Solution to IMO 2016 shortlisted problems.

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1 Algebra

A1 Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Thoughts. Non-homogeneous inequality, with the relation $\min(ab, bc, ca) \geq 1$ that we aren't sure how to use it (at its first sight). How should we solve it, then? The way, when desperate, is to brute force the whole thing by cubing the left-hand side and trying to expand $\left(\frac{a+b+c}{3}\right)^6 + 3\left(\frac{a+b+c}{3}\right)^4 + 3\left(\frac{a+b+c}{3}\right)^2 + 1$, but let's not make our lives miserable with (probably) hundreds of terms on the right.

Instead, we start with the following observation:

1. What happens when $a = b = c$? Then the equality holds! This motivates us to show that whenever $a+b+c$ is fixed, the maximal possible value of $(a^2+1)(b^2+1)(c^2+1)$ can be attained whenever $a = b = c$. With this in mind...
2. We want to see what happens to $(a^2+1)(b^2+1)$ by ranging all possible pairs of (a, b) such that $ab \geq 1$ and $a+b$ is fixed. As it turns out, $(a^2+1)(b^2+1) = a^2b^2+a^2+b^2+1$ (there are only four terms so it doesn't hurt to expand). Since $a+b$ is fixed in our context, we can write this in terms of $a+b$, giving $a^2b^2+a^2+b^2+1 = (a+b)^2+(ab-1)^2$. Now it's easy to see that this value increases with ab (as $ab \geq 1$).

Solution. We start with a preliminary observation: given that $k \geq 2$, and given the set of pairs $K = \{(a, b) : a+b = k, ab \geq 1\}$, then for any $(a_1, b_1), (a_2, b_2) \in K$, $(a_1^2+1)(b_1^2+1) \geq (a_2^2+1)(b_2^2+1)$ iff $|a_1 - b_1| \leq |a_2 - b_2|$. Indeed, for $(a, b) \in K$, $(a^2+1)(b^2+1) = (a+b)^2 + a^2b^2 - 2ab + 1 = k^2 + (ab-1)^2 = k^2 + (\frac{(a+b)^2-(a-b)^2}{2} - 1)^2 = k^2 + (\frac{k^2-(a-b)^2}{2} - 1)^2$, and given that $ab \geq 1$, this function is increasing in ab. In addition, with $a+b$ fixed, this function is also dereasing in $(a-b)^2$, which turns out to also be decreasing in $|a-b|$.

Now let $a+b+c = 3k$, and let $f(a, b, c) = (a^2+1)(b^2+1)(c^2+1)$. W.l.o.g. assume that $a \leq b \leq c$. Let $b \leq k$, then by above, $(b^2+1)(c^2+1) \leq (k^2+1)((b+c-k)^2+1)$ because $b+c \geq 2k$ ($a \leq k$ and $c \geq k$), and $(b+c-k) - k = (b+c) - 2k \leq (b+c) - 2b = c - b$ (since $b \leq k$), which follows that $k(b+c-k) \geq bc \geq 1$. Likewise, if $b \geq k$ then by above, $(a^2+1)(b^2+1) \leq ((a+b-k)^2+1)(k^2+1)$ because $b+a \leq 2k$ ($a \leq k$ and $c \geq k$), and $k - (a+b-k) = 2k - (b+c) \leq 2b - (a-b) = b - a$ (since $b \geq k$), which follows that $k(a+b-k) \geq ab = 1$. Additionally, after the operation, we have $a \leq k \leq b+c-k$ in the first case, and $a+b-k \leq k \leq c$ (a good question to ask might be: what if $a+b-k < 0$ in the second case? This is impossible because we only do this when $b \geq k$). So we only need to verify that the first two has product at least one. In case 1, $ak \geq ab = 1$ and in case 2 we have already verified that $k(a+b-k) \geq ab = 1$. Thus $f(a, b, c) \leq f(a, k, (b+c-k))$ in case

1, or $f(a,b,c) \leq f((a+b-k),k,c)$ in case 2. This means we can focus on the case where $b=k$. Nevertheless, when $b=k$ we have $a+c=2k$ and by similar procedure we have $(a^2+1)(c^2+1) \leq (k^2+1)^2$, therefore we have: $f(a,b,c) \leq f(a,k,(b+c-k)) \leq f(k,k,k)$ in case 1, $f(a,b,c) \leq f((a+b-k),k,c) \leq f(k,k,k)$ in case 2, and $f(k,k,k) = (k^2+1)^3$ is precisely the cube the right hand side. Q.E.D.

A2 Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \leq C.$$

Solution. The answer is $\frac{1}{2}$.

First, let $a_1 = 1, a_2 = a_3 = a_4 = 2$, $a_5 = n$ for some extremely large real number n. The possible numbers of $\frac{a_i}{a_j}$ are $\frac{1}{n}, \frac{2}{n}, \frac{1}{2}, 1, 2, \frac{n}{2}, n$. Observe that the ratios $\frac{1}{n}, \frac{2}{n}, \frac{n}{2}, n$ all involve a_5, so there cannot exist distinct i, j, k, l satisfying $\frac{a_i}{a_j}, \frac{a_k}{a_l} \in \{\frac{1}{n}, \frac{2}{n}, \frac{n}{2}, n\}$. In addition, both $\frac{1}{2}$ and 2 will involve a_1, meaning that there cannot exist distinct i, j, k, l satisfying $\frac{a_i}{a_j}, \frac{a_k}{a_l} \in \{\frac{1}{2}, 2\}$. Since there are three i's satisfying $a_i = 2$, there cannot be distinct i, j, k, l satisfying $\frac{a_i}{a_j} = \frac{a_k}{a_l} = 1$. We therefore know that $\frac{a_i}{a_j} = \frac{a_k}{a_l}$ is impossible, and same goes to $\frac{a_i}{a_j} = \frac{1}{n}, \frac{a_k}{a_l} = \frac{2}{n}$. We also have $n - \frac{n}{2} > \frac{n}{2} - 2 > 2 - 1 > 1 - \frac{1}{2} > \frac{1}{2} - \frac{2}{n}$ for sufficiently large real n, and $\lim_{n\to\infty} \frac{1}{2} - \frac{2}{n} = \frac{1}{2}$. Therefore $C \geq \frac{1}{2}$.

Now, we show that $C = \frac{1}{2}$ fits in all situations. W.L.O.G. let $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, and let $a = \frac{a_2}{a_1}$, $b = \frac{a_3}{a_2}$, $c = \frac{a_4}{a_3}$, $d = \frac{a_5}{a_4}$. Observe that $a, b, c, d \geq 1$. Suppose that for some a, b, c, d, $C = \frac{1}{2}$ does not fit for any of the four distinct subscripts. Now, considering $|\frac{a_1}{a_4} - \frac{a_2}{a_5}| = \frac{b}{c}|\frac{1}{a} - \frac{1}{d}|$ and from $0 \leq \frac{1}{a}, \frac{1}{d} \leq 1$ we have $\frac{b}{c} \geq \frac{b}{c}|\frac{1}{a} - \frac{1}{d}| > C = \frac{1}{2}$ so $b, c < 2$. Next, $C < |\frac{a_1}{a_3} - \frac{a_2}{a_4}| = \frac{b}{|}\frac{1}{a} - \frac{1}{c}| \leq |\frac{1}{a} - \frac{1}{c}|$ and from $\frac{1}{c} < \frac{1}{2}$ we must have $\frac{1}{a} > \frac{1}{2}$. Similarly, $C < |\frac{a_2}{a_4} - \frac{a_3}{a_5}| = \frac{c}{|}\frac{1}{b} - \frac{1}{d}| \leq |\frac{1}{b} - \frac{1}{d}|$ and from $\frac{1}{b} < \frac{1}{2}$ we must have $\frac{1}{d} > \frac{1}{2}$. Looking back, we have $\frac{1}{2} < |\frac{a_1}{a_4} - \frac{a_2}{a_5}| = \frac{b}{c}|\frac{1}{a} - \frac{1}{d}| \leq |\frac{1}{a} - \frac{1}{d}|$, yet $\frac{1}{2} < \frac{1}{a}, \frac{1}{d} \leq 1$, contradiction.

A3 Find all positive integers n such that the following statement holds: Suppose real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, each of which is either -1 or 1, such that

$$\left| \sum_{i=1}^{n} \varepsilon_i a_i \right| + \left| \sum_{i=1}^{n} \varepsilon_i b_i \right| \leq 1.$$

Solution. The answer is all odd n. We first find a counterexample for all $n = 2k$ for some $k \geq 1$ an integer. Consider $a_i = 1$, $b_i = 0$ for all $i \in [1, 2k-1]$, and $a_{2k} = 0, b_{2k} = 1$. Then $\sum_{i=1}^{n} \varepsilon_i b_i = \varepsilon_{2k}$, which has absolute value of 1. Also $\sum_{i=1}^{n} \varepsilon_i a_i = \sum_{i=1}^{2k-1} \varepsilon_i$. If x of the indices $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2k-1}$ are 1 and the rest -1, then the value would be $x - (2k-1-x) = 2x - 2k + 1$, which is an odd integer. Thus it has absolute value at least 1 too. Therefore we have $|\sum_{i=1}^{n} \varepsilon_i a_i| + |\sum_{i=1}^{n} \varepsilon_i b_i| \geq 1 + 1 = 2$.

Now let n be odd. We start with the following lemma: Let $(a_i, b_i), i \in [1, m]$ satisfy $0 \leq a_i, b_i \leq 1$ and $a_i + b_i = 1, \forall i \in [1, m]$. Then there exists $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m \in \{-1, 1\}$ satisfying

$$\begin{array}{rcll} 0 \leq \sum_{i=1}^{m} \varepsilon_i a_i, \sum_{i=1}^{m} \varepsilon_i b_i \leq 1 & \text{and} & \sum_{i=1}^{m} \varepsilon_i a_i + \sum_{i=1}^{m} \varepsilon_i b_i = 1 & \text{if } m \text{ is odd,} \\ -1 \leq \sum_{i=1}^{m} \varepsilon_i a_i, \sum_{i=1}^{m} \varepsilon_i b_i \leq 1 & \text{and} & \sum_{i=1}^{m} \varepsilon_i a_i + \sum_{i=1}^{m} \varepsilon_i b_i = 0 & \text{if } m \text{ is even.} \end{array}$$

In the first case, we notice that the second condition can be achieved whenever exactly $\frac{m+1}{2}$ of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ are 1 and the rest -1. For any combination of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ with exactly $\frac{m+1}{2}$ of them as 1 we have $\sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = \sum_{i=1}^m \varepsilon_i = \frac{m+1}{2} - \frac{m-1}{2} = 1$. The aim now is to assign $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ in a way that exactly $\frac{m+1}{2}$ of them are 1, and $0 \le \sum_{i=1}^m \varepsilon_i a_i \le 1$ (if this is true, $0 \le \sum_{i=1}^m \varepsilon_i b_i \le 1$ holds true too). W.L.O.G. assume that $a_1 \le a_2 \le \cdots \le a_m$, and consider the numbers $x_0, x_1, \cdots, x_{\frac{m-1}{2}}$ such that

$$x_k = \sum_{i=1}^{k} -a_i + \sum_{i=k+1}^{k+\frac{m+1}{2}} a_i + \sum_{i=k+\frac{m+1}{2}+1}^{m} -a_i$$

Notice, first, that for each x_i, exactly $\frac{m+1}{2}$ of the a_i's has coefficient 1 and the rest -1. Therefore if $0 \le x_k \le 1$ for some k we are done. Observe also that, $x_0 = \sum_{i=1}^{\frac{m+1}{2}} a_i + \sum_{i=\frac{m+1}{2}+1}^m -a_i \le \sum_{i=1}^{\frac{m+1}{2}} a_{\frac{m+1}{2}} + \sum_{i=\frac{m+1}{2}+1}^m -a_{\frac{m+1}{2}} = a_{\frac{m+1}{2}} \le 1$. $x_{\frac{m-1}{2}} = \sum_{i=1}^{\frac{m-1}{2}} -a_i + \sum_{i=\frac{m+1}{2}}^m a_i \ge \sum_{i=1}^{\frac{m-1}{2}} -a_{\frac{m+1}{2}} + \sum_{i=\frac{m+1}{2}}^m a_{\frac{m+1}{2}} = a_{\frac{m+1}{2}} \ge 0$. Moreover, $x_k - x_{k-1} = a_{k+\frac{m+1}{2}} - a_k - a_k + a_{k+\frac{m+1}{2}} = 2(a_{k+\frac{m+1}{2}} - a_k)$, so $x_{k-1} \le a_k \le x_{k-1} + 2$ (since $|a_i - a_j| \le 1$). If $0 \le x_0 \le 1$ or $0 \le x_{\frac{m-1}{2}} \le 1$ then we are done. Otherwise we have $x_0 < 0$ and $x_{\frac{m-1}{2}} > 1$, which means there exists k such that $x_k > 1$ and $x_{k-1} \le 1$. If $x_{k-1} \ge 0$ we are done. Otherwise, we have $x_k - x_{k-1} > 1$ and therefore $a_{k+\frac{m+1}{2}} - a_k > \frac{1}{2}$, meaning that $a_{k+\frac{m+1}{2}} > \frac{1}{2}$. Now let $y_k = x_k - 2a_{k+\frac{m+1}{2}}$ we have $y_k \le x_{k-1} < 0$ but $y_k \ge x_k - 2 > -1$. This means $0 \le -y_k \le 1$ and $y_k = \sum_{i=1}^k a_i + \sum_{i=k+1}^{k+\frac{m+1}{2}-1} -a_i + \sum_{i=k+\frac{m+1}{2}}^m a_i$. It is not hard to see that exactly $\frac{m+1}{2}$ of the a_i's has coefficient 1 and the rest -1, so we are done.

