1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Answer. $f \equiv 0$ (the zero function) or $f(n) = 2n + c$ for all $n \in \mathbb{Z}$ with c an arbitrary integer constant. It's east to verify that these functions work.

Solution. We consider the following for each $n \in \mathbb{Z}$, substituting $(a, b) = (0, n)$ and $(1, n-1)$:

$$f(2(0)) + 2f(n) = f(f(n)) = f(2(1)) + 2f(n-1)$$

which means $f(n) - f(n-1) = \frac{f(2)-f(0)}{2}$, which is fixed. As f is defined only on $\mathbb{Z}$, we can dedue right away that f is linear.

Now write $f(n) = mn + c$ with m being the gradient and c being the intercept of linear equation, the desired equation now becomes:

$$m(2a) + c + 2(mb + c) = f(m(a + b) + c) = m(m(a + b) + c) + c$$

or rather, $2m(a + b) + 3c = m^2(a + b) + (m + 1)c$ for all a and b. Pushing everything to one side we get

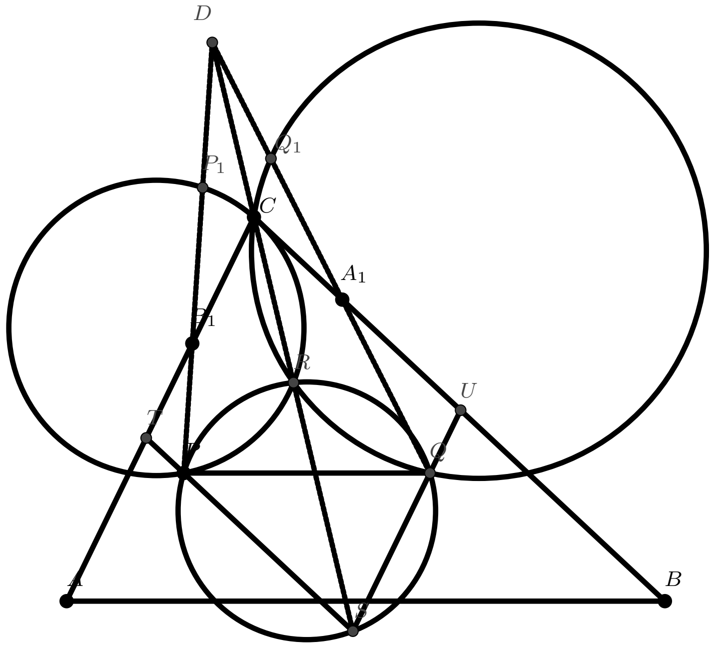
$$(2m - m^2)(a + b) + (3 - m - 1)c = 0$$

Since this must be true for all integers a and b, $m(2 - m) = 2m - m^2 = 0$ which means $m = 0$ or $m = 2$. If $m = 0$ then $0 = (3 - 1)c = 2c$ so $c = 0$. If $m = 2$ then there's no additional restriction on c. This completes the proof.

2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Solution. An equivalent thing would be proving that lines PP_1, QQ_1 and the radical axis of circumcircles of CPP_1 and CQQ_1 are either concurrent or parallel. Let R be the second intersection of circles CPP_1 and CQQ_1; CR is the radical axis of these two circles.



Now, let S be the second intersection of CR and circle PQR. We claim that $SP \parallel BC$ and $SQ \parallel AC$. For this part only, let's use directed angles for the sake of clarity. Observe that:

$$\angle(CA, AB) = \angle(CP_1, P_1P) = \angle(CR, RP) = \angle(RS, RP) = \angle(QS, QP) = \angle(QS, AB)$$

with the first equality following from $\angle PP_1C = \angle BAC$ and the last equality from $QP \parallel AB$. Thus CA and QS are parallel, and similarly so are lines SP and BC. Thus we will focus on proving that PP_1, QQ_1, CS are either concurrent or parallel.

