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On estimating the tail index and the spectral measure of multivariate α-stable distributions

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Abstract We propose estimators for the tail index and the spectral measure of multivariate α-stable distributions and derive their asymptotic properties. Simulation studies reveal the appropriateness of the estimators. Applications to financial data are also considered.

Keywords Asymptotic distribution · Multivariate α-stable distribution · Spectral measure · Tail index estimation · Generalized empirical likelihood estimation

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1 Introduction

Heavy tailed distributions are widely used for modeling in different scenarios such as finance, telecommunication, insurance and medicine. One of the popular classes

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of heavy tailed distributions is that of α-stable distributions. A univariate α-stable distribution has four parameters: tail index ($0 < \alpha \leq 2$), skewness intensity, scale, and location. These four parameters show the flexibility of α-stable distributions for modeling real data, and estimation of these parameters is of great interest. However, the classical estimation method, such as the method of maximum likelihood estimation, may fail for α-stable distributions, because explicit forms of probability density functions are not available.

Estimation of the tail index of univariate α-stable distributions has a long history. For a rich background on this topic, see: Nolan (1999), Fan (2001), and Paulauskas and Vaičiulis (2011), to name a few. For the multivariate α-stable distribution, Nolan et al. (2001) proposed a component wise based estimator for the tail index.

Several researchers have shown that some multivariate financial data can be fitted by multivariate α-stable distributions, see Nolan (2003), Lombardi and Veredas (2009), Kabašinskas et al. (2009), and Dominicy and Veredas (2013). Therefore, the estimation of multivariate α-stable distributions is an important issue. All information in an α-stable random vector lies in the spectral measure Γ, and two parameters, the tail index, α, and the location vector $\boldsymbol{\mu}$. Estimation of the spectral measure is considered in several articles. Nolan et al. (2001) presented a method for estimating the spectral measure, based on the sample characteristic function. They assume that α is known or can be estimated. Their method has some difficulties. The estimator is generally complex and when the spectral measure concentrates on symmetric points, the method fails. These problems are solved in the bivariate case for a special set of location of point masses. Using spherical harmonic analysis, estimation of the spectral measure was investigated in Pivato and Seco (2003). Indirect estimation of an elliptical α-stable distribution was considered in Lombardi and Veredas (2009). Ogata (2013) proposed using the generalized empirical likelihood (GEL) method with estimating function constructed by empirical and theoretical characteristic functions.

Here, we state the definition of a multivariate α-stable distribution. Let $0 < \alpha < 2$. Then the random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ is said to have an α-stable distribution in $\mathbb{R}^d$ if there exists a finite measure Γ on the unit sphere S^d of $\mathbb{R}^d$ and a vector $\boldsymbol{\mu}$ in $\mathbb{R}^d$ such that its characteristic function $\Phi_{\boldsymbol{X}}(\boldsymbol{t})$ has the form

$$\log \Phi_{\boldsymbol{X}}(\boldsymbol{t}) = -\int_{S^d} |(\boldsymbol{t}, \boldsymbol{s})|^{\alpha} \big(1 - i\,\mathrm{sign}((\boldsymbol{t}, \boldsymbol{s}))G(\alpha, \boldsymbol{t}, \boldsymbol{s})\big)\mathrm{d}\Gamma(\boldsymbol{s}) + i(\boldsymbol{t}, \boldsymbol{\mu}), \tag{1}$$

where

$$G(\alpha, \boldsymbol{t}, \boldsymbol{s}) = \begin{cases} \tan(\pi\alpha/2), & \alpha \neq 1 \\ -\frac{2}{\pi}\log|(\boldsymbol{t}, \boldsymbol{s})|, & \alpha = 1 \end{cases}.$$

In the sequel, we need another equivalent definition: a random vector $\boldsymbol{X}$ is said to have an α-stable distribution if there exist an $\alpha \in (0, 2]$ and a real vector $\boldsymbol{h}_m$ such that

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_m \stackrel{d}{=} m^{1/\alpha}\boldsymbol{X} + \boldsymbol{h}_m, \quad \text{for every} \quad m \geq 1, \tag{2}$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are independent copies of $\boldsymbol{X}$. For more details on different definitions of α-stable random vectors, see Samorodnitsky and Taqqu (1994).