The second case isn't much different from the first. The second condition can be achieved whenever exactly half of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ are 1 and the rest -1. To see why, we have $\sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = \sum_{i=1}^m \varepsilon_i = \frac{m}{2} - \frac{m}{2} = 0$. Again w.l.o.g. let $a_1 \le a_2 \le \cdots \le a_m$, and consider the numbers $x_0, x_1, \cdots, x_{\frac{m}{2}}$ such that

$$x_k = \sum_{i=1}^{k} -a_i + \sum_{i=k+1}^{k+\frac{m}{2}} a_i + \sum_{i=k+\frac{m}{2}+1}^{m} -a_i$$

Again each x_k satisfies exactly $\frac{m}{2}$ have coefficient 1 and the rest -1. Thus if $-1 \le x_k \le 1$ for some k we are done. Observe also that, $x_0 = \sum_{i=1}^{\frac{m}{2}} a_i + \sum_{i=\frac{m}{2}+1}^m -a_i \le \sum_{i=1}^{\frac{m}{2}} a_{\frac{m}{2}} + \sum_{i=\frac{m}{2}+1}^m -a_{\frac{m}{2}} = 0$. $x_{\frac{m}{2}} = \sum_{i=1}^{\frac{m}{2}} -a_i + \sum_{i=\frac{m}{2}+1}^m a_i \ge \sum_{i=1}^{\frac{m}{2}} -a_{\frac{m}{2}} + \sum_{i=\frac{m}{2}+1}^m a_{\frac{m}{2}} = 0$. If $-1 \le x_0 \le 1$ or $-1 \le x_{\frac{m}{2}} \le 1$ we are done. Otherwise we have $x_0 < -1$ and $x_{\frac{m}{2}} > 1$, which allows us to pick a k satisfying $x_k > 1$ and $x_{k-1} \le 1$. By the similar logic as in case 1 we have $0 \le x_k - x_{k-1} \le 2$. This means $x_{k-1} > -1$, which in turn gives $-1 \le x_{k-1} \le x_k$ and we are done.

To finish up the solution, we split the a_i's and b_i's into two groups: one with $a_i b_i \ge 0$, and the other one with $a_i b_i \le 0$ (if $a_i b_i = 0$ then we assign them arbitrarily). W.L.O.G. let the first group be (a_i, b_i); $i \in [1, m]$ and the second group be (a_i, b_i): $i \in [m+1, n]$. For $i \in [1, m]$, we can w.l.o.g. assume that $a_i, b_i \ge 0$ (indeed, if ε_i is a solution for (a_i, b_i) then $-\varepsilon_i$ is a solution for $(-a_i, -b_i)$). Similarly we can also assume that $i \in [m+1, n]$ we have $a_i \ge 0$ and $b_i \le 0$. Also we invoke another property: if $-1 \le x + y \le 1$ and $-1 \le x - y \le 1$ then $|x| + |y| \le 1$. Indeed, $|x| + |y| \in \{x+y, x-y, y-x, -x-y\}$ so $-1 \le x + y \le 1 \to -1 \le -x - y \le 1$, $-1 \le x - y \le 1 \to -1 \le -x + y \le 1$ so $-1 \le |x| + |y| \le 1$. (Bonus: try to show that this is an if and only if condition: convince yourself that $|x| + |y| = \max\{x+y, x-y, y-x, -x-y\}$.)

We split into two cases:

Case 1. m is odd. From our lemma, by some careful choices of ε_i's we have $0 \le \sum_{i=1}^m \varepsilon_i a_i, \sum_{i=1}^m \varepsilon_i b_i \le 1$ and $\sum_{i=1}^m \varepsilon_i a_i + \sum_{i=1}^m \varepsilon_i b_i = 1$. In addition, $-1 \le \sum_{i=m+1}^n \varepsilon_i a_i, \sum_{i=m+1}^n \varepsilon_i(-b_i) \le 1$

and $\sum_{i=m+1}^{n}\varepsilon_i a_i+\sum_{i=m+1}^{n}\varepsilon_i(-b_i)=0$. This means, $\sum_{i=m+1}^{n}\varepsilon_i a_i=\sum_{i=m+1}^{n}\varepsilon_i b_i=c$ for some c. Also let $\sum_{i=1}^{m}\varepsilon_i a_i=a$ and $\sum_{i=1}^{m}\varepsilon_i b_i=b$, from which we know $a+b=1$ and $0\le a,b\le 1$. Now our term of interest $|\sum_{i=1}^{n}\varepsilon_i a_i|+|\sum_{i=1}^{n}\varepsilon_i b_i|$ becomes $|a+c|+|b+c|=|a+c|+|1-a+c|$ If $-1\le c\le 0$, then we have $a+c+b+c=1+2c$ and from $-1\le c\le 0$ we have $-1\le c\le 1$. Also, $a+c-b-c=a-b=a-(1-a)=2a-1$ and from $0\le a\le 1$ we have $-1\le 2a-1\le 1$. Thus we are done for the case where $c\le 0$. If $c\ge 0$, we 'flip' the signs of ε_i for $i\in[m+1,n]$ so that $\sum_{i=m+1}^{n}(-\varepsilon_i a_i)=\sum_{i=m+1}^{n}(-\varepsilon_i)b_i=c$. Now our sum becomes $|a-c|+|1-a-c|$, and we get $-1=a-1+1-a-1\le a-c+1-a-c\le a-0+1-a-0=1$, and $a+c-b-c=a-b$ is also in the interval $[-1,1]$ as before. This settles our case 1.

Case 2. m is even. Again by our lemma we have $-1\le\sum_{i=1}^{m}\varepsilon_i a_i,\sum_{i=1}^{m}\varepsilon_i b_i\le 1$ and $\sum_{i=1}^{m}\varepsilon_i a_i+\sum_{i=1}^{m}\varepsilon_i b_i=0$. In addition, $0\le\sum_{i=m+1}^{n}\varepsilon_i a_i,\sum_{i=m+1}^{n}\varepsilon_i(-b_i)\le 1$ and $\sum_{i=m+1}^{n}\varepsilon_i a_i+\sum_{i=m+1}^{n}\varepsilon_i(-b_i)=1$. Now, let $\sum_{i=1}^{m}\varepsilon_i a_i=a$ and we have $\sum_{i=1}^{m}\varepsilon_i b_i=-a$ $(-1\le a\le 1)$, let $\sum_{i=m+1}^{n}\varepsilon_i a_i=c$ and we have $\sum_{i=m+1}^{n}\varepsilon_i b_i=c-1$ $(0\le c\le 1)$. Again, we need to consider $|a+c|+|-a+c-1|$. Observe that $a+c-a+c-1=2c-1$ and from $0\le c\le 1$ we have $-1\le 2c-1\le 1$. $a+c+a-c+1=2a+1$. If $-1\le a\le 0$ we have $-1\le 2a+a\le 1$, which serves our purpose. If $0\le a\le 1$, again we can reverse the signs of a_i $(i\in[1,m])$ as of above so that $\sum_{i=1}^{m}(-\varepsilon_i)a_i=-a,\sum_{i=1}^{m}(-\varepsilon_i)b_i=a$, meaning that we need to now consider $|-a+c|+|a+c-1|$. Again $-a+c+a+c-1=2c-1$, giving $-1\le 2c-1\le 1$, as usual. Moreover, $-a+c-a-c+1=1-2a$, which means $-1\le 1-2a\le 1$ for $0\le a\le 1$. This completes the proof as well.

A4 Find all functions $f:(0,\infty)\to(0,\infty)$ such that for any $x,y\in(0,\infty)$,

$$xf(x^2)f(f(y))+f(yf(x))=f(xy)\left(f(f(x^2))+f(f(y^2))\right)\cdots(*).$$

Solution. The only function is $f(x)\equiv\frac{1}{x}$, which works because $xf(x^2)f(f(y))+f(yf(x))=x\frac{1}{x^2}y+\frac{1}{y\frac{1}{x}}=\frac{x}{y}+\frac{y}{x}=\frac{x^2}{xy}+\frac{y^2}{xy}=f(xy)\left(f(f(x^2))+f(f(y^2))\right)$.

For the rest of the solution we proceed with the normal functional algorithmic procedure:

Step 1. Plugging $x=y=1$ gives $f(1)f(f(1))+f(f(1))=2f(1)f(f(1))$, and since $f>0$, we can factorize $f(f(1))$ out to get $f(1)+1=2f(1)$, giving $\boxed{f(1)=1}$.

Step 2. Plugging $x\leftarrow 1$ (and substituting $f(1)\leftarrow 1$ due to step 1) gives $f(f(y))+f(y)=f(y)(1+f(f(y^2)))$, giving $\boxed{f(f(y))=f(y)f(f(y^2))}$.

Step 3. Plugging $y\leftarrow 1$, on the other hand, gives $xf(x^2)+f(f(x))=f(x)(f(f(x^2))+1)$. From step 2, $f(f(x))=f(x)f(f(x^2))$, which gives rise to $\boxed{xf(x^2)=f(x)}$.

Step 4. Substitute $xf(x^2)\leftarrow f(x)$ (step 3), $f(f(y))\leftarrow f(y)f(f(y^2))$ (step 2), and $yf(x)\leftarrow xyf(x^2)$ (step 3) into (*) gives:

$$f(x)f(y)f(f(y^2))+f(xyf(x^2))=f(xy)\left(f(f(x^2))+f(f(y^2))\right)\cdots(**).$$

In the special case where $xy=1$ we have $f(x)f(y)f(f(y^2))+f(f(x^2))=1\left(f(f(x^2))+f(f(y^2))\right)$, so $f(x)f(y)=1$ whenever $xy=1$. In other words, for all $x\in\mathbb{R}^+$, $\boxed{f(\frac{1}{x})=\frac{1}{f(x)}}$.

Step 5. Having the results in Step 4 in mind, we do the following substitution:
- Substitute $\frac{1}{x}$ and $\frac{1}{x}$ in place of x and y into (**) to turn $f(\frac{1}{x})f(\frac{1}{x})f(f(\frac{1}{x^2}))+f(\frac{1}{x^2}f(\frac{1}{x^2}))=f(\frac{1}{x^2})\left(f(f(\frac{1}{x^2}))+f(f(\frac{1}{x^2}))\right)$ into $\frac{1}{f(x)f(x)f(f(x^2))}+\frac{1}{f(x^2f(x^2))}=\frac{2}{f(x^2)f(f(x^2))}\cdots(5a)$.
- Subsitute $x=y$ into (**) we get $f(x)f(x)f(f(x^2))+f(x^2f(x^2))=2f(x^2)f(f(x^2))\cdots(5b)$.

Comparing 5a and 5b gives $\frac{1}{f(x)f(x)f(f(x^2))}+\frac{1}{f(x^2f(x^2))}=\frac{2}{f(x^2)f(f(x^2))}=\frac{4}{f(x)f(x)f(f(x^2))+f(x^2f(x^2))}$, which we can cross multiply to get $(f(x)f(x)f(f(x^2))+f(x^2f(x^2)))^2=4(f(x)f(x)f(f(x^2))\cdot$

$f(x^2f(x^2)))$, a.k.a. $(f(x)f(x)f(f(x^2)) - f(x^2f(x^2)))^2 = 0$. Therefore $f(x)f(x)f(f(x^2)) = f(x^2f(x^2)) = f(x^2)f(f(x^2))$. Factorizing out $f(f(x^2))$ gives $f(x)^2 = f(x^2)$, and from (2), $f(x) = xf(x^2) = xf(x)^2$, or $xf(x) = 1$, or $f(x) = \frac{1}{x}$.

A5 Consider fractions $\frac{a}{b}$ where a and b are positive integers.
(a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n} + 1$.
(b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}$.

Solution. For part (a), we partion the set of positive integers according to their integer square roots, that is, the sets $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 5, 6, 7, 8\}$, $S_3 = \{9, 10, 11, 12, 13, 14, 15\}$, etc. Consider $S_k = \{k^2, k^2+1, \cdots k^2+2k\}$, and we claim that $b = k$ and $b = k+1$ alone will jointly work for the sets. Indeed, considering $c \in [0, k]$ we have $(k + \frac{a}{k})^2 = k^2 + 2a + (\frac{a}{k})^2$. With $(\frac{a}{k})^2 \le 1$, we have $\sqrt{k^2 + 2a} \le k + \frac{a}{k} \le \sqrt{k^2 + 2a + 1}$, so $b = k$ works for $k^2, k^2+2, \cdots, k^2+2k$. Meanwhile for $c \in [0, k+1]$ we have $(k+\frac{a}{k+1})^2 = k^2 + \frac{2ak}{k+1} + (\frac{a}{k+1})^2$ $= k^2 + 2a - \frac{2a}{k+1} + (\frac{a}{k+1})^2$. Notice that $-\frac{2a}{k+1} + (\frac{a}{k+1})^2 = \frac{a^2-2a(k+1)}{(k+1)^2} = \frac{(a-(k+1))^2-(k+1)^2}{(k+1)^2}$ $= \frac{(a-(k+1))^2}{(k+1)^2} - 1$, and with $0 \le a \le k+1$ we have $-1 \le \frac{(a-(k+1))^2}{(k+1)^2} - 1 \le 0$. Therefore $\sqrt{k^2 + 2a - 1} \le k + \frac{a}{k+1} \le \sqrt{k^2 + 2a + 1}$, and this works for $n = k^2 + 1$, $k^2 + 3$, $\cdots$, $k^2 + 2k - 1$. Therefore all elements in S_k are covered. As for part (b) we show that there's no fraction $\frac{a}{b}$ (with $b \le k$) lying in the interval $[\sqrt{k^2+1}, \sqrt{k^2+2}]$. Notice that, $k < \sqrt{k^2+1} < \sqrt{k^2+2} < \sqrt{k^2+2k+1} = k+1$. Assume that $\frac{a}{b}$ satisfies this property, then from $\frac{a}{b} > k$ and $b \le k$ we have $(\frac{a}{b})^2 \ge (k + \frac{1}{k})^2 = k^2 + 2 + \frac{1}{k^2} > k^2 + 2$, contradiction.