Let SP intersect AC at T, and QP intersect BC at U. THis means $CTSU$ is a parallelogram. The fact that PP_1, QQ_1, CS are parallel is actually the same as the below equality:

$$\frac{SP}{PT} \cdot \frac{TB_1}{B_1C} \cdot \frac{CA_1}{A_1U} \cdot \frac{UQ}{QS} = 1$$

notice that we need to be careful of the sign convention, though it's quite clear that P is between T and S and Q is between U and S, so the first and the last ratio can be easily made positive. We now let $\frac{TB_1}{B_1C}$ be postive if B_1 lies between T and C and negative otherwise, similarly $\frac{CA_1}{A_1U}$ be positve if A_1 between C and U and negative otherwise. This allows us to adopt the convention that $TB_1 = TC - B_1C$ and $UA_1 = UC - A_1C$.

Since $PQ \parallel AB$, $SP \parallel BC$ and $SQ \parallel AC$, the triangles SPQ and CBA are in fact similar. Thus $\frac{SP}{SQ} = \frac{BC}{AC}$. Since $CU = TS$ and $CT = US$, we in fact have $\frac{SP}{ST} \div \frac{SQ}{SU} = \frac{BC}{CU} \div \frac{AC}{CT}$. In addition, we have the following identity:

$$\frac{PT}{A_1C} = \frac{AT}{AC} \qquad \frac{UQ}{B_1C} = \frac{BU}{BC}$$

thus allowing us to write

$$A_1C = PT \cdot \frac{AC}{AT} = PT \cdot \frac{AC}{AC - TC} \qquad B_1C = UQ \cdot \frac{BC}{BU} = UQ \cdot \frac{BC}{BC - CU}$$

and therefore

$$
\begin{aligned}
\frac{SP}{PT}\cdot\frac{TB_1}{B_1C}\cdot\frac{CA_1}{A_1U}\cdot\frac{UQ}{QS} &= \frac{SP}{QS}\cdot\frac{CT-B_1C}{\frac{BC}{BC-CU}}\cdot\frac{\frac{AC}{AC-TC}}{CU-CA_1}\\
&= \frac{SP}{QS}\cdot\frac{CT(1-B_1C/CT)}{\frac{BC}{BC-CU}}\cdot\frac{\frac{AC}{AC-TC}}{CU(1-CA_1/CU)}\\
&= \frac{BC}{AC}\cdot\frac{CT(1-\frac{UQ}{US}\cdot\frac{BC}{BC-CU})}{\frac{BC}{BC-CU}}\cdot\frac{\frac{AC}{AC-TC}}{CU(1-\frac{PT}{ST}\cdot\frac{AC}{AC-TC})}\\
&= (\frac{BC}{CU}\div\frac{AC}{CT})\cdot\frac{1-(1-\frac{SQ}{SU})\cdot\frac{BC}{BC-CU}}{\frac{BC}{BC-CU}}\cdot\frac{\frac{AC}{AC-TC}}{1-(1-\frac{SP}{ST})\cdot\frac{AC}{AC-TC}}\\
&= \frac{CT}{CU}\cdot\frac{1-(1-\frac{SQ}{SU})\cdot\frac{BC}{BC-CU}}{\frac{1}{BC-CU}}\cdot\frac{\frac{1}{AC-TC}}{1-(1-\frac{SP}{ST})\cdot\frac{AC}{AC-TC}}\\
&= \frac{CT}{CU}\cdot\frac{1-(1-\frac{SQ}{SU})\cdot\frac{BC}{BC-CU}}{AC-TC}\cdot\frac{BC-CU}{1-(1-\frac{SP}{ST})\cdot\frac{AC}{AC-TC}}\\
&= \frac{CT}{CU}\cdot\frac{BC-CU-(1-\frac{SQ}{SU})\cdot BC}{AC-TC-(1-\frac{SP}{ST})\cdot AC}\\
&= \frac{CT}{CU}\cdot\frac{\frac{SQ}{SU}\cdot BC-CU}{\frac{SP}{ST}\cdot AC-TC}\\
&= \frac{\frac{SQ}{SU}\cdot\frac{BC}{CU}-1}{\frac{SP}{ST}\cdot\frac{AC}{CT}-1}
\end{aligned}
$$