In this paper, we propose new estimation methods for the tail index and the spectral measure of multivariate α-stable distributions. The structure of this paper is as follows. In Sect. 2, we focus on estimation methods for the tail index. We propose two estimators for the tail index of multivariate α-stable distributions with $\boldsymbol{h}_m = \boldsymbol{0}$, for every $m \geq 1$. Such a random vector is called strictly α-stable. In this case, every linear combination has a strictly α-stable distribution. Every symmetric α-stable distribution is strictly α-stable. The first method gives a class of estimators based on quantiles. The second method gives an estimator that converges to the true value, almost surely. In the univariate case, we show that the proposed estimators have smaller asymptotic variances in comparison to some popular estimators. We use the new estimators for α in estimating the spectral measure. In Sect. 3, two methods for estimating the spectral measure are presented. The new estimators are based on the real and imaginary parts of the natural logarithm of the characteristic function. The proposed estimators are generally real. When Γ concentrates on symmetric points the first estimator does not work, but the second method can be used in this case. It is shown that under appropriate conditions the presented estimators converge to the true value, almost surely. Section 4 is devoted to a simulation study through a few examples. We compared the proposed estimators with four other estimators. An illustrated study on daily returns of the SP500 market is done in Sect. 5.

2 Estimation of the tail index

In this section, we give two efficient estimators for the tail index of strictly α-stable random vectors. In the univariate case, asymptotic variances of the proposed estimators are compared to some known estimators.

2.1 The first estimator of α

In the following, $f_X(.)$ and $F_X(.)$ denote the probability density function and distribution function of a random variable X, respectively, and ξ_p^X satisfies

$$p = P(X \leq \xi_p^X).$$

Also, we need the following classical result in the one-dimensional case (see for example Mood et al. 1974): Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with probability density function $f_X(.)$. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics of $X_1, \ldots, X_n$. If $j/n \to p \in (0, 1)$, then

$$\sqrt{n}(X_{j:n} - \xi_p^X) \to_D N\left(0, \frac{p(1-p)}{(f_X(\xi_p^X))^2}\right), \tag{3}$$

as $n \to +\infty$. The notation "$\to_D$" denotes convergence in distribution.

Throughout this subsection $|\boldsymbol{X}|_{1:n} \le \cdots \le |\boldsymbol{X}|_{n:n}$ are the corresponding order statistics of the random variables $|\boldsymbol{X}_1|, \ldots, |\boldsymbol{X}_n|$, where $|\boldsymbol{x}| = (\sum_{k=1}^{d} x_k^2)^{1/2}$, for every $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Consider an i.i.d. sequence of strictly α-stable random vectors $\boldsymbol{X}_i, i = 1, 2, \ldots, m$ with dimension $d \ge 1$. Let $\boldsymbol{Y} = \sum_{i=1}^{m} \boldsymbol{X}_i$. From equality (2) we know that $\log|\boldsymbol{Y}|$ and $(\log m)/\alpha + \log|\boldsymbol{X}_1|$ have the same distributions. This implies that

$$p = P\big(\log|\boldsymbol{Y}| \le \xi_p^{\log|\boldsymbol{Y}|}\big) = P\big(\log|\boldsymbol{X}_1| \le \xi_p^{\log|\boldsymbol{Y}|} - (\log m)/\alpha\big),$$

for every $p \in (0, 1)$. Therefore,

$$\big(\xi_p^{\log|\boldsymbol{Y}|} - \xi_p^{\log|\boldsymbol{X}_1|}\big)/\log m = 1/\alpha. \tag{4}$$

This relation together with (3) helps us to construct an estimator for $1/\alpha$. In the following theorem, based on the relation (4), we find an estimator for $1/\alpha$ and we investigate its asymptotic behavior.

Theorem 2.1 *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$, $N = (m+1)n \in \mathbb{N}$, be an i.i.d. sequence of strictly α-stable random vectors. Let $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i = \sum_{j=1}^{m} \boldsymbol{X}_{n+j+(i-1)m}$, $i = 1, \ldots, n$ be two i.i.d. independent sequences that are constructed from the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$. Let*

$$\widehat{\frac{1}{\alpha_k(m,n)}} = \big(\log|\boldsymbol{Y}|_{k:n} - \log|\boldsymbol{X}|_{k:n}\big)/\log m.$$

Assume that $\frac{k}{n} \to p$, as $n \to +\infty$, where $0 < p < 1$. Then

$$\sqrt{n}\log m\Big(\widehat{\frac{1}{\alpha_k(m,n)}} - \frac{1}{\alpha}\Big) \to_D N(0, A), \tag{5}$$

as $n \to +\infty$, where $A = \frac{p(1-p)}{\big(f_{\log|\boldsymbol{Y}_1|}(\xi_p^{\log|\boldsymbol{Y}_1|})\big)^2} + \frac{p(1-p)}{\big(f_{\log|\boldsymbol{X}_1|}(\xi_p^{\log|\boldsymbol{X}_1|})\big)^2}$.

