Solution to APMO 2012 Problems

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1. Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of the triangle ABC must be 6 if the area of each of the triangles PFA, PDB and PEC is 1.

   Solution. Denote the areas of PDC, PEA and PFB as a, b, c respectively. Ceva's theorem say that $abc = 1$. We'll in fact claim that $a = b = c = 1$. Observe the following too:
$$a = \frac{[PDC]}{[PDB]} = \frac{DC}{BD} = \frac{[DAC]}{[ABD]} = \frac{a+b+1}{c+1+1}$$
   which follows that $\frac{b+1}{c+1} = a$. Similarly, $\frac{c+1}{a+1} = b$ and $\frac{a+1}{b+1} = c$.

   If $a > 1$, then $b > c$ and from $abc = 1$ we have $c < 1$. This means $a < b$ based on $\frac{a+1}{b+1} = c$. However, $a < b$ also implies that $b > 1$ and therefore $1 > c > a > 1$, contradiction. A similar contradiction can be attained if $a < 1$.

   This leaves with $a = 1$ and therefore $b = c$. But then $bc = 1$ so $b = c = 1$, and the conclusion then follows.

2. Into each box of a 2012×2012 square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the 2012×2012 numbers inserted into the boxes.

   Answer. It's 5.

   Solution. I will construct the example for 5 later as I need to brush up this part of my LaTeX.

   To show that 5 is the upper bound, consider sliding the vertical line from left to right. The left has sum increasing and right has sum decreasing in this process. By the problem condition, at one point (say when the line is at k with $1 \leq k \leq 2012$) the left part has sum ≤ 1 but after a slide to the right the right part has sum ≤ 1. This means:

   - The sum of columns $1, 2, \cdots, k$ is ≤ 1
   - The sum of columns $k+2, \cdots, n = 2012$ is ≤ 1

   We could have let $k = 0$ or 2012, but this will make the sum of the whole grid ≤ 1 which is no longer fun.

   Similarly, there's an m such that the sum of rows $1, \cdots, m$ is ≤ 1 and sum of rows $m+2, \cdots, 2012$ is ≤ 1. Thus, the sum of parts of grids that have columns $\leq k$, $\geq k+2$, or rows $\leq m$ or $\geq m+2$ is at most 4(some part of the grid is counted > 1 times but that's okay for this upper bound since all numebers are nonnegative). The only part not covered is $(k+1, m+1)$ which has sum ≤ 1, hence 5 is an upper bound.

3. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

   Answer. Whenever $p = n$, and also $(p, n) = (2, 4)$. In all these cases $n^p + 1 = p^n + 1$.

   Solution. One main requirement is $n^p \geq p^n$. We first isolate the case $p = 2$. For $n = 1, \cdots, 4$ we see that $n = 2$ and 4 work. We show that $n^2 < 2^n$ for $n \geq 5$. Well this is because $2^4 = 4^2$ and for all $n \geq 4$:

$$\frac{(n+1)^2}{n^2} = \frac{n^2+2n+1}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2} < 1 + \frac{1}{2} + \frac{1}{4^2} < 2$$

   and therefore for all $n \geq 5$:

$$n^2 = 4^2 \prod_{k=4}^{n-1} \frac{(k+1)^2}{k^2} < 4^2 \prod_{k=4}^{n-1} 2 = 4^2 2^{n-4} = 2^n$$

   as desired.