To show that this ratio is indeed 1, it suffices to show that $\frac{SQ}{SU}\cdot\frac{BC}{CU}=\frac{SP}{ST}\cdot\frac{AC}{CT}$. With $CT=SU$ and $CU=ST$, and that $\frac{SP}{SQ}=\frac{BC}{AC}$ (i.e. $SP\cdot AC=SQ\cdot BC$), we have

$$\frac{SQ}{SU}\cdot\frac{BC}{CU}=\frac{SP}{SU}\cdot\frac{AC}{CU}=\frac{SP}{CT}\cdot\frac{AC}{ST}$$

as desired.

3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time: Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged. Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

   Solution. (Reproduced from AoPS) For a connected component C of a graph, call it good if either $|C|\le 2$, or $|C|$ is not a clique and not all vertices have even degree. One can easily prove that the initial graph has a single connected component (otherwise, take the smaller component which has at most 1009 vertices and therefore having degree at most 1008), and obviously not a clique. For each good connected component of size at least 3, we show that we can make a move as described in the problem statement such that either the connected component is preserved, or the move splits it into two good components (in the latter case, the resulting component will still be good because as Swistak mentioned, the each move preserves the degree of each vertex, and the total number of edges decreases by 1, hence cannot be a clique).

   Consider any good component C of size at least 3. For each two vertices (u,v), denote the distance $d(u,v)$ as the length of the shortest path from u and v. Choose two of them, A_0, A_n such that $d(u,v) = n$ and is the maximal possible within that component. Moreover, let $A_0, A_1, \cdots, A_n$ to be a path of length n. By the definition of distance, A_i and A_j is not connected by an edge if $|i-j| > 1$. In addition, $n \geq 2$ since C is not a clique.

Now, one possible candidate (call this "candidate move") is to consider edges A_0A_1 and A_1A_2, and remove them, resulting in a new edge A_0A_2. If A_1 is still connected to A_0, or A_1 has no other vertices connected to itself, then we are good. Otherwise, A_1 will have another neighbour, B.

We first claim that in this "supposedly new" component, A_1 has distance 1 with all other vertices, Otherwise, choose B and C such that there are edges A_1B and BC but not A_1C. Now these B and C are isolated from $A_0, A_2, \cdots, A_n$. In the original graph, any path from A_n to C must pass through A_1 (length $n-1$), and from A_1 it takes exactly two steps to reach C. Thus $d(A_n, C)$ in the pre-move configuration is $n+1$, contradicting the maximality of n.

Back to the original configuration; we know that originally, the neighbours of A_1 are A_0, A_2, and a bunch of $B_1, \cdots, B_k$ $(k \geq 1)$ which will be isolated from $A_0, A_2, \cdots, A_n$ should we use the candidate move. In addition, by the immediate previous claim, B_i has no other neighbours other than B_j $(j \neq i)$ and A_1. There are three cases:

Case 1. $k = 1$. This means the degree of A_1 is 3. We now disconnect A_1B_1 and A_1A_0, resulting in new edge B_1A_0. If this were to split things up into two components, then B_1, A_0 are in the same component, while $A_1, A_2, \cdots, A_n$ are in the other. Both B_1 and A_1 now have degree 1, so both components are guaranteed to be good.

Case 2. $k \geq 2$ and each B_i has degree 1. The previous algorithm will still work: disconnect A_1B_1 and A_1A_0 with new edge B_1A_0. Again if this were to split things up into two components then B_1 and A_1 are in different components. Now B_1 has degree 1; B_2 has degree 1 too and is in A_1's component, so both components are good too.

Case 3. Some B_i has degree more than 1. WLOG let B_1 to have degree more than 1, and by before's observation, B_1 has no other neighbour other than some other B_j's and A_1. Therefore there's another B_j that's B_1's neighbour, say, B_2. Same algorithm, cut A_1B_1, A_1A_0, add B_1A_0. Now $A_0B_1B_2A_1$ is a path, so the original component remains connected.

