Putnam 2021

A1 A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Answer. 578.

Solution. The equality can be achieved by doing 288 times (4, 3) and 288 times (3, 4), which reaches (2016, 2016), and then (5, 0) and (0, 5). The lower bound is achieved by noting that at each iteration, the sum of coordinates can increase by no more than 7:

$$a + b \leq \sqrt{2(a^2 + b^2)} = \sqrt{50} < 8$$

so we need at least $\lceil \frac{4042}{7} \rceil = 578$ steps.

A2 For every positive real number x, let

$$g(x) = \lim_{r \to 0} ((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}.$$

Find $\lim_{x\to\infty} \frac{g(x)}{x}$.

Answer. e.

Solution. Let's consider the generalized binomial $\begin{pmatrix} r \\ k \end{pmatrix} = \frac{r(r-1)\cdots(r-k+1)}{k!}$ for r that doesn't have to be nonnegative integer. Then for $|x| < 1$ we have $(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$. We also have the property $\begin{pmatrix} r+1 \\ k \end{pmatrix} = \frac{r+1}{k} \begin{pmatrix} r \\ k-1 \end{pmatrix}$ for all $k \geq 1$.

Now, we have

$$(x+1)^{r+1} - x^{r+1} = x^{r+1} \left[(1 + \frac{1}{x})^{r+1} - 1 \right] = x^{r+1} \sum_{k=1}^{\infty} \begin{pmatrix} r+1 \\ k \end{pmatrix} \frac{1}{x^k} = x^r (r+1) \sum_{k=0}^{\infty} \begin{pmatrix} r \\ k \end{pmatrix} \frac{1}{x^k(k+1)}$$

Denote $h(r, x) = \left((r+1) \sum_{k=0}^{\infty} \binom{r}{k} \frac{1}{x^k(k+1)} \right)^{\frac{1}{r}}$ for all $r > 0$, and $h(x) = \lim_{r\to 0} h(x, r)$. Then

$$g(x) = \lim_{r\to 0} \left[x^r (r+1) \sum_{k=0}^{\infty} \begin{pmatrix} r \\ k \end{pmatrix} \frac{1}{x^k(k+1)} \right]^{1/r} = \lim_{r\to 0} x h(x, r) = x h(x)$$

And thus the quantity of interest now becomes $h(x)$.

We see that

$$r + 1 \leq (r+1) \sum_{k=0}^{\infty} \begin{pmatrix} r \\ k \end{pmatrix} \frac{1}{x^k(k+1)} \leq (r+1) \sum_{k=0}^{\infty} \begin{pmatrix} r \\ k \end{pmatrix} \frac{1}{x^k} = (r+1) \left(1 + \frac{1}{x} \right)^r$$

and so, taking limit $r \to 0$,

$$\lim_{r\to 0} (r+1)^{1/r} \leq h(x) \leq \lim_{r\to 0} (r+1)^{1/r} \left(1 + \frac{1}{x} \right)$$

But given that it's well-known that $\lim_{r\to 0} (r+1)^{1/r} = e$, we have $e \leq h(x) \leq e(1 + \frac{1}{x})$. Finally, as $\frac{1}{x} \to 0$ as $x \to \infty$, we have $h(x) \to e$ as $x \to \infty$.

A5 Let A be the set of all integers n such that $1 \le n \le 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let

$$S(j) = \sum_{n \in A} n^j.$$

Determine all values of j such that $S(j)$ is a multiple of 2021.

Answer. Any integer j that's divisible by neither 42 nor 46.

Solution. Let's be a bit more ambitious and replace 2021 with any general positive integer m. Write m into its prime factorization:

$$m = \prod_{i=1}^{k} p_i^{a_i}$$

with $a_i \ge 1$, and p_i primes. Now $m \mid S(j)$ if and only if $m \mid p_i^{a_i}$ for all i.

