

Putnam 2020 (No official contest)

A1 How many positive integers N satisfy all of the following three conditions?

- N is divisible by 2020.
- N has at most 2020 decimal digits.
- The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Answer. 508536.

Solution. Let N to have a 1's and b 0's, so $N = 10^b \times \underbrace{1 \cdots 1}_a$. Since N is divisible by 20, it must have at least 2 ending zeros. Moreover, $2020 = 101 \times 20$ and $\gcd(10, 101) = 1$, so $101 \mid \underbrace{1 \cdots 1}_a = \frac{10^a - 1}{9}$. Since $1111 = 101 \times 11$ but $101 \nmid 111, 11, 1$, we see that $\text{ord}_{101}(10) = 4$ so $4 \mid a$. This gives the complete characterization:

$$a + b = 2020, b \geq 2, 4 \mid a, a \geq 1$$

Now for each a we can pick $b = 2, 3, \dots, 2020 - a$ (thus giving $2019 - a$ choices). The maximum a is 2016. This gives the following:

$$\sum_{k=1}^{504} (2019 - 4k) = 2019 \times 504 - 4 \times 504 \times 505 \div 2 = 1009 \times 504 = 508536$$

A2 Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^k 2^{k-j} \binom{k+j}{j}.$$

Answer. 4^k .

Solution. Let our sum be S_k and do induction on it. Base case is given by $k = 0$ when we just have 1. Now suppose that our claim holds for some k , that is $S_k = 4^k$ for some $k \geq 0$. We have

$$S_{k+1} - 2S_k = \sum_{j=0}^{k+1} 2^{k+1-j} \binom{k+1+j}{j} - 2 \sum_{j=0}^k 2^{k-j} \binom{k+j}{j} = \binom{2(k+1)}{k+1} + 2 \sum_{j=1}^k 2^{k-j} \binom{k+j}{j-1}$$

where we used the fact that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$. Notice again that the equivalence

$$2 \sum_{j=1}^k 2^{k-j} \binom{k+j}{j-1} = 2 \sum_{j=0}^{k-1} 2^{k-j-1} \binom{k+1+j}{j} = \frac{1}{2} (S_{k+1} - 2 \binom{2k+1}{k} - \binom{2(k+1)}{k+1})$$

which gives

$$\frac{1}{2} S_{k+1} = 2S_k - \binom{2k+1}{k} + \frac{1}{2} \binom{2(k+1)}{k+1}$$

Given that $\binom{2k}{k} = 2 \binom{2k-1}{k-1}$, we also have $-\binom{2k+1}{k} + \frac{1}{2} \binom{2(k+1)}{k+1} = 0$ and therefore $S_{k+1} = 4S_k = 4 \cdot 4^k = 4^{k+1}$, as desired.

A3 Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Answer. This sequence diverges.

Solution. We first see that $a_n > 0$ all the while: if $0 < a_n \leq \frac{\pi}{2}$ for some n then $0 < a_{n+1} \leq 1$, so we have $0 < a_n \leq \frac{\pi}{2}$ for all n .

If $a_n \not\rightarrow 0$ then the series would diverge, so we may assume $a_n \rightarrow 0$. We first analyze the asymptotic behaviour of a_n^2 as a_n small:

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{(a_n - a_{n+1})(a_n + a_{n+1})}{a_n^2 a_{n+1}^2} = \frac{(a_n - \sin a_n)(a_n + \sin a_n)}{a_n^2 \sin^2 a_n} \underset{a_n \rightarrow 0}{\sim} \frac{\frac{a_n^3}{6} \cdot 2a_n}{a_n^4} = \frac{1}{3}$$

so as $n \rightarrow \infty$, we see that a_n^2 decays in the pace of $\frac{3}{n+c}$ for some constant c . Since $\sum \frac{1}{n}$ diverges, it follows that $\sum a_n^2$ diverges too.

A5 Let a_n be the number of sets S of positive integers for which

$$\sum_{k \in S} F_k = n,$$

where the Fibonacci sequence $(F_k)_{k \geq 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$ and begins $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$. Find the largest number n such that $a_n = 2020$.

Answer. $F_{4040} - 1$.

Solution. (TODO)

B3 Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \dots$, a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z , as a function of δ .

Answer. $1 + \frac{1}{\delta}$.

Solution. Let's consider the function $F_n(\delta) = \mathbb{P}[x_n < \delta]$, and $f_n(\delta) = \frac{d}{d\delta} F_n$. This has the following recursive formula:

$$F_1(\delta) = \delta \quad F_n(\delta) = F_{n-1}(\delta) + \int_{\delta}^1 \frac{\delta}{x} f_{n-1}(x) dx$$

Let's now show that

$$F_n(\delta) = \delta \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{\delta})^k}{k!} \right) \tag{1}$$

Using induction, we see that this holds for $n = 1$, and if this holds for some n , i.e.

$$F_n(x) = x \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{x})^k}{k!} \right) \quad F_n(x) = \sum_{k=0}^{n-1} \frac{\ln(\frac{1}{x})^k}{k!} - \sum_{k=1}^{n-1} \frac{\ln(\frac{1}{x})^{k-1}}{(k-1)!} = \frac{\ln(\frac{1}{x})^{n-1}}{(n-1)!}$$

and therefore

$$F_{n+1}(\delta) = \delta \left(\sum_{k=0}^{n-1} \frac{\ln(\frac{1}{\delta})^k}{k!} \right) + \int_{\delta}^1 \frac{\delta}{x} \frac{\ln(\frac{1}{x})^{n-1}}{(n-1)!} dx = \delta \left(\sum_{k=0}^n \frac{\ln(\frac{1}{\delta})^k}{k!} \right)$$

establishing 1.

