

Neural tangent kernel analysis of shallow α -Stable ReLU neural networks

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Abstract

There is a recent and growing literature on large-width properties of Gaussian neural networks (NNs), namely NNs whose weights are distributed or initialized according to Gaussian distributions. Two popular problems are: i) the study of the large-width behaviour of NNs, which provided a characterization of the infinitely wide limit of a rescaled NN in terms of a Gaussian process; ii) the study of the training dynamics of NNs, which set forth a large-width equivalence between training the rescaled NN and performing a kernel regression with a deterministic kernel referred to as the neural tangent kernel (NTK). In this paper, we consider these problems for α -Stable NNs, which generalize Gaussian NNs by assuming that the NN's weights are distributed as α -Stable distributions with $\alpha \in (0, 2]$, i.e. distributions with heavy-tails or infinite variance. For shallow α -Stable NNs with a ReLU activation function, we show that if the NN's width goes to infinity then a rescaled NN converges weakly to an α -Stable process, namely a stochastic process with α -Stable finite-dimensional distributions. As a novelty with respect to the Gaussian setting, it turns out that in the α -Stable setting the choice of the activation function affects the scaling of the NN, that is: to achieve the infinitely wide α -Stable process, the ReLU function requires an additional logarithmic scaling with respect to sub-linear functions. Then, our main contribution is the NTK analysis of shallow α -Stable ReLU-NNs, which leads to a large-width equivalence between training a rescaled NN and performing a kernel regression with an $(\alpha/2)$ -Stable random kernel. The randomness of such a kernel is a further novelty with respect to the Gaussian setting, that is: in the α -Stable setting the randomness of the NN at initialization does not vanish in the NTK analysis, thus inducing a distribution for the kernel of the underlying kernel regression. An extension of our results to deep α -Stable NNs is discussed.

Keywords: α -Stable stochastic process; gradient descent; infinitely wide limit; kernel regression; neural network; neural tangent kernel analysis; ReLU activation function

1 Introduction

There is a recent and growing literature on large-width properties of Gaussian neural networks (NNs), namely NNs whose weights are distributed or initialized according to Gaussian distributions (Neal, 1996; Williams,

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1997; Der and Lee, 2006; Garriga-Alonso et al., 2018; Jacot et al., 2018; Lee et al., 2018; Matthews et al., 2018; Novak et al., 2018; Arora et al., 2019; Lee et al., 2019; Yang, 2019,a,b; Bracale et al., 2021; Eldan et al., 2021; Klukowski, 2021; Yang and Littwin, 2021; Basteri and Trevisan, 2022). Consider the setting of shallow (fully connected) NNs, that is: i) for $d, k \geq 1$ let X be the $d \times k$ NN's input, with $x_j = (x_{j1}, \dots, x_{jd})^T$ being the j -th input (column vector); ii) let ϕ be an activation function; iii) for $m \geq 1$ let $W = (w_1^{(0)}, \dots, w_m^{(0)}, w)$ be the NN's weights, such that $w_i^{(0)} = (w_{i1}^{(0)}, \dots, w_{id}^{(0)})$ and $w = (w_1, \dots, w_m)$ with the $w_{ij}^{(0)}$'s and the w_i 's being i.i.d. according to a Gaussian distribution with zero mean and variance σ^2 . Then,

$$f_m(x_j) = \sum_{i=1}^m w_i \phi(\langle w_i^{(0)}, x_j \rangle)$$

is a shallow Gaussian ϕ -NN of width m for x_j , $j = 1, \dots, k$. For a general input X , we set $f_m(X) = (f_m(x_1), \dots, f_m(x_k))$. In his seminal work, Neal (1996) first investigated the large-width asymptotic behaviour of shallow Gaussian ϕ -NNs, providing a natural characterization of the infinitely wide limit of a rescaled NN in terms of a Gaussian process. In particular, under suitable assumptions on ϕ , an application of the central limit theorem provided the following result: if $m \rightarrow +\infty$ then the rescaled NN $m^{-1/2} f_m(X)$ converges weakly to a Gaussian process with covariance function $\Sigma_{X,\phi}$ such that $\Sigma_{X,\phi}[r, s] = \sigma^2 \mathbb{E}[\phi(\langle w_i^{(0)}, x_r \rangle) \phi(\langle w_i^{(0)}, x_s \rangle)]$. See Matthews et al. (2018) and Yang (2019a,b) for some extensions to deep NN and general architectures.

1.1 Neural tangent kernel analysis

Some recent works have investigated the training dynamics of Gaussian NNs, with the training being performed through gradient descent, and set forth an interesting large-width equivalence between training a rescaled NN and performing a kernel regression with a deterministic kernel referred to as the neural tangent kernel (NTK) (Jacot et al., 2018; Arora et al., 2019; Lee et al., 2019). This is typically known as the NTK analysis of the NN. In particular, let $f_m(X)$ be a shallow Gaussian ϕ -NN, and assume ϕ to be the ReLU activation function. We denote by (X, Y) the training set, where $Y = (y_1, \dots, y_k)$ is the (training) output, with y_j being the (training) output for the j -th input x_j . Moreover, we denote by

$$\tilde{f}_m(W, X) = \frac{1}{m^{1/2}} f_m(X)$$

the rescaled (model) output, and by $\tilde{f}_m(W, x_j) = m^{-1/2} f_m(x_j)$ the (model) output of x_j , for $j = 1, \dots, k$. Starting from a random initialization $W(0)$ for the NN's weights or parameters, we write the gradient flow, that is for $t \geq 0$

$$\frac{dW(t)}{dt} = -\eta_m \nabla_W \frac{1}{m} \sum_{j=1}^k \ell(y_j, \tilde{f}_m(W(t), x_j)), \quad (1)$$

with $\eta_m > 0$ being the (continuous) learning rate parameterized by the NN's width m , and $\ell(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ being a suitable loss function. Note that the learning rate parameterized by m is equivalent to a re-parameterization of the NN's weights with respect to m , for a learning rate η independent of m , e.g. $\eta = 1$ without loss of generality. See e.g. Yang and Hu (2021) for details.

Consider the gradient flow (1), with respect to the squared-error loss function. Then, an application of the chain rule leads to the NN's dynamics for $t \geq 0$, i.e.

$$\frac{d\tilde{f}_m(W(t), X)}{dt} = -(\tilde{f}_m(W(t), X) - Y) \eta_m H_m(W(t), X), \quad (2)$$

where the kernel $H_m(W(t), X)$ is a $k \times k$ matrix whose (j, j') entry is $\langle \partial \tilde{f}_m(W(t), x_j) / \partial W, \partial \tilde{f}_m(W(t), x_{j'}) / \partial W \rangle$. Jacot et al. (2018) showed that if $\eta_m = 1$ in the dynamics (2), then: i) at random initialization, if $m \rightarrow +\infty$ then $H_m(W(0), X)$ converges in probability to a deterministic kernel $H^*(X, X)$; ii) during training $t > 0$, if m is sufficiently large then the fluctuations of the squared Frobenius norm $\|H_m(W(t), X) - H_m(W(0), X)\|_F^2$

are vanishing. See [Arora et al. \(2019\)](#) for rigorous statements and proofs. These results suggested to replace $\eta_m H_m(W(t), X)$ with $H^*(X, X)$ in (2), and to write

$$\frac{df^*(t, X)}{dt} = -(f^*(t, X) - Y)H^*(X, X).$$

This is precisely the dynamics of a kernel regression under gradient flow, for which at $t \rightarrow +\infty$ the prediction formula for a generic test point $x \in \mathbb{R}^d$ is of the form

$$f^*(x) = YH^*(X, X)^{-1}H^*(X, x)^T.$$

In particular, [Arora et al. \(2019\)](#) showed rigorously that: if $\eta_m = 1$ in the dynamics (2), then the prediction of the NN $\tilde{f}_m(W(t), x)$ at $t \rightarrow +\infty$, for m sufficiently large, is equivalent to the kernel regression prediction $f^*(x)$. The kernel $H^*(X, X)$ is the NTK. See [Arora et al. \(2019\)](#), [Yang \(2019\)](#) and [Yang and Littwin \(2021\)](#) for some extensions to deep NNs and general architectures.

1.2 Our contributions

In this paper, we investigate large-width properties of shallow α -Stable ReLU-NNs, namely shallow NNs with a ReLU activation function and weights distributed as α -Stable distributions ([Samoradnitsky and Taqqu, 1994](#)). For $\alpha \in (0, 2]$, α -Stable distributions provide the most natural generalization of the Gaussian distribution, which is recovered for $\alpha = 2$, defining a broad class of heavy tails or infinite variance distributions. [Neal \(1996\)](#) first investigated the use of α -Stable distributions to initialize NNs, showing the following large-width behaviour at initialization: while the contribution of all Gaussian weights vanishes in the infinitely wide limit, some α -Stable weights retain a non-negligible contribution, allowing them to represent “hidden features” ([Der and Lee, 2006](#); [Favaro et al., 2020](#); [Fortuin et al., 2019](#); [Li et al., 2021b](#); [Hodgkinson and Mahoney, 2021](#)). Along this line of research, [Favaro et al. \(2020, 2021\)](#) considered the problem of characterizing the infinitely wide limits of deep α -Stable NNs. In particular, for a shallow α -Stable ϕ -NN $f_m(X; \alpha)$, their result may be stated as follows: for a sub-linear ϕ , if $m \rightarrow +\infty$ then the rescaled NN $m^{-1/\alpha} f_m(X; \alpha)$ converges weakly to an α -Stable process, that is a stochastic process with α -Stable (finite-dimensional) marginal distributions. Here, we extend this result to the ReLU activation, which is arguably the most popular linear activation function. That is, we show that: if $m \rightarrow +\infty$ then the shallow α -Stable ReLU-NN $(m \log m)^{-1/\alpha} f_m(X; \alpha)$ converges weakly to an α -Stable process. As a novelty with respect to the Gaussian setting, it turns out, that in the α -Stable setting the choice of the activation function ϕ affects the scaling of the NN. That is, in order to achieve the infinitely wide α -Stable process, the use of the ReLU function in place of a sub-linear function results in a change of the scaling $m^{-1/\alpha}$ of the NN through the additional $(\log m)^{-1/\alpha}$ term.

Our main contribution is the NTK analysis of shallow α -Stable ReLU-NNs, which generalizes the works of [Jacot et al. \(2018\)](#) and [Arora et al. \(2019\)](#) to the α -Stable setting. Let $\tilde{f}_m(W, X; \alpha) = (m \log m)^{-1/\alpha} f_m(X; \alpha)$ be the rescaled (model) output and, for a training set (X, Y) consider a training performed through gradient descent. That is, we write the dynamics of $\tilde{f}_m(W(t), X; \alpha)$, i.e. the equivalent of (2) in the α -Stable setting, with η_m being the continuous learning rate and $H_m(W(t), X)$ being the kernel. For such a dynamics, we show that if $\eta_m = (\log m)^{2/\alpha}$ then: i) at random initialization, if $m \rightarrow +\infty$ then $(\log m)^{2/\alpha} H_m(W(0), X)$ converges weakly to an $(\alpha/2)$ -Stable (almost surely) positive definite random kernel $\tilde{H}^*(X, X; \alpha)$; ii) during training $t > 0$, for every $\delta > 0$ the fluctuations of the squared Frobenius norm $\|(\log m)^{2/\alpha} (H_m(W(t), X) - H_m(W(0), X))\|_F^2$ are vanishing, for m sufficiently large, with probability at least $1 - \delta$. Then, for some $m \geq 1$ we set $H^*(X, X; \alpha) = (\log m)^{2/\alpha} H_m(W(0), X)$ and for a generic test point $x \in \mathbb{R}^d$ we show that: if $\eta_m = (\log m)^{2/\alpha}$ then for every $\delta > 0$ the prediction of the NN $\tilde{f}_m(W(t), x; \alpha)$ at $t \rightarrow +\infty$, for m sufficiently large, with probability at least $1 - \delta$ is equivalent to the kernel regression prediction formula

$$f^*(x; \alpha) = YH^*(X, X; \alpha)^{-1}H^*(X, x; \alpha)^T.$$

The kernel $H^*(X, X; \alpha)$ is referred to as the $(\alpha/2)$ -Stable NTK. The randomness of the $(\alpha/2)$ -Stable NTK is a further novelty with respect to the Gaussian setting. That is, within the α -Stable setting the randomness of the NN at initialization does not vanish in the NTK analysis, and it induces a distribution for the kernel of the underlying kernel regression, i.e. the $(\alpha/2)$ -Stable NTK. Such a phenomenon may be viewed as the counterpart, at the training level, of the large-width behaviour at initialization described in [Neal \(1996\)](#).

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we present our main contributions: i) the study of the large-width behaviour of shallow α -Stable ReLU-NNs, which provides a characterization of the infinitely wide limit of a rescaled NN in terms of an α -Stable process; ii) the study of the training dynamics of shallow α -Stable ReLU-NNs, which sets forth a large-width equivalence between training the rescaled NN and performing a kernel regression with the $(\alpha/2)$ -Stable NTK. In Section 3 we discuss our results and their possible extension to the context of deep α -Stable NNs, and then we present some directions for future work on α -Stable NNs.

2 NTK analysis of shallow α -Stable ReLU-NNs

We recall the definitions of one-dimensional α -Stable distribution and of multivariate α -Stable distribution (Samoradnitsky and Taqqu, 1994, Chapter 1 and Chapter 2). For $\alpha \in (0, 2]$, a random variable $S \in \mathbb{R}$ is distributed as a symmetric and centered α -Stable distribution with scale $\sigma > 0$ if its characteristic function is

$$\mathbb{E}(\exp\{izS\}) = \exp\{-\sigma^\alpha|z|^\alpha\},$$

and we write $S \sim \text{St}(\alpha, \sigma)$. The parameter α is referred to as the stability. If $\alpha = 2$ then S is distributed according to a Gaussian distribution with mean 0 and variance σ^2 . Let \mathbb{S}^{k-1} be the unit sphere in \mathbb{R}^k , with $k \geq 1$. A random variable $S \in \mathbb{R}^k$ is distributed as a symmetric and centered k -dimensional α -Stable distribution with spectral measure Γ on \mathbb{S}^{k-1} if its characteristic function is

$$\mathbb{E}(\exp\{i\langle z, S \rangle\}) = \exp\left\{-\int_{\mathbb{S}^{k-1}} |\langle z, s \rangle|^\alpha \Gamma(ds)\right\},$$

and we write $S \sim \text{St}_k(\alpha, \Gamma)$. Let 1_r be the r -dimensional (column) vector with 1 in the r -th entry and 0 elsewhere, for any $r = 1, \dots, k$. Then, the r -th element of S , that is $S1_r$ is distributed as an α -Stable distribution with scale

$$\sigma = \left(\int_{\mathbb{S}^{k-1}} |\langle 1_r, s \rangle|^\alpha \Gamma(ds)\right)^{1/\alpha}.$$

We deal mostly with k -dimensional α -Stable distributions with discrete spectral measure, that is a measure $\Gamma(\cdot) = \sum_{1 \leq i \leq n} \gamma_i \delta_{s_i}(\cdot)$ with $n \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ and $s_i \in \mathbb{S}^{k-1}$, for $i = 1, \dots, n$ (Samoradnitsky and Taqqu, 1994, Chapter 2). Throughout this section, it is assumed that all the random variables are defined on a common probability space, say $(\Omega, \mathcal{F}, \mathbb{P})$, unless otherwise stated.

