \frac{n+1}{2} \rfloor$.

To see this is achievable, we split into cases where n is even and n is odd. When n is odd, we can do $\{n\}, \{1, n-1\}, \{2, n-2\}, \cdots, \{\frac{n-1}{2}, \frac{n+1}{2}\}$. When n is even, the example $\{1, n\}, \{2, n-1\}, \cdots, \{\frac{n}{2} - 1, \frac{n}{2} + 1\}$ gives the example of $\frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$ boxes.

A2 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Answer. All linear functions $f(x) = mx + c$ in which case $f'(x) = \frac{f(x+n)-f(x)}{n} = m$
Solution. To show that f must be linear, we notice that for each x, $f'(x) = f(x+1) - f(x) = \frac{f(x+2)-f(x)}{2}$ so this gives the relation $f(x+2) - f(x) = 2(f(x+1) - f(x))$, and therefore $f'(x+1) = f(x+2) - f(x+1) = f(x+1) - f(x) = f'(x)$.

We now introduce the function $g(x) = f(x+1) - f(x)$. Notice that:

$$\begin{aligned}
g'(x) &= \lim_{\epsilon \to 0} \frac{g(x+\epsilon) - g(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{f(x+1+\epsilon) - f(x+\epsilon) - f(x+1) + f(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{f(x+1+\epsilon) - f(x+1)}{\epsilon} - \frac{f(x+\epsilon) - f(x)}{\epsilon} \\
&= f'(x+1) - f'(x) \\
&= 0
\end{aligned}$$

since $f'(x+1) = f'(x)$ for all x. Therefore g is a constant function, meaning that, $f(x+1) - f(x)$ is constant. But since $f(x+1) - f(x) = f'(x)$, $f'(x)$ is also a constant, which means that f has to be linear.

A3 Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all (x, y) in $\mathbb{R}^2$, then h is identically zero.

Solution. Fix a point $(x_0, y_0) \in \mathbb{R}^2$. Denote $g(z) = h(x_0 + az, y_0 + bz)$. We have the following:

$$\begin{aligned} g'(z) &= \lim_{\epsilon\to 0}\frac{g(z+\epsilon)-g(z)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{h(x_0+a(z+\epsilon),y_0+b(z+\epsilon))-h(x_0+az,y_0+bz)}{\epsilon} \\ &= a\frac{\partial h}{\partial x}(x_0+az,y_0+bz)+b\frac{\partial h}{\partial y}(x_0+az,y_0+bz) \\ &= h(x_0+az,y_0+bz) \\ &= g(z) \end{aligned}$$

So the condition $g'(z)=g(z)$ holds for all $z\in\mathbb{R}$. It follows from the identity of differential equation that the solution to g is $g(z)=Ae^z$ for all $z\in\mathbb{R}$. Since h us bounded in $\mathbb{R}^2$, so is g and the only possibility is $A=0$. Thus $g\equiv 0$ and in particular, $g(0)=h(x_0,y_0)=0$.

A4 Prove that for each positive integer n, the number $10^{10^{10^n}}+10^{10^n}+10^n-1$ is not prime.

Solution. Let k be the maximum positive integer such that $2^k\mid n$. We claim that the number $10^{10^{10^n}}+10^{10^n}+10^n-1$ is divisible by $10^{2^k}+1$. It suffices to show that $10^{10^{10^n}}$ and 10^{10^n} are congruent to 1 modulo p whereas 10^n congruent to 1.

Now, $n=d\cdot 2^k$ for some odd number d. Therefore $10^n=10^{d\cdot 2^k}\equiv(-1)^d=-1\pmod{10^{2^k}+1}$ since d is odd. On the other hand, $10^{10^n}=10^{2^n\cdot 5^n}$ and from $n=d\cdot 2^k\geq 2^k>k$ we have $n\geq k+1$, so $2^{k+1}\mid 2^n\cdot 5^n$. Since $10^{2^{k+1}}\equiv(-1)^2=1\pmod{10^{2^k}+1}$ we have $10^{10^n}\equiv 1\pmod{10^{2^k}+1}$, and similarly for $10^{10^{10^n}}$.