Finally, notice that we always perform such a move above until we run out of connected components of size ≥ 3. This will eventually happen since each move decreases the number of edges in the graph by 1, thus cannot happen forever. At this stage, each vertex will have degree at most 1, as desired.

4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Answer. $(1,1), (3,2)$.
Solution. It's obvious that the answers above satisfy the condition, and we will show that these are the only pairs. By considering the power of 2 dividing both sides (denote by $v_2(\cdot)$) we have the v_2 of right hand side as

$$= \sum_{i=0}^{n-1} v_2(2^n - 2^i) = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

On the other hand we have

$$v_2(k!) = \sum_{i=1}^{\infty} \lfloor \frac{k}{2^i} \rfloor < \sum_{i=1}^{\infty} \frac{k}{2^i} = k$$

hence $k > \frac{n(n-1)}{2}$.

Now we compare the size of both sides: on right hand side we have

$$\prod_{i=1}^{n-1}(2^n - 2^i) \leq \prod_{i=1}^{n-2} 2^n \cdot (2^{n-1}) = 2^{n^2-1}$$

while for $k > 16$ (valid for $n \geq 7$ as of the first identity) we have

$$k! = 15! \cdot \prod_{i=16}^{k} \geq 15! \cdot 2^4 = 15! \cdot 2^{4(k-15)}$$

and we have $15! = 7! \cdot 8 \cdot \dots \cdot 15 > 5040 \cdot 8^8 > 2^12 \cdot 8^8 = 2^36$ so we in fact have $k! > 2^{4(k-15)+36} = 2^{4(k-6)}$. This means, $2^{4(k-6)} < k! \leq 2^{n^2-1}$ and therefore $4(k-6) < n^2 - 1$. But then $k \geq \frac{n(n-1)}{2} + 1$ so

$$4(\frac{n(n-1)}{2} + 1 - 6) \leq 4(k-6) < n^2 - 1$$

Or rather, $2n(n-1) - 20 < n^2 - 1$ or $n^2 - 2n = n(n-2) < 19$. Since $n(n-2)$ is an increasing functin and when $n = 7$ we have $7(5) = 35 > 19$, this inequality is false for $n \geq 7$.

We therefore only need to consider $n \leq 6$. When $n \geq 5$, the factor 31 is present on the right hand side (via $2^n - 2^{n-5} = 2^{n-5}(2^5 - 1)$) and 31 is prime, so $k \geq 31$ here. This means $100 = 4(31-6) < n^2 - 1$, which is only true when $n \geq 11$ so this case is eliminated. To consider the rest manually:

- $n = 1$ gives $1 = 1!$ and $n = 2$ gives $2 \times 3 = 6 = 3!$.
- $n = 3$ gives $7 \times 6 \times 4 = 168$ which lies strictly between $5! = 120$ and $6! = 720$.
- $n = 4$ gives $15 \times 14 \times 12 \times 8 = 20160$ which lies strictly between $7! = 5040$ and $8! = 40320$.

Hence only $n = 1, 2$ work.

5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

   (a) Show that, for each initial configuration, Harry stops after a finite number of operations.

   (b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

   Answer. The expected value, $E(L(C)) = \frac{n(n+1)}{4}$.
   Solution. Let's induce on n and denote $f(n)$ as the answer for n. It's obvious that when $n = 1$, $E(C) = \frac{0+1}{2} = \frac{1}{2}$, corresponding to the configuration T and H, respectively. We'll show that $f(n+1) = f(n) + \frac{n+1}{2}$ which completes the solution. For clarity sake, denote $op(C)$ as the configuration by applying the operation on C.

   Let A_0 be the set of sequence of $n+1$ coins ending on a T, and A_1 be the set of sequence of $n+1$ coins ending on a H. Also, for each set S of sequences of coins let $L(S)$ be the expected value of number of operations needed (that is, $L(S) = \frac{1}{|S|} \sum_{C \in S} L(C)$). We have the relation $f(n+1) = \frac{L(A_0)+L(A_1)}{2}$ since A_0 and A_1 have the same size, each being 2^n.