2.1 α -Stable ReLU-NNs and their infinitely wide limits

We start by introducing the setting of shallow α -Stable NNs, with the assumption of a ReLU activation function, that is: i) for any $d, k \geq 1$ let X be the $d \times k$ NN's input, with $x_j = (x_{j1}, \dots, x_{jd})^T$ being the j -th input (column vector); ii) for $m \geq 1$ let $W = (w_1^{(0)}, \dots, w_m^{(0)}, w)$ be the NN's weights, such that $w_i^{(0)} = (w_{i1}^{(0)}, \dots, w_{id}^{(0)})$ and $w = (w_1, \dots, w_m)$ with the $w_{ij}^{(0)}$'s and the w_i 's being i.i.d. according to an α -Stable distribution with scale parameter $\sigma > 0$. Then,

$$f_m(x_j; \alpha) = \sum_{i=1}^m w_i \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0), \quad (3)$$

with $I(\cdot)$ being the indicator function, is a shallow α -Stable ReLU-NN of width m for x_j , with $j = 1, \dots, k$. For a general input X , we set $f_m(X; \alpha) = (f_m(x_1; \alpha), \dots, f_m(x_k; \alpha))$. Without loss of generality, throughout this section we assume $\sigma = 1$. As discussed in Neal (1996), with respect to the study of large-width properties of NNs, the class of α -Stable NNs provide the most natural and flexible (distributional) generalization of Gaussian NNs. In particular, the definition of shallow Gaussian ReLU-NN is recovered from (3) by setting $\alpha = 2$. Other examples may be obtained by suitable specifications of the stability parameter α , e.g. $\alpha = 1/2$ for the Lévy NN and $\alpha = 1$ for the Cauchy NN. See also Der and Lee (2006), and references therein, for a discussion of some classes of α -Stable NNs. The next theorem characterizes the large-width behaviour of shallow α -Stable ReLU-NNs in terms of an α -Stable process.

Theorem 1. Let $f_m(X; \alpha)$ be the shallow α -Stable ReLU-NN. If $m \rightarrow +\infty$ then

$$\frac{1}{(m \log m)^{1/\alpha}} f_m(X; \alpha) \xrightarrow{w} f(X),$$

where $f(X) \sim St_k(\alpha, \Gamma_X)$, with the spectral measure Γ_X being of the following form:

$$\Gamma_X = \frac{1}{4} \sum_{i=1}^d \left[\frac{(\delta(D_i^+(X)) + \delta(-D_i^+(X)))}{\|[x_{ji}I(x_{ji} > 0)]_j\|^{-\alpha}} + \frac{(\delta(D_i^-(X)) + \delta(-D_i^-(X)))}{\|[x_{ji}I(x_{ji} < 0)]_j\|^{-\alpha}} \right]$$

such that

$$D_i^+(X) = \frac{[x_{ji}I(x_{ji} > 0)]_j}{\|[x_{ji}I(x_{ji} > 0)]_j\|}$$

and

$$D_i^-(X) = \frac{[x_{ji}I(x_{ji} < 0)]_j}{\|[x_{ji}I(x_{ji} < 0)]_j\|}.$$

In particular, the stochastic process $f(X) = (f(x_1), \dots, f(x_k))$, as a process indexed by the NN's input X , is an α -Stable process with spectral measure Γ_X .

See Appendix A for the proof of Theorem 1. For a broad class of bounded or sub-linear activation functions, the main result of Favaro et al. (2021) characterizes the large-width behaviour of deep α -Stable NNs. In particular, let

$$f_m(x_j; \alpha) = \sum_{i=1}^m w_i \phi(w_i^{(0)}, x_j)$$

be the shallow α -Stable NN of width m for the input x_j , with the function ϕ being an arbitrary bounded activation function. Let $f_m(X; \alpha) = (f_m(x_1; \alpha), \dots, f_m(x_k; \alpha))$. According to Favaro et al. (2021, Theorem 1.2) if $m \rightarrow +\infty$ then

$$\frac{1}{m^{1/\alpha}} f_m(X; \alpha) \xrightarrow{w} f(X), \quad (4)$$

with $f(X)$ being an α -Stable process with spectral measure $\Gamma_{X, \phi}$. Theorem 1 may be viewed as an extension of Favaro et al. (2021, Theorem 1.2) to the ReLU activation, which is the most popular unbounded activation function. It is useful to discuss Theorem 1 with respect to the scaling $(m \log m)^{-1/\alpha}$, which is required to achieve the infinitely wide α -Stable process. Theorem 1 shows that the use of the ReLU activation in place of a bounded activation results in a change of the scaling $m^{-1/\alpha}$ in (4), through the inclusion of the $(\log m)^{-1/\alpha}$ term. This is a critical difference between the α -Stable setting and Gaussian setting, as in the latter the choice of the activation function ϕ does not affect the scaling $m^{-1/2}$ required to achieve the infinitely wide Gaussian process. We refer to Bordino et al. (2022) for infinitely wide limits of α -Stable NNs with classes of sub-linear, linear and super-linear activation functions.

2.2 NTK analysis

We study the large-width training dynamics of shallow α -Stable ReLU-NNs. Let $f_m(X; \alpha)$ be the NN (3). Let (X, Y) the training set, where $Y = (y_1, \dots, y_k)$ is the (training) output, with y_j being the (training) output for the j -th input x_j . We consider the rescaled (model) output

$$\tilde{f}_m(W, X; \alpha) = \frac{1}{(m \log m)^{1/\alpha}} f_m(X; \alpha),$$

and denote by $\tilde{f}_m(W, x_j; \alpha) = (m \log m)^{-1/\alpha} f_m(x_j; \alpha)$ the (model) output of x_j , for $j = 1, \dots, k$. In particular, along the same lines presented for the Gaussian setting, the use of gradient descent allows us to write the gradient flow for the parameter $W(t)$, for any $t \geq 0$. Then, by assuming the squared-error loss function $\ell(y_j, \tilde{f}_m(W, x_j; \alpha)) = 2^{-1} \sum_{1 \leq j \leq k} (\tilde{f}_m(W, x_j; \alpha) - y_j)^2$, an application of the chain rule leads to the NN's dynamics, that is for any $t \geq 0$

$$\frac{d\tilde{f}_m(W(t), X; \alpha)}{dt} = -(\tilde{f}_m(W(t), X; \alpha) - Y) \eta_m H_m(W(t), X), \quad (5)$$

where the kernel $H_m(W(t), X)$ is a $k \times k$ matrix whose (j, j') entry is of the form

$$H_m(W(t), X)[j, j'] = \left\langle \frac{\partial \tilde{f}_m(W(t), x_j; \alpha)}{\partial W}, \frac{\partial \tilde{f}_m(W(t), x_{j'}; \alpha)}{\partial W} \right\rangle, \quad (6)$$

and η_m denotes the continuous learning rate. The next theorem provides a characterization of the large-width training dynamics of shallow α -Stable ReLU-NNs, showing linear convergence of $\|Y - \tilde{f}_m(W(t), X; \alpha)\|_2^2$ toward 0 as $t \rightarrow +\infty$.

Theorem 2. *For any $k \geq 1$ let the collection of NN's inputs x_1, \dots, x_k be linearly independent, and such that $\|x_j\| = 1$. Under the dynamics (5), if $\eta_m = (\log m)^{2/\alpha}$ then for every $\delta > 0$ there exists $\lambda_0 > 0$ such that, for m sufficiently large and any $t > 0$, with probability at least $1 - \delta$ it holds true that*

$$\|Y - \tilde{f}_m(W(t), X; \alpha)\|_2^2 \leq \exp(-\lambda_0 t) \|Y - \tilde{f}_m(W(0), X; \alpha)\|_2^2.$$

To prove Theorem 2, let $\tilde{H}_m(W(t), X) = (\log m)^{2/\alpha} H_m(W(t), X)$ be the kernel $H_m(W(t), X)$ scaled by $\eta_m = (\log m)^{2/\alpha}$. Moreover, we denote by $\lambda_{\min}(\tilde{H}_m(W(t), X))$ the minimum eigenvalue of the kernel $\tilde{H}_m(W(t), X)$. If there exists $\lambda_0 > 0$ such that $\lambda_{\min}(\tilde{H}_m(W(s), X)) > \lambda_0/2$ for every $s \leq t$, then

$$\frac{d}{ds} \|Y - \tilde{f}_m(W(s), X; \alpha)\|_2^2 \leq -\lambda_0 \|Y - \tilde{f}_m(W(s), X; \alpha)\|_2^2,$$

and hence

$$\frac{d}{ds} \exp(\lambda_0 s) \|Y - \tilde{f}_m(W(s), X; \alpha)\|_2^2 \leq 0.$$

Since $\exp(\lambda_0 s) \|Y - \tilde{f}_m(W(s), X; \alpha)\|_2^2$ is a decreasing function of $s > 0$, then we write

$$\|Y - \tilde{f}_m(W(s), X; \alpha)\|_2^2 \leq \exp(-\lambda_0 s) \|Y - \tilde{f}_m(W(0), X; \alpha)\|_2^2.$$

The proof of Theorem 2 then consists in providing a positive lower bound for $\lambda_{\min}(\tilde{H}_m(W(t), X))$. Hereafter, we show that for every $\delta > 0$ there exists $\lambda_0 > 0$ such that with probability at least $1 - \delta$, m sufficiently large and any $t > 0$

$$\lambda_{\min}(\tilde{H}_m(W(t), X)) > \frac{\lambda_0}{2}. \quad (7)$$

We prove (7) by first providing a lower bound for the minimum eigenvalue at random initialization, i.e. $\lambda_{\min}(\tilde{H}_m(W(0), X))$, and then showing the stability of $\lambda_{\min}(\tilde{H}_m(W(t), X))$ with respect to $\lambda_{\min}(\tilde{H}_m(W(0), X))$ during training $t > 0$.

2.2.1 A lower bound for $\lambda_{\min}(\tilde{H}_m(W(0), X))$

We characterize the large-width behaviour of the kernel $\tilde{H}_m(W(0), X)$. In particular, by a straightforward calculation, it is easy to show that we can write $\tilde{H}_m(W(0), X)$ as follows

$$\tilde{H}_m(W(0), X) = \tilde{H}_m^{(1)} + \tilde{H}_m^{(2)}, \quad (8)$$

with $\tilde{H}_m^{(1)}$ and $\tilde{H}_m^{(2)}$ being random matrices whose (j, j') entries are of the form

$$\tilde{H}_m^{(1)}[j, j'] = \frac{1}{m^{2/\alpha}} \sum_{i=1}^m w_i^2 \langle x_j, x_{j'} \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) I(\langle w_i^{(0)}, x_{j'} \rangle > 0) \quad (9)$$

and

$$\tilde{H}_m^{(2)}[j, j'] = \frac{1}{m^{2/\alpha}} \sum_{i=1}^m \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) \langle w_i^{(0)}, x_{j'} \rangle I(\langle w_i^{(0)}, x_{j'} \rangle > 0), \quad (10)$$

respectively. The next lemmas characterize the large-width behaviour of $\tilde{H}_m^{(1)}$ and $\tilde{H}_m^{(2)}$, respectively. Here, we denote by $\|\cdot\|_F$ the Frobenius norm, and set

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)} & \alpha \neq 1 \\ \frac{2}{\pi} & \alpha = 1. \end{cases} \quad (11)$$

Lemma 3. Let $\tilde{H}_m^{(1)}$ be the random matrix defined in (9). If $m \rightarrow +\infty$ then

$$\tilde{H}_m^{(1)} \xrightarrow{w} \tilde{H}_1^*(\alpha),$$

where $\tilde{H}_1^*(\alpha)$ is an $(\alpha/2)$ -Stable non-negative definite random matrix with spectral measure

$$\Gamma_1^* = C_{\alpha/2} \mathbb{E} \left(\|G(w_1^{(0)})\|_F^{\alpha/2} \delta_{G(w_1^{(0)})/\|G(w_1^{(0)})\|_F} \right) \quad (12)$$

such that $G(w_1^{(0)})$ denotes the Gram matrix of the vector $(x_1 I(\langle w_1^{(0)}, x_1 \rangle > 0), \dots, x_k I(\langle w_1^{(0)}, x_k \rangle > 0))$, and $C_{\alpha/2}$ is the constant defined in (11), for $\alpha \in (0, 2]$.

Lemma 4. Let $\tilde{H}_m^{(2)}$ be the random matrix defined in (10). If $m \rightarrow +\infty$ then

$$\tilde{H}_m^{(2)} \xrightarrow{w} \tilde{H}_2^*(\alpha),$$

where $\tilde{H}_2^*(\alpha)$ is an $(\alpha/2)$ -Stable non-negative definite random matrix.

Then, in the next Theorem, we combine Lemma 3 and Lemma 4 in order to characterize the large-width behaviour of the random matrix $\tilde{H}_m(W(0), X)$.

Theorem 5. Let $\tilde{H}_m(W(0), X)$ be the random matrix (8). If $m \rightarrow +\infty$ then

$$\tilde{H}_m(W(0), X) \xrightarrow{w} \tilde{H}^*(X, X; \alpha),$$

where $\tilde{H}^*(X, X; \alpha)$ is an $(\alpha/2)$ -Stable non-negative definite random matrix.

See Appendix B for the proof of Lemma 3, Lemma 4 and Theorem 5, as well as for the characterization of the spectral measure Γ^* of $\tilde{H}^*(X, X; \alpha)$. Now, we exploit Lemma 3 in order to show that for every $\delta > 0$, there exists $\lambda_0 > 0$ such that, for m sufficiently large, with probability at least $1 - \delta$ it holds true

$$\lambda_{\min}(\tilde{H}_m(W(0), X)) > \lambda_0. \quad (13)$$

From (8), $\tilde{H}_m(W(0), X)$ is the sum of two non-negative definite random matrices, $\tilde{H}_m^{(1)}$ and $\tilde{H}_m^{(2)}$. Then, it is sufficient to show that for every $\delta > 0$, there exists $\lambda_0 > 0$ such that, for m sufficiently large, with probability at least $1 - \delta$ and any $t > 0$

$$\lambda_{\min}(\tilde{H}_m^{(1)}) > \lambda_0, \quad (14)$$

with the large-width behaviour of $\tilde{H}_m^{(1)}$ being characterized in Lemma 3, through an $(\alpha/2)$ -Stable limiting random matrix $\tilde{H}_1^*(\alpha)$ with spectral measure Γ_1^* of the form (12). The next two lemmas are required in order to prove (14).

Lemma 6. Under the assumptions of Theorem 2, the distribution of the random matrix $\tilde{H}_1^*(\alpha)$ is absolutely continuous in the subspace of the symmetric non-negative definite matrices with zero entries in the positions (j, j') such that $\langle x_j, x_{j'} \rangle = 0$, with $j, j' \in \{1, \dots, k\}$, with the topology of Frobenius norm.

We observe that the space of the symmetric non-negative definite matrices with zeros in the entries (j, j') such that $\langle x_j, x_{j'} \rangle = 0$ contains all the matrices with non-zero diagonal element since $\langle x_j, x_j \rangle = 1 \neq 0$ for every index j .

Lemma 7. Under the assumptions of Theorem 2, for every $\delta > 0$ there exists $\lambda_0 > 0$ such that with probability at least $1 - \delta$

$$\lambda_{\min}(\tilde{H}_1^*(\alpha)) > \lambda_0.$$

In the next Theorem, we make use of Lemma 6 and Lemma 7 in order to prove the lower bound (14). Such a result, in turn, proves the lower bound (13).