We now consider the set $A_{i,g} = A \cap \{a : a \equiv g \pmod{p_i^{a_i}}\}$. By Chinese Remainder Theorem, for each h there's exactly one solution a with $1 \le a \le m$ that satifies

$$a \equiv g \pmod{p_i^{a_i}} \qquad a \equiv h \pmod{\frac{m}{p_i^{a_i}}}$$

Therefore, from here we can deduce that $|A_{i,g}| = \phi\left(\frac{m}{p_i^{a_i}}\right)$, where ϕ is the Euler totient function. This is independent of g and therefore

$$S(j) \equiv \phi\left(\frac{m}{p_i^{a_i}}\right) \sum_{p_i \nmid n, 1 \le n \le p_i^{a_i}} n^j$$

Let's now dissect this property. For this purpose, we'll claim the following:

Lemma. The sum $\sum_{p_i \nmid n, 1 \le n \le p_i^{a_i}} n^j$ satisfies the following:

- If $p_i - 1 \nmid j$, then $p_i^{a_i} \mid \sum_{p_i \nmid n, 1 \le n \le p_i^{a_i}} n^j$.
- Otherwise, $v_p(\sum_{p_i \nmid n, 1 \le n \le p_i^{a_i}} n^j) = a_i - 1$.

with the only exception that none of these is fulfilled if $p_i = 2$ and $a_i = 1$.

Proof: consider, first, the case when p_i is odd. Take g as any primitive root of $p_i^{a_i}$, then

$$\sum_{p_i \nmid n, 1 \le n \le p_i^{a_i}} n^j \equiv \sum_{k=0}^{(p_i-1)p_i^{a_i-1}-1} g^{jk} = \frac{g^{j(p_i-1)p_i^{a_i-1}} - 1}{g^j - 1}$$

and therefore, if $p_i - 1 \nmid j$, then $p_i \nmid g^j$. Since $g^{j(p_i-1)p_i^{a_i-1}} \equiv 1 \pmod{p_i^{a_i}}$ by Euler's theorem, it follows that $p_i^{a_i} \mid \sum_{p_i \nmid n, 1 \le n \le p_i^{a_i}} n^j$.

Conversely, if $p_i - 1 \mid j$, let $\ell = \min(a_i - 1, v_p(\frac{j}{p_i - 1}))$. Then the numbers g^{jk} are in the form of $p^{\ell+1}x + 1$ for $x = 0, 1, \cdots, p^{a_i - \ell - 1} - 1$, each of them occuring $p^\ell(p_i - 1)$ times. To see why, viewing the group $\mathbb{Z}_{p_i^{a_i}}^{\otimes}$ (i.e. all numbers n with $p_i \nmid n$, $1 \le n \le p_i^{a_i}$) generated by g, the subgroup

$$\{g^0, g^j, \cdots, g^{(p_i-1)p_i^{a_i-1}j-1}\}$$

has $p^{a_i - \ell - 1}$ distinct elements. By Euler's theorem, again, each element g^{jk} must satisfy $p^{\ell+1} \mid g^{jk} - 1$, so

$$\{g^0, g^j, \cdots, g^{(p_i-1)p_i^{a_i-1}j-1}\} = \{p^{\ell+1}x + 1 : x = 0, 1, \cdots, p^{a_i-\ell-1} - 1\}$$

Now with this, it's easy to compute our desired sum:

$$\sum_{k=0}^{(p_i-1)p_i^{a_i-1}-1} g^{jk} = p^\ell(p_i-1)\sum_{k=0}^{(p_i-1)p^{a_i-\ell-1}-1}(p^{\ell+1}x+1)$$

$$\equiv p^\ell(p_i-1)(p_i-1)p^{a_i-\ell-1} = (p_i-1)p^{a_i-1}$$

as desired.

If $p_i = 2$, the case $a_i = 1$ will only leave us with the sum 1 regardless of j, so assume $a_i \geq 2$. we can still show that if j is odd, then

$$1^j + 3^j + \cdots + (2^{a_i}-1)^j \equiv 1 + 3 + \cdots + (2^{a_i}-1) = 2^{2(a_i-1)} \equiv 0 \pmod{2^{a_i}}$$

We also have $a^2 \equiv 1 \pmod 8$ for all odd a, and for odd a and $k \geq 3$,

$$(a2^k+1)^2 \equiv a2^{2k} + a2^{k+1} + 1 \equiv 2^{k+1} + 1 \pmod{2^{k+2}}$$

so if $j = a \cdot 2^k$ with a odd and $1 \leq k \leq a_i - 2$, we have

$$\{1^j, 3^j, \cdots, (2^{a_i}-1)^j\} \equiv \{b2^{k+2}+1 : b = 0, \cdots, 2^{a_i-k-2}-1\}$$