Let B_0 be the set of sequence of n coins. We first define a mapping γ from A_0 to B_0 as follows: for each $C \in A_0$, let C' be the sequence in B_0 that is C with the last T removed. Then $\gamma(C) = C'$. This mapping is clearly bijective. We claim that $\gamma(op(C)) = op(\gamma(C))$ if C has at least a head. Notice that C and C' each has the same number of heads, say k, so in the operation, the k-th coin from the left in each C and $C' = \gamma(C)$ are flipped. Since the first n coins of C and $\gamma(C)$ are the same, so are the first n coins of $op(C)$ and $op(\gamma(C))$. Since $k \leq n$, the last tail of C is not flipped in getting into $op(C)$, hence $\gamma(C) = op(\gamma(C))$. Continuing this iteration, we find that $L(C) = L(C')$ and therefore aggregating this for all configurations $C \in A_0$ we have $L(A_0) = L(B_0) = f(n)$.

Computing $L(A_1)$ is trickier. Again, define a mapping σ from A_1 to B_0 such that for each C in A_1, $\sigma(C)$ is obtainly by the following algorithm: take C, drop the last H, invert the coin sequence and flip all the coins (formally, $\sigma(s_0 s_1 \cdots s_{n-1} H) = \overline{s_{n-1} \cdots s_1 s_0}$ where $\overline{H} = T$ and $\overline{T} = H$). Again this mapping is bijective. We claim that if C has at least a tail, then $\sigma(op(C)) = op(\sigma(C))$. Let $1 \leq k \leq n$ be the number of heads in C, and let $C = s_0 s_1 \cdots s_{n-1} H$. Then $op(C) = s_0 \cdots \overline{s_{k-1}} s_k \cdots s_{n-1} H$. Excluding the last head, C actually has $k-1$ heads and $n-k+1$ tails, which means that $\sigma(C) = \overline{s_{n-1} \cdots s_1 s_0}$ has $n-k+1$ heads and $k-1$ tails, which means that $op(\sigma(C)) = \overline{s_{n-1} \cdots \overline{s_{k-1}} \cdots s_1 s_0} = \overline{s_{n-1} \cdots} s_{k-1} \overline{\cdots s_1 s_0}$, which establishes that $\sigma(op(C)) = op(\sigma(C))$. If m is the number operations needed for $\sigma(C)$ to reach all T, $op^m(\sigma(C)) = TT \cdots T$ and from above we can deduce that $op^m(\sigma(C)) = \sigma(op^m(C)) = TT \cdots T$, we have $op^m(C) = HH \cdots H$ (i.e. $n+1$ H's). It's now not hard to see that the subsequent moves on $op^m(C)$ are to flip the rightmost possible H, and there are $n+1$ of them, hence $L(C) = m + n + 1 = L(\sigma(C)) + n + 1$. Aggregating this over all C in A_1 we get $L(A_1) = L(B_0) + n + 1 = f(n) + n + 1$.

Summarizing above we have $f(n+1) = \frac{L(A_0)+L(A_1)}{2} = \frac{f(n)+f(n)+n+1}{2} = f(n) + \frac{n+1}{2}$, as desired.

6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

   Prove that lines DI and PQ meet on the line through A perpendicular to AI.

   Solution. (Reproduced from AoPS) Here's a solution using inversion and trigonometric bashing: by inverting in ω we turn the problem into the follows: keep A, D, E, F, P as they are, A_1, B_1, C_1 as midpoints of EF, DF, DE. Let the circumcircles of triangles PC_1E and PB_1F meet again on Q_1. Let γ be the circle with diameter A_1I. Prove that the second intersection of circumcircle of PQ_1I and line DI (or I if tangent) meet on γ. The last statement (to prove) is the same as proving that the radical axis of γ, circumcircles of triangles PC_1E and PB_1F lie on line DI. Equivalently, the radical axis of PC_1E and γ, and the radical axis of PB_1F and γ concur on DI. This last statement is our focus. W.L.O.G. assume $AB < AC$, so $DF < DE$ too and we know DEF is acute. In this setting (details skipped), P will lie on minor arc DF, and in particular P, D, F are different points.