A6/IMO 5 The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

Solution. The answer is 2016. Anything fewer doesn't work, because for some i, the factor $x - i$ appears on both sides, so i is itself a root.

It remains to show that 2016 is good to go. We claim that the equation $\prod_{i=1}^{504}(x - (4i - 3))(x - (4i)) = \prod_{i=1}^{504}(x - (4i-2))(x - (4i-1))$ has no real solution by showing that the left-hand side is always strictly smaller than the right hand side. We first eliminate the obvious cases where $LHS > 0$ while $RHS < 0$. Observe that whenever $x \in (4i+1, 4i+2)$ for some $i \in [0, 503]$, there are $2i+1$ negative factors (and the rest $1007 - 2i$ positive) on the left (hence negative) while $2i$ negative factors (and the rest $1008 - 2i$ positive) on the right (hence positive). Also whenever $x \in (4i-1, 4i)$ for some $i \in [1, 504]$, there are $2i - 1$ negative factors (and the rest $1009 - 2i$ positive) on the left (hence negative) while $2i$ negative factors (and the rest $1008 - 2i$ positive) on the right (hence positive). Thus in both of the cases the left is less than 0 while the right is more than 0. As for the endpoints $x \in \{1, 2, \cdots, 2016\}$, if $x = 4i$ or $x = 4i+1$ then LHS=0 while RHS has $2i$ negative factors (while the rest positive) hence positive. If $x = 4i-1$ or $x = 4i-2$ then the right is 0 while the left has $2i-1$ negative factors (while the rest positive) hence negative.

If $x > 2016$ then we have LHS and RHS both greater than 0 (since all remaining 2016 factors are positive). Nevertheless, in light of the relation $(x-(4i-2))(x-(4i-1)) - (x-(4i-3))(x-(4i)) = (4i-1)(4i-2) - (4i-3)(4i) = 2$ we have $|(x-(4i-2))(x-(4i-1))| >$

$|(x-(4i-3))(x-(4i))|$, and thus $\prod_{i=1}^{504}|(x-(4i-3))(x-(4i))| < \prod_{i=1}^{504}|(x-(4i-2))(x-(4i-1))|$. Since each side is positive, $\prod_{i=1}^{504}(x-(4i-3))(x-(4i)) < \prod_{i=1}^{504}(x-(4i-2))(x-(4i-1))$. The case $x < 1$ is symmetrical and hence analogous.

We are thus left with the trickiest case: $x \in (4i-2, 4i-1)$ for some $i \in [1, 504]$, whereby both sides are negative. The goal is therefore to show that $|LHS| > |RHS|$. We still want to keep in mind that $(x-(4i-2))(x-(4i-1))-(x-(4i-3))(x-(4i)) = (4i-1)(4i-2)-(4i-3)(4i) = 2$, and that both $(x-(4i-2))(x-(4i-1))$ and $(x-(4i-3))(x-(4i))$ are positive for $x \notin [4i-3, 4i]$. Now, let $x \in (4i-2, 4i-1)$ for some $i \in [1, 504]$, then from $(x-(4i-2))(x-(4i-1)) = (x-(4i-1.5))-\frac{1}{4} \geq -\frac{1}{4}$ we get $\frac{|(x-(4i-2))(x-(4i-1))|}{|(x-(4i))(x-(4i-3))|} = \frac{c}{c+2} = 1-\frac{2}{c+2} \leq 1-\frac{2}{2+\frac{1}{4}} = \frac{1}{9}$ where $c = |(x-(4i-2))(x-(4i-1))|$. Next, let's investigate $\frac{|(x-(4j-2))(x-(4j-1))|}{|(x-(4j))(x-(4j-3))|}$ for some $j < i$. We know that $x > 4i+1$, so $(x-(4j-2))(x-(4j-1)) > (4i-4j-1)(4i-4j)$ and therefore $\frac{|(x-(4j-2))(x-(4j-1))|}{|(x-(4j))(x-(4j-3))|}$ $\frac{(x-(4j-2))(x-(4j-1))}{(x-(4j-2))(x-(4j-1))-2} = 1 + \frac{2}{(x-(4j-2))(x-(4j-1))-2} < 1 + \frac{2}{(4(i-j)-1)(4i-4j)-2} = 1 + \frac{1}{2(4(i-j)-1)(i-j)-1}$. It's also not hard to verify that $2(4(i-j)-1)(i-j)-1 < (i-j+1)^2-1$ for $j \leq i-1$, so we in turn have $1 + \frac{1}{2(4(i-j)-1)(i-j)-1} < 1 + \frac{1}{(i-j+1)^2-1}$ Thus $\dfrac{\prod_{j=1}^{i-1}(x-(4j-2))(x-(4j-1))}{\prod_{j=1}^{i-1}(x-(4j-3))(x-(4j))} < \prod_{j=1}^{i-1}(1+\dfrac{2}{(4i-4j-1)(4i-4j)-2})$ $< \prod_{j=-\infty}^{i-1}(1+\dfrac{2}{(4i-4j-1)(4i-4j)-2}) < \prod_{j=-\infty}^{i-1}(1+\dfrac{1}{(i-j+1)^2-1}) = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdots$ $= \lim_{x\to\infty} 2 \cdot \frac{x-1}{x} = 2$. Likewise, $\dfrac{\prod_{j=i+1}^{504}(x-(4j-2))(x-(4j-1))}{\prod_{j=i+1}^{504}(x-(4j-3))(x-(4j))} < 2$, (we can drop the modulus since they are all greater than 0. Thus $\dfrac{\prod_{i=1}^{504}|(x-(4i-2))(x-(4i-1))|}{\prod_{i=1}^{504}|(x-(4i-3))(x-(4i))|}$ $< 2 \times \frac{1}{9} \times 2 < \frac{4}{9}$, and we are done. (OMG the long proof...)

2 Combinatorics

C1 The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leaders in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leaders string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Solution. The answer is 2 for $n = 2k$ and 1 otherwise.

Notice that there are $\binom{n}{k}$ strings the deputy leader can write. For the i-th digit (for any $i \in [0, n-1]$), there are $\binom{n-1}{k-1}$ such strings with i-th digit differing from the original, $\binom{n-1}{k}$ such strings with i-th digit equal to the original. If $\frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1} \neq \binom{n-1}{k} = \frac{(n-1)!}{(k)!(n-k-1)!}$, the contestant can determine that digit by counting the number of strings with 0 in it (and the number of strings with 1 in it). This happens when $(k-1)!(n-k)! \neq k!(n-k-1)!$, or $n - k \neq k$ (fatorizing factors out) or $n \neq 2k$. No further guesses is needed and the contestant can get it in one try.

If $n = 2k$, then for each digit, half of the strings have one's and half have zero's. The student then considers the strings with 0 on the leading digit. If, the correct string has 0 on that leading digit, then for each of the written strings (with leading 0), among the remaining $2k-1$ digits there are $k-1$ being changed from the original. By the claim above the student can determine the remaining $2k-1$ digits. Similar conclusion can be reached for he case with 1 as leading digit. This gives the student the correct answer after 2 guesses. To see why 2 guesses is necessary, let $a_0a_1 \cdots a_{2k-1}$ be the string given by the leader, $b_0b_1 \cdots b_{2k-1}$ be a string with $b_i = 1 - a_i$ for each i, $c_0c_1 \cdots c_{2k-1}$ be any string written by the deputy leader. Now, we have $c_i = a_i$ or $c_i = b_i$ but not both. With $c_0c_1 \cdots c_{2k-1}$ having k same digits and k different digits as $a_0a_1 \cdots a_{2k-1}$, it must have $2k - k = k$ same digits and $2k - k = k$ different digits as $b_0b_1 \cdots b_{2k-1}$ too. Thus $b_0b_1 \cdots b_{2k-1}$ is actually another possibility.

C2 Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:
- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

Solution. The answer is $n = 1$, which works with 1 being placed in a 1×1 table. To show that this fails for other n, first prime factorize it into $\prod_{i=1}^{k} p_i^{a_i}$. If r is the number of rows and c is the number of columns then $rc = \prod_{i=1}^{k}(a_i+1)$, the number of divisors of n. W.l.o.g. $r \geq c$ and therefore $r \geq \sqrt{\prod_{i=1}^{k}(a_i+1)} = \prod_{i=1}^{k}\sqrt{(a_i+1)}$. We have also known that the sum of divisors is $\prod_{i=1}^{k}\frac{p_i^{a_i+1}-1}{p_i-1}$. Knowing that one of the cells contains n, the sum of each row must be greater than n, (n cannot be the only cell in that row, otherwise all cells would have to contain the same number which is absurd for $n > 1$). This means that the sum of each column is greater than rn, giving the following inequality:

$$\prod_{i=1}^{k}\frac{p_i^{a_i+1}-1}{p_i-1} > rn \geq \prod_{i=1}^{k}\sqrt{(a_i+1)}p_i^{a_i}$$

or equivalently, $\prod_{i=1}^{k}\frac{p_i - \frac{1}{p_i^{a_i}}}{p_i-1} > \prod_{i=1}^{k}\sqrt{(a_i+1)}$

Now for each prime p, we are interested to investigate the ratio $\frac{1}{p-1}(p - \frac{1}{p^{a_i}}) : \sqrt{(a_i+1)}$. For $p = 2$ we have $(2 - \frac{1}{2^{a_i}}) : \sqrt{(a_i+1)}$. Notice that for $a_i \geq 3$, $(2 - \frac{1}{2^{a_i}}) < 2$ while $\sqrt{(a_i+1)} \geq 2$. so the ratio is smaller than 1. for $a_i = 1$, $(2-\frac{1}{2^{a_i}}) = \frac{3}{2}$ and $\sqrt{(a_i+1)} = \sqrt{2}$ so the ratio is $\frac{3}{2\sqrt{2}}$, for $a_i = 2$ we have $\frac{7}{4} \div \sqrt{3} = \frac{7}{4\sqrt{3}}$. Knowing that $\frac{3}{2\sqrt{2}} = \sqrt{\frac{9}{8}} > \sqrt{\frac{49}{16}} =$

$\frac{7}{4\sqrt{3}}$ the maximum ratio is $\sqrt{\frac{9}{8}}$. For $p \geq 3$ we have $\frac{1}{p-1}(p - \frac{1}{p^{a_i}})$ decreasing with p with a_i fixed because $\frac{1}{p-1}(p - \frac{1}{p^{a_i}}) = 1 + \frac{1}{p} + \cdots + \frac{1}{p^{a_i}} \leq 1 + \frac{1}{3} + \cdots + \frac{1}{3^{a_i}} = \frac{1}{2}(3 - \frac{1}{3^{a_i}})$. When $a_i = 1$ the ratio is $\frac{4}{3\sqrt{2}} = \sqrt{\frac{8}{9}}$, when $a_i \geq 2$ the ratio is at most $\frac{1}{2}(3 - \frac{1}{3^{a_i}}) \div \sqrt{a_i + 1}$ $\leq \frac{1}{2}(3) \div \sqrt{2+1} = \frac{3}{2\sqrt{3}} = \sqrt{\frac{3}{4}} < \sqrt{\frac{8}{9}}$. Thus the maximum possible ratio is $\sqrt{\frac{8}{9}}$.

Summing up, for n consisting at least two distinct prime factors the ratio $\prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1} \div \prod_{i=1}^{k} \sqrt{(a_i + 1)}$ cannot exceed $\sqrt{\frac{9}{8}} \times \sqrt{\frac{8}{9}}^{\,i-1} \leq 1$, contradicting that $\prod_{i=1}^{k} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i - 1} > \prod_{i=1}^{k} \sqrt{(a_i + 1)}$. Hence $i = 1$ and from the previous paragraph, $p < 3$ and thus $p = 2$. However, this implies n is a power of 2 and from $a_i \geq 1$, at least two rows must be used (we assumed $r \geq c$). The row containing n must therefore have sum at least $2n$, but for n a power of two the sum of divisors is $2n - 1$, contradiction.