Theorem 8. Under the assumptions of Theorem 2, for every $\delta > 0$ there exists $\lambda_0 > 0$ such that, for m sufficiently large, with probability at least $1 - \delta$

$$\lambda_{\min}(\tilde{H}_m(W(0), X)) > \lambda_0.$$

See Appendix B for the proof of Lemma 6, Lemma 7 and Theorem 8

2.2.2 Stability of $\lambda_{\min}(\tilde{H}_m(W(t), X))$ during training $t > 0$

We recall that $W = (w^{(0)}, w)$ are the NN's weights, where $w^{(0)} = (w_{1,1}^{(0)}, \dots, w_{m,d}^{(0)})$ and $w = (w_1, \dots, w_m)$, and let $H_m(W, X)$ be a $k \times k$ matrix whose (j, j') entry is

$$H_m(W, X)[j, j'] = \left\langle \frac{\partial f_m(W, x_j; \alpha)}{\partial W}, \frac{\partial f_m(W, x_{j'}; \alpha)}{\partial W} \right\rangle,$$

and set $\tilde{H}_m(W, X) = (\log m)^{2/\alpha} H_m(W, X)$. Moreover, we denote by $W(0) = (w^{(0)}(0), w(0))$ the NN's weights at random initialization and by $W(t) = (w^{(0)}(t), w(t))$ the NN's weights during training $t > 0$. Let $H_m(W(t), X)$ be the $k \times k$ matrix whose (j, j') entry is defined in (6), and recall that $\tilde{H}_m(W(t), X) = (\log m)^{2/\alpha} H_m(W(t), X)$ for any $t \geq 0$. In the next lemmas, we show the stability of $\lambda_{\min}(\tilde{H}_m(W(t), X))$ during training $t > 0$, that is we show that $\lambda_{\min}(\tilde{H}_m(W(t), X))$ remains bounded from below by $\lambda_0/2$ during training $t > 0$.

Lemma 9. *Let $\gamma \in (0, 1/2)$ and $c > 0$ be fixed numbers. For every $\delta > 0$ the following property holds true, for m sufficiently large, with probability at least $1 - \delta$:*

$$(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F^2 < cm^{-2\gamma/\alpha},$$

for every W such that $\|W - W(0)\|_F^2 \leq \log m$ and every NN's input x_j , with $j = 1, \dots, k$.

Lemma 10. *For every $\delta > 0$ there exists $\lambda_0 > 0$ such that the following two properties hold true, for m sufficiently large, with a probability at least $1 - \delta$:*

i)

$$\|\tilde{H}_m(W, X) - \tilde{H}_m(W(0), X)\|_2 < \frac{\lambda_0}{2};$$

ii)

$$\lambda_{\min}(\tilde{H}_m(W, X)) > \frac{\lambda_0}{2};$$

for every W such that $\|W - W(0)\|_F^2 \leq \log m$.

Lemma 11. *Let $\gamma \in (0, 1/2)$ and $c > 0$ be fixed numbers. For every $\delta > 0$ the following property holds true, for m sufficiently large, with probability at least $1 - \delta$:*

$$\|W(t) - W(0)\|_F^2 < \log m.$$

if

$$(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F^2 \leq cm^{-2\gamma/\alpha}$$

for every NN's input x_j , with $j = 1, \dots, k$, and for every $s \leq t$.

See Appendix B for the proof of Lemma 9, Lemma 10 and Lemma 11. Now, we show that for every $\delta > 0$ there exists $\lambda_0 > 0$ such that, for m sufficiently large, the lower bound (7) holds true for every $t \geq 0$ with probability at least $1 - \delta$. In particular, let $m \in \mathbb{N}$ and $N \in \mathcal{F}$ be such that $\mathbb{P}(N) > 1 - \delta$ and Lemma 9, Lemma 10 and Lemma 11 hold true for every $\omega \in N$. Therefore, by means of Lemma 10, it is sufficient to show that

$$\|W(t) - W(0)\|_F^2(\omega) < \log m$$

for every $t > 0$ and $\omega \in N$. Suppose that there exists, for some $\omega \in N$, $t_0(\omega)$ finite with

$$t_0(\omega) := \inf_{t \geq 0} \{t : \|W(t) - W(0)\|_F^2(\omega) \geq \log m\}.$$

Since $W(t)(\omega)$ is a continuous function of t , then $\|W(t_0(\omega)) - W(0)\|_F^2(\omega) = \log m$. Then

$$(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F^2(\omega) < cm^{-2\gamma/\alpha},$$

for every $s \leq t_0$ and every j follows by Lemma 9. Therefore, by Lemma 11 it holds true that $\|W(t_0(\omega)) - W(0)\|_F^2(\omega) < \log m$, which contradicts the definition of t_0 . This proves (7), thus completing the proof of Theorem 2.

2.3 A kernel regression from the NTK analysis

In addition to $X = (x_1, \dots, x_k)$ we consider a generic d -dimensional input x , and then we define the NN's input $[X, x] = (x_1, \dots, x_k, x)$. Let $f_m(W, [X, x]; \alpha)$ be the shallow α -Stable ReLU-NN for the input $[X, x]$ and let $\tilde{f}_m(W, [X, x]; \alpha) = (m \log m)^{-1/\alpha} f_m(W, [X, x]; \alpha)$ be the rescaled (model) output. Now, along the same lines discussed above, we consider the NTK analysis of the NN $\tilde{f}_m(W, [X, x]; \alpha)$ over the training set (X, Y) , with the training being performed through gradient descent with respect to the squared-error loss function. In particular, by means of Theorem 1 and Theorem 5, if $m \rightarrow +\infty$ then

$$\tilde{f}_m(W(0), [X, x]; \alpha) \xrightarrow{w} f([X, x])$$

and

$$\tilde{H}_m(W(0), [X, x]) \xrightarrow{w} \tilde{H}^*([X, x], [X, x]; \alpha),$$

respectively. Since Lemma 9 and Lemma 10 do not depend on the training of W , then their conclusions hold true if the NN's input X is replaced by the NN's input $[X, x]$. Also Lemma 11 holds true with X substituted by $[X, x]$, since

$$\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x'; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x'; \alpha) \right\|_F^2 \leq c(\log m)^{-2/\alpha} m^{-2\gamma/\alpha} \quad (15)$$

for every $x' = x_1, \dots, x_k, x$ implies that the same property holds for every $x' = x_1, \dots, x_k$. Following the same reasoning as in Section 2.2.2, it can be proved that, for every δ , the inequality in (15) holds true for every $s \geq 0$ and for fixed $c > 0$ and $\gamma \in (0, 1/2)$, with probability at least $1 - \delta$, if m is sufficiently large. Next lemma states the stability of $\tilde{H}_m(W(t), [X, x])$ during training $t > 0$.

Lemma 12. *Let $c > 0$ and $\gamma \in (0, 1/2)$ be two fixed numbers. Then, under the assumptions of Theorem 2, for every $\delta > 0$ the following property holds true, for m sufficiently large and for any $s \geq 0$, with probability at least $1 - \delta$:*

$$\|\tilde{H}_m(W(s), [X, x]) - \tilde{H}_m(W(0), [X, x])\|_F \leq cm^{-2\gamma/\alpha}.$$

See Appendix C for the proof of Lemma 12. Now, for some $m \geq 1$, set $H^*(X, X; \alpha) = \tilde{H}_m(W(0), X; \alpha)$ and $H^*(X, x; \alpha) = \tilde{H}_m(W(0), X, x)$, where we denote by $\tilde{H}_m(W(t), X, x)$ the $(k+1)$ -th row of $\tilde{H}_m(W(t), [X, x])$. Lemma 12 suggests to set $\eta_m = (\log m)^{2/\alpha}$ in the dynamics (5), and then replace $(\log m)^{2/\alpha} H_m(W(t), X)$ with $H^*(X, X; \alpha)$ in such a dynamics. This leads to write

$$\frac{df^*(t, X; \alpha)}{dt} = -(f^*(t, X; \alpha) - Y)H^*(X, X; \alpha). \quad (16)$$

This is precisely the dynamics of a kernel regression under gradient flow, for which at $t \rightarrow +\infty$ the prediction formula for a generic test point $x \in \mathbb{R}^d$ is of the form

$$f^*(x; \alpha) = Y(H^*(X, X; \alpha))^{-1}H^*(X, x; \alpha)^T.$$

provided that $f^*(0, x; \alpha) = 0$. To establish the large-width equivalence between the prediction of the NN $\tilde{f}_m(W, x; \alpha)$ at $t \rightarrow +\infty$ and the kernel regression prediction $f^*(x; \alpha)$, we would like the initial output of the NN to be small. This is because $f^*(x; \alpha)$ corresponds to the linear dynamics (16) with zero initialization. Therefore, we apply a small multiplier κ to $\tilde{f}_m(W, x; \alpha)$, that is

$$\tilde{f}_{NN}(W, x; \alpha) = \kappa \tilde{f}_m(W, x; \alpha).$$

Then, we set $\tilde{f}_{NN}(x; \alpha) = \lim_{t \rightarrow \infty} \tilde{f}_{NN}(W(t), x; \alpha)$ in order to denote the prediction of the NN at the end of the training. The next theorem shows that, for m sufficiently large, the NN prediction $\tilde{f}_{NN}(x; \alpha)$ is equivalent to $f^*(x; \alpha)$.

Theorem 13. *Under the assumption of Theorem 2, for every $\epsilon > 0$, $\delta > 0$ and a test point $x \in \mathbb{R}^d$ such that $\|x\| = 1$, the following property holds true, for m is sufficiently large and $|\tilde{f}_{NN}(W(0), x; \alpha)| < \epsilon/4$, with probability at least $1 - \delta$:*

$$|\tilde{f}_{NN}(x; \alpha) - f^*(x; \alpha)| < \epsilon.$$

See Appendix C for the proof of Theorem 13. The kernel $H^*(X, X; \alpha)$ is referred to as the $(\alpha/2)$ -Stable NTK. Theorem 13 completes our NTK analysis of shallow α -Stable ReLU-NNs, thus providing a generalization of the works of Jacot et al. (2018) and Arora et al. (2019) to the more general α -Stable setting. Differently from the NTK arising in the Gaussian setting, which is a deterministic kernel, the $(\alpha/2)$ -Stable NTK is a random kernel, and its distribution depends on the α -Stable distribution assigned to the NN's weights. That is, within the α -Stable setting the randomness of the NN at initialization does not vanish in the NTK analysis, and it induces a distribution for the kernel of the underlying kernel regression, i.e. the $(\alpha/2)$ -Stable NTK.

3 Discussion

We investigated large-width properties of a shallow α -Stable ReLU-NNs, with respect to two popular problems: i) the study of the large-width behaviour of NNs; ii) the study of the large-width training dynamics of NNs. With regards to the first problem, we showed that, as the NN's width goes to infinity, a rescaled α -Stable ReLU-NN converges weakly to an α -Stable process, for which we have characterized the (finite-dimensional) marginal distributions. Such a result provides an extension to the ReLU activation function of a result in Favaro et al. (2020, 2021), which have considered infinitely wide limits of α -Stable NNs with sub-linear activation functions. As a novelty with respect to the Gaussian setting, it turns out that in the α -Stable setting the choice of the activation function affects the scaling of the NN, that is: to achieve the infinitely wide α -Stable process, the ReLU function requires an additional logarithmic scaling with respect to sub-linear functions. With regards to the second problem, we developed the NTK analysis of shallow α -Stable ReLU-NNs, setting forth a large-width equivalence between training the rescaled NN and performing a kernel regression with the $(\alpha/2)$ -Stable NTK. Such a result generalizes to the α -Stable setting the main results of Jacot et al. (2018) and Arora et al. (2019), which have considered the NTK analysis in the Gaussian setting. In particular, the randomness of the $(\alpha/2)$ -Stable NTK is a further novelty with respect to the Gaussian setting, that is: within the α -Stable setting the randomness of the NN at initialization does not vanish in the NTK analysis, thus inducing a distribution for the kernel of the underlying kernel regression. Such a peculiar phenomenon may be viewed as the counterpart, at the training level, of the large-width behaviour at initialization described in Neal (1996).

The most interesting avenue for future research would be to extend our results to deep α -Stable NNs. In particular, consider the setting of a deep α -Stable NN, with the assumption of a ReLU activation function, that is: i) for $d, k \geq 1$ let X be the $d \times k$ NN's input, with $x_j = (x_{j1}, \dots, x_{jd})^T$ being the j -th input (column vector); ii) for $D, m \geq 1$ and $n \geq 1$ let: i) $(W^{(1)}, \dots, W^{(D)})$ be the NN's weights such that $W^{(1)} = (w_{1,1}^{(1)}, \dots, w_{m,d}^{(1)})$ and $W^{(l)} = (w_{1,1}^{(l)}, \dots, w_{m,m}^{(l)})$ for $2 \leq l \leq D$, where the $w_{i,j}^{(l)}$'s are i.i.d. according to an α -Stable distribution with scale parameter $\sigma > 0$, e.g. assume $\sigma = 1$. Then,

$$f_i^{(1)}(X; \alpha) = \sum_{j=1}^d w_{i,j}^{(1)} x_j$$

and

$$f_{i,m}^{(l)}(X; \alpha) = \sum_{j=1}^m w_{i,j}^{(l)} f_j^{(l-1)}(X, m) I(f_j^{(l-1)}(X, m) > 0)$$

with $f_{i,m}^{(1)}(X; \alpha) := f_i^{(1)}(X; \alpha)$, is a deep α -Stable ReLU-NN of depth D and width m . Under the assumption that the NN's width grows sequentially over the NN's layers, i.e. $m \rightarrow +\infty$ one layer at a time, it is easy to

extend Theorem 1 to $f_{i,m}^{(l)}(X; \alpha)$. Similarly, under such an assumption for the growth of the NN's layers, we expect the NTK analysis of deep α -Stable ReLU-NN to be a straightforward extension of our NTK analysis. A more challenging task would be to extend our results to deep α -Stable ReLU-NNs under the assumptions that the NN's width grows simultaneously over the NN's layers.

Some further open problems are: i) the NTK analysis for shallow α -Stable NNs with a general activation function, e.g. sub-linear, linear and super-linear activation (Bordino et al., 2022); ii) the NTK analysis for α -Stable NN with more general architecture, e.g. convolutional NNs. The problem of quantifying large-width properties, i.e. providing rates of convergence, is also open, both in the study of the large-width behaviour of NNs and in the NTK analysis. See, e.g. Eldan et al. (2021), Klukowski (2021) and Basteri and Trevisan (2022) for Gaussian NNs, and in (Favaro et al., 2021) for α -Stable NNs. A further interesting line of research is related to the “lazy training” phenomenon discussed in Chizat et al. (2019), which is known to arise for Gaussian NNs rescaled by $m^{-1/2}$. Our NTK analysis shows an analogous “lazy training” phenomenon for shallow α -Stable ReLU-NNs rescaled by $(m \log m)^{-1/\alpha}$. In general, for NN's weights distributed according to a distribution with finite moments, Chizat et al. (2019) provided a sufficient condition for the “lazy training” phenomenon to hold, and showed that Gaussian NNs rescaled by m^{-1} do not satisfy such a condition. The scaling m^{-1} allows the NN's dynamics to converge to a non degenerate dynamic described by a partial differential equation and referred to as the mean-field limit (Chizat and Bach, 2018; Rotskoff and Vanden-Eijnden, 2018; Mei et al., 2018; Sirignano and Spiliopoulos, 2020). Then, one may consider the problem of finding a sufficient condition for the “lazy training” phenomenon in the α -Stable setting, and then look for possible scaling that violate such a condition.

A

Throughout this section, it is assumed that all the random variables are defined on a common probability space, say $(\Omega, \mathcal{F}, \mathbb{P})$, unless otherwise stated. We make use several times of the following characterization of the spectral measure of α -stable distributions: if $S \sim \text{St}_k(\alpha, \Gamma)$, then for every Borel set B of \mathbb{S}^{k-1} such that $\Gamma(\partial B) = 0$, it holds true that

$$\lim_{r \rightarrow \infty} r^\alpha \mathbb{P} \left(\|S\| > r, \frac{S}{\|S\|} \in B \right) = C_\alpha \Gamma(B),$$

where

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)} & \alpha \neq 1 \\ \frac{2}{\pi} & \alpha = 1. \end{cases}$$

The proof is reported in Appendix D for completeness.