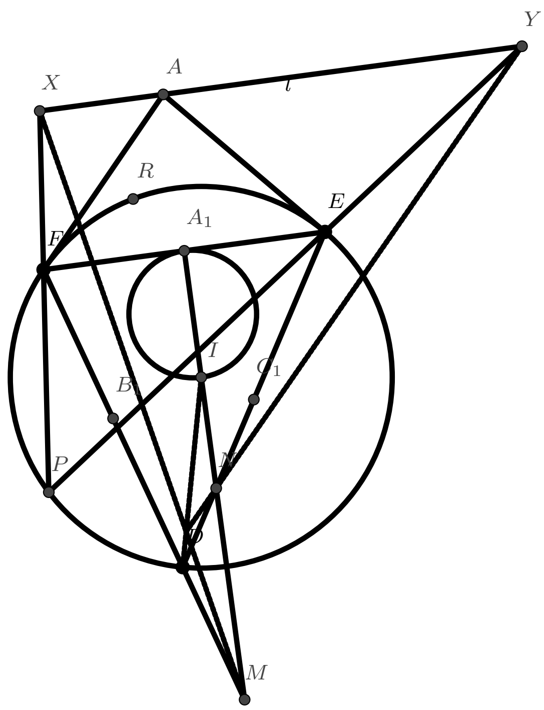
   We need to identify those radical axes, and for each of them it's defined based on two points. We first focus on finding the radical axis of PB_1F and γ. We need the following lemma:

   Lemma: A is on the radical axis of ω and γ.
   Proof: The power of point of A to ω is $AE = AF$; to γ is $AA_1 \cdot AI$. But then $AA_1 \perp EF$ and $\angle AFI = 90°$, the conclusion follows by similarity of triangles. This means the radical

axis is actually line through A perpendicular to AA_1 (parallel to EF, in other words), let's name it ℓ.

Consider, now, the circles PB_1F, γ and ω. The radical axis of ω and γ has been established above; the one for ω and PB_1F is PF, so one such point must be $X = \ell \cap PF$. Consider, now, the circles PB_1F, γ and the circle B_1FA_1I (the four points are concylic bcz $\angle FB_1I = \angle FA_1I$). The radical axis of γ and B_1FA_1I is A_1I; the one for PB_1F and B_1FA_1I is B_1F which is DF. Thus $M = DF \cap A_1I$ is the radical centre of the three circles, hence on the radical axis of γ and PB_1F. The radical axis, therefore, is XM (we need to be careful in showing $X \neq M$; suppose $X = M$, then the fact that both pass through F via FP and FD and that F, D, P are not collinear because they are different point on ω means $X = M = F$. But the power of point of F to FPB_1 is 0 while to γ is $A_1F \neq 0$, assuming nondegenerate here). In a similar fashion, if $Y = \ell \cap PE$ and $N = DE \cap A_1I$ then YN is the radical axis of γ and PC_1E.

We are left with proving that YN, XM, DI are concurrent, which we will use bashing (trigonometric) here! Extend line MFD to meet ℓ at X_1 and NED to meet ℓ at Y_1. We now have $MF > DF$ and $NE < DE$ According to our assumption, X will be between P and X_1 and MX will be in the angle domain of $\angle AMX_1$. So goes to segment DI so MX will intersect segment DI (and not anything outside). Similarly, Y_1 will be between A and Y, which also means NY is outside angle domain $\angle ANY_1$. But then segment DI won't be on the angle domain either so YN intersects DI in its segment. These realization are here to free us from using signed convention later (well we could but I am lazy now).