Finally, it's not hard to see that $10^{10^{10^n}}+10^{10^n}+10^n-1>10^{2^k}+1$ because $n\geq 2^k$ and $10^{10^{10^n}}$ and 10^{10^n} are both strictly greater than 1 (because n is positive so $10^n>1$ and so is 10^{10^n} and $10^{10^{10^n}}$). Therefore $10^{10^{10^n}}+10^{10^n}+10^n-1>1+1+\cdot 10^n-1=10^n+1=10^{2^k}+1$.

A5 Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a},\mathbf{b}\in G$, either $\mathbf{a}\times\mathbf{b}=\mathbf{a}*\mathbf{b}$ or $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ for all $\mathbf{a},\mathbf{b}\in G$.

Solution. Let $\mathbf{e}\in\mathbb{R}^3$ be the identity element in the group. If $\mathbf{e}\times\mathbf{a}\neq\mathbf{0}$ for some $\mathbf{a}\in G$, then $\mathbf{e}\times\mathbf{a}=\mathbf{e}*\mathbf{a}=\mathbf{a}$. Since both $\mathbf{e}$ and $\mathbf{a}$ are not parallel and nonzero, $\mathbf{a}=\mathbf{e}\times\mathbf{a}$ is perpendicular to $\mathbf{a}$, which is impossible for $\mathbf{a}$ nonzero. Hence, for all $\mathbf{a}\in G$ we have $\mathbf{e}\times\mathbf{a}=\mathbf{0}$.

Now if $\mathbf{e}$ is not the zero vector, then from $\mathbf{e}\times\mathbf{a}=\mathbf{0}$ for all $\mathbf{a}\in G$ all such $\mathbf{a}$'s are either 0 or parallel to $\mathbf{e}$. In this case, the condition $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ for all $\mathbf{a},\mathbf{b}\in G$. We can now assume that $\mathbf{e}=\mathbf{0}$.

Suppose there are $\mathbf{a}$ and $\mathbf{b}\in G$ such that $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$. From (ii), $\mathbf{a}*\mathbf{b}=\mathbf{a}\times\mathbf{b}\in G$ and is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, i.e. there exists a vector in G that is perpendicular to $\mathbf{a}$. Thus we can choose $\mathbf{a}$ and $\mathbf{b}$ (in the beginning) such that they are perpendicular to each other, and we will now assume that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.

Let $\mathbf{a}\times\mathbf{b}=\mathbf{c}=-\mathbf{b}\times\mathbf{a}$. Since $\mathbf{e}=0$, we have $(\mathbf{a}\times\mathbf{b})*(\mathbf{b}\times\mathbf{a})=\mathbf{0}$ and with $\mathbf{a}\times\mathbf{b}=\mathbf{a}*\mathbf{b}$ as of above (and similarly $\mathbf{b}\times\mathbf{a}=\mathbf{b}*\mathbf{a}$) we get $\mathbf{0}=\mathbf{a}*\mathbf{b}*\mathbf{b}*\mathbf{a}$ (notice the omission of brackets because $*$ is associative). "Multiplying" both sides by $-\mathbf{a}$ from the left and $\mathbf{a}$ from the right gives $\mathbf{b}^2*\mathbf{a}^2=\mathbf{0}$ (second power means multiplication by itself), i.e. $\mathbf{b}^2$ and $\mathbf{a}^2$ are inverses of each other. But since $\mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, a similar conclusion yields $\mathbf{c}^2$ and $\mathbf{a}^2$ are inverses of each other, and so are $\mathbf{c}^2$ and $\mathbf{b}^2$. With $\mathbf{a}^2$, $\mathbf{b}^2$ and $\mathbf{c}^2$ being inverses of one another, the only possibility is all of them being $\mathbf{0}$.