Thus we can compute our desired sum as

$$2^{k+2}\sum_{b=0}^{2^{a_i-k-2}-1} b2^{k+2} + 1 \equiv 2^{a_i-1} \pmod{2^{a_i}}$$

Denote $v_p(n)$ by the highest power of p dividing n, then

$$v_{p_i}\left(\phi\left(\frac{m}{p_i^{a_i}}\right)\right) = \sum_{j\neq i} v_{p_i}(p_j^{a_j-1}(p_j-1)) = \sum_{j\neq i} v_{p_i}(p_j-1)$$

so this gives us the following:

- For each p_i, if there exists j with $p_i \mid p_j - 1$ then all the j would fulfill $p_i^{a_i} \mid \sum_{n\in A} n^j$.
- Otherwise, we will require $j \nmid p_i - 1$ whenever $p_i^{a_i} \neq 2$. For $p_i^{a_i} = 2$ there's no such j that can satisfy.

Finally, $2 \mid p_j - 1$ for all p_j odd prime, hence we can make a firm conclusion on m:

- if $m = 2$, no j would satisfy our condition.
- otherwise, for each p dividing m, either $p \mid \phi(m)$ or $p \nmid j$.

□

Now going back to the problem where $m = 2021 = 43 \times 47$, we have $43 \nmid 46 = \phi(47)$ and $47 \nmid 42 = \phi(43)$. Therefore j is suitable if and only if j is divisible by neither 42 nor 46.

B1 Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

Answer. $2 - \frac{6}{\pi}$.

Solution. Let the event of interest be E, and the orientation of the dropped unit square w.r.t. the checkerboard be θ (which we can fix $0 \leq \theta \leq \frac{\pi}{2}$). We first claim the following:

Lemma. $\mathbb{P}[E|\theta] = (\cos\theta + \sin\theta - 1)^2$.

Proof: let the center O of dropped unit square d to be contained inside the square $ABCD$ of the checkerboard, and let P be the center of $ABCD$. Consider, also, points A_1, B_1, C_1, D_1 such that $A_1B_1C_1D_1$ is a square with sides parallel to dropped square d, and A, B, C, D lie on segments $A_1D_1, A_1B_1, B_1C_1, C_1D_1$, respectively. Since the center O of d lies in $ABCD$, $A_1B_1C_1D_1$ and d cannot be disjoint. We consider the following cases:

Case 1. d lies entirely inside $A_1B_1C_1D_1$.
Now, d cannot intersect A, B, C, D since they lie on the sides of $A_1B_1C_1D_1$. The other lattice points (corners of original checkerboard squares) are outside the square $A_1B_1C_1D_1$, hence disjoint from d. We therefore have this d fulfilling the criteria of E.

Case 2. d and $A_1B_1C_1D_1$ overlap at exactly one edge of $A_1B_1C_1D_1$. Notice that the side length of $A_1B_1C_1D_1$ is $\cos\theta + \sin\theta$.
W.l.o.g. suppose that d intersects only A_1D_1. Since opposite sides of d are parallel to A_1D_1, they segment of A_1D_1 that's in d has length 1. On the other hand, A partitions A_1D_1 into lengths of $\cos\theta$ and $\sin\theta$. It follows that d necessarily contains A (hence violating E).

Case 3. d and $A_1B_1C_1D_1$ overlap at exactly one corner of $A_1B_1C_1D_1$.
W.l.o.g. let A_1 be the corner covered by d. If d contains none of A, B, then from side lengths of d parallel to A_1A and A_1B we have $OA_1 < O_A$ and $OA_1 < O_B$. The above inequality determines a region (via perpendicular bisectors of AA_1 and A_1B) with corner at midpoint of B, and otherwise outside of the checkerboard square $ABCD$. This contraidcts that O is inside $ABCD$. We therefore cannot have E realized in this case.

This means that we have E fulfilled if and only if Case 1 applies. Given that $A_1B_1C_1D_1$ has side length $\cos\theta+\sin\theta$, we have O inside the square of center P, parallel to $A_1B_1C_1D_1$, and has side length $\cos\theta + \sin\theta - 1$, proving the lemma. $\square$

Now given that θ is uniform in $[0, \frac{\pi}{2}]$, we have

$$\mathbb{P}[E] = \int_0^{\frac{\pi}{2}} \mathbb{P}[E|\theta]\mathbb{P}[\theta]d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (\cos\theta+\sin\theta-1)^2 d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (2+\sin 2\theta - 2\cos\theta - 2\sin\theta)d\theta$$
$$= \frac{2}{\pi}(\pi + 1 - 2 - 2) = 2 - \frac{6}{\pi}$$

B3 Let $h(x, y)$ be a real-valued function that is twice continuously differentiable throughout $\mathbb{R}^2$, and define
$$\rho(x, y) = yh_x - xh_y.$$
Prove or disprove: For any positive constants d and r with $d > r$, there is a circle S of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of S is zero.