Now a not-so-well-known trigonometric identity says that considering the triangle AFX_1

and the cevian FX we get $\frac{AX}{XX_1} = \frac{AF}{FX_1} \cdot \frac{\sin \angle AFX}{\sin \angle XFX_1}$. Considering the triangle AMX_1 and cevian MX we get $\frac{AX}{XX_1} = \frac{AM}{MX_1} \cdot \frac{\sin \angle AMX}{\sin \angle XMX_1}$. Finally, if MX intersects DI at Z_1, considering triangle MDI and the cevian MZ_1 we get, $\frac{IZ_1}{Z_1D} = \frac{MI}{MD} \cdot \frac{\sin \angle AMX}{\sin \angle XMX_1}$. Thus we have

$$\frac{IZ_1}{Z_1D} = \frac{MI}{MD} \cdot \frac{\sin \angle AMX}{\sin \angle XMX_1} = \frac{MI}{MD} \cdot \frac{AX}{XX_1} \div \frac{AM}{MX_1} = \frac{MI}{MD} \frac{AF}{FX_1} \cdot \frac{\sin \angle AFX}{\sin \angle XFX_1} \div \frac{AM}{MX_1}$$

and similarly if NY intersects DI at Z_2 we get

$$\frac{IZ_2}{Z_2D} = \frac{NI}{ND} \frac{AE}{EY_1} \cdot \frac{\sin \angle AEY}{\sin \angle YEY_1} \div \frac{AN}{NY_1}$$

so we need to prove the two ratios are equal (and this would be sufficient since we know that Z_1 and Z_2 are both on segment DI, i.e.

$$\frac{MI}{MD} \frac{AF}{FX_1} \cdot \frac{\sin \angle AFX}{\sin \angle XFX_1} \div \frac{AM}{MX_1} = \frac{NI}{ND} \frac{AE}{EY_1} \cdot \frac{\sin \angle AEY}{\sin \angle YEY_1} \div \frac{AN}{NY_1}$$

First, notice that $AF = AE$, so these can be cancelled out. Next, $\angle AFX = \angle FEP$ and $\angle AEY = \angle EFP$, but then by sine rule $\sin \angle FEP / \sin \angle EFP = FP/EP$. These two angles are on numerators of two different sides so they can be replaces with FP and EP, respectively. Then, $\angle XFX_1 = \angle FPD = \angle PED = \angle YEY_1$, again can be cancelled. Also, since $\ell \parallel EF$ we have $FX_1/EY_1 = DF/DE$. Thus we now need to check the following:

$$\frac{MI}{MD} \frac{1}{DF} \cdot \frac{FP}{1} \div \frac{AM}{MX_1} = (?) \frac{NI}{ND} \frac{1}{DE} \cdot \frac{EP}{1} \div \frac{AN}{NY_1}$$

Next, AM/MX_1 is actually $\cos \angle AMX_1 = \sin \angle DFA_1 = \sin \angle DFE$ and similarly $AN/NY_1 = \sin \angle DEA_1 = \sin \angle DEF$ (notice the implicit use of the fact that $A_1I \perp EF$ and $A_1I \perp XY = \ell$. But then by sine rule $\sin \angle DFE / \sin \angle DEF = DE/DF$. Thus we have the equation to prove above becomes the following:

$$\frac{MI}{MD} \frac{1}{DF} \cdot \frac{FP}{1} \div DE = (?) \frac{NI}{ND} \frac{1}{DE} \cdot \frac{EP}{1} \div DF$$

so now it suffices to show that

$$\frac{MI}{MD} \cdot FP = (?) \frac{NI}{ND} \cdot EP$$

Using sine rule again, $\frac{MI}{MD} = \frac{\sin \angle MDI}{\sin \angle MID}$ and $\frac{NI}{ND} = \frac{\sin \angle NDI}{\sin \angle NID}$. But then both $\angle MID$ and $\angle NID$ are angle between DI and IM so they must be either equal or supplementary, hence having equal since. We now reduce everything to the following: $\sin \angle MDI \cdot FP = (?) \sin \angle NDI \cdot EP$. But now, $\sin \angle MDI = \sin \angle IDF$ and notice that $\angle IDF = 90° - \angle DEF = \angle EDR = \angle EFR$ so $\sin \angle MDI = \angle EFR$ and $\sin \angle NDI = \sin \angle FER$ for the similar reason. But then $\sin \angle EFR / \sin \angle FER = ER/FR$ so we are left with proving that $ER \cdot FP = FR/cdotEP$. But this follows from the fact that $PEFR$ is a harmonic quadrilateral!