MATH 135: Extra Practice Set 9

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Write $z = \dfrac{9+i}{5-4i}$ in the form $r(\cos\theta + i\sin\theta)$ with $r \geq 0$ and $0 \leq \theta < 2\pi$.
2. Write $(\sqrt{3}+i)^4$ in standard form.

Recommended Problems

1. Find all $z \in \mathbb{C}$ which satisfy
   (a) $z^2 + 2z + 1 = 0$,
   (b) $z^2 + 2\overline{z} + 1 = 0$,
   (c) $z^2 = \dfrac{1+i}{1-i}$.
2. (a) Find all $w \in \mathbb{C}$ satisfying $w^2 = -15 + 8i$,
   (b) Find all $z \in \mathbb{C}$ satisfying $z^2 - (3+2i)z + 5 + i = 0$.
3. Let $z, w \in \mathbb{C}$. Prove that if $zw = 0$ then $z = 0$ or $w = 0$.
4. Let $a, b, c \in \mathbb{C}$. Prove: if $|a| = |b| = |c| = 1$, then $\overline{a+b+c} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.
5. Find all $z \in \mathbb{C}$ satisfying $z^2 = |z|^2$.
6. Find all $z \in \mathbb{C}$ satisfying $|z+1|^2 \leq 3$ and shade the corresponding region in the complex plane.
7. Prove that if $|z| = 1$ or $|w| = 1$ and $\overline{z}w \neq 1$, then $\left|\dfrac{z-w}{1-\overline{z}w}\right| = 1$.
8. Show that $|\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}\,|z|$.
9. Prove that $\forall z, w \in \mathbb{C}$, $|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2)$ (This is the Parallelogram Identity).
10. Use *De Moivre's Theorem (DMT)* to prove that $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$.
11. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Show that $z = (a+bi)^n + (a-bi)^n$ is real.

Challenge(s)

1. Let $z, w \in \mathbb{C}$.
   (a) Prove that $|z+w| \leq |z| + |w|$.
   (b) Prove that $\big||z| - |w|\big| \leq |z-w| \leq |z| + |w|$.
2. Let $a, b, c \in \mathbb{C}$. Show that if $\dfrac{b-a}{a-c} = \dfrac{a-c}{c-b}$ then $|b-a| = |a-c| = |c-b|$.