A.1 Proof of Theorem 1

First, we will prove that $[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j$ belongs to the domain of attraction of an α -stable law with spectral measure

$$\Gamma_1 = C_\alpha \mathbb{E}_{u \sim \Gamma_0} \left(\|[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j\|^\alpha \delta \left(\frac{[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j}{\|[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j\|} \right) \right),$$

where Γ_0 is the spectral measure of $w_i^{(0)}$. For this, it is sufficient to show that

$$\begin{aligned} r^\alpha \mathbb{P} \left(\frac{[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j}{\|[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j\|} \in B, \|[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j\| > r \right) \\ \rightarrow \Gamma_1(B), \end{aligned}$$

for every Borel set B of \mathbb{S}^{k-1} such that $\Gamma_1(\partial B) = 0$. Let $T : \mathbb{S}^{k-1} \mapsto [0, 1]^k$ and $C : \mathbb{R}^k \setminus \{0\} \rightarrow \mathbb{S}^{k-1}$ be defined as $T(u) = [\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j$ and $C(v) = v/\|v\|$, respectively. Fix a Borel set B of \mathbb{S}^{k-1} such that $\Gamma_1(\partial B) = 0$. This condition implies that

$$\Gamma_0 \left(\{u \in \mathbb{S}^{k-1} : \|T(u)\| \neq 0, T(u) \in C^{-1}(\partial B)\} \right)$$

$$= \Gamma_0 \left(\left\{ u \in \mathbb{S}^{k-1} : \|T(u)\| \neq 0, \frac{T(u)}{\|T(u)\|} \in \partial B \right\} \right) = 0.$$

Hence

$$\begin{aligned} & \Gamma_0 \left(T^{-1} \left(\{z \in [0, 1]^k : \|z\| \neq 0, z \in \partial C^{-1}(B)\} \right) \right) \\ &= \Gamma_0 \left(T^{-1} \left(\{z \in [0, 1]^k : \|z\| \neq 0, z \in C^{-1}(\partial B)\} \right) \right) = 0. \end{aligned}$$

Now, let $Z = T(w_i^{(0)} / \|w_i^{(0)}\|) I(\|w_i^{(0)}\| \neq 0)$. We can write that

$$\begin{aligned} & r^\alpha \mathbb{P} \left(\frac{[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j}{\|[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j\|} \in B, \|[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j\| > r \right) \\ &= r^\alpha \mathbb{P} \left(\|Z\| \neq 0, \frac{Z}{\|Z\|} \in B, \|w_i^{(0)}\| \|Z\| > r \right) \\ &= \int_{C^{-1}(B) \cap [0, 1]^k} r^\alpha \mathbb{P}(\|w_i^{(0)}\| > r \|z\|^{-1}, Z \in dz) \\ &= \int_{C^{-1}(B) \cap [0, 1]^k} \|z\|^\alpha (r \|z\|^{-1})^\alpha \mathbb{P}(\|w_i^{(0)}\| > r \|z\|^{-1}, \frac{w_i^{(0)}}{\|w_i^{(0)}\|} \in T^{-1}(dz)). \end{aligned}$$

Since $\Gamma_0 \left(T^{-1} \left(\{z \in [0, 1]^k : z \neq 0, z \in \partial(C^{-1}(B))\} \right) \right) = 0$, then the points of discontinuity of the function $\|z\|^\alpha I(C^{-1}(B))(z)$ have zero $\Gamma_0(T^{-1}(\cdot))$ -measure. It follows that

$$\begin{aligned} & \int_{C^{-1}(B) \cap [0, 1]^k} \|z\|^\alpha (r \|z\|^{-1})^\alpha \mathbb{P}(\|w_i^{(0)}\| > r \|z\|^{-1}, w_i^{(0)} \in T^{-1}(dz)) \\ & \rightarrow C_\alpha \int_{C^{-1}(B) \cap [0, 1]^k} \|z\|^\alpha \Gamma_0(T^{-1}(dz)) \\ &= C_\alpha \int_{\mathbb{S}^{k-1}} I(B) \left(\frac{T(u)}{\|T(u)\|} \right) \|T(u)\|^\alpha \Gamma_0(du) \\ &= C_\alpha \Gamma_1(B), \end{aligned}$$

as $r \rightarrow \infty$, which completes the proof that $[\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j$ belongs to the domain of attraction of an α -stable law with spectral measure Γ_1 . Then, for every k -dimensional vector s ,

$$\frac{1}{m^{1/\alpha}} \sum_{i=1}^m \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0),$$

as a sequence of random variables in m , converges in distribution, as $m \rightarrow +\infty$, to a random variable with α -stable distribution and characteristic function

$$\exp \left(-|t|^\alpha \mathbb{E}_{u \sim \Gamma_0} \left(\left| \sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0) \right|^\alpha \right) \right).$$

Thus, the distribution of $\sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)$ belongs to the domain of attraction of an α -stable law. In particular, this implies that as $m \rightarrow +\infty$

$$\begin{aligned} & r^\alpha \mathbb{P} \left(\left| \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) \right| > r \right) \\ & \rightarrow C_\alpha \mathbb{E}_{u \sim \Gamma_0} \left(\left| \sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0) \right|^\alpha \right). \end{aligned}$$

By [Cline \(1986, Theorem 4\)](#) with $\beta = \gamma = 0$,

$$\begin{aligned} & \mathbb{P}\left(|w_i| \mid \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) \mid > e^t\right) \\ & \sim C_\alpha^2 \mathbb{E}_{u \sim \Gamma_0}(|\sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0)|^\alpha) \alpha t e^{-\alpha t} \end{aligned}$$

as $t \rightarrow \infty$. Thus, for $r \rightarrow \infty$,

$$\begin{aligned} & r^\alpha \mathbb{P}\left(|w_i| \mid \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) \mid > r\right) \\ & \sim C_\alpha^2 \mathbb{E}_{u \sim \Gamma_0}(|\sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0)|^\alpha) \alpha \log r. \end{aligned}$$

Let $\tilde{L}(r) = C_\alpha^2 \mathbb{E}_{u \sim \Gamma_0}(|\sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0)|^\alpha) \alpha \log r$. Since the distribution of $w_i \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)$ is symmetric, then we can write that

$$\frac{1}{a_m} \sum_{i=1}^m w_i \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0),$$

as a sequence of random variables in m , converges in distribution, as $m \rightarrow +\infty$, to a random variable with symmetric α -stable law with scale 1 provided $(a_m)_{m \geq 1}$ satisfies

$$\frac{m \tilde{L}(a_m)}{a_m^\alpha} \rightarrow C_\alpha$$

as $m \rightarrow \infty$. The condition is satisfied if

$$a_m = \left(C_\alpha \mathbb{E}_{u \sim \Gamma_0}(|\sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0)|^\alpha) m \log m \right)^{1/\alpha}.$$

It follows that

$$\frac{1}{(m \log m)^{1/\alpha}} \sum_{i=1}^m w_i \sum_{j=1}^k s_j \langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0),$$

as a sequence of random variables in m , converges in distribution, as $m \rightarrow +\infty$, to a random variable with symmetric α -stable distribution with scale of the form

$$\left(C_\alpha \mathbb{E}_{u \sim \Gamma_0}(|\sum_{j=1}^k s_j \langle u, x_j \rangle I(\langle u, x_j \rangle > 0)|^\alpha) \right)^{1/\alpha}.$$

Since this holds for every vector s , then

$$\frac{1}{(m \log m)^{1/\alpha}} \sum_{i=1}^m w_i [\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0)]_j,$$

as a sequence of random variables in m , converges in distribution, as $m \rightarrow +\infty$, to a random vector with symmetric α -stable law with the spectral measure

$$\Gamma_X = \frac{1}{2} C_\alpha \mathbb{E}_{u \sim \Gamma_0} \left(\| [\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j \|^\alpha \right)$$

$$\delta\left(\frac{[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j}{\|[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j\|}\right) + \delta\left(-\frac{[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j}{\|[\langle u, x_j \rangle I(\langle u, x_j \rangle > 0)]_j\|}\right).$$

Since $\Gamma_0 = \frac{1}{2} \sum_{i=1}^d (\delta(e_i) + \delta(-e_i))$, where $e_{ij} = 1$ if $j = i$ and 0 otherwise, then

$$\begin{aligned} \Gamma_X = \frac{1}{4} \sum_{i=1}^d & \left(\| [x_{ji} I(x_{ji} > 0)]_j \|^\alpha \left(\delta\left(\frac{[x_{ji} I(x_{ji} > 0)]_j}{\| [x_{ji} I(x_{ji} > 0)]_j \|}\right) + \delta\left(-\frac{[x_{ji} I(x_{ji} > 0)]_j}{\| [x_{ji} I(x_{ji} > 0)]_j \|}\right) \right) \right. \\ & \left. + \| [x_{ji} I(x_{ji} < 0)]_j \|^\alpha \left(\delta\left(\frac{[x_{ji} I(x_{ji} < 0)]_j}{\| [x_{ji} I(x_{ji} < 0)]_j \|}\right) + \delta\left(-\frac{[x_{ji} I(x_{ji} < 0)]_j}{\| [x_{ji} I(x_{ji} < 0)]_j \|}\right) \right) \right). \end{aligned}$$

B

Throughout this section, it is assumed that all the random variables are defined on a common probability space, say $(\Omega, \mathcal{F}, \mathbb{P})$, unless otherwise stated.

B.1 Proof of Lemma 3

The proof follows from by a direct application of results in [Cline \(1986\)](#). In particular, by [Cline \(1986, Lemma 1\)](#), as $m \rightarrow +\infty$

$$\tilde{H}_m^{(1)} \xrightarrow{w} \tilde{H}_1^*(\alpha),$$

where $\tilde{H}_1^*(\alpha)$ is an $(\alpha/2)$ -Stable random matrix with spectral measure Γ_1^* of the form

$$\Gamma_1^* = C_{\alpha/2} \mathbb{E} \left(\|G(w_1^{(0)})\|_F^{\alpha/2} \delta_{G(w_1^{(0)})/\|G(w_1^{(0)})\|_F} \right) \quad (17)$$

such that $G(w_1^{(0)})$ denotes the Gram matrix of the vector $(x_1 I(\langle w_1^{(0)}, x_1 \rangle > 0), \dots, x_k I(\langle w_1^{(0)}, x_k \rangle > 0))$, and $C_{\alpha/2}$ is the constant defined in (11), for $\alpha \in (0, 2]$. We will now prove that $\tilde{H}_1^*(\alpha)$ is non-negative definite. By definition, $\tilde{H}_m^{(1)}(\omega)$ is non-negative definite for every ω and every m . By Portmanteau Theorem, for every vector $u \in \mathbb{S}^{k-1}$,

$$\mathbb{P} \left(u^T \tilde{H}_1^*(\alpha) u \geq 0 \right) \geq \limsup_m \mathbb{P} \left(u^T \tilde{H}_m^{(1)} u \geq 0 \right) = 1.$$

Let \mathcal{A} be a countable dense subset of \mathbb{S}^{k-1} . Then, with probability one, $a^T \tilde{H}_1^*(\alpha) a \geq 0$ for every $a \in \mathcal{A}$. By continuity, this implies that the same property holds true with probability one for every $u \in \mathbb{S}^{k-1}$, which proves that $\tilde{H}_1^*(\alpha)$ is almost surely non-negative definite. By eventually modifying $\tilde{H}_1^*(\alpha)$ on a null set, we obtain a non-negative definite random matrix.

B.2 Proof of Lemma 4

Consider a random vector Z defined as follows

$$Z = \begin{pmatrix} \langle w_i^{(0)}, x_1 \rangle I(\langle w_i^{(0)}, x_1 \rangle > 0) \\ \vdots \\ \langle w_i^{(0)}, x_k \rangle I(\langle w_i^{(0)}, x_k \rangle > 0) \end{pmatrix}.$$

Then Z belongs to the domain of attraction of an α -stable distribution with some Lévy measure ν . Let $g : \mathbb{R}^k \rightarrow \mathbb{R}^{k^2}$ be defined as $g(z) = zz^T$. By the connection of the spectral measure of an α -stable distribution with its Lévy measure (see [Appendix D](#)), we can say that, for every Borel set $B \subset \overline{\mathbb{R}^{k^2}} \setminus \{0\}$, such that $\nu(\partial g^{-1}(B)) = 0$,

$$m \mathbb{P} \left(\frac{1}{m^{2/\alpha}} Z Z^T \in B \right)$$

$$= m\mathbb{P}\left(\frac{1}{m^{1/\alpha}}Z \in g^{-1}(B)\right) \rightarrow \nu(g^{-1}(B)).$$

By the properties of multivariate Stable laws (see Appendix D), as $m \rightarrow +\infty$

$$\tilde{H}_m^{(2)} \xrightarrow{w} \tilde{H}_2^*(\alpha),$$

where $\tilde{H}_2^*(\alpha)$ denotes a random matrix with $\alpha/2$ -stable distribution and Lévy measure $\mu(\cdot) = \nu(g^{-1}(\cdot))$, that is an $(\alpha/2)$ -Stable random matrix. To prove that $\tilde{H}_2^*(\alpha)$ is non-negative definite, we can proceed as in the proof of Lemma 3, with $\tilde{H}_2^*(\alpha)$ and $\tilde{H}_m^{(2)}$ in the place of $\tilde{H}_1^*(\alpha)$ and of $\tilde{H}_m^{(1)}$, respectively.

B.3 Proof of Theorem 5

We start by introducing two random matrices, say $Z^{(1)}$ and $Z^{(2)}$, such that their (j, j') entries are defined as follows

$$w_i^2 \langle x_j, x_{j'} \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) I(\langle w_i^{(0)}, x_{j'} \rangle > 0)$$

and

$$\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) \langle w_i^{(0)}, x_{j'} \rangle I(\langle w_i^{(0)}, x_{j'} \rangle > 0),$$

respectively. We observe that both $Z^{(1)}$ and $Z^{(2)}$ belong to the domain of attraction of $\alpha/2$ -stable distributions. Accordingly it is sufficient to show that there exists a measure ν such that, for every A such that $\nu(\partial A) = 0$, as $m \rightarrow +\infty$

$$m\mathbb{P}\left(\left(\frac{Z^{(1)}}{m^{1/\alpha}}, \frac{Z^{(2)}}{m^{1/\alpha}}\right) \in A\right) \rightarrow \nu(A).$$

See Cline (1988). We can write

$$\begin{aligned} & m\mathbb{P}\left(\left(\frac{Z^{(1)}}{m^{1/\alpha}}, \frac{Z^{(2)}}{m^{1/\alpha}}\right) \in A\right) \\ &= m \sum_{u_1, \dots, u_k \in \{0,1\}} \mathbb{P}\left(\left(\left(\frac{Z^{(1)}}{m^{1/\alpha}}, \frac{Z^{(2)}}{m^{1/\alpha}}\right) \in A\right) \cap A_1^{u_1} \cap \dots \cap A_k^{u_k}\right), \end{aligned}$$

where we set $A_j = (\langle w_i^{(0)}, x_j \rangle > 0)$ and $A_j^1 = A_j$, $A_j^0 = A_j^c$. Since as $m \rightarrow +\infty$

$$\begin{aligned} & m\mathbb{P}\left(\left(\left(\frac{Z^{(1)}}{m^{1/\alpha}}, \frac{Z^{(2)}}{m^{1/\alpha}}\right) \in A\right) \cap A_1^{u_1} \cap \dots \cap A_k^{u_k}\right) \\ &= m\mathbb{P}\left(\left(\left(\frac{1}{m^{1/\alpha}}[\langle x_j, x_{j'} \rangle w_i^2 u_j u_{j'}]_{j,j'}, \frac{1}{m^{1/\alpha}}[\langle w_i^{(0)}, x_j \rangle \langle w_i^{(0)}, x_{j'} \rangle u_j u_{j'}]\right) \in A\right) \cap A_1^{u_1} \cap \dots \cap A_k^{u_k}\right) \end{aligned}$$

converges to $(\nu_{u_1, \dots, u_k}^{(1)} \times \nu_{u_1, \dots, u_k}^{(2)})(A_{u_1, \dots, u_k})$, for some set A_{u_1, \dots, u_k} , where we defined

$$\nu_{u_1, \dots, u_k}^{(1)}(B) = \lim_{m \rightarrow \infty} \sqrt{m} \mathbb{P}\left(\frac{1}{m^{1/\alpha}}[\langle x_j, x_{j'} \rangle w_i^2 u_j u_{j'}]_{j,j'} \in B\right)$$

and

$$\nu_{u_1, \dots, u_k}^{(2)}(B) = \lim_{m \rightarrow \infty} \sqrt{m} \mathbb{P}\left(\left(\left(\frac{1}{m^{1/\alpha}}[\langle w_i^{(0)}, x_j \rangle \langle w_i^{(0)}, x_{j'} \rangle u_j u_{j'}]\right) \in B\right) \cap A_1^{u_1} \cap \dots \cap A_k^{u_k}\right),$$

then the random matrices $Z^{(1)}$ and $Z^{(2)}$ belong to the joint domain of attraction of an $(\alpha/2)$ -Stable distribution. Therefore, it follows that the random matrix $Z^{(1)} + Z^{(2)}$ is in the domain of attraction of and $\alpha/2$ -Stable distribution. To prove that $\tilde{H}^*(X, X; \alpha)$ is non-negative definite, we can proceed as in the proof of Lemma 3, with $\tilde{H}_1^*(\alpha) + \tilde{H}_2^*(\alpha)$ and $\tilde{H}_m^{(1)} + \tilde{H}_m^{(2)}$ in the place of $\tilde{H}_1^*(\alpha)$ and of $\tilde{H}_m^{(1)}$, respectively.

