MATH 135: Extra Practice Set 7

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. For each linear congruence, determine the complete solution, if a solution exists.
   (a) $3x \equiv 11 \pmod{18}$
   (b) $4x \equiv 5 \pmod{21}$
2. Complete a multiplication table for $\mathbb{Z}_5$.
3. What is the remainder when 14^{43} is divided by 41?

Recommended Problems

1. How many integers x where $0 \leq x < 1000$ satisfy $42x \equiv 105 \pmod{56}$?
2. State whether the given statement is true or false and prove or disprove accordingly.
   (a) For all $a, b, c, x \in \mathbb{Z}$ such that $c, x > 0$, if $a \equiv b \pmod{c}$ then $a + x \equiv b + x \pmod{c + x}$.
   (b) For all $m \in \mathbb{N}$ and for all $[a] \in \mathbb{Z}_m$ there exists a $[b] \in \mathbb{Z}_m$ such that $[b]^2 = [a]$.
3. In each of the following cases, find all values of $[x] \in \mathbb{Z}_m$, $0 \leq x < m$, that satisfy the equation.
   (a) $[4][3] + [5] = [x] \in \mathbb{Z}_{10}$
   (b) $[7]^{-1} - [2] = [x] \in \mathbb{Z}_{10}$
   (c) $[2][x] = [4] \in \mathbb{Z}_8$
   (d) $[3][x] = [9] \in \mathbb{Z}_{11}$
4. Which elements of $\mathbb{Z}_6$ have multiplicative inverses?
5. What are the integer solutions to $x^2 \equiv 1 \pmod{15}$?
6. What are the last two digits of 43^{201}?

7. What is the remainder when 3141^{2001} is divided by 17?
8. Solve $49x^{177} + 37x^{26} + 3x^2 + x + 1 \equiv 0 \pmod 7$.
9. Suppose that p is a prime and $a \in \mathbb{Z}$. Prove using induction that $a^{(p^n)} \equiv a (\text{mod } p)$ for all $n \in \mathbb{N}$.
10. Prove that a prime p divides $ab^p - ba^p$ for all integers a and b.
11. Suppose p is a prime greater than five. Prove that the positive integer consisting of $p-1$ digits all equal to one $(111\ldots1)$ is divisible by p. (Hint: $111111 = \frac{10^6-1}{9}$.)
12. Determine all $k \in \mathbb{N}$ such that $n^k \equiv n \pmod 7$ for all integers n. Prove that your answer is correct

Challenge(s)

1. Let a_1 be any number. Create a sequence a_1, a_2, a_3, where a_n is formed by appending a digit from 0 to 8 to the end of a_{n-1}. Show that this sequence contains infinitely many composite numbers. What happens if you allow for 9? Can you still guarantee that infinitely many composite number must occur?
2. Prove that for any $k \in \mathbb{N}$, there is exists $n \in \mathbb{N}$ such that $2^k \mid (3^n + 5)$.

   (This is may be very difficult. Try using both induction and even-odd factorization discussed in Chapter 39 of the notes. It might also help to look up The Binomial Theorem.)