   For prime $p \geq 3$, let's also prove that $n^p < p^n$ for all $n > p$. Well taking the np-th root of both sides and taking log gives the similar inequality $\frac{\ln n}{n} < \frac{\ln p}{p}$ and Calculus tells us that the function $\frac{\ln x}{x}$ is decreasing for $x > e$. However an elementary solution exists using the trick above:

$$\frac{(n+1)^p}{n^p} = (1 + \frac{1}{n})^p \leq (1 + \frac{1}{p})^p = \sum_{k=0}^{p} \binom{p}{k} p^{-k}$$

   Each term $\binom{p}{k} \cdot p^{-k} = p^{-k} \frac{p(p-1)\cdots(p-k+1)}{k!} \leq \frac{1}{k!}$ and again we can use the fact that $\sum \frac{1}{k!} < e < p$ to finish but it suffices to use

$$\sum_{k=0}^{p} \binom{p}{k} p^{-k} \leq \sum_{k=0}^{p-1} \binom{p}{k} p^{-k} + p^{-p} \leq \sum_{k=0}^{p-1} 1 + p^{-p} = p + p^{-p}$$

   so $(p+1)^p \leq p(p^p) + 1$. But the equality above cannot all hold (we have $\frac{1}{2!} < 1$ when $k = 2 < p$) and with $(p+1)^p \equiv 1 \pmod{p}$, $(p+1)^p \leq p(p^p) - p + 1 < p(p^p)$. It thus follows that

$$\frac{(n+1)^p}{n^p} = (1 + \frac{1}{n})^p \leq (1 + \frac{1}{p})^p < p$$

   as desired.

   Now that we have restricted our attention to $n \leq p$, since p is odd, $p^n + 1$ is even and so is $n^p + 1$. Therefore n is odd and this then follows $p + 1 \mid p^n + 1$. This means $p + 1 \mid n^p + 1$, i.e. $p + 1 \mid n^{2p} - 1$. This means $ord_{p+1}(n) \mid 2p$ and since p is prime, we have $ord_{p+1}(n) \in \{1, 2, p, 2p\}$. Since $p + 1 \nmid n^p - 1$ (as $p \geq 3$), $p + 1 \nmid n - 1$ as well and so $ord_{p+1}(n) \in \{2, 2p\}$.

   If $ord_{p+1}(n) = 2p$, then $2p \mid \phi(p+1)$, which is impossible because $\phi(p+1) < p + 1 \leq 2p$. Therefore, $ord_{p+1}(n) = 2$ and therefore $p + 1 \mid n^2 - 1$ and since $p + 1 \mid n^p + 1$, we have $n \equiv -1 \pmod{p+1}$ and so $p + 1 \mid n + 1$. This means $p \leq n$ and with $n \leq p$, we have $p = n$.

4. Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

   (Here we denote XY the length of the line segment XY.)

Solution. By sine rule on BCF we have $\frac{BF}{CF} = \frac{\sin \angle BCF}{\sin \angle CBF} = \frac{\sin \angle BED}{\sin \angle CED}$, and by considering the cevian ED on the triangle EBC we have

$$\frac{BD}{DC} = \frac{EB}{EC} \cdot \frac{\sin \angle BED}{\sin \angle CED} = \frac{EB}{EC} \cdot \frac{BF}{CF}$$

and similarly let EM to intersect Γ again at G, we have

$$1 = \frac{BM}{MC} = \frac{EB}{EC} \cdot \frac{BG}{CG}$$

so combining these two equations give

$$\frac{BF}{CF} = \frac{BD}{DC} \div \frac{EB}{EC} = \frac{BD}{DC} \cdot \frac{BG}{CG}$$

Now, $\angle BHC = \angle BGC = 180^\circ = \angle BAC$ and HG passes through the midpoint M of BC. Therefore $HBGC$ is a parallelogram and so $BG = CH$ and $CG = BH$. So we have

$$\frac{BF}{CF} = \frac{BD}{DC} \cdot \frac{BG}{CG} = \frac{BD}{DC} \cdot \frac{CH}{BH} = \frac{BD}{BH} \cdot \frac{CH}{CD} = \cos HBD \div \cos HCD = \frac{\cos DAC}{\cos DAB} = \frac{\sin ACD}{\sin ABD} = \frac{AB}{AC}$$

as desired.

5. Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \le i < j \le n} \frac{1}{n - a_i a_j} \le \frac{n}{2}$$

must hold.

Solution. Too hard. TODO.