MATH 135: Extra Practice Set 6

These problems are for extra practice and are not to be handed in. Solutions will not be posted but, unlike assignment problems, they may discussed in depth on Piazza.

- The warm-up exercises are intended to be fairly quick and easy to solve. If you are unsure about any of them, then you should review your notes and possibly speak to an instructor before beginning the corresponding assignment.
- The recommended problems supplement the practice gained by doing the corresponding assignment. Some should be done as the material is learned and the rest can be left for exam preparation.
- A few more challenging extra problems are also included for students wishing to push themselves even harder. Do not worry if you cannot solve these more difficult problems.

Warm-up Exercises

1. Let $a, b, c \in \mathbb{Z}$. Disprove the statement: If $a \mid (bc)$, then $a \mid b$ or $a \mid c$.
2. Find the complete solution to $7x + 11y = 3$.
3. Find the complete solution to $28x + 60y = 10$.
4. What is the smallest non-negative integer x such that $2000 \equiv x \pmod{37}$?

Recommended Problems

1. How many positive divisors does 33480 have?
2. Find all non-negative integer solutions to $12x + 57y = 423$.
3. Prove or disprove the following statements. Let a, b, c be fixed integers.
   (a) If there exists an integer solution to $ax^2 + by^2 = c$, then $\gcd(a, b) \mid c$.
   (b) If $\gcd(a, b) \mid c$, then there exists an integer solution to $ax^2 + by^2 = c$.
4. Prove or disprove: If $7a^2 = b^2$ where $a, b \in \mathbb{Z}$, then 7 is a common divisor of a and b.
5. Prove that if p is prime and $p \leq n$, then p does not divide $n! + 1$.
6. For what values of c does $8x + 5y = c$ have exactly one solution where both x and y are strictly positive?
7. Consider the following statement:
   Let $a, b, c \in \mathbb{Z}$. For every integer x_0, there exists an integer y_0 such that $ax_0 + by_0 = c$.
   (a) Determine conditions on a, b, c such that the statement is true if and only if these conditions hold. State and prove this if and only if statement.
   (b) Carefully write down the negation of the given statement and prove that this negation is true.
8. Suppose a and b are integers. Prove that $\{ax + by \mid x, y \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$.

9. Is $27^{129} + 61^{40}$ is divisible by 14? Show and justify your work.
10. Is 7386458999999992324343123 divisible by 11? Show and justify your work.
11. Prove or disprove: A prime number can be formed using each of the digits from 0 to 9 exactly once.

Challenge(s)

1. An integer n is perfect if the sum of all of its positive divisors (including 1 and itself) is $2n$.
   (a) Is 6 a perfect number? Give reasons for your answer.
   (b) Is 7 a perfect number? Give reasons for your answer.
   (c) Prove the following statement:
   If k is a positive integer and $2^k - 1$ is prime, then $2^{k-1}(2^k - 1)$ is perfect.
2. Prove that $\gcd(a^n, b^n) = \gcd(a, b)^n$ for all $n \in \mathbb{N}$.