B.4 Spectral measure Γ^*

From Theorem 5, the distribution of the limiting random matrix $\tilde{H}^*(X, X; \alpha)$ is an $\alpha/2$ -Stable distribution on the space of non-negative symmetric matrices. Now, we find the spectral measure Γ^* of the distribution of $\tilde{H}^*(X, X; \alpha)$. First, observe that we can write $\tilde{H}_m(W(0), X)$ as

$$\tilde{H}_m(W(0), X) = \frac{1}{m^{2/\alpha}} \sum_{i=1}^m \left(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X \right) \circ U_i U_i^T,$$

where $U_i = [I(\langle w_i^{(0)}, x_j \rangle > 0)]_j$, and \circ denotes the element-wise multiplication. By the properties of the multivariate stable distribution, we can write

$$\begin{aligned} C_{\alpha/2} \Gamma^*(\cdot) &= \lim_{R \rightarrow \infty} R^{\alpha/2} \mathbb{P} \left(\|(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X) \circ U_i U_i^T\|_F > R, \right. \\ &\quad \left. \frac{(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X) \circ U_i U_i^T}{\|(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X) \circ U_i U_i^T\|_F} \in \cdot \right) \\ &= \sum_u \lim_{R \rightarrow \infty} R^{\alpha/2} \mathbb{P} \left(\|(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X) \circ uu^T\|_F > R, \right. \\ &\quad \left. \frac{(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X) \circ uu^T}{\|(w_i^2 X^T X + X^T w_i^{(0)T} w_i^{(0)} X) \circ uu^T\|_F} \in \cdot \mid U_i = u \right) \mathbb{P}(U_i = u), \end{aligned}$$

where u denotes any vector of zeroes and ones. Thus we can write Γ^* as follows

$$\Gamma^*(dm) = \sum_u \left(\prod_{u_j u_{j'} = 0} \delta_0(dm_{jj'}) \right) \times \Gamma_u^* \left(\prod_{u_j, u_{j'} \neq 0} dm_{j,j'} \right) \mathbb{P}(U = u)$$

where Γ_u^* is a measure on the unit sphere in the space of symmetric, non-negative definite matrices of dimension equal to the number of non-zero elements in u , with the metric of Frobenius norm. Moreover, the probability $\mathbb{P}(U = u)$ provides the probability that $w_i^{(0)}$ takes values in the set

$$\{w \in \mathbb{R}^d : \langle w, x_j \rangle > 0 \text{ if } u_j = 1, \langle w, x_j \rangle < 0 \text{ if } u_j = 0, j = 1, \dots, k\}$$

Now, we find Γ_u^* for a fixed u . This is the spectral measure of the $\alpha/2$ -stable distribution

$$X_u^T X_u \lim_{m \rightarrow \infty} \frac{1}{m^{2/\alpha}} \sum_{i=1}^m w_i^2 + X_u^T \left(\lim_{m \rightarrow \infty} \frac{1}{m^{2/\alpha}} \sum_{i=1}^m (w_{ui}^{(0)})^T w_{ui}^{(0)} \right) X_u,$$

where: i) X_u is the restriction of X to the columns j such that $u_j = 1$; ii) $w_{ui}^{(0)}$ is independent of w_i and its distribution coincides with the conditional distribution, given that $U_i = u$, of the coordinates j of $w_i^{(0)}$ corresponding to $u_j = 1$. The limit is the sum of the independent random matrices:

$$X_u^T X_u Z$$

and

$$X_u^T M_u X_u,$$

where Z is a random variable with $\alpha/2$ -Stable distribution and M_u is a random matrix with $\alpha/2$ -Stable distribution. In particular, the spectral measure $\underline{\Gamma}_u$ of M_u satisfies

$$C_{\alpha/2} \underline{\Gamma}_u(\cdot) = \lim_{R \rightarrow \infty} R^{\alpha/2} \mathbb{P} \left(\frac{(w_{ui}^{(0)})^T w_{ui}^{(0)}}{\|(w_{ui}^{(0)})^T w_{ui}^{(0)}\|_F} \in \cdot, \|(w_{ui}^{(0)})^T w_{ui}^{(0)}\|_F > R \right).$$

Notice that $\|(w_{ui}^{(0)})^T w_{ui}^{(0)}\|_F = \|w_{ui}^{(0)}\|_F^2$. The spectral measure $\underline{\Gamma}_u$ is concentrated on the space of matrices that can be written as $a^T a$ for some row vector a with $\|a\|_F = 1$. Moreover,

$$\begin{aligned} C_{\alpha/2} \underline{\Gamma}_u(\{a^T a : a \in B\}) &= \lim_{R \rightarrow \infty} R^{\alpha/2} \mathbb{P} \left(\frac{w_{ui}^{(0)}}{\|w_{ui}^{(0)}\|_F} \in B, \|w_{ui}^{(0)}\|_F > \sqrt{R} \right) \\ &= \lim_{R \rightarrow \infty} R^{\alpha} \mathbb{P} \left(\frac{w_{ui}^{(0)}}{\|w_{ui}^{(0)}\|_F} \in B, \|w_{ui}^{(0)}\|_F > R \right), \end{aligned}$$

determining the spectral measure Γ^* .

B.5 Proof of Lemma 6

From Nolan (2010), it is sufficient to show that

$$\inf_{s \in \mathbb{S}_0^{k^2-1}} \int |\langle s, u \rangle|^{\alpha/2} \Gamma_1^*(du) \neq 0,$$

where Γ_1^* is the spectral measure (12), $\mathbb{S}_0^{k^2-1}$ is the unit sphere in the space of the $k \times k$ symmetric matrices such that $s_{j,j'} = 0$ if $\langle x_j, x_{j'} \rangle = 0$, with the Frobenius metric. Now, since

$$\begin{aligned} &\int |\langle s, u \rangle|^{\alpha/2} \Gamma_1^*(du) \\ &= C_{\alpha/2} \mathbb{E} \left(\left| \sum_{j,j'} s_{j,j'} \langle x_j, x_{j'} \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) I(\langle w_i^{(0)}, x_{j'} \rangle > 0) \right|^{\alpha/2} \right) \end{aligned}$$

is a continuous function of s that takes value in a compact set, then the minimum is attained. Thus it is sufficient to show that for every $s \in \mathbb{S}_0^{k^2-1}$,

$$\mathbb{E} \left(\left| \sum_{j,j'} s_{j,j'} \langle x_j, x_{j'} \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) I(\langle w_i^{(0)}, x_{j'} \rangle > 0) \right|^{\alpha/2} \right) \neq 0.$$

For every j and every $u_j \in \{0, 1\}$, let $A_j^{u_j}$ be the event $(\langle w_i^{(0)}, x_j \rangle > 0)$ if $u_j = 1$ and its complement if $u_j = 0$. Then

$$\begin{aligned} &\mathbb{E} \left(\left| \sum_{j,j'} s_{j,j'} \langle x_j, x_{j'} \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) I(\langle w_i^{(0)}, x_{j'} \rangle > 0) \right|^{\alpha/2} \right) \\ &= \sum_{u_1, \dots, u_k} \mathbb{P}(A_1^{u_1} \cap \dots \cap A_k^{u_k}) \left| \sum_{j,j'} u_j u_{j'} s_{j,j'} \langle x_j, x_{j'} \rangle \right|^{\alpha/2}. \end{aligned}$$

Since x_1, \dots, x_k are linearly independent, then for every u_1, \dots, u_k , $\mathbb{P}(A_1^{u_1} \cap \dots, A_k^{u_k}) > 0$. To prove it, assume, without loss of generality, that $u_i = 1$ for every i . Since x_1, \dots, x_k are linearly independent, then we can complete the matrix $X = [x_1 \dots x_k]$ by adding $k - d$ columns in such a way that the completed matrix \tilde{X} is non-singular. For every d -dimensional vector v such that $v_1 > 0, \dots, v_k > 0$ there exists a vector u such that $u = (\tilde{X}^T)^{-1}v$. Thus,

$$\{u \in \mathbb{R}^d : \langle u, x_1 \rangle > 0, \dots, \langle u, x_k \rangle > 0\} = \{(\tilde{X}^T)^{-1}v : v_1 > 0, \dots, v_k > 0\}$$

is an open non-empty set. Since $w_i^{(0)}$ has independent and identically distributed components, with stable distribution, then

$$\mathbb{P}(w_i^{(0)} \in \{(\tilde{X})^{-1}v : v_1 > 0, \dots, v_k > 0\}) > 0.$$

This concludes the proof that $\mathbb{P}(A_1^{u_1} \cap \dots, A_k^{u_k}) > 0$ for every $(u_1, \dots, u_k) \in \{0, 1\}^k$. It follows that $\int |\langle s, u \rangle|^{\alpha/2} \Gamma_1^*(du)$ is zero if and only if, for every $(u_1, \dots, u_k) \in \{0, 1\}^k$, it holds

$$\sum_{j, j'} u_j, u_{j'} \langle x_j, x_{j'} \rangle s_{j, j'} = 0.$$

The only solution of the above system of equations in the space of symmetric matrices s such that $s_{j, j'} = 0$ if $\langle x_j, x_{j'} \rangle = 0$ is $s = 0$, which is not consistent with $\|s\|_F = 1$.

B.6 Proof of Lemma 7

Since the distribution of $\tilde{H}_1^*(\alpha)$ is absolutely continuous in the space of symmetric non-negative definite matrices with zero entries in the positions j, j' such that $\langle x_j, x_{j'} \rangle = 0$, and since this space contains all the symmetric non-negative definite matrices with non-zero diagonal entries, then we can write that $\mathbb{P}(\det(\tilde{H}_1^*(\alpha)) = 0) = 0$. Moreover, since $\tilde{H}_1^*(\alpha)$ is non-negative definite, then $\mathbb{P}(\lambda_{\min}(\tilde{H}_1^*(\alpha)) > 0) = 1$. Thus, for every $\delta > 0$, the exists $\lambda_0 > 0$ such that $\mathbb{P}(\lambda_{\min}(\tilde{H}_1^*(\alpha)) > \lambda_0) > 1 - \delta$.

B.7 Proof of Theorem 8

The random matrix $\tilde{H}_m(W(0), X)$ is defined as the sum of the two non-negative random matrices $\tilde{H}_m^{(1)}$ and $\tilde{H}_m^{(2)}$. Thus the minimum eigenvalue is greater than or equal to the minimum of the eigenvalues of the two matrices. As $m \rightarrow +\infty$, the matrix $\tilde{H}_m^{(1)}$ converges in distribution to the random matrix $\tilde{H}_1^*(\alpha)$. Let $\delta > 0$ be a fixed number. By Lemma 7, there exists λ_0 such that $\mathbb{P}(\lambda_{\min}(\tilde{H}_1^*(\alpha)) > \lambda_0) \geq 1 - \delta$. Since the minimum eigenvalue map is continuous with respect to Frobenius norm then, by Portmanteau theorem, for m sufficiently large,

$$\liminf_m \mathbb{P}(\lambda_{\min}(\tilde{H}_m^{(1)}(W(0), X)) > \lambda_0) \geq \mathbb{P}(\lambda_{\min}(\tilde{H}_1^*(\alpha)) > \lambda_0) \geq 1 - \delta.$$

Hence,

$$\begin{aligned} & \liminf_m \mathbb{P}(\lambda_{\min}(\tilde{H}_m(W(0), X)) > \lambda_0) \\ & \geq \liminf_m \mathbb{P}(\lambda_{\min}(\tilde{H}_m^{(1)}(W(0), X)) > \lambda_0) \geq 1 - \delta, \end{aligned}$$

thus completing the proof.

B.8 Proof of Lemma 9

The following decomposition holds:

$$\begin{aligned} & \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F^2 \\ & = \left\| \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W(0), x_j; \alpha) \right\|_F^2 \\ & \quad + \left\| \frac{\partial \tilde{f}_m}{\partial w}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial w}(W(0), x_j; \alpha) \right\|_F^2. \end{aligned}$$

Let $\delta > 0$ be a fixed number. We will prove the following facts:

- i) If m is sufficiently large, then $(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W(0), x_j; \alpha) \right\|_F^2 \leq \frac{\epsilon}{2} m^{-2\gamma/\alpha}$ for every W such that $\|W - W(0)\|_F^2 \leq \log m$;

ii) With probability at least $1 - \delta$, $(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial w}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial w}(W(0), x_j; \alpha) \right\|_F^2 < \frac{c}{2} m^{-2\gamma/\alpha}$ for every W such that $\|W - W(0)\|_F^2 \leq \log m$, if m is sufficiently large.

Proof of i).

