Solution to APMO 2021 Problems

Anzo Teh

Problem 1. Prove that for each real number $r > 2$, there are exactly two or three positive real numbers x satisfying the equation $x^2 = r\lfloor x \rfloor$.

Solution. We consider the ratio $\frac{x^2}{\lfloor x \rfloor}$ for each $x \geq 1$, and use the fact that $x = \lfloor x \rfloor + \{x\}$ where $0 \leq \{x\} < 1$ is the fractional part of x. Now,

$$\frac{x^2}{\lfloor x \rfloor} = \frac{(\lfloor x \rfloor + \{x\})^2}{\lfloor x \rfloor} = \lfloor x \rfloor + 2\{x\} + \frac{\{x\}^2}{\lfloor x \rfloor}$$

Fixing $\lfloor x \rfloor$, this ratio lies in the range $[\lfloor x \rfloor, \lfloor x \rfloor + 2 + \frac{1}{\lfloor x \rfloor})$, and strictly increases with $\{x\}$. Therefore for each r in $[\lfloor x \rfloor, \lfloor x \rfloor + 2 + \frac{1}{\lfloor x \rfloor})$ there is exactly one x satisfying the quantity.

Therefore for each r, if integer k is such that $k \leq r < k + 1$, or $k + 1 \leq r < k + 2$, then there's exactly one x with $k \leq x < k + 1$ with $x^2 = r\lfloor x \rfloor$ (in other words, $k = \lfloor r \rfloor$ or $\lfloor r \rfloor - 1$). For $k + 2 \leq r < k + 3$ (i.e. $k = \lfloor r \rfloor - 2$) there may or may not be solution in this range). k in other range will not work. This means that two or three positive real x satisfies the given condition.

Problem 2. For a polynomial P and a positive integer n, define P_n as the number of positive integer pairs (a, b) such that $a < b \leq n$ and $|P(a)| - |P(b)|$ is divisible by n. Determine all polynomial P with integer coefficients such that $P_n \leq 2021$ for all positive integers n.

Answer. The two families of solutions are:

- $-x + d$, $d \leq 2022$
- $x + d$, $d \geq -2022$

Solution. The key is to show that $|P(n + 1) - P(n)| = 1$ for all n. There are two main facts of polynomials with integer coefficients that we'll use:

- Whenever $m \neq n$ are integers, we have $m - n \mid P(m) - P(n)$.
- There's a constant n_0 such that either $P(n) > 0$ for all $n \geq n_0$, or $P(n) < 0$ for all $n \geq n_0$ (depending on the sign of the leading coefficient), with the exception of the zero polynomial.

Suppose there's n and $n+1$ such that $|P(n+1) - P(n)| \neq 1$, then there exists $d > 1$ such that $d|P(n+1) - P(n)$. By replacing n with $n - dx$ for any x, and using the first property about integer polynomials, we can assume that $0 \leq n < d$.

Consider an arbitrary k and suppose that $P(n)$ and $P(n + 1)$ are both congruent to m mod d. Then

$$\{P(dx+n), P(dx+n+1)|x = 0, 1, \cdots, k-1\} \equiv \{m, d+m, \cdots, (k-1)d+m\} \pmod{kd}$$

W.l.o.g. assume that the leading coefficient of P is positive, which means that there exists an x_0 such that $P(dx + n)$ and $P(dx + n + 1)$ are both positive for all $x \geq x_0$. This will give us

$$\{|P(dx+n)|, |P(dx+n+1)| : x = x_0, x_0+1, \cdots, k-1\} \equiv \{m, d+m, \cdots, (k-1)d+m\} \pmod{kd}$$

which would make sense so long as $k \geq x_0$.

