MATH 135: Extra Practice Set 10

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Compute all the fifth roots of unity and plot them in the complex plane.
2. Find all complex numbers z solutions to $z^2 = \frac{1+i}{1-i}$.
3. Find a real cubic polynomial whose roots are 1 and i.
4. Divide $f(x) = x^3 + x^2 + x + 1$ by $g(x) = x^2 + 4x + 3$ to find the quotient $q(x)$ and remainder $r(x)$ that satisfy the requirements of the *Division Algorithm for Polynomials (DAP)*.

Recommended Problems

1. Let $z \in \mathbb{C}$. Prove that $(x - z)(x - \overline{z}) \in \mathbb{R}[x]$.
2. Prove that there exists a polynomial in $\mathbb{Q}[x]$ with the root $2 - \sqrt{7}$.
3. For each of the following polynomials $f(x) \in \mathbb{F}[x]$, factor $f(x)$ into factors with degree as small as possible over $\mathbb{F}[x]$. Cite appropriate propositions to justify each step of your reasoning.
   (a) $x^2 - 2x + 2 \in \mathbb{C}[x]$
   (b) $x^2 + (-3i + 2)x - 6i \in \mathbb{C}[x]$
   (c) $2x^3 - 3x^2 + 2x + 2 \in \mathbb{R}[x]$
   (d) $3x^4 + 13x^3 + 16x^2 + 7x + 1 \in \mathbb{R}[x]$
   (e) $x^4 + 27x \in \mathbb{C}[x]$
4. Prove that if w is an n^{th} root of unity, then $\frac{1}{w}$ is also an n^{th} root of unity.
5. Let $g(x) = x^3 + bx^2 + cx + d \in \mathbb{C}[x]$ be a cubic polynomial whose leading coefficient is 1 (such polynomials are called *monic*). Let z_1, z_2, z_3 be three roots of $g(x)$, such that

$$g(x) = (x - z_1)(x - z_2)(x - z_3).$$

   Prove that

$$\begin{aligned} z_1 + z_2 + z_3 &= -b, \\ z_1z_2 + z_2z_3 + z_3z_1 &= c, \\ z_1z_2z_3 &= -d. \end{aligned}$$

6. Let $n \geq 2$ be an integer. Prove that

$$\sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0 = \sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right)$$

7. A complex number z is called a *primitive* n-th root of unity if $z^n = 1$ and $z^k \neq 1$ for all $1 \leq k \leq n-1$.

   (a) For each $n = 1, 2, 3, 6$, list all the primitive n-th roots of unity.

   (b) Let z be a primitive n-th root of unity. Prove the following statements.

      i. For any $k \in \mathbb{Z}$, $z^k = 1$ if and only if $n \mid k$.

      ii. For any $m \in \mathbb{Z}$, if $\gcd(m, n) = 1$, then z^m is a primitive n-th root of unity.

8. Let u and v be fixed complex numbers. Let ω be a non-real cube root of unity. For each $k \in \mathbb{Z}$, define $y_k \in \mathbb{C}$ by the formula

$$y_k = \omega^k u + \omega^{-k} v.$$

   (a) Compute y_1, y_2 and y_3 in terms of u, v and ω.

   (b) Show that $y_k = y_{k+3}$ for any $k \in \mathbb{Z}$.

   (c) Show for any $k \in \mathbb{Z}$,

$$y_k - y_{k+1} = \omega^k(1-\omega)(u - \omega^{k-1}v).$$

9. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$. We say $f(x)$ is *palindromic* if the coefficients a_j satisfy

$$a_{n-j} = a_j \quad \text{for all} \quad 0 \leq j \leq n.$$

   Prove that

   (a) If $f(x)$ is a palindromic polynomial and $c \in \mathbb{C}$ is a root of $f(x)$, then c must be non-zero, and $\frac{1}{c}$ is also a root of $f(x)$.

   (b) If $f(x)$ is a palindromic polynomial of odd degree, then $f(-1) = 0$.

   (c) If $\deg(f) = 1$ and $f(x)$ is a monic, palindromic polynomial, then $f(x) = x + 1$.

Challenge(s)

1. Show that the exact value of $\cos\left(\frac{2\pi}{5}\right)$ is $\frac{\sqrt{5}-1}{4}$.

2. We call a polynomial primitive if the greatest common divisor of all of its coefficients is 1. Show that the product of two primitive polynomials is again primitive.