Answer. The statement is true.

Solution. We first show that integrating ρ over any circle with center $(0, 0)$ is 0. Consider $R > 0$ and $(x, y) = (R\cos\theta, R\sin\theta)$. Also let $g(\theta) = h(R\cos\theta, R\sin\theta)$. Then

$$g'(\theta) = -R\sin\theta h_x(R\cos\theta, R\sin\theta) + R\cos\theta h_y(R\cos\theta, R\sin\theta) = -\rho(R\cos\theta, R\sin\theta)$$

And therefore

$$0 = g(2\pi) - g(0) = \int_0^{2\pi} g'(\theta)d\theta = -\int_0^{2\pi} \rho(R\cos\theta, R\sin\theta)d\theta = -\int_{||(x,y)||=R} \rho(x, y)d(x, y)$$

as claimed. $\square$

Now consider the circle $S_{\theta,d,r}$ with cneter at $(d\cos\theta, d\sin\theta)$ and radius r. We claim that

$$\int_{\theta=0}^{\theta=2\pi}\int_{x,y\in S_{\theta,d,r}} \rho(x, y)d(x, y)d\theta = 0 \tag{1}$$

To see this, we first see that for any function f, there exists a constant $p(d,r,R)$ such that

$$\int_{\theta=0}^{\theta=2\pi}\int_{x,y\in S_{\theta,d,r}} f(x,y)d(x,y)d\theta = \int_{R=d-r}^{R=d+r} p(d,r,R)\int_{||x,y||=R} f(x,y)d(x,y)dR$$

Indeed, let $(x,y)=(R\cos\alpha, R\sin\alpha)$, then the circle $S_{\theta,d,r}$ containing (x,y) must satisfy

$$r^2 \geq (x-d\cos\theta)^2+(y-d\sin\theta)^2 = R^2+d^2-2d(x\cos\theta+y\sin\theta)$$

$$= R^2+d^2-2dR(\cos\theta\cos\alpha+\sin\theta\sin\alpha) = R^2+d^2-2dR\cos(\theta-\alpha)$$

i.e.

$$|\theta-\alpha| \leq \arccos\left(\frac{R^2+d^2-r^2}{2dR}\right)$$

(angle taken modulo 2π). Hence the allowable angle is $[\alpha-\arccos\left(\frac{R^2+d^2-r^2}{2dR}\right), \alpha+\arccos\left(\frac{R^2+d^2-r^2}{2dR}\right)]$, which has a window of $2\arccos\left(\frac{R^2+d^2-r^2}{2dR}\right)$. Thus we can take $p(d,r,R) = 2\arccos\left(\frac{R^2+d^2-r^2}{2dR}\right)$ and so

$$\int_{\theta=0}^{\theta=2\pi}\int_{x,y\in S_{\theta,d,r}} \rho(x,y)d(x,y)d\theta = \int_{R=d-r}^{R=d+r} p(d,r,R)\int_{||x,y||=R} \rho(x,y)d(x,y)dR = 0$$

establishing Identity 1.

Finally, since h is twice continuously differentiable, it follows that ρ is continuous in (x,y). This means the integral $\int_{(x,y)\in S_{\theta,d,r}} \rho(x,y)d(x,y)$ is also continuous w.r.t. θ and since they integrate to 0 over all θ, it follows that for some θ we have $\int_{(x,y)\in S_{\theta,d,r}} \rho(x,y)d(x,y)=0$.