Now let $a_y = |\{x_0 \leq x \leq k-1 : |P(dx+n)| \equiv dy+m\} \cup \{x_0 \leq x \leq k-1 : |P(dx+n+1)| \equiv dy+m\}|$ (notice that the two sets $dx+n$ and $dx+n+1$ are disjoint since $d>1$). We have $\sum_{y=0}^{k-1} a_y = 2(k-x_0)$, and the number of pairs with similar congruence modulo dk is given by

$$\sum_{y=0}^{k-1} \binom{a_y}{2} = \sum_{y=0}^{k-1} \frac{a_y^2 - a_y}{2} = \sum_{y=0}^{k-1} \frac{a_y^2}{2} - (k-x_0) \geq \frac{2(k-x_0)^2}{k} - (k-x_0) \geq k - 3x_0$$

with the use of AM-GM insequality. Since this works for any $k \geq x_0$, choosing k with $k > 3x_0 + 2021$ yields the number of such pairs more than 2021.

Thus we're now restricted to $|P(n+1) - P(n)| = 1$, and since P is a polynomial, it follows that P must be linear and must have form $x+d$ or $-x+d$. W.l.o.g. consider the case $x+d$. If $d \geq 0$ then $P(1), \cdots, P(n) > 0$ and $|P(k)|$ will leave n distinct remainders modulo n for $k = 1, 2, \cdots, n$. Otherwise, the numbers $1, 2, \cdots, |d|-1$ will appear twice in $|P(1)|, \cdots, |P(n)|$ for n sufficiently large, which follows that the polynomial is valid iff $d \geq -2022$. A similar conclusion can be drawn to case $-x+d$.

Problem 3. Let $ABCD$ be a cyclic convex quadrilateral and Γ be its circumcircle. Let E be the intersection of the diagonals of AC and BD. Let L be the center of the circle tangent to sides AB, BC, and CD, and let M be the midpoint of the arc BC of Γ not containing A and D. Prove that the excenter of triangle BCE opposite E lies on the line LM.

Solution. We have $MB = MC$, so there's a circle Ω that's tangent to both MB and MC. Let I be the excenter of BCE opposite E. We claim that IB and CL, as well as IC and BL, intersect on Ω. For clarity let IB and CL intersect at X, and IC and BL intersect at Y.

Now, we have the following:

$$\angle LBI = \angle LBC + \angle CBI = \frac{1}{2}\angle ABC + 90^\circ - \frac{1}{2}\angle EBC = 90^\circ + \frac{1}{2}\angle ABD$$

and similarly $\angle LCI = 90^\circ + \frac{1}{2}\angle DCA$.

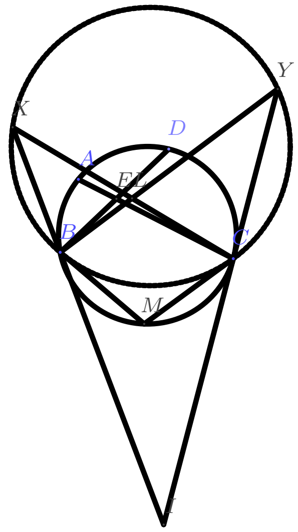
By the definition of L, we also have

$$\angle BLC = 180^\circ - \angle LBC - \angle LCB = 180^\circ - \frac{1}{2}\angle ABC - \angle DCB$$

So

$$\angle LBI + \angle BLC = 270^\circ - \frac{1}{2}(\angle ABC + \angle DCB - \angle DCA) = 270^\circ - \frac{1}{2}(\angle ABC + ACB) = 180^\circ + \frac{1}{2}BAC$$

which means that X is on the other side of BL than I with $\angle BXL = \frac{1}{2}BAC$ and similarly $\angle CYL = \frac{1}{2}BDC$, But since $\angle BAC = \angle BDC$, these angles are also equal. With $\angle MBC = \angle MCB = 90^\circ - \frac{1}{2}\angle BMC = \frac{1}{2}BAC = \frac{1}{2}BDC$, the claim follows.



Now with the claim, we can finish the solution: now that $BCYX$ is cyclic with circumcircle Ω, we consider the line IL, which, by Brokard's theorem, its pole is the intersection of XY and BC (say, Z). Meanwhile, MB and MC are tangeent to Ω so M's polar is BC. We have Z lies on BC by definition, so by La Hire's theorem, Z's polar (i.e. LI) contains BC's pole (i.e. M).