MATH 135: Extra Practice Set 1

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Prove that if k is an odd integer, then $4k + 7$ is an odd integer.
2. Prove that $(\neg A) \vee B$ is logically equivalent to $\neg(A \wedge \neg B)$.
3. Consider the following proposition about integers a and b.

   If $a^3 \mid b^3$, then $a \mid b$.

   We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

   (a) Consider $a = 2, b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.

   (b) Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, $a^3 \mid b^3$.

   (c) Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.

Recommended Problems

1. Determine whether $A \implies B$ is logically equivalent to $(\neg A) \vee B$.
2. Let n, a and b be positive integers. Negate the following implication without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine). *Implication:* If $a^3 \mid b^3$, then $a \mid b$.
3. Assume that it has been established that the following implication is true:

   If I don't see my advisor today, then I will see her tomorrow.

   For each of the statements below, determine if it is true or false, or explain why the truth value of the statement cannot be determined.

   (a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using either or both interpretations.)

   (b) I meet my advisor both today and tomorrow.

   (c) I meet my advisor either today or tomorrow (but not on both days).

4. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

5. Prove the following statement using a chain of logical equivalences as in Chapter 3 of the notes.

$$(A \wedge C) \vee (B \wedge C) \equiv \neg((A \vee B) \implies \neg C)$$

6. Suppose r is some (unknown) real number, where $r \neq -1$ and $r \neq -2$. Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r\,(2^r)}{(r+1)(r+2)}.$$

7. Let a, b, c and d be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$. Prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

8. Prove that $x^2 + 9 \geq 6x$ for all real numbers x.

9. Let n be an integer. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

10. Let a and b be two integers. Prove each of the following statements about a and b.

    (a) If $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$.

    (b) If a and b are positive, then $a^2(b + 1) + b^2(a + 1) \geq 4ab$.

11. Let a, b, c be integers. Prove that if $a \mid b$ then $ac \mid bc$.

12. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

13. Prove that the product of any four consecutive integers is one less than a perfect square.

Challenge(s)

1. Let n be an integer. Prove that if $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

2. Given that $a^2 + b^2 + c^2 = 1$ for real numbers a, b and c, prove that $-1/2 \leq ab + bc + ca \leq 1$.