Since $\mathbf{a}$ and $\mathbf{c}$ are also perpendicular to each other, we get $\mathbf{a}\times\mathbf{c} = \mathbf{a}*\mathbf{c} = \mathbf{a}*\mathbf{a}*\mathbf{b} = \mathbf{0}*\mathbf{b} = \mathbf{b}$. Similarly, $\mathbf{c} \times \mathbf{b} = \mathbf{a}$. Denote $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as the vectors parallel to x, y, z axes, respectively. By rotating $\mathbf{a}$ and $\mathbf{b}$, we can assume that $\mathbf{a} = x\mathbf{i}$ and $\mathbf{b} = y\mathbf{j}$, making $\mathbf{c} = xy(\mathbf{i}\times\mathbf{j}) = (xy)\mathbf{k}$. But then $y\mathbf{j} = \mathbf{b} = \mathbf{a} \times \mathbf{c} = x^2y\mathbf{i} \times \mathbf{k} = -x^2y\mathbf{j}$, so $y = -x^2y$, or $x^2 = -1$ because $y \neq 0$. This is impossible since this forces $x = \sqrt{-1}$, which is imaginary. This contradiction shows that there cannot be $\mathbf{a}$ and $\mathbf{b}$ with nonzero cross products.

B1 Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Answer. No.

Solution. Denote the sum $a_1^m + a_2^m + a_3^m + \cdots$ as S_m. If $\sup_i\{a_i\} \leq 1$ then we have $S_1 \geq S_2 \geq \cdots$ which immediately invalidates the equation. Therefore we can choose k such that $a_k > 1$. However, we have $a_k^n > n$ for all n sufficiently large: this is because the function $\log n$ is $o(n)$ and so we can find n such that $\log n < cn$ where $c = \log a_k$.

B2 Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Answer. 3.

Solution. The above configuration is given by $A = (0, 0), B = (0, 3), C = (4, 0)$, giving a 3-4-5 right triangle.

Now consider $AB = 1$ or $AB = 2$. If $AB = 1$ then it has to be parallel to one of the axis, so we name it $A = (0, 0), B = (0, 1)$. If $C = (x, y)$ with $x \neq 0$ then we have $x^2 + y^2$ and $x^2 + (y - 1)^2$ both positive integers. W.l.o.g., set $y \geq 1$, and let $x^2 + y^2 = z^2$ for some integer z, then $z > y > 0$ and

$$x^2 + y^2 - (2y - 1) = x^2 + (y - 1)^2 \leq (z - 1)^2 = z^2 - (2z - 1) = x^2 + y^2 - (2z - 1)$$

i.e. $y \geq z$, contradiction.

If $AB = 2$, since the only way to write $x^2 + y^2 = 4$ is where $(x, y) = (\pm 2, 0)$ and $(0, \pm 2)$, AB again has to be parallel to one of the axes, say, $(0, 0)$ and $(0, 2)$. Again let $C = (x, y)$ with $x \neq 0$, and let $x^2 + y^2 = z^2$ for some z. Now that z^2 and $x^2 + (y - 2)^2$ must have the same parity, consider the following cases:

- If $y = 1$, then we have $x^2 + 1 = z^2$, which is impossible unless $x = 0$.
- If $y \neq 1$, by symmetry we can consider $y > 1$ case only. Then $(y - 2)^2 < y^2$. Then

  $$x^2 + y^2 - (4y - 4) = x^2 + (y - 2)^2 \leq (z - 2)^2 = z^2 - (4z - 4) = x^2 + y^2 - (4z - 4)$$

  i.e. $y \geq z$, which is also contradiction.

B3 There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Answer. $n = 1005$.

Solution. If $n \leq 1004$, then $2010n \leq \frac{2010\times 2009}{2} = 0 + 1 + \cdots + 2009$, so we can distribute the balls such that B_i contains $\leq i - 1$ balls. For example, do it greedily by distributing $i - 1$ balls to B_i for $i = 1, 2, \cdots$. Then no redistribution move is possible.

Now consider $n \geq 1005$, i.e. $2010n > \frac{2010 \times 2009}{2} = 0 + 1 + \cdots + 2009$. This means at each stage, there's at least one box B_i with at least i balls. Now we consider the following iteration:

- Let B_i to have at least i balls. If $i > 1$, then move those i balls to B_1.
- Otherwise, if the only i with B_i at least i balls is B_1, then choose $j > 1$ such that B_j has at least 1 ball (but fewer than j balls). Now move 1 ball from B_1 to B_j, and repeat until B_j has exactly j balls, and now move all the j balls from B_j back to B_1.
- Repeat all of the above until B_1 has all the balls.

Once this is done, we can now distribute the balls one-at-a-time from B_1 to the other boxes, such that each has n balls.