C4/IMO 2 Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Solution. The answer is all n divisible by 9. We start by showing an example for $n = 9$, given below:

I	M	O	M	O	I	O	I	M
M	M	M	O	O	O	I	I	I
I	M	O	M	O	I	O	I	M
O	I	M	I	M	O	M	O	I
I	I	I	M	M	M	O	O	O
O	I	M	I	M	O	M	O	I
M	O	I	O	I	M	I	M	O
O	O	O	I	I	I	M	M	M
M	O	I	O	I	M	I	M	O

For $n = 9k$ for some k we just have to split the grid into k^2 9×9 grids, and fill each one with the letters above. For sake of verification, observe that there are exactly 3 I's, 3 M's and 3 O's in each column or each row of a single 9×9 grid. Also, each diagonal is in the form of either $R_m = \{(i,j) : i + j = m\}$, or $L_m = \{(i,j) : i - j = m\}$, for some m satisfying $1 \leq (i,j) \leq n$. Now for R_m, the size $|R_m|$ is $m - 1$ for $m \leq n + 1$, and $2n + 1 - m$ for $m \geq n + 1$. Notice that 3 divides $|R_m|$ iff $m \equiv 1 \pmod{n}$ (first case), or iff $m \equiv 1 \pmod{n}$ (second case). Thus it is not hard to see that the diagonals are in the form of $(1, m-1), (2, m-2), \cdots (m-1, 1)$ in the first case, and $(m-n, n), (m-n+1, n-1), \cdots (n, m-n)$ in the second case. In each of the cases we can group them into groups of three, such that, if we further split each 9×9 grids into 3×3 grids, each group contains three cells along the main diagonal. Nevertheless, from the construction above we see that each main diagonal in the 3×3 grids have one I, one M and one O. Thus this set of diagonal works too. A similar conclusion can be yielded for diagonals in the form of L_m.

To show that $9|n$ is necessary, observe from the first condition that $3|n$. Let $n = 3k$ and let's split the table into k^2 3×3 cells. Notice from the logic (of diagonals characterization) as of above, the center of each 3×3 cell ((i,j) where $i, j \equiv 2 \pmod{3}$) lie on both R_m

and L_m with both size divisible by 3; the four corners $(((i, j)$ where $i, j \not\equiv 2 \pmod 3))$ lie on exactly one of the sets satifying the properties; the four sides $(((i, j)$ where exactly one of i and j is congruent to 2 mod 3) lie on none of them. Thus, when we mark the cells in each column, each row, and each diagonal with size divisible by 3, the center cells are marked 4 times, the corners thrice, and the sides twice (as illustrated below).

3	2	3
2	4	2
3	2	3

Let c be the number of M's on the center cells. Considering just the $3i-1$-th column for $i \in [1, k]$ and the $3j-1$-th row for $j \in [1, k]$ yields $2k^2$ M's being counted. Each cell on the "side" is being counted once, each cell on the "center" twice, an each cell on the "corner" none. This gives the number of M's on the side as $2k^2 - c$, which follows that there must be $k^2 + c$ M's at the corner. Now let's see what happens as we consider all such markings (all columns, all rows, and all diagonals of size divisible by 3). Observe that for each 3×3 cells we have $3+2+3+2+4+2+3+2+3 = 24$ markings, so each letter (M, in particular) has $8k^2$ markings. This means $8k^2 = 4c + 2(2k^2 - c) + 3(k^2 + c) = 3c + 7k^2$, or $c = \frac{k^2}{3}$. Hence $3|k^2$, or $3|k$, or $9|n$.

C7/IMO 6 There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Feridun has to choose an endpoint of each segment and place a goose on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each goose will immediately jump forward to the next intersection point on its segment. Geese never change the direction of their jumps. Feridun wishes to place the geese in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Feridun can always fulfill his wish if n is odd.

(b) Prove that Feridun can never fulfill his wish if n is even.

Solution. (a) Let the segments be $\ell_1, \ell_2, \cdots, \ell_n$. Let P_{ij} be the intersection of line ij. For each segment ℓ_i we aim to investigate the number of points on each side of P_{ij} (other than P_{ij}). Since there are $n-2$ such points (which is odd), one side has even number of points and the other side odd. We call this odd side of ℓ_i w.r.t. point P_{ij}.

Now place the first goose arbitrarily on ℓ_1. For $i \in [2, n]$ we do the following: if the goose corresponding to ℓ_1 is place on the odd side of ℓ_1 w.r.t. P_{1i}, Feridun places one goose at the even side of ℓ_i w.r.t. P_{1i} (and vice versa). We now proceed to the following claim: using the procedure detailed above, for each two distinct integers $i, j \in [1, n]$, the geese corresponding to l_i and l_j lie on different parity of l_i and l_j, respectively, both w.r.t. P_{ij}. Indeed, consider the triangle formed by lines ℓ_1, ℓ_i and ℓ_j. Menelaus' theorem says that any line either intersects none or two of the segments $P_{ij}P_{1i}$, $P_{1j}P_{1i}$, $P_{ij}P_{1j}$. Thus considering lines l_k with $k \notin \{1, i, j\}$ we know that it has even number of total intersection points with segments $P_{ij}P_{1i}$, $P_{1j}P_{1i}$, $P_{ij}P_{1j}$. If this number is even on $P_{1j}P_{1i}$, then each endpoint is on the odd side of l_1 w.r.t. one of P_{1j} and P_{1i}, and even on the other. Thus according of our choice of placing the geese , either one goose come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}, or vice versa. The intersection with $P_{ij}P_{1i}$ and $P_{ij}P_{1j}$ will be both odd or both even. If it's both odd and in the first case (one goose come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}), then the goose corresponding to i come from the odd side of l_i w.r.t. P_{1i} and the other from even side of l_j w.r.t. P_{1j}, which works for this pair of (i, j). The other three subcases can be treated equally. If this number is odd on $P_{ij}P_{1i}$, then each endpoint is on the odd side of l_i w.r.t. both P_{1i} and P_{1j}, or vice versa (both even). According to our choice again, both geese come from the odd side of l_i w.r.t. P_{1i} and the and of

l_j w.r.t. P_{1j}, or both from the even side of their respective lines. The intersection with $P_{ij}P_{1i}$ and $P_{ij}P_{1j}$ will be one odd and one even, for the same endpoint w.r.t the lines l_i and l_j, exactly one of them will change sign when switching from P_{1i} to P_{ij} and from P_{1j} to P_{ij}. Again this (i,j) works.

Finally, to see why the geese won't intersect at the same time, observe that if this happens for some of (i,j), then the geese must have encountered the same number of points before. This implies that they have to come both from the odd side or the even side of the line, contradiction.

(b) Let $\ell_1, \ell_2, \cdots, \ell_n$ be the segments, and let $\theta_i \in [0, \pi)$ be the angle need for x-axis to rotate counterclockwise to reach ℓ_i. W.L.O.G let $0 = \theta_1 < \theta_2 < \cdots < \theta_n$. For convinience we introduce $\ell_{n+1} = \ell_1$ with $\theta_{n+1} = \pi$.

For each segment Feridun has the choice of placing the goose in the direction of θ_i or $\theta_i + \pi$ compared to the positive x-direction. Let's investigate ℓ_i and ℓ_{i+1} together. Now, suppose that ℓ_i and ℓ_j intersect at P_{ij} for some j; define $\ell_{(i+1)j}$ similarly. We know that for each j different from i, $i+1$, ℓ_i, ℓ_{i+1} and ℓ_j are in that order (in anticlockwise angle, cycles allowed). This means that $P_{ij}P_{i(i+1)}, P_{i(i+1)}P_{(i+1)j}, P_{(i+1)j}P_{ij}$ must also be in that order, forcing $P_{ij}, P_{i(i+1)}, P_{(i+1)j}$ to be in clockwise order. From here we infer that the vectors $P_{i(i+1)}P_{ij}$ and $P_{i(i+1)}P_{(i+1)j}$ either have directions $(\theta_i,\, \theta_{i+1})$ or $(\theta_i + \pi,\, \theta_{i+1} + \pi)$, and considering all such j's, we know that there are equal number of intersection points lying on the half-line starting from $P_{i(i+1)}$ and extending in the θ_i direction, and on the half-line starting from $P_{i(i+1)}$ and extending in the θ_{i+1} direction. This means that the geese will collide when both placed in the $\theta_i, theta_{i+1}$ direction, or $\theta_i + \pi, theta_{i+1} + \pi$ direction, forcing the directions to be $\theta_i, \theta_{i+1} + \pi$ or $\theta_i + \pi, \theta_{i+1}$.

Summarizing above, if we let θ_1 to be the direction headed by the first goose the directions must be $\theta_1, \theta_2 + \pi, \theta_3, \theta_4 + \pi, \cdots, \theta_n + \pi$ (n is even). Recall that we can also compare it with the "$n+1$-th" line (which is the first) and it has to have direction $\theta_{n+1} = \theta_1 + \pi$, which contradicts our choice of making the first *goose* facing the direction θ_1. (The case where $\theta_1 + \pi$ is chosen as the direction is completely analogous: if one configuration works, then the completely opposite configuration works too).

3 Geometry

G1/IMO 1 Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

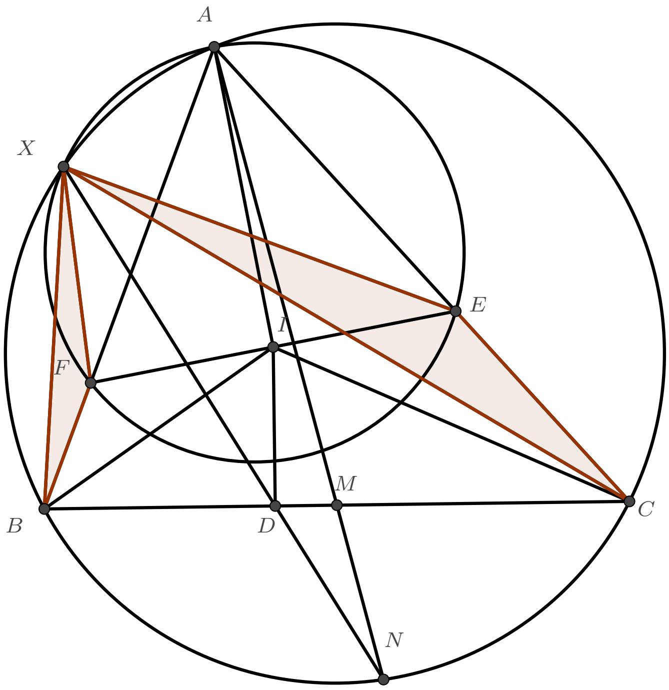
Solution. The fact that $\angle CBF = 90^\circ$ and M being the midpoint of CF should very well suggest us to draw the circumcircle of $\triangle BCF$. As it turns out, D and X seems to lie on this circle (and that's almost everything we need). Now $\angle DCF = \angle DCA = \angle DAC = \angle BAF = \angle ABF = 90^\circ - \frac{1}{2}\angle BFC$. $DC = DA = \frac{CA}{2\cos\angle DCA} = \frac{CF - AF}{2\cos(90^\circ - \frac{1}{2}\angle BFC)}$ $= \frac{CF - BF}{2\sin\frac{1}{2}\angle BFC} = \frac{CF - CF\cos\angle BFC}{2\sin\frac{1}{2}\angle BFC} = CF \cdot \frac{1 - (1 - 2\sin^2\frac{1}{2}\angle BFC)}{2\sin\frac{1}{2}\angle BFC} = CF \cdot \sin\frac{1}{2}\angle BFC = CF \cdot \cos(90^\circ - \frac{1}{2}\angle BFC) = CF \cdot \cos\angle DCF$. If D' is on ray CD satisfying $\angle CD'F = 90^\circ$ we have $CD' = CF\cos\angle D'CF = CF\cos\angle DCF = CD$, so $D = D'$ and D lies on the circumcircle of BCF. Moreover, $\angle DFC = 90^\circ - \angle DCF = \frac{1}{2}\angle BFC = \angle BFD$, so BD and DC subtend the same angle and $BD = DC$.

Now $EA = ED$ and $\angle CAD = \angle EAD$ so $\angle EDA = \angle EAD = \angle CAD$, so $ED \parallel CM$. With $EX \parallel AM$ we have E, X, D collinear, too. Moreover, $DE = \frac{DA}{2\cos\angle EDA} = \frac{DA}{2\cos\angle CAD}$ $= \frac{DC}{2\cos\angle DCA} = \frac{CF\cos\angle DCF}{2\cos\angle DCF} = \frac{CF}{2} = CM$, so $CMED$ is a parallelogram. Thus $\angle DEM =$

$\angle DCM = \angle DCA = \angle CAD$, meaning that $DEMA$ is an isoceles trapezoid (so $\angle EAM = \angle EDM$). With $AMXE$ parallelogram we also have $\angle DXM = \angle EXM = angleEAM = \angle EDM = \angle XDM$, so $MX = MD$ and X lies on the circumcircle of BCF, too. With $DX \parallel CF$ we conclude that $DXFC$ is an isoceles trapezoid so $CD = FX$, from previous identity we have $BD = DC$ so $FX = BD$.

Finally, we already had F, E, X collinear and $\angle DBA = \angle DBC + \angle CBA = \angle DFC + (90° - \angle FBA) = \angle DFC + (\frac{1}{2}\angle BFC) = \angle DFC + (\angle DFC) = \angle DMC$, so B lies on the circle containing D, E, A, M too. This means $\angle BED = \angle BMD$, and since BD and FX subtend the same angle on circle BCF we have $\angle BMD = \angle FMX = \angle FEX$ (the last equality follows from that $EXFM$ is isoceles trapezoid, hence cyclic). Therefore, B, E, F are in fact collinear, and E is the intersection of DX and BF. Hence ME is the common perpendicular bisector of segments BX and DF (since $BXFD$ is an isoceles trapezoid), and the intersection of BD and FX will lie on this perpendicular bisector too.