To finish the solution, we have $\mathbb{P}[Z = k] = \mathbb{P}[x_n < \delta \wedge x_{n-1} \geq \delta] = F_k(\delta) - F_{k-1}(\delta) = \delta \frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!}$. Therefore,

$$\begin{aligned}\mathbb{E}(Z) &= \sum_{k=1}^{\infty} k \delta \frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} \\ &= \sum_{k=1}^{\infty} \delta \frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} + (k-1) \delta \frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} \\ &= \sum_{k=1}^{\infty} \delta \frac{\ln(\frac{1}{\delta})^{k-1}}{(k-1)!} + \sum_{k=2}^{\infty} \delta \frac{\ln(\frac{1}{\delta})^{k-2}}{(k-2)!} \\ &= \delta e^{\ln(\frac{1}{\delta})} + \delta \ln(\frac{1}{\delta}) e^{\ln(\frac{1}{\delta})} \\ &= 1 + \ln \frac{1}{\delta}\end{aligned}$$

as claimed.

- B4** Let n be a positive integer, and let V_n be the set of integer $(2n+1)$ -tuples $\mathbf{v} = (s_0, s_1, \dots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \dots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

Answer. $\frac{1}{4040}$.

Solution. In general, we show that the average of $1 + \sum_{j=1}^{2n-1} \alpha^{s_j}$ over V_n is $\frac{1}{2n}$ for any $\alpha > 0$.

Consider $\mathbf{v} = (s_0, s_1, \dots, s_{2n-1}, s_{2n})$ and $t(\mathbf{v}) := (t_0, \dots, t_{2n-1})$ be such that $t_k = s_{k+1} - s_k$. We say $\mathbf{v} \sim \mathbf{v}'$ if $t(\mathbf{v}')$ can be obtained from $t(\mathbf{v})$ via cyclic shift (hence we could also say $t(\mathbf{v}) \sim t(\mathbf{v}')$). Now t maps V_n to $T_n := \{(t_0, \dots, t_{2n-1}) \subseteq \{-1, 1\}^{2n}, \sum t_i = 0\}$, and is a bijection. Moreover, relation defined via cyclic shift is both symmetric (just reverse cycle) and transitive, and also $\mathbf{v} \sim \mathbf{v}$ holds for all \mathbf{v} . Thus \sim is an equivalence relation.

Denote, now, the equivalence class of each \mathbf{v} :

$$E_{\mathbf{v}} = \{\mathbf{v}' : \mathbf{v}' \sim \mathbf{v}\}$$

We'll show that the average of $\frac{1}{q}$ in $E_{\mathbf{v}}$ is $\frac{1}{2n}$. Let $t(\mathbf{v}) = (t_0, t_1, \dots, t_{2n-1})$ and for each $\mathbf{v}' \sim \mathbf{v}$ can be written as $t(\mathbf{v}') = (t_j, t_{j+1}, \dots, t_{2n+j-1})$ for some $j \geq 0$ (indices taken modulo $2n$). Thus if $\mathbf{v} = (0, s_1, \dots, s_{2n-1}, 0)$ we have $\mathbf{v}' = (0, s_{j+1} - s_j, \dots, s_{2n} - s_j, s_1 - s_j, \dots, s_{j-1} - s_j, 0)$, and $\frac{1}{q(\mathbf{v}')} = \frac{\alpha^{s_j}}{q(\mathbf{v})}$. Now considering $j = 0, \dots, 2n-1$ we see that the average of $\frac{1}{q}$ is now

$$\frac{1}{2n} \sum_{j=0}^{2n-1} \frac{\alpha^{s_j}}{q(\mathbf{v})} = \frac{1}{2n}$$

since $\frac{\alpha^{s_j}}{q(\mathbf{v})}$ is just 1. But we're not done yet – we need to show that averaging over all $j = 0, \dots, 2n-1$ is the *true* average of this equivalence class. This is to show that as we loop over $j = 0, \dots, 2n-1$ each $\mathbf{v}' \in E_{\mathbf{v}}$ shows up equally many times. If we extend t_0, \dots, t_{2n-1} infinitely (hence having $2n$ as period) and let g as its *minimal* period, we see that each $\mathbf{v}' \in E_{\mathbf{v}}$ shows up $\frac{2n}{g}$ times, proving our claim.