For a fixed $W(0)$, let W be such that $\|W - W(0)\|_F^2 \leq \log m$. Then it holds $\|w^{(0)} - w^{(0)}(0)\|_F^2 \leq \|W - W(0)\|_F^2 \leq \log m$. Accordingly, we can write the following

$$\begin{aligned} & (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial w}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial w}(W(0), x_j; \alpha) \right\|_F^2 \\ & \leq \frac{1}{m^{2/\alpha}} \sum_{i=1}^m \left(\langle w_i^{(0)}, x_j \rangle I(\langle w_i^{(0)}, x_j \rangle > 0) - \langle w_i^{(0)}(0), x_j \rangle I(\langle w_i^{(0)}(0), x_j \rangle > 0) \right)^2 \\ & \leq \frac{2}{m^{2/\alpha}} \sum_{i=1}^m \left(\langle w_i^{(0)}, x_j \rangle - \langle w_i^{(0)}(0), x_j \rangle \right)^2 I(\langle w_i^{(0)}, x_j \rangle > 0) \\ & \quad + \frac{2}{m^{2/\alpha}} \sum_{i=1}^m \langle w_i^{(0)}(0), x_j \rangle^2 \left(I(\langle w_i^{(0)}, x_j \rangle > 0) - I(\langle w_i^{(0)}(0), x_j \rangle > 0) \right)^2. \end{aligned}$$

We will bound the two terms of the sum separately. First, we define $r_i = |\langle w_i^{(0)} - w_i^{(0)}(0), x_j \rangle|$ for $i = 1, \dots, m$. Then, we can write that

$$\sum_{i=1}^m r_i^2 \leq \sum_{i=1}^m \|w_i^{(0)} - w_i^{(0)}(0)\|^2 \cdot \|x_j\|^2 \leq \|w^{(0)} - w^{(0)}(0)\|_F^2 \leq \log m.$$

Since $\gamma < 1$,

$$\begin{aligned} & \frac{2}{m^{2/\alpha}} \sum_{i=1}^m \left(\langle w_i^{(0)}, x_j \rangle - \langle w_i^{(0)}(0), x_j \rangle \right)^2 I(\langle w_i^{(0)}, x_j \rangle > 0) \\ & \leq 2m^{-2/\alpha} \log m < \frac{c}{4} m^{-2\gamma/\alpha}, \end{aligned}$$

for m sufficiently large. In order to bound the second term, we observe that the following set

$$\{w^{(0)}(0) : \exists w^{(0)} \text{ s.t. } |\langle w_i^{(0)} - w_i^{(0)}(0), x_j \rangle| = r_i, I(\langle w^{(0)}, x_j \rangle > 0) \neq I(\langle w^{(0)}(0), x_j \rangle > 0)\}$$

is included in the set $\{w_i^{(0)}(0) : |\langle w_i^{(0)}(0), x_j \rangle| \leq r_i\}$. Therefore, we can write that

$$\begin{aligned} & \sup_{\sum_i r_i^2 \leq \log m} \sup_{|w_i^{(0)} - w_i^{(0)}(0)| \leq r_i} \frac{2}{m^{2/\alpha}} \sum_{i=1}^m \langle w_i^{(0)}(0), x_j \rangle^2 \left(I(\langle w_i^{(0)}, x_j \rangle > 0) - I(\langle w_i^{(0)}(0), x_j \rangle > 0) \right)^2 \\ & \leq \sup_{\sum_i r_i^2 \leq \log m} \sup_{|w_i^{(0)} - w_i^{(0)}(0)| \leq r_i} \frac{2}{m^{2/\alpha}} \sum_{i=1}^m \langle w_i^{(0)}(0), x_j \rangle^2 I(\langle w_i^{(0)}(0), x_j \rangle < r_i) \\ & \leq \sup_{\sum_i r_i^2 \leq \log m} \sup_{|w_i^{(0)} - w_i^{(0)}(0)| \leq r_i} \frac{2}{m^{2/\alpha}} \sum_{i=1}^m r_i^2 \\ & \leq \frac{1}{m^{2/\alpha}} \log m < \frac{c}{4} m^{-2\gamma/\alpha}, \end{aligned}$$

for m sufficiently large.

Proof of ii).

We define $r_i = |\langle w_i^{(0)} - w_i^{(0)}(0), x_j \rangle|$ for $i = 1, \dots, m$. Now, since $\|x_j\| = 1$ by assumption, for $j = 1, \dots, k$, then we can write

$$\sum_i r_i^2 \leq \|x_j\|^2 \cdot \|w_i^{(0)} - w^{(0)}(0)\|_F^2 \leq \|W - W(0)\|_F^2 \leq \log m.$$

It holds

$$\begin{aligned}
& (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W(0), x_j; \alpha) \right\|_F^2 \\
& \leq \frac{1}{m^{2/\alpha}} \sum_{i=1}^m \left(w_i I(\langle w_i^{(0)}, x_j \rangle > 0) - w_i(0) I(\langle w_i^{(0)}(0), x_j \rangle > 0) \right)^2 \\
& \leq \frac{2}{m^{2/\alpha}} \sum_{i=1}^m (w_i - w_i(0))^2 I(\langle w_i^{(0)}, x_j \rangle > 0) \\
& \quad + \frac{2}{m^{2/\alpha}} \sum_{i=1}^m w_i(0)^2 |I(\langle w_i^{(0)}, x_j \rangle > 0) - I(\langle w_i^{(0)}(0), x_j \rangle > 0)|.
\end{aligned}$$

We will bound the tow terms separately. First,

$$\begin{aligned}
& \frac{2}{m^{2/\alpha}} \sum_{i=1}^m (w_i - w_i(0))^2 I(\langle w_i^{(0)}, x_j \rangle > 0) \\
& \leq \frac{1}{m^{2/\alpha}} \sum_{i=1}^m (w_i - w_i(0))^2 \\
& \leq \frac{2}{m^{2(1-\gamma)/\alpha}} \|w - w(0)\|_F^2 \\
& \leq \frac{2}{m^{2/\alpha}} \log m < \frac{c}{4} m^{-2\gamma/\alpha},
\end{aligned}$$

if m is sufficiently large. In order to bound the second term, we observe that the following set

$$\{w^{(0)}(0) : \exists w^{(0)} \text{ s.t. } |\langle w_i^{(0)} - w_i^{(0)}(0), x_j \rangle| = r_i, I(\langle w^{(0)}, x_j \rangle > 0) \neq I(\langle w^{(0)}(0), x_j \rangle > 0)\}$$

is included in the set $\{w_i^{(0)}(0) : |\langle w_i^{(0)}(0), x_j \rangle| \leq r_i\}$. Therefore, we can write that

$$\begin{aligned}
& \sup_{\sum_i r_i^2 \leq \log m} \sup_{|w_i^{(0)} - w_i^{(0)}(0)| \leq r_i} \frac{2}{m^{2/\alpha}} \sum_{i=1}^m w_i(0)^2 |I(\langle w_i^{(0)}, x_j \rangle > 0) - I(\langle w_i^{(0)}(0), x_j \rangle > 0)| \\
& \leq \sup_{\sum_i r_i^2 \leq \log m} \frac{2}{m^{2/\alpha}} \sum_{i=1}^m w_i(0)^2 I(|\langle w_i^{(0)}(0), x_j \rangle| \leq r_i).
\end{aligned}$$

Let $N_m(r) = \sum_{i=1}^m I(|\langle w_i^{(0)}(0), x_j \rangle| \leq r_i)$ and $\bar{N}_m = \sup_{\sum_i r_i^2 \leq \log m} N_m(r)$. Then

$$\sup_{\sum_i r_i^2 \leq \log m} \frac{2}{m^{2/\alpha}} \sum_{i=1}^m w_i(0)^2 I(|\langle w_i^{(0)}(0), x_j \rangle| \leq r_i)$$

has the same distribution as

$$\frac{2}{m^{2/\alpha}} \sum_{i=1}^{\bar{N}_m} w_i(0)^2 = \left(\frac{\bar{N}_m}{m} \right)^{2/\alpha} \frac{2}{\bar{N}_m^{2/\alpha}} \sum_{i=1}^{\bar{N}_m} w_i(0)^2$$

Since the sequence \bar{N}_m is non decreasing, then $\bar{N}_m^{-2/\alpha} \sum_{i=1}^{\bar{N}_m} w_i(0)^2$ converges in distribution. Therefore there exist M and m_0 such that, for every $m \geq m_0$

$$\mathbb{P} \left(\frac{2}{\bar{N}_m^{2/\alpha}} \sum_{i=1}^{\bar{N}_m} w_i(0)^2 \geq M \right) < \frac{\delta}{2}.$$

The proof will be completed by showing that, for m sufficiently large

$$\mathbb{P}\left(\left(\frac{\bar{N}_m}{m}\right)^{2/\alpha} < \frac{c}{4Mm^{2\gamma/\alpha}}\right) > 1 - \frac{\delta}{2}.$$

For a fixed m , the set $\{(r_1, \dots, r_m) : \sum_{i=1}^m r_i^2 \leq \log m\}$ is compact. Hence there exists a sequence $(r^{(m)})_m$ of random vectors of increasing dimensions $(m)_m$ such that $\bar{N}_m = N_m(r^{(m)})$ and $\sum_{i=1}^m (r_i^{(m)})^2 \leq \log m$. We can write

$$\begin{aligned} & \mathbb{P}\left(\frac{\bar{N}_m}{m} \geq \frac{c^{\alpha/2}}{(4M)^{\alpha/2}}\right) \\ &= \mathbb{P}\left(\frac{1}{m^{1-\gamma}} \sum_{i=1}^m I(|\langle w_i^{(0)}(0), x_j \rangle| \leq r_i^{(m)}) \geq \frac{c^{\alpha/2}}{(4M)^{\alpha/2}}\right) \\ &\leq \frac{(4M)^{\alpha/2}}{c^{\alpha/2}m^{1-\gamma}} \sum_{i=1}^m \mathbb{E}(\mathbb{P}(\langle w_i^{(0)}(0), x_j \rangle^2 \leq (r_i^{(m)})^2 \mid r^{(m)})) \\ &\leq \frac{(4M)^{\alpha/2}}{c^{\alpha/2}m^{1-\gamma}} \mathbb{E}\left(\sum_{i=1}^m F((r_i^{(m)})^2)\right), \end{aligned}$$

where we denoted by F the distribution function of the random variable $\langle w_i^{(0)}(0), x_j \rangle^2$. Now, let us find $\max_{\sum_{i=1}^m r_i^2 \leq \log m} \sum_{i=1}^m F(r_i^2)$. First, solving, for $a > 0$

$$\frac{\partial}{\partial r_i^2} \left(\sum_{j=1}^m F(r_j^2) - c \left(\sum_{j=1}^m r_j^2 - a \right) \right) = 0,$$

subject to the constraint $\sum_{j=1}^m r_j^2 = a$, we find $r_i^2 = a/m$ for $i=1, \dots, m$. The function $\sum_{i=1}^m F(a/m)$ is an increasing function of a . Therefore, with respect to a , the maximum is in $r_j^2 = \log m/m$. Finally, by comparing the value of $\sum_{j=1}^m F(r_j^2)$ for $r_j^2 = \log m/m$ with the values on the boundary of the set

$$\left\{ \sum_{i=1}^m r_i^2 \leq \log m \right\}$$

we find that, for m large, the maximum is attained in $r_1^2 = \dots = r_m^2 = \log m/m$. Thus,

$$\begin{aligned} \frac{(4M)^{\alpha/2}}{c^{\alpha/2}m^{1-\gamma}} \mathbb{E}\left(\sum_{i=1}^m F((r_i^{(m)})^2)\right) &\leq \frac{(4M)^{\alpha/2}}{c^{\alpha/2}m^{1-\gamma}} \sum_{i=1}^m F\left(\frac{\log m}{m}\right) \\ &\leq \frac{(4M)^{\alpha/2}}{c^{\alpha/2}} m^\gamma F\left(\frac{\log m}{m}\right). \end{aligned}$$

Since $w_i^{(0)}(0)$ has stable distribution, then F is absolutely continuous. Denoting by g the symmetric α -stable density function with scale parameter $\|x_j\|_\alpha$, we can write

$$m^\gamma F\left(\frac{\log m}{m}\right) = 2m^\gamma \int_0^{(\log m/m)^{1/2}} g(u) du < \frac{c^{\alpha/2}}{(4M)^{\alpha/2}}$$

for m sufficiently large, since $\gamma < 1/2$ and g is bounded.

B.9 Proof of Lemma 10

We start by observing that we can write that

$$|\tilde{H}_m(W, X)[i, j] - \tilde{H}_m(W(0), X)[i, j]|$$

$$\begin{aligned}
&= (\log m)^{2/\alpha} \left| \left\langle \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha), \frac{\partial \tilde{f}_m}{\partial W}(W, x_j; \alpha) \right\rangle - \left\langle \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha), \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\rangle \right| \\
&\leq (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha) \right\|_F \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F \\
&\quad + (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F \\
&\leq (\log m)^{2/\alpha} \left(\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F + \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha) \right\|_F \right) \\
&\quad \times \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F \\
&\quad + (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F.
\end{aligned}$$

Notice that

$$\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F^2 = \left\| \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W(0), x_i; \alpha) \right\|_F^2 + \left\| \frac{\partial \tilde{f}_m}{\partial w}(W(0), x_i; \alpha) \right\|_F^2.$$

Moreover, for every $i = 1, \dots, k$,

$$\begin{aligned}
(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial w^{(0)}}(W(0), x_i; \alpha) \right\|_F^2 &= \frac{1}{m^{2/\alpha}} \sum_{i=1}^m w_i^2 I(|\langle w_i^{(0)}(0), x_i \rangle| > 0) \\
&\leq \frac{1}{m^{2/\alpha}} \sum_{i=1}^m w_i^2,
\end{aligned}$$

which converges in distribution, as $m \rightarrow \infty$. Analogously, for every $i = 1, \dots, k$, the following quantity

$$\begin{aligned}
(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial w}(W(0), x_i; \alpha) \right\|_F^2 &= \frac{1}{m^{2/\alpha}} \sum_{i=1}^m \langle w_i^{(0)}(0), x_i \rangle^2 I(|\langle w_i^{(0)}(0), x_i \rangle| > 0) \\
&\leq \frac{1}{m^{2/\alpha}} \sum_{i=1}^m \langle w_i^{(0)}(0), x_i \rangle^2,
\end{aligned}$$

which converges in distribution, as $m \rightarrow \infty$. Thus there exist $M > 0$ and m_0 such that for every $m \geq m_0$ and every $i = 1, \dots, k$,

$$\mathbb{P} \left((\log m)^{1/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F > M \right) < \frac{\delta}{8k^2}.$$

By Lemma 9, for m sufficiently large, with probability at least $1 - \delta/(4k^2)$

$$(\log m)^{1/\alpha} \left(\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F + \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha) \right\|_F \right) < 2M$$

whenever $\|W - W(0)\|_F^2 < \log m$. Lemma 9 also implies that, for every $i = 1, \dots, k$, with probability at least $1 - \delta/(8k^2)$

$$(\log m)^{1/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_i; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F < \frac{\lambda_0}{4Mk^2} m^{-2\gamma/\alpha}$$

for m sufficiently large, whenever $\|W - W(0)\|_F^2 < \log m$. Thus, with probability at least $1 - \delta$, if m is sufficiently large

$$\max_{i,j} |\tilde{H}_m(W, X)[i, j] - \tilde{H}_m(W(0), X)[i, j]| < \frac{\lambda_0}{k^2} m^{-2\gamma/\alpha},$$

whenever $\|W - W(0)\|_F^2 < \log m$. Thus

$$\begin{aligned} & \|\tilde{H}_m(W, X) - \tilde{H}_m(W(0), X)\|_2 \\ & \leq \|\tilde{H}_m(W, X) - \tilde{H}_m(W(0), X)\|_F < \lambda_0 m^{-2\gamma/\alpha} < \frac{\lambda_0}{2} \end{aligned}$$

for m sufficiently large, whenever $\|W - W(0)\|_F^2 < \log m$. The last inequality and Theorem 8 imply that, with probability at least $1 - \delta$, if m is sufficiently large, then

$$\|\tilde{H}_m(W, X)\|_2 > \lambda_0/2,$$

for every W such that $\|W - W(0)\|_F^2 < \log m$.