G2 Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.



Solution. W.L.O.G. let $AB < AC$. First, well-known spiral similarity property should dictate the similarity of triangles BXF an CXE, so $\frac{CX}{CE} = \frac{BX}{BF}$. Also, let's also invoke an identity for triangles (feel free to verify it; I'm not gonna do this):

$$\frac{BX}{XC} \cdot \frac{\sin \angle BXD}{\sin \angle CXD} = \frac{BD}{DC}.$$

Denoting N_1 as the other intersection of XD and Γ gives $\frac{\sin \angle BXD}{\sin \angle CXD} = \frac{BN_1}{CN_1}$. Similarly we have $\frac{AB}{AC} \cdot \frac{\sin \angle ABM}{\sin \angle ACM} = \frac{BM}{CM} = 1$. ALso let N_2 as the other intersection of AM and Γ and we have $\frac{\sin \angle ABM}{\sin \angle ACM} = \frac{BN_2}{CN_2}$. Therefore all we need is $\frac{\sin \angle ABM}{\sin \angle ACM} = \frac{\sin \angle BXD}{\sin \angle CXD}$, and it's not hard to see that $\frac{\sin \angle ABM}{\sin \angle ACM} = \frac{AC}{AB}$, so we are left with proving the fact $\frac{BF}{EC} \cdot \frac{AC}{AB} = \frac{BD}{DC}$.

Now, $\frac{BD}{DC} = \frac{\tan \frac{1}{2}\angle C}{\tan \frac{1}{2}\angle B}$, $\frac{AC}{AB} = \frac{\sin \angle B}{\sin \angle C} = \frac{2 \sin \frac{1}{2}\angle B \cos \frac{1}{2}\angle B}{2 \sin \frac{1}{2}\angle C \cos \frac{1}{2}\angle C}$. Also $IE = IF$, and by angle chasing we have $\angle FIB = \angle ICE = \frac{1}{2}\angle C$, $\angle EIC = \angle IBF = \frac{1}{2}\angle B$. Therefore BIF and ICE

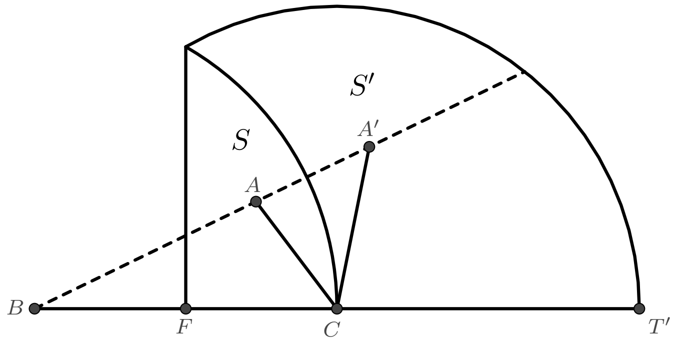
similar, yielding $\frac{BF}{EC} = (\frac{BF}{FI})^2 = (\frac{\sin\frac{1}{2}\angle C}{\sin\frac{1}{2}\angle B})^2$, now it's no longer difficult to prove that $(\frac{\sin\frac{1}{2}\angle C}{\sin\frac{1}{2}\angle B})^2 \cdot \frac{2\sin\frac{1}{2}\angle B\cos\frac{1}{2}\angle B}{2\sin\frac{1}{2}\angle C\cos\frac{1}{2}\angle C} = \frac{\tan\frac{1}{2}\angle C}{\tan\frac{1}{2}\angle B}$.

G3 Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be nice if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.



Solution. We show that the following works: $S = \{(x, y) : x \geq 0, y \geq 0, (x+1)^2+y^2 \leq 4\}$ and $S' = \{x \geq 0, y \geq 0, (x+1)^2 + y^2 \geq 4, (x-1)^2 + y^2 \leq 4\}$. We claim that $BA \cdot BA' = 4$ for those choices. Denote triangles ABC and DEF as quasi-similar if there exists a permutation σ of $\{D, E, F\}$ with ABC and $\sigma(D)\sigma(E)\sigma(F)$ as similar.

We first start with the following claim: for every point A above x-axis, $A \in S$ iff $AC \leq AB \leq BC$ and $A' \in S'$ iff $A'C \leq BC \leq A'B$.
Proof: let's first investigate all points $A \in S$ and $A' \in S'$. Let $A = (x, y)$ be arbitrary. Now, $AC \leq AB \leftrightarrow (x-1)^2 + y^2 \leq (x+1)^2 + y^2 \leftrightarrow 0 \leq x$. $BC \geq AB \leftrightarrow 2^2 \geq (x+1)^2 + y^2 \leftrightarrow (x+1)^2 + y^2 \leq 4$. Therefore $AC \leq AB \leq BC$ iff (x, y) satisfies both $x \geq 0$ and $(x+1)^2+y^2 \leq 4$ iff $(x, y) \in S$. Next, $AC \leq BC$ iff $AC \leq 2$ or $(x-1)^2+y^2 \leq 4$ and $BC \leq AB$ iff $(x-1)^2 + y^2 \geq 4$, therefore $AC \leq BC \leq AB$ iff the two conditions are satisfied (and from here $(x-1)^2 + y^2 \leq (x-1)^2 + y^2$ so $x \geq 0$ is implied), and this is equivalent to $A \in S'$.

We are now ready to justify our selection:

(i) In S we can simply take any point as T, since the boundaries x-axis with $x \in [0, \sqrt{3}]$, y-axis with $y \in [0, 1]$, and the arc of the circle $(x-1)^2 + y^2 = 4$ with $x \geq 0$ are concave. In S' we take $T = (3, 0)$.

(ii) Observe that the objective is equivalent to: for each triangle $P_1P_2P_3$ there is a unique point A with triangle ABC quasisimilar to $P_1P_2P_3$ satisfying $AC \leq AB \leq BC$, and another unique point A' with $A'BC$ quasisimilar to $P_1P_2P_3$ and $A'C \leq BC \leq A'B$. Moreover we want to prove that $BA \cdot BA' = BC^2 = 4$. We will also use the fact that for each triangle DEF there is a unique A above x-axis that is similar to DEF (with B, C fixed). We split into the following cases:

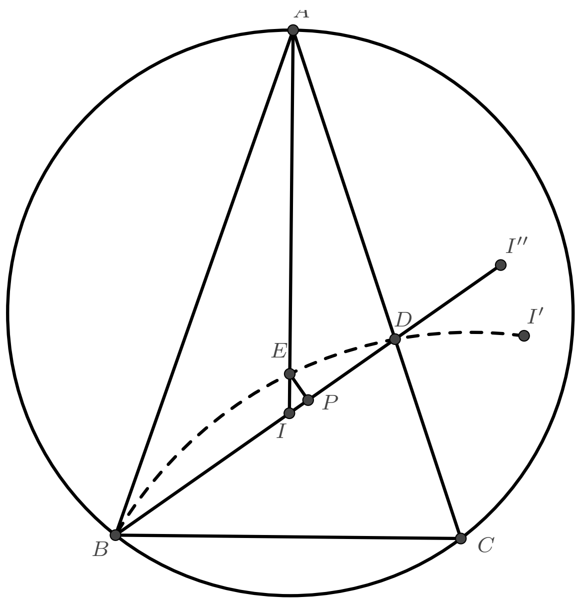
Case 1. $P_1P_2P_3$ equilateral. The only point A with $y \geq 0$ satisfying this is $A = (0, \sqrt{3})$. Since we have $(x+1)^2 + y^2 = (x-1)^2 + y^2 = 1 + 3 = 4$, A lies in both S and S' and we have $BA \cdot BA = 2 \times 2 = 4$.

Case 2. $P_1P_2P_3$ is isoceles, with the two equal sides longer than the other. Now, let $P_1P_2 = P_1P_3 > P_2P_3$. This means, if AC is the shortest and if triangle ABC is quasisimilar to $P_1P_2P_3$ then AC corresponds to P_2P_3, and AB, BC correspond to P_1P_2 and P_1P_3, which implies that $AB = BC = 2$. Such A can be uniquely constructed, and with $AC < AB = BC = 2$ we have A lies in S. Similarly, if $A'C$ is the shortest side of $A'BC$ and if triangle $A'BC$ is quasisimilar to $P_1P_2P_3$ then $A'C$ corresponds to P_2P_3, and $A'B, BC$ corresponds to P_1P_2 and P_1P_3, so A' can also be uniquely constructed (which turns out to be equal to A in this case). Therefore, $A'C < BC = A'B = 2$, which implies that A' is in S', and moreover $BA \cdot BA' = 2 \times 2 = 4$.

Case 3. $P_1P_2P_3$ is isoceles with the two equal sides longer than the other. Now, let $P_1P_2 = P_1P_3 < P_2P_3$. This means AC and $A'C$ corresponds to P_1P_2. In ABC, we know that $AB \leq BC$ implies AB corresponds to P_1P_3, BC corresponds to P_2P_3; In ABC, we know that $BC \leq A'B$ implies BC corresponds to P_1P_3, $A'B$ corresponds to P_2P_3. Therefore $\frac{AB}{BC} = \frac{P_1P_3}{P_2P_3} = \frac{BC}{A'B}$ and $BC^2 = AB \cdot A'B$.

Case 4. $P_1P_2P_3$ scalene, and let $P_1P_2 < P_1P_3 < P_2P_3$. This means AC and $A'C$ corresponds to P_1P_2. In ABC, we know that $AB \leq BC$ implies AB corresponds to P_1P_3, BC corresponds to P_2P_3; In ABC, we know that $BC \leq A'B$ implies BC corresponds to P_1P_3, $A'B$ corresponds to P_2P_3. Therefore $\frac{AB}{BC} = \frac{P_1P_3}{P_2P_3} = \frac{BC}{A'B}$ and $BC^2 = AB \cdot A'B$.

G4 Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

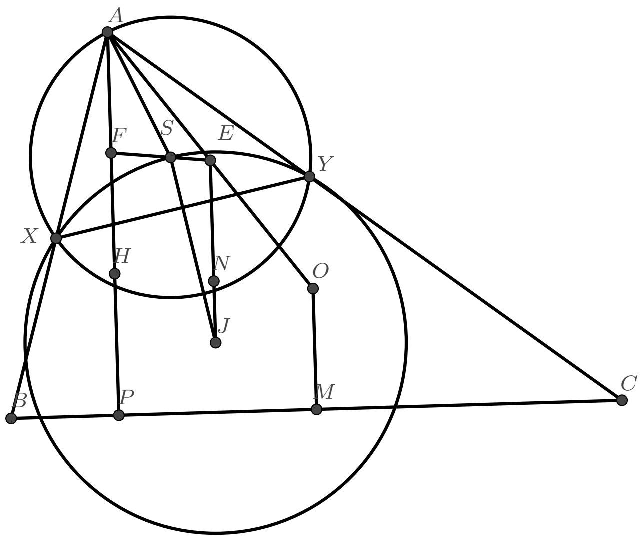


Solution. Let I' be the reflection of I in AC. Observe that AC is an angle bisector of $\angle BDI'$ by the definition of I', and since $DE \perp AC$, DE is another angle bisector of this angle. This implies that the intersection of DE and the circumcircle of BDI' (other than D) is equidistant from B and I', i.e. on the perpendicular bisector of BI'. It therefore suffices to prove that E lies on this perpendicular bisector, or $BE = EI'$.

Let I'' be the image of I when reflected in D, then $DI = DI' = DI''$. Moreover, I'' lies on line BD, which entails that $I'I''$ is parallel to AC and perpendicular to DE. Therefore, DE is the perpendicular bisector of $I'I''$ and $EI'' = EI'$. The problem is now reduced to proving $BE = EI''$. Let P be the foot of perpendicular from E to BD, then the problem is now equivalent to proving that P is the midpoint of BI''. Knowing that $BP = BI + ID - PD$ and $PI'' = PD + DI'' = PD + DI$ it suffices to prove that $BI = 2PD$.

Denote the common angles $\angle ABI, \angle IBC, \angle ICB, \angle ACI$ as α. Then, $\angle ADB = 3\alpha$, and $\angle IDE = |90° - 3\alpha|$ (as we will see, we are only interested in the cosine of this angle so don't worry about the sign). So $\frac{PD}{BI} = \frac{DE\cos\angle IDE}{BI} = \frac{AD\tan\angle DAI\cos\angle IDE}{BI} = \frac{AB\tan\angle DAI\cos\angle IDE\sin\angle ABD}{BI\sin\angle ADB} = \frac{BI\tan\angle DAI\cos\angle IDE\sin\angle ABD\sin\angle AIB}{BI\sin\angle ADB\sin\angle BAI} = \frac{\tan(90°-2\alpha)\cos|90°-3\alpha|\sin\alpha\sin(90°+\alpha)}{\sin(3\alpha)\sin(90°-2\alpha)} = \frac{\sin(90°-2\alpha)\sin(3\alpha)\sin\alpha\cos\alpha}{\sin(3\alpha)\sin(90°-2\alpha)\cos(90°-2\alpha)} = \frac{\sin\alpha\cos\alpha}{\sin(2\alpha)} = \frac{\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha} = \frac{1}{2}$, as $\cos|90°-x| = \cos(90°-x) = \sin x$, $\sin(90° + \alpha) = \cos\alpha$ and $\tan x = \frac{\sin x}{\cos x}$.

G5 Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.



