Solutions to Tournament of Towns, Spring 2018, Senior

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O-Level

1.

A-Level

2. Do there exist 2018 positive irreducible fractions, each with a different denominator, so that that the denominator of the difference of any two (after reducing the fraction) is less than the denominator of any of the initial 2018 fractions?

   Solution. Let $n = 2018$. Define:

   $$a_k = \frac{1}{2^k} + \frac{1}{2^n+1} = \frac{2^n + 2^k + 1}{2^k(2^n+1)} \quad \forall k = 1, 2, \cdots, n$$

   Then $a_k - a_\ell$ has denominator $2^{\max(k,\ell)} \leq 2^n$. We now show that $\gcd(2^n + 2^k + 1, 2^k(2^n + 1)) = 1$. Otherwise, there exists a prime p dividing both numerator and denominator, and therefore p divides either 2^k or $2^n + 1$. In the former case, $p = 2$ but $2^n + 2^k + 1$ is odd; in the latter case, $p \mid 2^n + 1$ so p is odd but then $2^n + 2^k + 1 \equiv 2^k \pmod{2^n + 1}$ so $p \mid 2^k$, contradicting that p is odd. So

   $$a_k = \frac{2^n + 2^k + 1}{2^k(2^n+1)}$$

   is already in its lowest term, with denominator $2^k(2^n+1) \geq 2(2^n+1) > 2^n$ (hence different for each of them), and certainly greater than 2^n.