Proof The random variables $\log|\boldsymbol{Y}|_{1:n} \le \cdots \le \log|\boldsymbol{Y}|_{n:n}$ are the corresponding order statistics of $\log|\boldsymbol{Y}_1|, \ldots, \log|\boldsymbol{Y}_n|$. Therefore, from (3) we have

$$\sqrt{n}\big(\log|\boldsymbol{Y}|_{k:n} - \xi_p^{\log(|\boldsymbol{Y}_1|)}\big) \to_D N(0, \sigma_1^2)$$

and

$$\sqrt{n}\big(\log|\boldsymbol{X}|_{k:n} - \xi_p^{\log(|\boldsymbol{X}_1|)}\big) \to_D N(0, \sigma_2^2),$$

as $n \to +\infty$, where

$$\sigma_1^2 = \frac{p(1-p)}{\big(f_{\log|\boldsymbol{Y}_1|}(\xi_p^{\log|\boldsymbol{Y}_1|})\big)^2} \quad \text{and} \quad \sigma_2^2 = \frac{p(1-p)}{\big(f_{\log|\boldsymbol{X}_1|}(\xi_p^{\log|\boldsymbol{X}_1|})\big)^2}.$$

Since $\boldsymbol{Y}_i, i = 1, \ldots, n$ and $\boldsymbol{X}_i, i = 1 \ldots, n$ are independent, we have

$$\sqrt{n}\log m\left(\frac{\log|\boldsymbol{Y}|_{k:n} - \log|\boldsymbol{X}|_{k:n}}{\log m} - \frac{\xi_p^{\log|\boldsymbol{Y}_1|} - \xi_p^{\log|\boldsymbol{X}_1|}}{\log m}\right) \rightarrow_D N(0, \sigma_1^2 + \sigma_2^2),$$

as $n \to +\infty$. The proof is complete by (4).

For estimation of $1/\alpha$, it is not necessary to construct two independent sequences. This is shown in the following corollary. Its proof is similar to Theorem 2.1.

Corollary 2.2 *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$, $N = mn \in \mathbb{N}$, be an i.i.d. sequence of strictly α-stable random vectors. Let $\boldsymbol{Y}_i = \sum_{j=1}^m \boldsymbol{X}_{j+(i-1)m}$, $i = 1, \ldots, n$ be a sequence that is constructed from the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$. Let*

$$\widehat{\frac{1}{\alpha_k(m,n)}} = \left(\log|\boldsymbol{Y}|_{k:n} - \log|\boldsymbol{X}|_{mk:N}\right)/\log m.$$

Assume that $\frac{k}{n} \to p$, as $n \to +\infty$, where $p \in (0, 1)$. Then $\widehat{\frac{1}{\alpha_k(m,n)}} \to_P \frac{1}{\alpha}$, as $n \to +\infty$.

2.2 The second estimator of α

Every α-stable random variable, X, satisfies $E|X|^\beta < +\infty$ for $0 < \beta < \alpha$. Let $\boldsymbol{X}_i$, $i = 1, \ldots, m$ be an i.i.d. sequence of α-stable random vectors. It can be shown that $E|\log|\boldsymbol{X}_1||^2$ and $E|\log|\sum_{j=1}^m \boldsymbol{X}_j||^2$ are finite. Therefore, from property (2) we have

$$\frac{E\left(\log|\sum_{j=1}^m \boldsymbol{X}_j| - \log|\boldsymbol{X}_1|\right)}{\log m} = \frac{1}{\alpha}. \tag{6}$$

This relation leads us to an estimator that is described in the following theorem.

Theorem 2.3 *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$, $N = mn$, be an i.i.d. sequence of strictly α-stable random vectors. Let $\boldsymbol{Y}_i = \sum_{j=1}^m \boldsymbol{X}_{j+(i-1)m}$, $i = 1, \ldots, n$, be a sequence that is constructed from the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$. Let*

$$\widehat{\frac{1}{\alpha(m,n)}} = \frac{1}{n\log m}\sum_{i=1}^n\left(\log|\boldsymbol{Y}_i| - \log|\boldsymbol{X}_{im}|\right).$$

Then

$$\widehat{\frac{1}{\alpha(m,n)}} \to \frac{1}{\alpha}, \tag{7}$$

almost surely as $n \to +\infty$ and

$$\sqrt{n/B}\left(\widehat{\frac{1}{\alpha(m,n)}} - \frac{1}{\alpha}\right) \to_D N(0,1), \tag{8}$$

as $n \to +\infty$, where $B = Var\Big(\big(\log|\boldsymbol{Y}_1| - \log|\boldsymbol{X}_m|\big)/\log m\Big)$.

