MATH 135: Extra Practice Set 2

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Suppose S and T are two sets. Prove that if $S \cap T = S$, then $S \subseteq T$. Is the converse true?
2. Give an example of three sets A, B, and C such that $B \neq C$ and $B - A = C - A$.
3. Prove the following two quantified statements.
   (a) $\forall\, n \in \mathbb{N},\ n + 1 \geq 2$
   (b) $\exists\, n \in \mathbb{Z},\ \frac{(5n-6)}{3} \in \mathbb{Z}$

Recommended Problems

1. Let S and T be any two sets in universe $\mathcal{U}$. Prove that $(S \cup T) - (S \cap T) = (S - T) \cup (T - S)$.
2. Let $A = \{n \in \mathbb{Z} \ : \ 2 \mid n\}$ and $B = \{n \in \mathbb{Z} \ : \ 4 \mid n\}$. Prove that $n \in (A - B)$ if and only if $n = 2k$ for some odd integer k.
3. Let $A = \{1, \{1, \{1\}\}\}$. List all the elements of $A \times A$.
4. Let $a, b, c \in \mathbb{Z}$. Is the following statement true? Prove that your answer is correct.

   $a \mid b$ if and only if $ac \mid bc$.
5. For each of the following statements, identify the four parts of the quantified statement (quantifier, variable, domain, and open sentence). Next, express the statement in symbolic form and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq, \nmid$ are okay).
   (a) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.
   (b) For every even integer a and odd integer b, a rational number c can always be found such that either $a < c < b$ or $b < c < a$.
   (c) There is some $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, $y \mid x$.
   (d) There exist sets of integers X, Y such that for all sets of integers Z, $X \subseteq Z \subseteq Y$.
   (You may use $\mathcal{P}(\mathbb{Z})$ to denote the set of all sets of integers. This is called *power set notation.*)

6. Prove or disprove each of the following statements.
   (a) $\forall\ n \in \mathbb{Z}$, $\frac{(5n-6)}{3}$ is an integer.
   (b) For every prime number p, $p + 7$ is composite.
   (c) There exists an integer $m < 123456$ such that 123456m is a perfect square.
   (d) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.

7. Prove or disprove each of the following statements involving nested quantifiers.
   (a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.
   (b) For every positive integer a, there exists an integer b with $|b| < a$ such that b divides a.
   (c) There exists an integer n such that $m(n - 3) < 1$ for every integer m.
   (d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$.

8. Prove that the converse of *Divisibility of Integer Combinations (DIC)* is true.

9. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

Challenge(s)

1. Show that if p and $p^2 + 2$ are prime, then $p^3 + 2$ is also prime.

2. Express the following statement in symbolic form and prove that it is true.

   There exists a real number L such that for every positive real number ε, there exists a positive real number δ such that for all real numbers x, if $|x| < \delta$, then $|3x - L| < \varepsilon$.