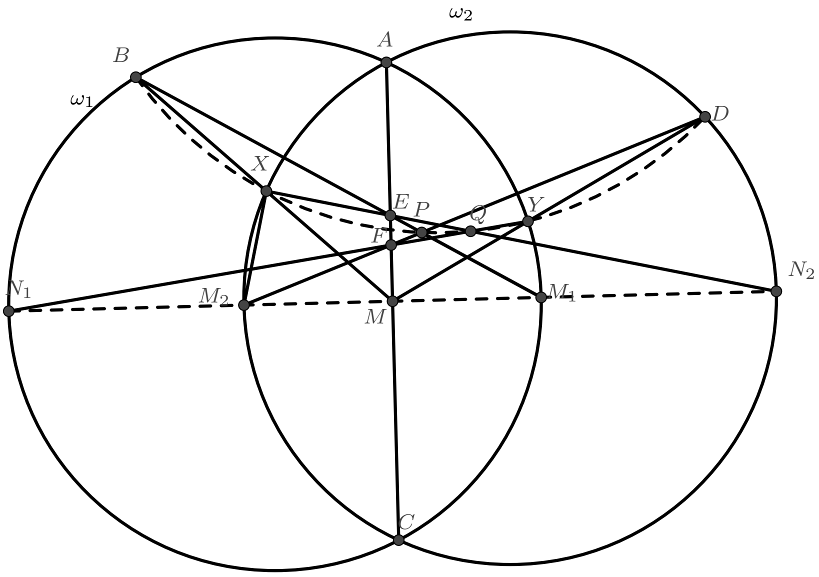
Solution. We first investigate the locus of S. Denote by O the circumcenter and H the orthocenter of triangle ABC Denote also by E the midpoint of AO, F the midpoint of AH, and N the midpoint of OH (the nine-point-center). Obviously S passes through the perpendicular bisector of AD, so this locus is a line. In the case there the circle passes through H, from the fact that $\angle ADH = 90°$ we know that $S = F$. Similarly, if the circle passes through O, $S = E$ in this case with $\angle ADO = 90°$. Thus the locus is actually EF, i.e. parallel to OH.

We proceed to prove that the circumcenter of triangle XSY (namely J) lies on the perpendicular bisector of PM. That is, the line passing through the midpoint of OH (a.k.a. nine-point-center) and perpendicular to BC (or parallel to AH). We first show this in the special cases that S is the midpoint of AO or AH. In the first case ($E = S$), X is the midpoint of AB and Y is the midpoint of AC (imagine the homothety centered at A with factor $\frac{1}{2}$ which brings ABC to AXY and point O to the midpoint of AO). J lies on the perpendicular bisector of XY Notice that, with $XY \parallel BC$, this perpendicular bisector of XY is also perpendicular to BC. Moreover, nine-point-circle passes through the midpoints of AB and AC, so this perpendicular bisector passes through the nine-point center. Therefore the perpendicualr bisector of XY is the perpendicular bisector of PM itself, and with $SX = SY$, S (the midpoint of AO, a.k.a. E in this case) lies on this perpendicular bisector too. In the second case, X and Y are going to be the altitude from C to AB, and B to AC, respectively. Since the nine-point circle passes through the midpoint of AH (S in this case), X, Y, P, M, J, the circumcenter of $XSYPM$ is the nine-point center itself.

Now let's do the general case. Observe that with E (midpoint of AO) and N (midpoint of OH) both equidistant from PM the conclusion now becomes J lies on EN. We invoke the following two lemmas:
• $\frac{SJ}{AS} = \frac{AE}{EN}$.
Proof: we have $AE = \frac{1}{2}AO$ and $EN = AF = \frac{1}{2}AH$, and $AS = SX = SY$, so we just have to prove that $\frac{SJ}{SX} = \frac{AO}{AH}$. First, it is well notice that SJ is the circumradius of SXY, so knowing that $\angle SXY = \angle XSY = 90^\circ - \frac{1}{2}\angle XSY = 90^\circ - \angle XAY = 90^\circ - \angle BAC$ we have $SX = 2SJ \sin\angle SXY = 2SJ\cos\angle BAC$, yielding $\frac{SJ}{SX} = \frac{1}{2\cos\angle BAC}$. Let T be the reflection of H in M, then $HBMT$ is a parallelogram with $\angle ABT = \angle ABC + \angle CBT$ $\angle ABC + \angle HCB = 90^\circ$, and similarly $\angle ACT = 90^\circ$. Therefore A, O, T collinear and with $AH \parallel OM$ we have $\frac{OM}{AH} = \frac{TH}{MT} = \frac{1}{2}$. We also know that $\frac{OM}{AO} = \frac{OM}{BO} = \cos\angle BOM = \cos\angle BAC$ so $\frac{AO}{AH} = \frac{AO}{2OM} = \frac{1}{2\cos\angle BAC} = \frac{SJ}{AS}$.
• $\angle(HA, AS) = \angle(SJ, AO)$. (This would also imply $\angle(AO, AS) = \angle(SJ, AH)$).
Proof: we use the well-known fact that the circumcenter and orthocenter of each triangle are the isogonal conjugates of each other. In particular, if ℓ is the perpendicular from A to XY then AS and ℓ are the images of each other in the reflection of the internal angle bisector of $\angle AXY$. This gives $\angle(AB, AS) = \angle(\ell, AC)$. Same goes for the relation between AH and AO, and therefore $\angle(AB, AH) = \angle(AO, AC)$. Moreover, $SJ \perp XY$ (since $SX = SY$ and $JX \perp JY$ we know that SJ must be the perpendicular bisector of XY). Therefore $SJ \parallel \ell$. Now we have $\angle(SJ, AO) = \angle(\ell, AO) = \angle(\ell, AC) + \angle(AC, AO) = \angle(AB, AS) + \angle(AH, AB) = \angle(AH, AS)$.

To complete the proof denote J' by the intersection of SJ and EN and we shall prove that $J = J'$ by proving that $SJ = SJ'$. From the first lemma it suffices to prove that $\frac{SJ'}{AS} = \frac{AE}{EN}$. Now $\frac{AS}{SE} = \frac{\sin\angle AES}{\sin\angle SAE} = \frac{\sin\angle AOH}{\sin\angle SAO}$ and $\frac{SJ'}{SE} = \frac{\sin\angle SEJ'}{\sin\angle EJ'S} = \frac{\sin\angle AFE}{\sin\angle EJ'S} = \frac{\sin\angle AHO}{\sin\angle EJ'S}$. Now, $\angle(EJ', J'S) = \angle(EN, SJ) = \angle(AH, SJ) = \angle(AO, AS)$ so the angles $\angle SEJ'$ and $\angle SAO$ are either equal or supplementary, hence $\sin\angle SEJ' = \sin\angle SAO$. Therefore, $\frac{SJ'}{AS} = \frac{SJ'}{SE} \div \frac{AS}{SE} = \frac{\sin\angle AHO}{\sin\angle EJ'S} \div \frac{\sin\angle AOH}{\sin\angle SAO} = \frac{\sin\angle AHO}{\sin\angle AOH} = \frac{AO}{AH} = \frac{EN}{AE}$, Q.E.D.

G6 Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.



Solution. Let ω_1 be the circumcircle of ABC and ω_2 the circumcircle of ADC, then these two circles are symmetric w.r.t. AC. Also notice that BP passes through M_1, the midpoint of arc AC of ω_1 not containing B, and DP passes through M_2, the midpoint of arc AC of ω_2 not containing D.

We first start with a preliminary observation: X lies on ω_2 and Y lies on ω_1. W.L.O.G. for this section we assume that $AB \leq AC$. Indeed, let X' be on BM satisfying $MX' \cdot MB = MA^2 = MC^2$. Then $\angle X'AC = \angle MBA$ and $\angle X'CA = \angle MBC$. Thus $\angle ADC = \angle ABC = \angle MBA + \angle MBC = \angle X'AC + \angle X'CA = \pi - \angle AX'C$, so X' lie on ω_2. In addition, let BM intersect ω_1 again at X'', then X' and X'' are symmetrical w.r.t. AC. Combining with the fact that M_1 and M_2 are also symmetrical w.r.t. AC (being the midpoint of arc) we have $X'M_2 = X''M_1$. Knowing that the two circles have the same radius further allows us to assert $\angle X'BP = \angle X''BM_1 = \angle X'DM_2 = \angle X'DP$, showing that D, B, P, X' cyclic hence $X' = X$. Similarly, Y lies on ω_1.

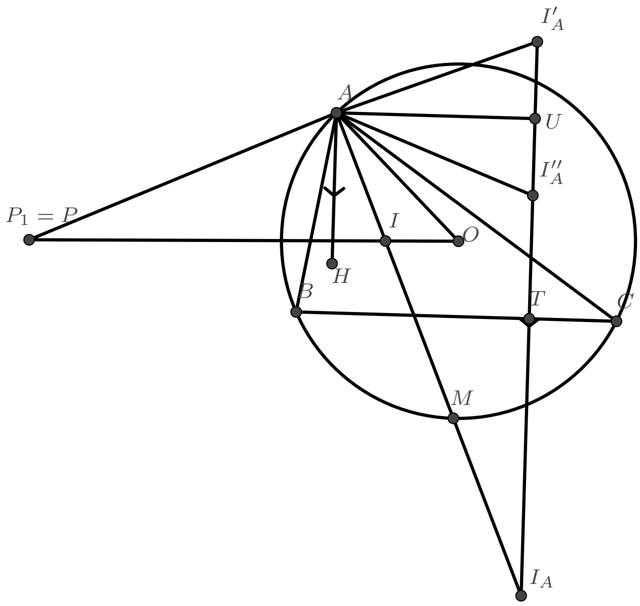
Next, let N_1 be diametrically opposite M_1 w.r.t. ω_1 and define similarly for N_2. We claim that XE passes through N_2 by claiming that XE is the internal angle bisector of $\angle AXC$. Indeed, by angle bisector theorem we have $\frac{AE}{EC} = \frac{AB}{BC}$. Invoking our X'' from the previous section (i.e. the other intersection of BM and ω_1) gives $AXCX''$ parallelogram. Now invoking a little bit more trigonometric bashing we have $1 = \frac{AM}{CM} = \frac{AB}{BC} \cdot \frac{\sin \angle ABM}{\sin \angle CBM} = \frac{AB}{BC} \cdot \frac{AX''}{CX''} = \frac{AB}{BC} \cdot \frac{CX}{AX}$, so $\frac{AX}{CX} = \frac{AB}{BC} = \frac{AE}{EC}$, and the conclusion follows by the angle bisector theorem. Analogously, YF passes through N_1.

Finally, considering triangle PEF, and letting the perpendicular from P to reach AC at P_1 we have (considering signed length) $\frac{EP_1}{FP_1} = \frac{\cot \angle FEP}{\cot \angle EFP}$. Similarly if letting perpendcular from Q to reach AC at Q_1 we have $\frac{EQ_1}{FQ_1} = \frac{\cot \angle FEQ}{\cot \angle EFQ}$. Now $\cot \angle FEP = \cot \angle MEM_1 = \frac{MM_1}{EM}$, $\cot \angle EFP = \cot \angle MFM_2 = \frac{MM_2}{FM}$. Considering $MM_2 = MM_1$ we have $\frac{\cot \angle FEP}{\cot \angle EFP} = \frac{FM}{EM}$. Analogously, $\cot \angle FEQ = \cot \angle FEQ = \cot \angle MEN_2 = \frac{MN_2}{EM}$, and $\cot \angle EFQ = \cot \angle N_1FM = \frac{MN_1}{FM}$. Therefore we have $\frac{\cot \angle FEQ}{\cot \angle EFQ} = \frac{FM}{EM}$ since again it is not hard to verify that $MN_2 = MN_1$. (For signed convention we can say that $ME < 0$ if it's nearer to A than B, and > 0 otherwise). Therefore, $\frac{EP_1}{FP_1} = \frac{EQ_1}{FQ_1}$, so $P_1 \equiv Q_1$ and the two perpendicular lines coincide.

G7 Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

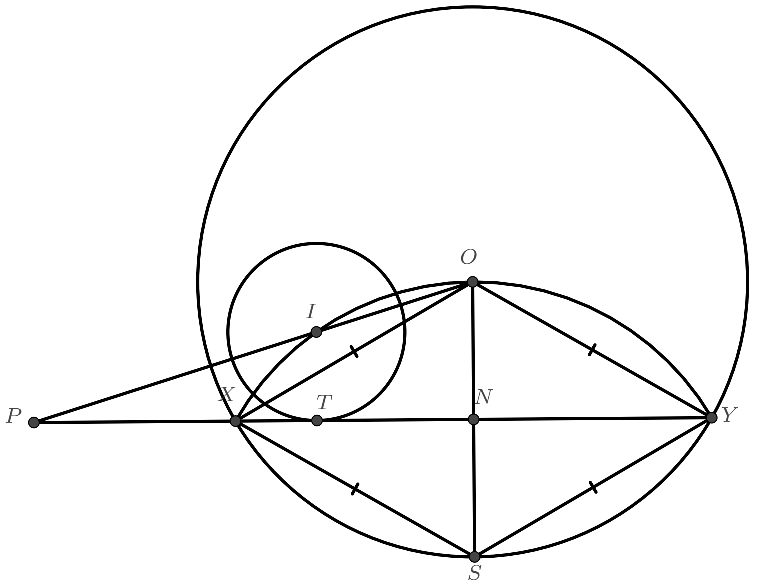


Solution.

