

**MATH 135: Extra Practice Set 1**

February 18<sup>th</sup> 2016

beg2119

**Question 1.** *Using direct proof, show that the sum of two consecutive perfect squares is odd.*

*Proof.* 1. Suppose that the sum of two consecutive perfect squares is odd

2. The definition of consecutive squares is  $(x)^2 + (x + 1)^2$

3.  $2x^2 + 2x + 1$

4.  $2(x^2 + x) + 1$

5. Set  $k$  to the value of the parentheses,

6.  $k = (x^2 + x)$

7. Thus,  $2k + 1$

8. The definition of an odd number is  $(\forall k \in \mathbb{Z})(2k + 1)$

9.  $\therefore$  The sum of two consecutive perfect squares is odd

□

**Question 2.** *Use proof by contrapositive to prove the following proposition:  
If  $x^3$  is odd, then  $x$  is odd.*

*Proof.* 1. Suppose that  $x^3$  is odd and  $x$  is odd

2. the contrapositive must also be true: "if  $x$  is not odd, then  $x^3$  is not odd"

3. OR, "if  $x$  is even, then  $x^3$  is even"

4. The definition of an even number is  $(\exists k \in \mathbb{Z})x = 2k$

5. Cube both sides,

6.  $x^3 = 8k^3$

7. Let  $m = 4k^3$  thus  $x^3 = 2m$

8.  $\therefore$  If  $x$  is even, then  $x^3$  is even

□

**Question 3.** *Prove that an integer is odd if and only if it is the sum of two consecutive integers.*

*Proof.* 1. Formally,  $(\forall x \in \mathbb{Z})(2 \nmid (x + (x + 1)) \equiv 2x + 1$

2. The definition of an odd number is  $2x + 1$

3.  $\therefore$  If it is the sum of two consecutive integers then it is odd

$$* p \Leftrightarrow q *$$

4. Suppose that  $x$  is odd,

5.  $x = 2k + 1$

6. Can be expressed as  $2k = k + k + 1$  which signifies consecutive integers such that  $(\forall x \in \mathbb{Z})$

7.  $\therefore$  If it is odd then it is the sum of two consecutive integers

□

**Question 4.** *Prove or disprove the following proposition: any prime number greater than 2 can be expressed as 1 less than a power of 2. More formally, this means that for every prime number  $p > 2$ , there exists a natural number  $n$  such that  $p = 2^n - 1$ .*

*Proof.* 1. Let  $\mathbb{P}$  denote the set of all prime numbers

2.  $(\forall p > 2 \in \mathbb{P})(\exists n \in \mathbb{N}) p = 2^n - 1$

3. Suppose that the previous proposition is true

4. Let  $p = 11$ , which clearly satisfies the condition

5.  $11 = 2^n - 1$ ;  $(2^3 = 8)$  &  $(2^4 = 16)$

6. There is no natural value of  $n$  that satisfies this theorem when  $p = 11$

7.  $\therefore$  This proposition is disproven by counterexample

□

**Question 5.** *Use direct proof for parts 1 and 2.*

1. Let  $x$  be an integer. Prove the following proposition:

If  $x \geq 3$ , then  $x^2 > 2x + 1$ .

*Proof.* (a) Suppose that  $x \geq 3$

(b) Multiply by  $x$  on both sides

(c)  $x^2 \geq 3x \equiv x^2 \geq 2x + x$

(d) From the original conditional clause we see  $x^2 > 2x + 1$

(e) While  $x \geq 3$ ,

(f) It must be the case that  $2x + x > 2x + 1$

(g)  $\therefore x^2 > 2x + 1$ .

□

2. Let  $a$  and  $b$  be integers, and let  $c$  be a negative integer. Prove the following proposition:

If  $a > b$ , then  $a^2c^2 - 2abc^2 + b^2c^2$  is positive.

*Proof.* (a)  $a^2c^2 - 2abc^2 + b^2c^2 = (ac - bc)^2$

(b) Take  $a > b$

(c) Multiply by  $c$ , (a negative integer)

(d)  $ac < bc$

(e)  $ac - bc < 0$

(f) Multiply by negative (square)

(g)  $(ac - bc)^2 > 0$

(h)  $\therefore ((ac - bc)^2)$ , or  $(a^2c^2 - 2abc^2 + b^2c^2)$ , is positive

□

**Question 6.** Let  $x$  and  $y$  be real numbers. Using proof by cases, show that the following property holds:

$$|x + y| \leq |x| + |y|$$

*Proof.* • CASE 1

Let both  $x$  &  $y$  be positive

$$|x + y| \leq |x| + |y|$$

$$x + y \leq x + y$$

- TRUE

- CASE 2

Let  $x$  be positive and  $y$  be negative  $|x - y| \leq |x| + |-y|$

$$x - y \leq x + y$$

- TRUE

- CASE 3

Let both  $x$  &  $y$  be negative

$$|-x - y| \leq |-x| + |-y|$$

$$|-(x + y)| \leq x + y$$

$$x + y \leq x + y$$

- TRUE

- CASE 4

Case 4:

Let  $x$  be 0 and  $y$  be positive

$$|0 + y| \leq |0| + |y|$$

$$|y| \leq y$$

$$y \leq y$$

- TRUE

□

**Question 7.** Using proof by contradiction, show that there are no integers  $x, y$  that satisfy the equation  $5x + 25y = 1723$ .

*Proof.* 1. Suppose that  $(x, y \in \mathbb{Z}) 5x + 25y = 1723$

2.  $5(x + 5y) = 1723$

3.  $x + 5y = 1723/5$

4. The result is not in the set of integers, and two integers cannot add to create something outside of that set

5.  $\therefore$  There are no integers  $x$  &  $y$  that satisfy the equation  $5x + 25y = 1723$

□

**Question 8.** Prove, using any method you'd like, that the sum of any three consecutive integers is divisible by 3.

*Proof.* 1. Assume that the sum of any three consecutive integers is divisible by 3

2.  $(x, y, z \in \mathbb{Z}) \ x < y < z$
3. As  $x, y, z$  are consecutive,  $y = x + 1, z = x + 1 + 1$
4. Thus,  $x + y + z \equiv x + (x + 1) + (x + 1 + 1)$
5.  $3x + 3 = 3(x + 1) = 3y$
6. Because  $(y \in \mathbb{Z}), 3|3y$
7.  $\therefore$  The sum of any three consecutive integers is divisible by 3

□

**Question 9.** *Prove, using any method you'd like, that the difference between distinct, nonconsecutive perfect squares is composite. Recall that an integer  $x$  is composite if and only if there exists some integer  $y$  such that  $1 < y < x$  and  $y|x$ . In other words,  $x$  is composite if it has some positive factor other than 1 and itself, i.e.  $x$  is not prime.*

*Proof.* 1. Let  $a$  &  $b$  be nonconsecutive perfect squares

2.  $(\exists x, y \in \mathbb{Z}^+) \text{ such that } a = x^2 \text{ \& } b = y^2$
3.  $x$  &  $y$  are non consecutive, so  $(x - y) \neq 1$
4. The difference between distinct, nonconsecutive perfect squares should be composite
5.  $x^2 - y^2 = (x + y)(x - y)$
6. Neither factor  $(x + y)$  nor factor  $(x - y)$  equals 1 or  $x^2 - y^2$
7.  $\therefore$  The difference between distinct, nonconsecutive perfect squares is composite

□

**Question 10.** *Convert the following statements into the formal notation of propositional logic (i.e. using variables and logical operators). Make sure to explain what each variable you introduce represents.*

- Whenever we add a rational number and an irrational number, the sum is irrational.

Let  $\mathbb{R}$  represent the set of all real numbers,  $\mathbb{Q}$  represent the set of all rational numbers, &  $\mathbb{R} \setminus \mathbb{Q}$  represent the set of all irrational numbers

$$\mathbb{Q} + \mathbb{R} \setminus \mathbb{Q} \Rightarrow \mathbb{R} \setminus \mathbb{Q}$$

- Two integers are odd only if their sum is even.

Let  $\mathbb{Z}$  represent the set of all integers and  $(x, y, z \in \mathbb{Z})$

$$2|x, y \text{ but } 2 \nmid z$$

$$x + y \Rightarrow z$$

- It is necessary that  $a|(b + c)$  be true for  $a|b$  and  $a|c$  to be true.

$$a|(b + c) \Rightarrow a|b \wedge a|c$$

- For  $(ac)|(bd)$  to be true, it is sufficient that  $a|b$  and  $c|d$ .

$$a|b \wedge c|d \Rightarrow (ac)|(bd)$$

- For  $x$  to be an odd number, it is necessary and sufficient that  $x - 1$  is even.

$$(x \in \mathbb{Z})$$

$$2|(x - 1) \Rightarrow x$$

- An integer is even if and only if its square is even.

$$(x \in \mathbb{Z})$$

$$2|x \Leftrightarrow 2|x^2$$

## BONUS

**Question 11.** Let  $x$  and  $y$  be two numbers. State whether the following proposition is true or not:

$$\text{If } x > y \text{ and } x < y, \text{ then } x = y.$$

True. Contradictions imply everything. (on the contrary tautologies are implied by everything)