B.10 Proof of Lemma 11

By Lemmas 9 and 10, there exists $N_1 \in \mathcal{F}$ with probability at least $1 - \delta/2$ such that, for every $\omega \in N_1$,

$$(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W, x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0)(\omega), x_j; \alpha) \right\|_F^2 < cm^{-2\gamma/\alpha},$$

for arbitrarily fixed $c > 0$ and $\gamma \in (0, 1/2)$, and

$$\lambda_{\min}(\tilde{H}_m(W, X)) > \frac{\lambda_0}{2},$$

for some $\lambda_0 > 0$, for every W such that $\|W - W(0)(\omega)\|_F^2 \leq \log m$ and every $j = 1, \dots, k$, provided m is sufficiently large. On the other hand, $\tilde{f}_m(W(0), x_i; \alpha)$ converge in distribution for $i = 1, \dots, k$, as $m \rightarrow \infty$. Also

$$\left\| \frac{\partial}{\partial W} \tilde{f}_m(W(0), x_i; \alpha) \right\|_F^2 = (\log m)^{-2/\alpha} \tilde{H}(W(0), X; \alpha)[i, i]$$

converges in distribution for every $i = 1, \dots, k$, as $m \rightarrow \infty$. Hence, there exist $N_2 \in \mathcal{F}$ and M with $P(N_2) > 1 - \delta/2$ such that, for every $\omega \in N_2$ and for m sufficiently large,

$$\|\tilde{f}_m(W(0)(\omega), X; \alpha) - Y\|_F < M,$$

and

$$\max_{1 \leq i \leq k} \left\| \frac{\partial}{\partial W} \tilde{f}_m(W(0)(\omega), x_i; \alpha) \right\|_F < M.$$

We will prove, by contradiction, that for every $\omega \in N_1 \cap N_2$, $\|W(t) - W(0)\|_F < \sqrt{\log m}$ for every $t > 0$. In the following we will write $W(s)$ in the place of $W(s)(\omega)$ and always assume that ω belongs to $N_1 \cap N_2$. The proof is by contradiction. Suppose that there exists t such that $\|W(t) - W(0)\|_F \geq \sqrt{\log m}$, and let

$$t_0 = \operatorname{argmin}_{t \geq 0} \{t : \|W(t) - W(0)\|_F \geq \sqrt{\log m}\}.$$

Since $\|W(s) - W(0)\|_F^2 \leq \sqrt{\log m}$ for every $s \leq t_0$, then, for every $s \leq t_0$,

$$\lambda_{\min}(\tilde{H}_m(W(s), X)) > \frac{\lambda_0}{2},$$

and

$$(\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F^2 < cm^{-2\gamma/\alpha}.$$

Let us now consider the gradient descent dynamic, with continuous learning rate $\eta = (\log m)^{2/\alpha}$:

$$\begin{aligned}\frac{dW(s)}{ds} &= -(\log m)^{2/\alpha} \nabla_W \frac{1}{2} \sum_{i=1}^k \left(\tilde{f}_m(W(s), x_i; \alpha) - y_i \right)^2 \\ &= -(\log m)^{2/\alpha} \sum_{i=1}^k \left(\tilde{f}_m(W(s), x_i) - y_i \right) \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_i; \alpha).\end{aligned}$$

This expression allows to write

$$\begin{aligned}\|W(t_0) - W(0)\|_F &\leq \left\| \int_0^{t_0} \frac{d}{ds} W(s) ds \right\|_F \\ &\leq (\log m)^{2/\alpha} \left\| \int_0^{t_0} \sum_{i=1}^k \left(\tilde{f}_m(W(s), x_i; \alpha) - y_i \right) \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_i; \alpha) ds \right\|_F \\ &\leq (\log m)^{2/\alpha} \max_{0 \leq s \leq t_0} \sum_{i=1}^k \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_i; \alpha) \right\|_F \int_0^{t_0} \|\tilde{f}_m(W(s), X; \alpha) - Y\| ds.\end{aligned}$$

To bound the term $\|\tilde{f}_m(W(s), X; \alpha) - Y\|$ we will exploit the dynamics of the NN output

$$\begin{aligned}\frac{d\tilde{f}_m(W(s), X; \alpha)}{ds} &= \frac{\partial \tilde{f}_m}{\partial W}(W(s), X; \alpha) \frac{dW^T(s)}{ds} \\ &= -(\log m)^{2/\alpha} (\tilde{f}_m(W(s), X; \alpha) - Y) H_m(W(s), X) \\ &= -(\tilde{f}_m(W(s), X; \alpha) - Y) \tilde{H}_m(W(s), X),\end{aligned}$$

that gives

$$\frac{d}{ds} \|\tilde{f}_m(W(s), X; \alpha) - Y\|_2^2 = -2 \left(\tilde{f}_m(W(s), X; \alpha) - Y \right) \tilde{H}_m(W(s), X) \left(\tilde{f}_m(W(s), X; \alpha) - Y \right)^T.$$

Since $\lambda_{\min}(\tilde{H}_m(W(s), X)) > \lambda_0/2$ for every $s \leq t_0$, then

$$\frac{d}{ds} \|\tilde{f}_m(W(s), X; \alpha) - Y\|_2^2 \leq -\lambda_0 \|\tilde{f}_m(W(s), X; \alpha) - Y\|_2^2,$$

which implies that

$$\frac{d}{ds} \left(\exp(\lambda_0 s) \|\tilde{f}_m(W(s), X; \alpha) - Y\|_2^2 \right) \leq 0.$$

It follows that $\exp(\lambda_0 s) \|\tilde{f}_m(W(s), X; \alpha) - Y\|_2^2$ is a decreasing function of s , and therefore

$$\|\tilde{f}_m(W(s), X; \alpha) - Y\|_2^2 \leq \exp(-\lambda_0 s) \|\tilde{f}_m(W(0), X; \alpha) - Y\|_2^2,$$

for every $s \leq t_0$. Substituting in the integral, we can write that

$$\begin{aligned}\|W(t_0) - W(0)\|_F &\leq \max_{0 \leq s \leq t_0} \sum_{i=1}^k \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_i; \alpha) \right\|_F \int_0^{t_0} \exp(-\lambda_0 s) ds \cdot \|\tilde{f}_m(W(0), X; \alpha) - Y\| \\ &\leq \frac{2}{\lambda_0} \max_{0 \leq s \leq t_0} \sum_{i=1}^k \left(\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F + \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_i; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_i; \alpha) \right\|_F \right) \\ &\quad \times \|\tilde{f}_m(W(0), X; \alpha) - Y\| \\ &\leq \left(\frac{2M}{\lambda_0} + cm^{-2\gamma/\alpha} (\log m)^{2/\alpha} \right) M,\end{aligned}$$

which, for m large, contradicts $\|W(t_0) - W(0)\|_F \geq \sqrt{\log m}$.

C

Throughout this section, it is assumed that all the random variables are defined on a common probability space, say $(\Omega, \mathcal{F}, \mathbb{P})$, unless otherwise stated.

C.1 Proof of Lemma 12

Let us denote x by x_{k+1} . For every $j, j' = 1, \dots, k+1$, we can write that

$$\begin{aligned} & |\tilde{H}_m(W(s), [X, x])[j, j'] - \tilde{H}_m(W(s), [X, x])[j, j']| \\ &= (\log m)^{2/\alpha} \left| \left\langle \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha), \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_{j'}; \alpha) \right\rangle \right. \\ &\quad \left. - \left\langle \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha), \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_{j'}; \alpha) \right\rangle \right| \\ &\leq (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_2 \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_{j'}; \alpha) \right\|_2 \\ &\quad + (\log m)^{2/\alpha} \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_2 \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_{j'}; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_{j'}; \alpha) \right\|_2. \end{aligned}$$

Since $\frac{\partial \tilde{f}_m}{\partial W}(W(0), [X, x]; \alpha)$ converges in distribution, then there exists M such that, for m sufficiently large

$$\mathbb{P} \left(\bigcup_{j=1}^{k+1} \left(\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_2 > M \right) \right) < \frac{\delta}{2}.$$

Let m be such that, with probability at least $1 - \delta/2$,

$$\left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_F^2 \leq \frac{c}{4(k+1)^2 M} (\log m)^{-2/\alpha} m^{-2\gamma/\alpha},$$

for every $j = 1, \dots, k+1$ and every $s \geq 0$. Then, with probability at least $1 - \delta$, the following properties hold for every $s \geq 0$ and every $j = 1, \dots, k+1$:

$$\begin{aligned} & \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_j; \alpha) - \frac{\partial \tilde{f}_m}{\partial W}(W(0), x_j; \alpha) \right\|_2 \leq \frac{c}{4(k+1)^2 M} (\log m)^{-2/\alpha} m^{-2\gamma/\alpha}, \\ & \left\| \frac{\partial \tilde{f}_m}{\partial W}(W(s), x_{j'}; \alpha) \right\|_2 \leq 2M. \end{aligned}$$

Then, with probability at least $1 - \delta$, for every j, j' and every $s \geq 0$,

$$|\tilde{H}_m(W(s), [X, x])[j, j'] - \tilde{H}_m(W(s), [X, x])[j, j']| \leq \frac{c}{(k+1)^2} m^{-2\gamma/\alpha},$$

which implies that

$$\|\tilde{H}_m(W(s), [X, x]) - \tilde{H}_m(W(0), [X, x])\|_F \leq cm^{-2\gamma/\alpha}.$$

C.2 Proof of Theorem 13

Let g be the feature map of the kernel regression, that is

$$g(x) = \frac{\partial \tilde{f}_m}{\partial W}(W(0), x; \alpha),$$

and let β^* be such that $f^*(x; \alpha) = \kappa g(x) \beta^*$. Then it holds that

$$\begin{cases} f^*(x; \alpha) = \kappa g(x) \beta^* \\ \beta^* = \operatorname{argmin}_{\beta \in A(X, y)} \|\beta\|_2, \end{cases}$$

where $A(X, y) = \{\beta : \kappa g(x_i) \beta = y_i \text{ for every } i = 1, \dots, k\}$. The vector β^* can be found by applying gradient flow to solve

$$\min_{\beta} L^*(\beta(t), X),$$

with initialization $\beta(0) = 0$, where

$$L^*(\beta, X) = \frac{1}{2} \sum_{i=1}^k (\kappa g(x_i) - y_i)^2.$$

Let $\beta(t)$ denote the dynamics of β during gradient flow, and, for every d -dimensional vector x , let $f^*(\beta(t), x; \alpha) = \kappa g(x) \beta(t)$. Since $f^*(\beta(0), x; \alpha) = 0$, then we can write that

$$f^*(x; \alpha) = \int_0^\infty \frac{df^*(\beta(t), x; \alpha)}{dt} dt.$$

We can rewrite the integrand as

$$\begin{aligned} \frac{df^*(\beta(t), x; \alpha)}{dt} &= \left\langle \frac{\partial f^*}{\partial \beta}(\beta(t), x; \alpha), \frac{d\beta(t)}{dt} \right\rangle \\ &= \left\langle \frac{\partial f^*}{\partial \beta}(\beta(t), x; \alpha), -\frac{\partial L^*(\beta(t), X)}{\partial \beta(t)} \right\rangle \\ &= - \left\langle \frac{\partial f^*}{\partial \beta}(\beta(t), x; \alpha), \sum_{i=1}^k (f^*(\beta(t), x_i; \alpha) - y_i) \frac{\partial f^*}{\partial \beta}(\beta(t), x_i; \alpha) \right\rangle \\ &= - \left\langle \kappa g(x), \sum_{i=1}^k (f^*(\beta(t), x_i; \alpha) - y_i) \kappa g(x_i) \right\rangle \\ &= -\kappa^2 (f^*(\beta(t), X; \alpha) - Y) H^*(X, x)^T, \end{aligned}$$

where $f^*(\beta(t), X; \alpha) = [f^*(\beta(t), x_i; \alpha)]_i$. In particular,

$$\frac{df^*(\beta(t), X; \alpha)}{dt} = -\kappa^2 (f^*(\beta(t), X; \alpha) - Y) H^*(X, X).$$

Let us now consider the dynamics of $\tilde{f}_{NN}(W(t), x, \alpha)$. We can write that

$$\begin{aligned} &\frac{d\tilde{f}_{NN}(W(t), x, \alpha)}{dt} \\ &= \left\langle \frac{\partial \tilde{f}_{NN}}{\partial W}(W(t), x; \alpha), \frac{dW(t)}{dt} \right\rangle \\ &= \left\langle \frac{\partial \tilde{f}_{NN}}{\partial W}(W(t), x; \alpha), -\frac{\partial L}{\partial W}(W(t), X) \right\rangle \\ &= - \left\langle \frac{\partial \tilde{f}_{NN}}{\partial W}(W(t), x; \alpha), \sum_{i=1}^k (\tilde{f}_{NN}(W(t), x_i; \alpha) - y_i) \frac{\partial \tilde{f}_{NN}}{\partial W}(W(t), x_i; \alpha) \right\rangle \\ &= -\kappa^2 (\tilde{f}_{NN}(W(t), X; \alpha) - Y) \tilde{H}_m(W(t), X, x)^T. \end{aligned}$$

By Lemmas 10 and 11, there exist $\lambda_0 > 0$ and $N_1 \in \mathcal{F}$ such that $\mathbb{P}(N_1) > 1 - \delta/8$ for m sufficiently large, and, for every $\omega \in N_1$,

$$\lambda_{\min}(H^*(X, X)(\omega)) > \lambda_0/2 \text{ and } \lambda_{\min}(\tilde{H}_m(W(t)X, X)(\omega)) > \lambda_0/2$$

hold true for every $t \geq 0$. It follows that, for every $\omega \in N_1$,

$$\begin{aligned} & \|f^*(\beta(t), X; \alpha)(\omega) - Y\|_2 \\ & \leq \exp(-\kappa^2 \lambda_0 t) \|f^*(\beta(0), X; \alpha)(\omega) - Y\|_2 \leq kC \exp(-\kappa^2 \lambda_0 t). \end{aligned}$$

and

$$\|\tilde{f}_{NN}(W(t), X; \alpha)(\omega) - Y\|_2 \leq \exp(-\kappa^2 \lambda_0 t) \|\tilde{f}_{NN}(W(0), X; \alpha)(\omega) - Y\|_2.$$

By Theorem 1 there exist $M_1 > 0$ and $N_2 \in \mathcal{F}$ such that $\mathbb{P}(N_2) > 1 - \delta/8$ for m sufficiently large and for every $\omega \in N_2$,

$$\|Y - \tilde{f}_{nn}(W(0), X; \alpha)(\omega)\|_2 \leq M_1.$$

Furthermore, since $\tilde{H}_m(W(0), [X, x]; \alpha)$ converges in distribution, then there exist $M_2 > 0$ and $N_3 \in \mathcal{F}$ such that $\mathbb{P}(N_3) > 1 - \delta/8$ for m sufficiently large and, for every $\omega \in N_3$,

$$\kappa^2 \|H^*(X, x)(\omega)\|_2 \leq M_2.$$

In the following, we will restrict to $\omega \in N_1 \cap N_2 \cap N_3$. Based on the previous inequalities, we can give an upper bound for the difference between the NN and the tangent kernel prediction at the end of the optimization process.

$$\begin{aligned} & |\tilde{f}_{NN}(x; \alpha) - f^*(x; \alpha)| \\ & = \left| \tilde{f}_{NN}(W(0), x; \alpha) + \int_0^\infty \left(\frac{d\tilde{f}_{NN}(W(t), x; \alpha)}{dt} - \frac{df^*(\beta(t), x; \alpha)}{dt} \right) dt \right| \\ & \leq \frac{\epsilon}{4} + \kappa^2 \left| \int_0^\infty ((\tilde{f}_{NN}(W(t), X; \alpha) - Y) \tilde{H}_m(W(t), X, x)^T - (f^*(\beta(t), X; \alpha) - Y) H^*(X, x)^T) dt \right| \\ & \leq \frac{\epsilon}{4} + \kappa^2 \left| \int_0^\infty (\tilde{f}_{NN}(W(t), X; \alpha) - Y) (\tilde{H}_m(W(t), X, x)^T - H^*(X, x)^T) dt \right| \\ & \quad + \kappa^2 \left| \int_0^\infty (\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)) H^*(X, x)^T dt \right| \\ & \leq \frac{\epsilon}{4} + \kappa^2 \sup_{t \geq 0} \|\tilde{H}_m(W(t), X, x) - H^*(X, x)\|_2 \int_0^\infty \|\tilde{f}_{NN}(W(t), X; \alpha) - Y\|_2 dt \\ & \quad + \kappa^2 \|H^*(X, x)\|_2 \int_0^\infty \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2 dt. \end{aligned}$$

