MATH 135: Extra Practice Set 4

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Evaluate $\displaystyle\sum_{i=3}^{8} 2^i$ and $\displaystyle\prod_{j=1}^{5} \frac{j}{3}$.

Recommended Problems

1. Prove the following statements by simple induction.

   (a) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n}(2i-1) = n^2$.

   (b) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ where r is any real number such that $r \neq 1$. .

   (c) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$.

   (d) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

   (e) For all $n \in \mathbb{N}$ where $n \geq 4$, $n! > n^2$.

2. Prove the following statements by strong induction.

   (a) A sequence $\{x_n\}$ is defined recursively by $x_1 = 8$, $x_2 = 32$ and $x_i = 2x_{i-1} + 3x_{i-2}$ for $i \geq 3$. For all $n \in \mathbb{N}$, $x_n = 2 \times (-1)^n + 10 \times 3^{n-1}$.

   (b) A sequence $\{t_n\}$ is defined recursively by $t_n = 2t_{n-1} + n$ for all integers $n > 1$. The first term is $t_1 = 2$. For all $n \in \mathbb{N}$, $t_n = 5 \times 2^{n-1} - 2 - n$.

3. The Fibonacci sequence is defined as the sequence $\{f_n\}$ where $f_1 = 1$, $f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 3$. Use induction to prove the following statements.

   (a) For $n \geq 2$,
   $$f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$$

   (b) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, $f_n = \frac{a^n - b^n}{\sqrt{5}}$

4. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.

   (a) A sequence $\{x_n\}$ is defined by $x_1 = 3$, $x_2 = 20$ and $x_i = 5x_{i-1}$ for $i \geq 3$. Then, for all $n \in \mathbb{N}$, $x_n = 3 \times 5^{n-1}$.

   Let $P(n)$ be the statement: $x_n = 3 \times 5^{n-1}$.
   When $n = 1$ we have $3 \times 5^0 = 3 = x_1$ so $P(1)$ is true. Assume that $P(k)$ is true for some integer $k \geq 1$. That is, $x_k = 3 \times 5^{k-1}$ for some integer $k \geq 1$. We must show that $P(k+1)$ is true, that is, $x_{k+1} = 3 \times 5^k$. Now
   $$x_{k+1} = 5x_k = 5(3 \times 5^{k-1}) = 3 \times 5^k$$
   as required. Since the result is true for $n = k+1$, and so holds for all n by the Principle of Mathematical Induction.

   (b) For all $n \in \mathbb{N}$, $1^{n-1} = 2^{n-1}$.

   Let $P(n)$ be the statement: $1^{n-1} = 2^{n-1}$.
   When $n = 1$ we have $1^0 = 1 = 2^0$ so $P(1)$ is true. Assume that $P(i)$ is true for all integers $1 \leq i \leq k$ where $k \geq 1$ is an integer. That is, $1^{i-1} = 2^{i-1}$ for all $1 \leq i \leq k$.
   We must show that $P(k+1)$ is true, that is, $1^{(k+1)-1} = 2^{(k+1)-1}$ or $1^k = 2^k$. By our inductive hypothesis, $P(2)$ is true so $1^1 = 2^1$. Also by our inductive hypothesis, $P(k)$ is true so $1^{k-1} = 2^{k-1}$. Multiplying these two equations together gives $1^k = 2^k$. Since the result is true for $n = k+1$, and so holds for all n by the Principle of Strong Induction.

5. In a strange country, there are only 4 cent and 7 cent coins. Prove that any integer amount of currency greater than 17 cents can always be formed.

Challenge(s)

1. Prove that for every positive integer, there exists a unique way to write the integer as the sum of distinct non-consecutive Fibonacci numbers.

2. You are given three pegs. On one of the pegs is a tower made up of n rings placed on top of one another so that as you move down the tower each successive ring has a larger diameter than the previous ring. The object of this puzzle is to reconstruct the tower on one of the other pegs by moving one ring at a time, from one peg to another. However, you can never have a ring above any smaller ring on any of the three pegs. A demonstration of this can be found at http://wiki.sagemath.org/animate courtesy of Pablo Angulo. Find a formula for the minimum steps required to solve the Tower of Hanoi puzzle and prove that your answer is correct.