(a) We first prove the following lemma: let I''_A the point on $I_AI'_A$ satisfying $AI''_A = AI'_A$, and $I''_A \neq I'_A$ unless AI'_A and $I_AI'_A$ are perpendicular to each other. Then triangles AOI and $I_AAI''_A$ are similar. Indeed, let H be the orthocenter of triangle ABC, then $AH \parallel I_AI''_A$. Also, $\angle OAI = \angle IAH = \angle AI_AI''_A$, so it suffices to prove that $\frac{AI}{AO} = \frac{I_AI''_A}{I_AA}$. To do so, define I_C has how I_A and I_B are defined. We first notice that I is the orthocenter of $I_AI_BI_C$, so I_AA is an altitude of triangle $I_AI_BI_C$. In addition, we know that triangles $I_AI_BI_C$ are I_ACB are similar with a similitude of $\frac{I_AB}{I_AI_B} = \cos\angle BI_AC$. Denote by T the altitude from I_A to BC, then by the similarities of the triangles mentioned above we have $\frac{I_AI'_A}{I_AA} = \frac{2I_AT}{I_AA} = 2\cos\angle BI_AC$. In addition, if we let U to be the perpendicular from A to $I_AI''_A$ we have I'_A and I''_A symmetric to each other w.r.t. U. Therefore, $I_AI''_A + I_AI'_A = 2I_AU = 2I_AA\cos\angle AI_AI'_A = 2I_AA\cos\angle AOI$. So $\frac{I_AI''_A}{I_AA} = \frac{2I_AA\cos\angle AOI - I_AI'_A}{I_AA}$ $= \frac{2I_AA\cos\angle AOI - 2I_AA\cos\angle BI_AC}{I_AA} = 2(\cos\angle AOI - \cos\angle BI_AC)$. On the other hand, denoting M as the other intersection of AI and BC we have $MB = MC = MI = 2AO\cos BAM$ (because AO is the circumradius of triangle ABC), and $AM = AO\sin\angle ABM$, so $\frac{AI}{AO} = \frac{AM - MI}{AO} = \frac{2AO\cos\angle ABM - 2AO\cos BAM}{AO} = 2(\cos\angle ABM - \cos\angle BAM) = 2(\cos\angle AOI - \cos\angle BI_AC)$ (the equality $\angle ABM = \angle AOI$ and $\angle BAM = \angle BI_AC$ can be established via angle chasing). This establishes the desired equality.

Now denote P_1 on OI such that I lies between O and P_1 and $OI \cdot OP_1 = AO$. Then, triangles AOI anf AP_1O are similar. Therefore, by the fact that $\triangle AOI \sim \triangle I_AAI''_A$ $\angle IAP_1 = \angle OAP_1 - \angle OAI = \angle AIO - \angle OAI = \angle I_AI''_AA - \angle OAI = 180° - \angle I_AI'_AA - \angle HAI$ $= 180° - \angle I_AI'_AA - \angle AI_AI'_A = \angle I'_AAI_A$. This means that l_A passes through P_1 and similarly, l_B passes through P_1. Hence $P_1 = P$ and lies on OI.

(b)



We first establish a relation between the length OI and the inradius r. Let the circumradius be $R = AO = BO = CO$. Using fresh new label than (a), we denote M as the second intersection of AI and the circumcircle of ABC. This M is also the midpoint of arc BC not containing A. Now, $MI = MB = MC = 2R\sin\angle BAI = 2R\sin\frac{\angle A}{2}$, $AM = AB\frac{\sin\angle ABI}{\angle AIB} = 2R\sin\angle C\frac{\sin\frac{1}{2}B}{\sin(90^\circ+\frac{1}{2}C)} = 2R(2\sin\angle\frac{\angle C}{2}\cos\angle\frac{\angle C}{2})\frac{\sin\frac{1}{2}\angle B}{\cos\frac{1}{2}\angle C}$ $= 4R\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$ so $MI\cdot AI = 8R^2\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$. Meanwhile, letting D be the point of tangency of the incircle to BC we have $r = ID = IB\sin\angle IBC = BC\frac{\sin\angle ICB}{\sin\angle CIB}\sin\frac{1}{2}\angle B = 2R\sin A\frac{\sin\frac{1}{2}C}{\sin(90^\circ+\frac{1}{2}A)}\sin\frac{1}{2}\angle B = 2R(2\sin\angle\frac{\angle A}{2}\cos\angle\frac{\angle A}{2})\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}A)}\sin\frac{1}{2}\angle B$ $= 4R\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$. Considering the fact that $-MI\cdot AI$ is the power of point of I w.r.t. the circumcircle we have $OI^2 = R^2 - MI\cdot AI = R^2 - 8R^2\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$, or $\frac{OI^2}{R^2} = 1 - 8\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$ while $\frac{r}{R} = 4\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C$ so $\frac{OI^2}{R^2} = 1 - \frac{2r}{R}$, or $\frac{r}{R} = \frac{1}{2} - \frac{OI^2}{2R^2}$, or $r = \frac{1}{2}R(1 - \frac{OI^2}{R^2})$ (this is actually a well-known identity, the purpose of including the proof is to show the power of trigonometry in solving problems).

Now, let T be the tangency point of the incircle to XY, and N be the midpoint of XY. Keeping in mind that $OP\cdot OI = R^2$, we now have $\frac{PI}{OP} = 1 - \frac{OI}{OP} = 1 - \frac{OI}{R^2\div OI} = 1 - \frac{OI^2}{R^2}$. Therefore $ON = IT\frac{PO}{PI} = r\frac{1}{1-\frac{OI^2}{R^2}} = \frac{1}{2}R(1 - \frac{OI^2}{R^2})\frac{1}{1-\frac{OI^2}{R^2}} = \frac{1}{2}R$. Moreover, letting S be the midpoint of arc XY lying on the opposite side as I w.r.t. XY we have O, N, S collinear, $ON = NS$, and $ON \perp XY$. Therefore, $OX = OY = OS = XS = YS$, yielding OXS and OYS both equilateral and $\angle XOY = 60^\circ + 60^\circ = 120^\circ$. Additionally, $PI\cdot PO = PO^2 - (IO\cdot OP) = PO^2 - R^2$, which is the power of point of P w.r.t. the circumcircle. This, in turn, is equal to the value $PX\cdot PY$, so $IOYX$ is cyclic. Thus $\angle XIY = \angle XOY = 120^\circ$.

4 Number Theory

N1 For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Solution. The answer is the constant polynomial $P(x) = c$ where $c \in \{1, 2, \cdots, 9\}$, or the identity polynomial $P(x) = x$. In the first case we have $S(P(n)) = S(c) = c = P(\text{anything}) = P(S(n))$, and in the second case $S(P(n)) = S(n) = P(S(n))$.

Now let $P(x) = \sum_{i=0}^{k} a_i x^i$, then for sufficiently large n (in particular, $10^n > max\{a_i(9^i) : i \in [0,k]\}$) and for each $c \in \{1, 2, \cdots, 9\}$ we have $P(c \cdot 10^n) = \sum_{i=0}^{k} a_i(c^i)(10^{ni})$. Since $a_i(c^i)(10^{ni}) < 10^{(n+1)i}$ (because $a_i(c^i) < 10^n$ by our choice of n), the number $P(c \cdot 10^n)$ are in the form of $(a_k c^k)(0 \cdots 0)(a_{k-1}c^{k-1})(0 \cdots 0) \cdots (0 \cdots 0)(a_0 c^0)$ when laid in decimal form. Therefore $S(P(c \cdot 10^n)) = \sum_{i=0}^{k} S(a_i(c^i))$, and $P(S(c \cdot 10^n)) = P(c) = \sum_{i=0}^{k} a_i(c^i)$. Knowing that $S(x) \le x$ with equality holds if and only if $0 \le x \le 9$ (indeed, if $k = \sum_{i=0}^{k} b_i(10^i)$ then $S(k) = \sum_{i=0}^{k} b_i$, so $k - S(k) = \sum_{i=0}^{k} b_i(10^i - 1) \ge 0$, with equality holds iff $b_i = 0$ for $i \ge 1$,) we have $a_i(c^i) \le 9$ for all $c \in \{0, 1, \cdots 9\}$. This means $k \le 1$ (if we assume that $a_k > 0$). If $k = 0$ then we get $a_0 \le 9$, yielding the constant solution. If $k = 1$, then $9a_1 \le 9$ (when $c = 9$) and $a_1 = 1$, yielding $P(x) = x + c$ for some constant c (and since $c = a_0$ we have $c = a_0 \le 9$ too). This entails $S(P(n)) = S(n+c)$ and $P(S(n)) = S(n)+c$ for all $n \ge 2016$, and letting $n = 10^d - 1$ we have $S(n) = 9d$, and for $c \ge 1$, $S(n + c) = S(10^k - 1 + c) = c$, which doesn't hold for $d = 5$. Therefore $c = 0$ and we get the identity polynomial.

N2 Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Solution. The answer is 2 and all composite numbers. Let $m = 10n$, with $m = 3^y \cdot \prod_{i=1}^{k} p_i^{a_i} \cdot \prod_{i=1}^{l} q_i^{b_i}$ with $p_i \equiv 1 \pmod 3$ and $q_i \equiv 2 \pmod 3$. Notice that $\tau(m) = (y + 1) \cdot \prod_{i=1}^{k}(a_i + 1) \cdot \prod_{i=1}^{l}(b_i + 1)$.

Now we want to investigate all the divisors that is congruent to 1 mod 3, observe that such divisors fulfill $\prod_{i=1}^{k} p_i^{c_i} \cdot \prod_{i=1}^{l} q_i^{d_i}$ with $c_i \le a_i$, $d_i \le b_i$ and $\sum_{i=1}^{l} d_i$ even. We proceed with the following claim: the number of combinations $(d_1, d_2, \cdots, d_l)$ satisfying $\sum_{i=1}^{l} d_i$ even and $d_i \le b_i$ is $\lfloor \frac{\prod_{i=1}^{l}(b_i+1)}{2} \rfloor$.

Case 1. b_i odd for some i, and w.l.o.g. let this i be l. Now, let x be the number of combinations $(d_1, d_2, \cdots, d_{l-1})$ $(d_i \le b_i)$ satisfying $\sum_{i=1}^{l-1} d_i$ even, and z be the number of combinations with corresponding odd sums. Considering $d_i = \{0, 2, \cdots, b_l - 1\}$ and $d_i = \{1, 3, \ldots, b_l\}$ we have: the number of combinations $(d_1, d_2, \cdots, d_{l-1})$ $(d_i \le b_i)$ satisfying $\sum_{i=1}^{l} d_i$ even is $x + z + x + z + \cdots + x + z = (x + z) \cdot \frac{b_i+1}{2}$, and similarly $z + x + + \cdots + z + x = (x + z) \cdot \frac{b_i+1}{2}$ for odd-sum combinations. Therefore there is equally many odd and even sum combinations, and we are done.

Case 2. Now let b_i even for all i. Let O be number of combinations with $\sum_{i=1}^{l} d_i$ odd and E be combinations with $\sum_{i=1}^{l} d_i$ even. The claim is $E - O = 1$. We induct on l. Base case $l = 0$ yield 1 combination for even sum and 0 combination for odd sum, vacuously. Now let $l = k$ for some k and we have O' as the number of combinations $(d_1, d_2, \cdots, d_k)$ with $\sum_{i=1}^{k} d_i$ odd, and E' as the number of combinations with $\sum_{i=1}^{l} d_i$ even. Now that b_{k+1} is even, using the logic above the number of even combination is $E' + O' + E' + O' + \cdots + E' = E'(\frac{b_{k+1}}{2} + 1) + O'(\frac{b_{k+1}}{2})$, and similarly the number of combinations yielding odd sum is $O'(\frac{b_{k+1}}{2} + 1) + E'(\frac{b_{k+1}}{2})$. This yields $E - O = E' - O'$ and by induction hypothesis this number is 1, so we are done.

Summing above, $\tau_1(m) = \prod_{i=1}^{k}(a_i + 1)\lfloor \frac{\prod_{i=1}^{l}(b_i + 1)}{2} \rfloor$, so the ratio now becomes $(y + 1)\frac{\prod_{i=1}^{l}(b_i + 1)}{\lfloor \frac{\prod_{i=1}^{l}(b_i+1)}{2} \rfloor}$. Equivalently, $2(y + 1)$ when b_i odd for some b_i, or $(y + 1)\frac{2k+1}{k}$ otherwise (where $2k + 1 = \prod_{i=1}^{l}(b_i + 1)$ here). The first case yields that the ratio must be even; in the second case, we have $\gcd(2k + 1, k) = 1$ so $k|y + 1$. In other words, the ratio must be divisible by $2k + 1$. Notice, also, that $l \geq 2$ ($m = 10n$ contains prime factors 2 and 5) so $2k + 1 = \prod_{i=1}^{l}(b_i + 1)$ must be composite. So our integer ratio cannot be an odd prime.

It remains to show that any even or composite numbers work. For even numbers $2k$, simply take $10 \cdot 3^{k-1}$ and by our proof the ratio is $2k$. For odd composite number xz with $x, z \geq 3$, take $m = 2^{x-1}5^{z-1}$.

N3/IMO 4 A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a + 1), P(a + 2), \ldots, P(a + b)\}$$

is fragrant?

