MATH 135: Extra Practice Set 3

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$ then $x \neq 5$.
2. In the proof of *Prime Factorization (PF)* given in the course notes, why is it okay to write $r \leq s$?
3. What is the remainder when -98 is divided by 7?

Recommended Problems

1. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.
2. Let a and b be integers. Prove that $(a \mid b \ \wedge \ b \mid a) \iff a = \pm b$.
3. Prove that an integer is even if and only if its square is an even integer.
4. Let x and y be integers. Prove or disprove each of the following statements.
   (a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.
   (b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.
   (c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.
   (d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.
5. Consider the following statement.

   For all $x \in \mathbb{R}$, if $x^6 + 3x^4 - 3x < 0$, then $0 < x < 1$.

   (a) Rewrite the given statement in symbolic form.
   (b) State the hypothesis of the implication within the given statement.
   (c) State the conclusion of the implication within the given statement.
   (d) State the converse of the implication within the given statement.
   (e) State the contrapositive of the implication within the given statement.
   (f) State the negation of the given statement without using the word “not” or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine).
   (g) Prove or disprove the given statement.

6. Prove the following statements.

   (a) There is no smallest positive real number.

   (b) For every even integer n, n cannot be expressed as the sum of three odd integers.

   (c) If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.

   (d) For every integer m with $2 \mid m$ and $4 \nmid m$, there are no integers x and y that satisfy $x^2 + 3y^2 = m$.

   (e) The sum of a rational number and an irrational number is irrational.

   (f) Let x be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

7. Prove there is a unique set T such that for every set S, $S \cup T = S$.

8. The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$. Prove that there is a unique real number x such that $\lfloor x \rfloor = \lceil x \rceil = 7$.

9. Are the following functions onto? Are they 1-1? Justify your answer with a proof.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 2n + 1$.

   (b) $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 + 4x + 9$.

   (c) $f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{5\})$, defined by $f(x) = \frac{5x+1}{x-2}$.

Challenge(s)

1. Prove that there are no positive integers a and b such that $b^4 + b + 1 = a^4$.

2. Prove that the length of at least one side of a right-angled triangle with integer side lengths must be divisible by 3.