We will bound the second and third terms of the sum, separately. For second term we can write that

$$\begin{aligned} & \int_0^\infty \|\tilde{f}_{NN}(W(t), X; \alpha) - Y\|_2 dt \\ & \leq \|\tilde{f}_{nn}(W(0), X; \alpha) - Y\|_2 \int_0^\infty \exp(-\kappa^2 \lambda_0 t) dt \\ & \leq \frac{M_1}{\lambda_0 \kappa^2} \end{aligned}$$

By Lemma 12, for m sufficiently large, with probability at least $1 - \delta/8$,

$$\sup_{t \geq 0} \|\tilde{H}_m(W(t), X, x) - H^*(X, x)\|_2 \leq \frac{\epsilon \lambda_0}{4M_1}.$$

It follows that, for m sufficiently large, with probability at least $1 - \delta/2$,

$$\kappa^2 \sup_{t \geq 0} \|\tilde{H}_m(W(t), X, x) - H^*(X, x)\|_2 \int_0^\infty \|\tilde{f}_{NN}(X, W(t); \alpha) - Y\|_2 dt < \frac{\epsilon}{4}.$$

Let us now consider the third term. We will prove that, with probability at least $1 - \delta/8$

$$\int_0^\infty \|\tilde{f}_{NN}(X, W(t); \alpha) - f^*(X, \beta(t); \alpha)\|_2 dt < \frac{\epsilon}{4M_2}.$$

For every $t_0 > 0$, we can write that

$$\begin{aligned} & \int_{t_0}^\infty \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2 dt \\ & \leq \int_{t_0}^\infty \|\tilde{f}_{NN}(W(t), X; \alpha) - Y\|_2 dt + \int_{t_0}^\infty \|f^*(\beta(t), X; \alpha) - Y\|_2 dt \\ & \leq \frac{M_1 + kC}{\lambda_0 \kappa^2} \exp(-\kappa^2 \lambda_0 t_0), \end{aligned}$$

which is smaller than $\epsilon/(8M_2)$ if t_0 is large. Let us now consider

$$\int_0^{t_0} \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2 dt \leq \max_{0 \leq t \leq t_0} \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2.$$

We can write that

$$\begin{aligned} & \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2 \\ & \leq \|\tilde{f}_{NN}(W(0), X; \alpha)\|_2 + \int_{s=0}^t \left\| \frac{d(\tilde{f}_{NN}(W(s), X; \alpha) - f^*(\beta(s), X; \alpha))}{ds} \right\|_2 ds \\ & \leq \frac{\epsilon}{4} + \int_{s=0}^t \left\| \frac{d(\tilde{f}_{NN}(W(s), X; \alpha) - f^*(\beta(s), X; \alpha))}{ds} \right\|_2 ds. \end{aligned}$$

Moreover,

$$\begin{aligned} & \frac{d(\tilde{f}_{NN}(W(s), X; \alpha) - f^*(\beta(s), X; \alpha))}{ds} \\ & = -\kappa^2 \tilde{H}_m(W(t), X)(\tilde{f}_{NN}(W(s), X; \alpha) - Y) \\ & \quad + \kappa^2 H^*(X, X; \alpha)(f^*(\beta(s), X; \alpha) - Y) \\ & \leq \kappa^2 (\tilde{f}_{NN}(W(s), X; \alpha) - Y)(H^*(X, X; \alpha) - \tilde{H}_m(W(t), X)) \\ & \quad - \kappa^2 H^*(X, X; \alpha)(\tilde{f}_{NN}(W(s), X; \alpha) - f^*(\beta(s), X; \alpha)). \end{aligned}$$

Since $H^*(X, X; \alpha)$ is non-negative definite, then

$$\begin{aligned} & \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2 \\ & \leq \kappa^2 \int_0^t \|\tilde{f}_{NN}(W(s), X; \alpha) - Y\| \|H^*(X, X; \alpha) - \tilde{H}_m(W(s), X; \alpha)\|_2 ds \\ & \leq \kappa^2 \sup_{t \geq 0} \|H^*(X, X; \alpha) - \tilde{H}_m(W(t), X)\|_2 \int_0^t \|\tilde{f}_{NN}(W(s), X; \alpha) - Y\| ds \\ & \leq \kappa^2 \sup_{t \geq 0} \|H^*(X, X; \alpha) - \tilde{H}_m(W(t), X)\|_2 \|\tilde{f}_{NN}(W(0), X; \alpha) - Y\| \frac{1}{\lambda_0 \kappa^2} \\ & \leq \frac{M_1}{\lambda_0} \sup_{t \geq 0} \|H^*(X, X; \alpha) - \tilde{H}_m(W(t), X)\|_2. \end{aligned}$$

By Lemma 12, for m sufficiently large, with probability at least $1 - \delta/8$,

$$\sup_{t \geq 0} \|H^*(X, X; \alpha) - \tilde{H}_m(W(t), X)\|_2 < \frac{\epsilon \lambda_0}{8M_1}.$$

It follows that, for m sufficiently large, with probability at least $1 - \delta/2$,

$$\kappa^2 \|H^*(X, x)\|_2 \int_0^\infty \|\tilde{f}_{NN}(W(t), X; \alpha) - f^*(\beta(t), X; \alpha)\|_2 dt < \frac{\epsilon}{4}.$$

D

The distribution of a random vector ξ is said to be infinitely divisible if, for every n , there exist some i.i.d. random vectors $\xi_{n1}, \dots, \xi_{nn}$ such that $\sum_k \xi_{nk} \stackrel{d}{=} \xi$. A k -dimensional random vector ξ is infinitely divisible if and only if its characteristic function admits the representation $e^{\psi(u)}$, where

$$\psi(u) = iu^T b - \frac{1}{2}u^T a u + \int \left(e^{iu^T x} - 1 - iu^T x I(\|x\| \leq 1) \right) \nu(dx) \quad (18)$$

where ν is a measure on $\mathbb{R}^k \setminus \{0\}$ satisfying $\int (\|x\|^2 \wedge 1) \nu(dx) < \infty$, a is a $k \times k$ non negative definite, symmetric matrix and b is a vector. The measure ν is called the Lévy measure of ξ and (a, b, ν) are called the characteristics of the infinitely divisible distribution. We will write $\xi \sim i.d.(a, b, \nu)$. Other kinds of truncation can be used for the term $iu^T x$. This affects only the vector of centering constants b . An i.i.d. array of random vectors is a collection of random vectors $\{\xi_{nj}, j \leq m_n, n \geq 1\}$ such that, for every n , $\xi_{n1}, \dots, \xi_{nm_n}$ are i.i.d. The class of infinitely divisible distributions coincides with the class of limits of sums of i.i.d. arrays (Kallenberg, 2002, Theorem 13.12).

To state a general criterion of convergence, we first introduce some notations. Let $\xi \sim i.d.(a, b, \nu)$. Define, for each $h > 0$,

$$\begin{aligned} a^{(h)} &= a + \int_{\|x\| < h} x x^T \nu(dx), \\ b^{(h)} &= b - \int_{h < \|x\| \leq 1} x \nu(dx), \end{aligned}$$

where $\int_{h < \|x\| \leq 1} = -\int_{1 < \|x\| \leq h}$ if $h > 1$. Denote by \xrightarrow{v} vague convergence, that is convergence of measures with respect to the topology induced by bounded, measurable functions with compact support. Moreover, let $\overline{\mathbb{R}^k}$ be the one-point compactification of \mathbb{R}^k . The following criterion for convergence holds (Kallenberg, 2002, Corollary 13.16).

Theorem 14. Consider in \mathbb{R}^k an i.i.d. array $(\xi_{nj})_{j=1, \dots, m_n, n \geq 1}$ and let ξ be $i.d.(a, b, \nu)$. Let $h > 0$ be such that $\nu(\|x\| = h) = 0$. Then $\sum_j \xi_{nj} \xrightarrow{d} \xi$ if and only if the following conditions hold:

- (i) $m_n \mathbb{P}(\xi_{n1} \in \cdot) \xrightarrow{v} \nu(\cdot)$ on $\overline{\mathbb{R}^k} \setminus \{0\}$
- (ii) $m_n \mathbb{E}(\xi_{n1} \xi_{n1}^T I(\|\xi_{n1}\| < h)) \rightarrow a^{(h)}$
- (iii) $m_n \mathbb{E}(\xi_{n1} I(\|\xi_{n1}\| < h)) \rightarrow b^{(h)}$

Inside the class of infinitely divisible distribution, we can distinguish the subclass of stable distributions. A k -dimensional random vector ξ has stable distribution if, for every independent random vectors ξ_1 and ξ_2 with $\xi_1 \stackrel{d}{=} \xi_2 \stackrel{d}{=} \xi$ and every $a, b \in \mathbb{R}$, there exists $c \in \mathbb{R}$ and $d \in \mathbb{R}^k$ such that $a\xi_1 + b\xi_2 \stackrel{d}{=} c\xi + d$. This is equivalent to the condition: for every $n \geq 1$,

$$\xi_1 + \dots + \xi_n \stackrel{d}{=} n^{1/\alpha} \xi + d_n \quad (19)$$

where $\alpha \in (0, 2]$, ξ_1, \dots, ξ_n are i.i.d. copies of ξ and d_n is a vector. The random vector ξ is said to be strictly stable if (19) holds with $d_n = 0$. A stable vector ξ is strictly stable if and only if all its components are strictly stable. The coefficient α is called the index of stability of ξ and the law of ξ is called α -stable. A stable vector ξ is symmetric stable if $\mathbb{P}(\xi \in A) = \mathbb{P}(-\xi \in A)$ for every Borel set A . A symmetric stable vector is strictly stable. The class of stable distributions coincides with the class of limit laws of sequences $((\sum_{k=1}^n X_k - b_n)/a_n)$, where (X_n) are i.i.d. random variables.

A stable distribution is infinitely divisible. Thus its characteristic function admits the Lévy representation (18). If $\alpha = 2$, then the Lévy measure is the null measure and, therefore, the stable distribution coincides with the multivariate normal distribution with covariance matrix a and mean vector b . If $\alpha < 2$, then $a = 0$ (the zero matrix) and the α -stability implies that there exists a measure σ on the unit sphere \mathbb{S}^{k-1} such that $\nu(dx) = r^{-(\alpha+1)} dr \sigma(ds)$, where $r = \|x\|$ and $s = x/\|x\|$. Substituting in (18), we obtain

$$\psi(u) = iu^T b + \int_S \int_0^\infty \left(e^{iru^T s} - 1 - iru^T s I(r \leq 1) \right) \frac{1}{r^{1+\alpha}} dr \sigma(ds)$$

For $\alpha < 1$, the centering $iru^T s I(r \leq 1)$ is not needed, since the function (of r) is integrable, and we can write

$$\psi(u) = iu^T b' + \int_S \int_0^\infty \left(e^{iru^T s} - 1 \right) \frac{1}{r^{1+\alpha}} dr \sigma(ds),$$

for some vector b' . After evaluating the inner integrals as in [Feller \(1968, Example XVII.3\)](#), we obtain

$$\begin{aligned} \psi(u) &= iu^T b' - \int_S |u^T s|^\alpha \Gamma(1-\alpha) \left(\cos(\pi\alpha/2) - i \operatorname{sign}(u^T s) \sin(\pi\alpha/2) \right) \sigma(ds) \\ &= iu^T b' - \int_S |u^T s|^\alpha \left(1 - i \operatorname{sign}(u^T s) \tan(\pi\alpha/2) \right) \Gamma(1-\alpha) \cos(\pi\alpha/2) \sigma(ds). \end{aligned}$$

For $\alpha > 1$, using the centering $iru^T s$, we can write

$$\psi(u) = iu^T b'' + \int_S \int_0^\infty \left(e^{iru^T s} - 1 - iru^T s \right) \frac{1}{r^{1+\alpha}} dr \sigma(ds),$$

for some b'' . After evaluating the inner integrals as in [Feller \(1968, Example XVII.3\)](#), we obtain

$$\begin{aligned} \psi(u) &= iu^T b'' + \int_S |u^T s|^\alpha \frac{\Gamma(2-\alpha)}{\alpha-1} \left(\cos(\pi\alpha/2) - i \operatorname{sign}(u^T s) \sin(\pi\alpha/2) \right) \sigma(ds) \\ &= iu^T b'' - \int_S |u^T s|^\alpha \left(1 - i \operatorname{sign}(u^T s) \tan(\pi\alpha/2) \right) \frac{\Gamma(2-\alpha)}{1-\alpha} \cos(\pi\alpha/2) \sigma(ds). \end{aligned}$$

Since, for $\alpha < 1$, $\Gamma(2-\alpha) = (1-\alpha)\Gamma(1-\alpha)$, we can encompass the above results in one equation, and write, for $\alpha \neq 1$,

$$\psi(u) = iu^T b''' - \int_S |u^T s|^\alpha \left(1 - i \operatorname{sign}(u^T s) \tan(\pi\alpha/2) \right) \frac{\Gamma(2-\alpha)}{1-\alpha} \cos(\pi\alpha/2) \sigma(ds),$$

for some b''' . Finally, for $\alpha = 1$, using the centering $ir \sin ru^T s$, we can write

$$\psi(u) = iu^T b'''' + \int_S \int_0^\infty \left(e^{iru^T s} - 1 - ir \sin ru^T s \right) \frac{1}{r^2} dr \sigma(ds),$$

for some b'''' . Evaluating the inner integral as in [Feller \(1968, Example XVII.3\)](#), we obtain

$$\begin{aligned} \psi(u) &= iu^T b'''' - \int_S |u^T s| \left(\frac{\pi}{2} + i \operatorname{sign}(u^T s) \log |u^T s| \right) \sigma(ds) \\ &= iu^T b'''' - \int_S |u^T s| \left(1 + i \frac{2}{\pi} \operatorname{sign}(u^T s) \log |u^T s| \right) \frac{\pi}{2} \sigma(ds). \end{aligned}$$

Considering the spectral representation $e^{\psi(u)}$ of the multivariate stable characteristic function

$$\psi(u) = \begin{cases} - \int_S |u^T s|^\alpha \left(1 - i \operatorname{sign}(u^T s) \tan(\pi\alpha/2) \right) \Gamma(ds) + iu^T \mu^{(0)} & \alpha \neq 1 \\ - \int_S |u^T s| \left(1 + i \frac{2}{\pi} \operatorname{sign}(u^T s) \log |u^T s| \right) \Gamma(ds) + iu^T \mu^{(0)} & \alpha = 1, \end{cases}$$

we can establish the following relationship between the Lévy measure σ and the spectral measure Γ :

$$\sigma = C_\alpha \Gamma,$$

where

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)} & \alpha \neq 1 \\ 2/\pi & \alpha = 1 \end{cases}$$

A Stable random vector ξ is strictly stable if and only if

$$\begin{cases} \mu^{(0)} = 0 & \alpha \neq 1 \\ \int_S s_j \Gamma(ds) = 0 \text{ for every } j & \alpha = 1. \end{cases}$$

(see e.g. [Samoradnitsky and Taqqu \(1994, Theorem 2.4.1\)](#)). We close, noticing that, by Theorem 14, σ satisfies

$$\lim_{n \rightarrow \infty} n \mathbb{P} \left(\|\xi\| > n^{1/\alpha} x, \frac{\xi}{\|\xi\|} \in A \right) = x^{-\alpha} \sigma(A) \quad (20)$$

for every Borel set A of S such that $\sigma(\partial A) = 0$.

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