B4 Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0=0, F_1=1$, and $F_n = F_{n-1}+F_{n-2}$ for $n\geq 2$. For $m>2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m-1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

Solution. We note that $F_m \geq 2$ for all $m\geq 3$. If F_m is composite, then either one of these holds:

- F_m is divisible by more than one primes, and therefore for any prime $p \mid F_m$ with $v_p(F_m)=a$ as the highest power of p dividing F_m, we have $p^a < F_m$ and so $p^a \mid \prod_{k=1}^{F_m-1} k^k$. Since this holds for all p, it follows that $R_m=0$.
- $F_m = p^k$ for some positive integer $k>1$ (i.e. a prime power). Consider the factor $(F_m-p)^{F_m-p}$. We have $p \mid F_m-p$, and since $k>1$, $F_m=p^k\geq p+k$ with equality iff $p=k=2$ (can be verified manually). Thus $F_m \mid (F_m-p)^{F_m-p}$ and we have $R_m=0$.

We therefore have $R_m=0$ whenever F_m is composite.

Now we focus on the case where F_m is prime. Notice for $F_m=2$ we have $R_m=1$ which is also Fibonacci number, so we may assume F_m is odd prime. Rewrite the product into the following:

$$\prod_{k=1}^{F_m-1} k^k \equiv \prod_{k=1}^{\frac{F_m-1}{2}} k^k \prod_{k=\frac{F_m+1}{2}}^{F_m-1} k^k \equiv \prod_{k=1}^{\frac{F_m-1}{2}} k^k (F_m-k)^{F_m-k}$$

$$= \prod_{k=1}^{\frac{F_m-1}{2}} (-1)^{F_m-k} k^{k+F_m-k} \equiv (-1)^{\sum_{(F_m+1)/2}^{F_m-1} j} \prod_{k=1}^{\frac{F_m-1}{2}} k$$

where the last equality we used $k^p \equiv k \pmod p$ for all integer k and prime p, due to Fermat's Little Theorem. Let's ignore the factor $(-1)^{\sum_{(F_m+1)/2}^{F_m-1} j}$ for now and focus on $\prod_{k=1}^{\frac{F_m-1}{2}} k$ instead, which we denote as S_m. This means $R_m = \pm S_m$.

By Wilson's theorem, we have:

$$-1 \equiv \prod_{k=1}^{F_m-1} k = \prod_{k=1}^{(F_m-1)/2} k(-k) = (-1)^{\frac{F_m(F_m-1)}{2}} S_m^2 \Rightarrow S_m^2 \equiv \begin{cases} 1 & p \equiv 3 \pmod 4 \\ -1 & p \equiv 1 \pmod 4 \end{cases}$$

We now proceed to the following facts about Fibonacci numbers:

Lemma 1. If F_m is odd prime, then either $m = 4$ or m is prime.

Proof: It suffices to show that $a \mid b$ implies $F_a \mid F_b$. Indeed, consider F_m modulo F_a. Then $F_a \equiv 0$ and since $F_1 = F_2 = 1$, we have $F_{k+a} \equiv F_k F_{a+1}$ for all $k \geq 0$ (which can be established iteratively). This means $F_{na} \equiv F_a F_{a+1}^{n-1} \equiv 0 \pmod{F_a}$, establishing the claim. In particular, if m is composite and $a \mid m$ with $a < m$, then $F_a \mid F_m$. If $a > 2$ this would immediately imply that F_m is composite. Such a can be chosen when $m > 2^2$, i.e. $m \geq 5$. Hence either m is prime or $m = 4$ (in which case $F_m = 3$).

Lemma 2. For $m \geq 0$, $F_m F_{m+2} - F_{m+1}^2 = (-1)^{m-1}$.

Proof: this holds for $m = 0$, and for $m > 1$:

$$F_m F_{m+2} - F_{m+1}^2 = F_m(F_m + F_{m+1}) - F_{m+1}^2 = F_m^2 - F_{m+1}(F_{m+1} - F_m) = F_m^2 - F_{m+1}F_{m-1}$$

and by inductive hypothesis, $F_m^2 - F_{m+1}F_{m-1} = (-1)(-1)^m = (-1)^{m-1}$, establishing the proof.

In particular, $F_m^2 \equiv (-1)^{m-1} \pmod{F_{m+1}}$.