Solution. The answer is $b = 6$. Observe that this solution works because the set $\{P(197), P(198), P(199), P(200), P(201), P(202)\}$ has $P(199) \equiv P(202) \equiv P(1) = 3 \equiv 0 \pmod 3$, $P(198) \equiv P(2) = 7 \equiv 0 \equiv 21 = P(4) \equiv P(200) \pmod 7$, $P(197) \equiv P(7) = 57 \equiv 0 \equiv 133 = P(11) \equiv P(201) \pmod{19}$.

First, notice that $P(n) - P(n - 1) = n^2 + n + 1 - (n^2 - n - 1) = 2n$, and knowing that $n^2 + n + 1 \equiv n + n + 1 = 2n + 1 \equiv 1 \pmod 2$, we know that if $p|P(n)$ and $p|2n$ then $p|n$ (since $P(n)$ is relatively prime to 2), and consequently $p|n^2 + n$ and $p|1$, showing that $P(n)$ and $P(n - 1)$ are relatively prime. This means, $b = 2$ fails, and $b = 3$ fails too sine $P(a+1)$ and $P(a+3)$ are both relatively prime to $P(a+2)$. (We will use profusely the fact that $P(a)$ and $P(a + 1)$ cannot have any common prime factor throughout the solution).

Now, for $b = 4$ and $b = 5$ our strategy is to determine an upper bound for $\gcd(P(n), P(n+c))$ for $c = 2, 3$. Observe that $P(n + c) - P(n) = 2cn + c^2 + c = c(2n + c + 1)$. For $c = 2$ this is the same as $2(2n + 3)$. If $p|P(n + 2)$ and $p|P(n)$ then $p|2(2n + 3)$, and therefore $p|2n+3$ with P being odd at all times. This entails $2n \equiv -3 \pmod p$, and $0 \equiv 4P(n) = 4n^+4n+1 = (2n)^2 + 2(2n) + 1 \equiv (-3)^2 - 3 + 1 = 7 \pmod 7$. Hence $p = 7$ and $n \equiv 2 \pmod 7$. Now for $b = 4$, knowing that $P(a+2)$ is relatively prime with $P(a+1)$ and $P(a+3)$ it must have a common prime factor with $P(a+4)$, and by the previous step this prime factor has to be 7. Similarly $P(a+1)$ and $P(a+3)$ must both be divisible by 7. This means $P(a+1), P(a+2), P(a+3), P(a+4)$ are all divisible by 7 for some a, contradicting that any two neighbouring elements are coprime.

Finally for $b = 5$ we investigate $c = 3$ as in the previous paragraph. Now $3(2n+3+1) = 3(2n+4) = 3(2)(n+2)$. If a prime p satisfies $p|P(n)$ and $p|P(n+3)$ simultaneously then either $p = 3$ or $p|n+2$ (again p must be relatively prime to 2 so this can be easily factored out). In the second case we have $n \equiv= 2 \pmod p$, so $P(n) \equiv P(-2) = 4 - 2 + 1 = 3 \equiv \pmod p$, forcing $p = 3$ (no choice!) Thus viewing the set $\{P(a+1), \cdots, P(a+5)\}$ we know that $P(a+3)$ must have a common factor with $P(a+1)$ or $P(a+5)$, and by previous paragraph this common factor has to be 7. Thus neither of $P(a+2)$ nor $P(a+4)$ can be divisible by 7, and they cannot have common prime factor (again by previous paragraph). This entails $P(a+1)$ and $P(a+4)$ must have common factor, and by what we established earlier this factor must be 3. Similarly, $P(a+2)$ and $P(a+5)$ must both be divisible by 3. However, $P(a+1)$ and $P(a+2)$ are both divisible by 3, contradiction.

N4 Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

Solution. We split our solution into two cases:

Case 1. $l \leq k$. Now from $n^k + mn^l + 1 | n^{k+l} - 1$, and from the fact that $(n^l - 1)(n^k + mn^l + 1)$ $= n^{k+l} + mn^{2l} + n^l - n^k - mn^l - 1$ we have $(n^{k+l} + mn^{2l} + n^l - n^k - mn^l - 1) - (n^{k+l} - 1)$ $= mn^{2l} + n^l - n^k - mn^l = n^l(mn^l + 1 - n^{k-l} - m) = n^l(m(n^l - 1) - (n^{k-l} - 1))$ is divisible by $n^k + mn^l + 1$. Knowing that $\gcd(n, n^k + mn^l + 1) = \gcd(n, 1) = 1$ we have $\gcd(n^l, n^k + mn^l + 1) = 1$ so $m(n^l - 1) - (n^{k-l} - 1)$ is itself divisible by $n^k + mn^l + 1$. Now, $m(n^l - 1) < mn^l < n^k + mn^l + 1$ and $n^{k-l} - 1 \leq n^k - 1 < n^k + mn^l + 1$, meaning that $0 < \frac{m(n^l-1)}{n^k+mn^l+1}, \frac{(n^{k-l}-1)}{n^k+mn^l+1} < 1$. Therefore $|\frac{m(n^l-1)-(n^{k-l}-1)}{n^k+mn^l+1}| < 1$, and therefore has to be 0. We thus have $m(n^l - 1) = (n^{k-l} - 1)$ and since $n > 1$, $m = \frac{n^{k-l}-1}{n^l-1}$. Let $k - l = cl + d$ with $0 \leq d < l$, then $n^{k-l} = n^{cl} \cdot n^d \equiv (n^l)^c \cdot n^d \equiv 1 \cdot n^d = n^d \pmod{n^l - 1}$, and from $n^d < n^l$ we have $n^d \not\equiv 1 \pmod{n^l - 1}$ unless $d = 0$. Therefore $l|k - l$, or $l|k$.

Case 2. $l \geq k$. Similar to above we have $(n^k - 1)(n^k + mn^l + 1) - (n^{k+l} - 1) = n^{2k} + mn^{k+l} + n^k - n^k - mn^l - 1 - (n^{k+l} - 1) = n^{2k} + mn^{k+l} - mn^l - n^{k+l} = n^k(n^k - mn^{l-k} + (m-1)n^l)$ is divisible by $n^k + mn^l + 1$. Again by the logic above, $\gcd(n^k, n^k + mn^l + 1) = 1$, which very well means that $n^k + mn^l + 1 | n^k - mn^{l-k} + (m-1)n^l$. Again we have $n^k + (m-1)n^l < n^k + mn^l + 1$ and $mn^{l-k} < n^k + mn^l + 1$ so by the logic above, again, $n^k - mn^{l-k} + (m-1)n^l = 0$. Rearranging the terms give: $m = \frac{n^l - n^k}{n^l - n^{l-k}}$. Now, if $m \geq 2$, then we have $n^l - n^k \geq 2n^l - 2n^{l-k}$, or $2n^{l-k} \geq n^l + n^k > n^l = n^k(n^{l-k})$, or $2 > n^k$, forcing $n = 1$ (contradiction since $k \geq 1$). Thus $m = 1$ (m must be positive) and we have $k = l - k$, or $l = 2k$.

N5 Let a be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let A be the set of positive integers k for which the equation admits a solution with $x > \sqrt{a}$, and let B be the set of positive integers for which the equation admits a solution with $0 \leq x < \sqrt{a}$. Show that $A = B$.

Solution. To show that $k \in A \to k \in B$, let $x > \sqrt{a}$ for some x satisfying the equation. It follows that $|y| < |x|$. Denote $y = x - c$ and we have $a = x^2 - k(x^2 - y^2) = x^2 - kc(2x - c) = x^2 - 2kcx + kc^2$. Let $x_1 = x - 2kc$ and $y_1 = x_1 + c$ and we have $\dfrac{x_1^2 - a}{x_1^2 - y_1^2} = \dfrac{(x - 2kc)^2 - (x^2 - kc(2x - c))}{(x_1 - y_1)(x_1 + y_1)} = \dfrac{-2kc(2x - 2kc) + kc(2x - c)}{-c(2(x - 2kc) + c)} = \dfrac{-kc(4x - 4kc - 2x + c)}{-c(2x - 4kc) + c)} = k$. This means k admits (x_1, y_1) as well, and from $x_1 = y_1 + c < y_1$ we have $x_1 < \sqrt{a}$. Also notice that $x \geq 2kc$ because... so $x_1 \geq 0$. Therefore $k \in B$ too.

Conversely, we want to show that $k \in B \to k \in A$. Let $x < \sqrt{a}$ for some x satisfying the equation. It follows that $|y| > |x|$. Denote $y = x + c$ and we have $a = x^2 - k(x^2 - y^2) = x^2 - k(-c)(2x + c) = x^2 + 2kcx - kc^2$. Let $x_2 = x + 2kc$ and $y_2 = x_1 0c$ and we have $\dfrac{x_2^2 - a}{x_1^2 - y_2^2} = \dfrac{(x + 2kc)^2 - (x^2 + kc(2x + c))}{(x_2 - y_2)(x_2 + y_2)} = \dfrac{2kc(2x + 2kc) - kc(2x + c)}{c(2(x + 2kc) - c)} = \dfrac{kc(4x + 4kc - 2x - c)}{c(2x + 4kc) - c)} = k$. This means k admits (x_2, y_2) as well, and from $x_2 = y_2 + c > y_2$ we have $x_1 > \sqrt{a}$. Therefore $k \in A$ too.

N6 Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Solution. The only function is $f(x) = x^2, \forall x \in \mathbb{N}$. In fact, $mf(m) + nf(n) = m^3 + n^3 = (m + n)(m^2 - mn + n^2) = (m + n)(f(m) - mn + n^2)$.

Substituting $m = n = 1$ gives $2f(1) - 1 | 2f(1)$, so $2f(1) - 1 = \pm 1$. Since $f(1) > 0$ ($f(1) \in \mathbb{N}$) we have $f(1) = 1$. Next, letting $n = 1$ gives $f(m) - (m - 1) | mf(m) + 1 = m(f(m) - (m - 1)) + m^2 - m + 1$, so with $f(m) - (m - 1) | m^2 - m + 1$ and $m^2 - m + 1 = (m - \frac{1}{2})^2 + \frac{3}{4} > 0$ we have $|f(m) - (m - 1)| \leq m^2 - m + 1$ and $f(m) \leq m^2$.

The next step is to show that $f(p) = p^2$ for all sufficiently large prime p. Substituting $m = n = p$ gives $2f(p) - p^2 | 2pf(p) = p(2f(p) - p^2) + p^3$, so $2f(p) - p^2 | p^3$ and from $f(p) \leq p^2$ we have $2f(p) - p^2 \in \{p^2, p, 1, -1, -p\}$ (again it this value cannot be $-p^2$ or lower because $f(p) > 0$. Therefore $f(p) \in \{p^2, \frac{p^2+p}{2}, \frac{p^2+1}{2}, \frac{p^2-1}{2}, \frac{p^2-p}{2}\}$. Now we check $n = 1, m = p$ again and we have (from above) $f(p) - (p - 1) | p^2 - p + 1$. We investigate the following cases:

(a) $f(p) = \frac{p^2+p}{2}$, then $\frac{p^2+p}{2} - (p-1) | p^2 - p + 1 = 2(\frac{p^2+p}{2} - (p-1)) - 1$, so $\frac{p^2+p}{2} - (p-1) \leq 1$, which doesn't hold for $p \geq 2$.

(b) $f(p) = \frac{p^2+1}{2}$, then $\frac{p^2+1}{2} - (p - 1) | p^2 - p + 1 = 2(\frac{p^2+1}{2} - (p - 1)) + p - 2$, which means $\frac{p^2+1}{2} - (p - 1) \leq p - 2$, not true for $p \geq 3$.

(c) $f(p) = \frac{p^2-1}{2}$, then $\frac{p^2-1}{2} - (p - 1) | p^2 - p + 1 = 2(\frac{p^2-1}{2} - (p - 1)) + p$, meaning $\frac{p^2+1}{2} - (p - 1) \leq p$, not true for $p \geq 3$.

(d) $f(p) = \frac{p^2-p}{2}$, then $\frac{p^2-p}{2} - (p-1) | p^2 - p + 1$. Observe that $2(\frac{p^2-p}{2} - (p-1)) = p(p-1) - 2(p-1) = (p-1)(p-2)$, so $p - 1 | 2(p^2 - p + 1)$. Now $2(p^2 - p + 1) \equiv 2(1^2 - 1 + 1) = 2 \pmod{p - 1}$, so $p - 1 \leq 2$¡ or $p \leq 3$.

We therefore know that all four cases cannot hold for $p \geq 5$, so $f(p) = p^2$ for $p \geq 5$.

Now, let m be arbitrary and let $n = p$ for some prime p we have $f(m) + p^2 - mp | mf(m) + p^3 = m(f(m) + p^2 - mp) + p^3 - mp^2 + m^2p = m(f(m) + p^2 - mp) + p(p^2 - pm + m^2)$. Consider the ratio $\dfrac{p(p^2 - pm + m^2)}{f(m) + p^2 - mp} = p(1 + \dfrac{m^2 - f(m)}{f(m) + p^2 - mp})$, and therefore $\dfrac{p(m^2 - f(m))}{f(m) + p^2 - mp}$ must be an integer. Choosing any $p > f(m)$ gives $p \nmid f(m)$, and hence $p \nmid f(m) + p^2 - mp$, hence p and $f(m) + p^2 - mp$ are relatively prime. Therefore $\dfrac{m^2 - f(m)}{f(m) + p^2 - mp}$ is itself an integer, and with $f(m) + p^2 - mp$ approaching infinity as p approaching infinity we know that $f(m) - m^2$ must be zero. (Remember, there are infinitely many primes).