To complete the proof, if $m = 4$ and $F_m = 3$, then $R_m = 1$. Otherwise, if m is odd prime ≥ 4, then considering F_m modulo 4 we have $0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, \cdots$ i.e. period 6. Here $F_m \equiv 3 \pmod 4$ iff $m \equiv 4 \pmod 6$, implying m is composite. We thus have m odd and $F_m \equiv 1 \pmod 4$, which means:

$$F_{m-1}^2 \equiv F_{m-2}^2 \equiv -1 \pmod{F_m} \qquad S_m^2 \equiv -1 \pmod{F_m}$$

and since the solution to $S_m^2 \equiv -1$ has exactly two solutions among $1, 2, \cdots, F_m - 1$ given that F_m is prime, we have $S_m \in \{F_{m-1}, F_{m-2}\}$. It follows that $R_m \in \{F_{m-1}, F_{m-2}\}$, too.

B5 Say that an n-by-n matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ with integer entries is very odd if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \geq 1$.

Solution. Consider the directed graph $G = (V, E)$, corresponding to A, with $V = \{1, 2, \cdots, n\}$ and $(i, j) \in E$ if and only if A_{ij} is odd. We start with the following claim:

Lemma. A is very odd if and only if both the following hold:

- $(i, i) \in E$ for all i (that is, self loop for each vertex);
- Apart from the self-loops described above, G has no directed cycle.

Only-if: if $S = \{i\}$ for any i, we need A_{ii} odd, so $(i, i) \in E$. To show the second point, suppose for sake of contradiction that there's a cycle of length at least 2. Choose a cycle with minimal length (which is still ≥ 2):

$$a_1 \to a_2 \to \cdots a_k \to a_1$$

Consider the subgraph with vertices in $\{a_1, \cdots, a_k\}$ and also $S = \{a_1, \cdots, a_k\}$. We first show that the only edges in this subgraph is (a_i, a_i) and (a_i, a_{i+1}) for $i = 1, 2, \cdots, k$

(indices taken modulo k). Suppose that there's another edge (i, j) such that $j - i \neq 1 \pmod{k}$. This means we have another cycle:

$$a_i \to a_j \to \cdots \to a_k \to a_1 \cdots \to a_i$$

which has length $k - (j - i) + 1$ for $j > i$, and $i - j + 1$ for $j < i$. This gives a cycle of length $2 \leq \ell < k$, contradicting the minimality of k.

Now we consider $|S|$. Denote P_S as the set of permutations on S, then we have

$$|S| = \sum_{\sigma \in P_S} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{k} A_{a_i \sigma(a_i)} \equiv \sum_{\sigma \in P_S} \prod_{i=1}^{k} A_{a_i \sigma(a_i)}$$

$$\equiv |\{\sigma \in P_S : A_{a_i \sigma(a_i)} \text{ odd}, \forall i = 1, \cdots, k\}| \pmod 2$$

In our case, given the nature of our graph, we have

$$A_{a_i \sigma(a_i)} \text{ odd}, \forall i = 1, \cdots, k$$

if and only if σ is identity, or σ sends a_i to a_{i+1}. This means $|S|$ will be even, hence A isn't very odd.

If: Now consider where G has no directed cycle with all the self-loops. Consider any set S, and we still have

$$|S| \equiv |\{\sigma \in P_S : A_{i\sigma(i)} \text{ odd } \forall i \in S\}| \equiv |\{\sigma \in P_S : (i, \sigma(i)) \in E, \forall i \in S\}|$$

Given that a permutation can be seen as disjoint union of cycles, the only $\sigma \in P_S$ that satisfies above is when σ is identity in S, which means $|S|$ is odd. $\square$

Now we're left with proving that A^k also satisfies the condition above. Now consider the following:

$$A^k_{ij} = \sum_{a_1, \cdots, a_{k-1}} A_{ia_1} A_{a_1 a_2} \cdots A_{a_{k-1} j}$$

for $k \geq 2$. This is equivalent to the number of ways to go from i to j in exactly k steps, including self-loops.

If $i = j$, the only possible way above is for $a_1 = a_2 = \cdots = a_{k-1} = i$ given that A has no directed cycle, so A^k_{ii} is odd for each i. To show that A^k cannot have any directed cycle of length ≥ 2, consider

$$a_1 \to a_2 \to \cdots \to a_\ell \to a_1$$

which each $\to$ meaning a path of length $\leq k$. This would imply a cycle somewhere in the path that's more than just self-loop (since $a_\ell \neq a_1$). It therefore follows that either $A_{a_i} A_{a_{i+1}} = 0$ for some i, or $A_{a_\ell a_1} = 0$. This would imply that A^k must not have nontrivial directed cycle, too, so A^k is also very odd.