Proof The strong law of large numbers and the central limit theorem yield Eqs. (7) and (8), respectively.

In the following corollary, it is shown that for finding (7) we can use all information in the sample $\boldsymbol{X}_i, i = 1, \ldots, N$.

Corollary 2.4 *Assume the conditions of Theorem 2.3. Let*

$$\widehat{\frac{1}{\alpha(m,n)}} = \frac{1}{n\log m}\sum_{i=1}^{n}\log|\boldsymbol{Y}_i| - \frac{1}{N\log m}\sum_{i=1}^{N}\log|\boldsymbol{X}_i|.$$

Then

$$\widehat{\frac{1}{\alpha(m,n)}} \to \frac{1}{\alpha}$$

almost surely, as $n \to +\infty$.

Proof Using the strong law of large numbers, we have

$$\frac{1}{n\log m}\sum_{i=1}^{n}\log|\boldsymbol{Y}_i| \to E\big(\log|\boldsymbol{Y}_1|/\log m\big)$$

and

$$\frac{1}{N\log m}\sum_{i=1}^{N}\log|\boldsymbol{X}_i| \to E\big(\log|\boldsymbol{X}_1|/\log m\big),$$

almost surely, as $n \to +\infty$. Now from (6) we have the desired result.

Remark 2.5 For constructing the presented estimators, it seems that if we take $m = 2$, then we can use more information from the sample.

Remark 2.6 Theorems 2.1 and 2.3 can be used for constructing a confidence interval for α. Since we have to calculate $f_{\log|\boldsymbol{X}|}(.)$ and $f_{\log|\boldsymbol{Y}|}(.)$ in Theorem 2.1 it seems that Theorem 2.3 is more applicable for constructing a confidence interval.

In the univariate case, Paulauskas and Vaičiulis (2011) compared four popular estimators for estimating $1/\alpha$. Their convergence rates are greater than $\sqrt{n}$. Therefore, in comparison of these estimators, two relations (5) and (8) show that the presented estimators in this article have smaller asymptotic variances.

3 Estimation of the spectral measure

In this section, we give two methods for estimating the spectral measure. They are based on the real and imaginary parts of the natural logarithm of the characteristic function. We obtain asymptotic properties of these estimators.

Consider an α-stable random vector $\boldsymbol{X}$ with characteristic function (1). Let $\boldsymbol{\mu} = 0$ and α be known or α can be estimated from some methods. Let the spectral measure Γ be discrete and concentrated on given points $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_L$ in S^d. In other words,

$$\Gamma(.) = \sum_{l=1}^{L} \gamma_l \eta_{s_l}(.), \tag{9}$$

where γ_l's are weights and $\eta_{s_l}(.)$'s are point mass functions at $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_L$. Now, from (1) and (9) we have

$$\log \Phi_{\boldsymbol{X}}(\boldsymbol{t}) = -\sum_{l=1}^{L} |(\boldsymbol{t}, \boldsymbol{s}_l)|^{\alpha} \big(1 - i\,\mathrm{sign}((\boldsymbol{t}, \mathbf{s}_l))G(\alpha, \boldsymbol{t}, \boldsymbol{s}_l)\big)\gamma_l. \tag{10}$$

If the spectral measure is not discrete, it can be approximated by a discrete spectral measure. Byczkowski et al. (1993) stated how a spectral measure can be approximated. They showed that, for any given $\epsilon > 0$ and α-stable random vector $\boldsymbol{X}$ with a spectral measure Γ, there exists an α-stable random vector $\boldsymbol{Y}$ with a discrete spectral measure $\tilde{\Gamma}$ such that

$$\sup_{A \in \mathcal{B}^d} |P(\boldsymbol{X} \in A) - P(\boldsymbol{Y} \in A)| \leq \epsilon,$$

where $\mathcal{B}^d$ denotes the Borel sigma field on S^d.

3.1 The first method

Consider L vectors
$\boldsymbol{t}_k = (t_{k1}, \ldots, t_{kd}) \in \mathbb{R}^d$, $k = 1, \ldots, L$. Let

$$\Delta_1 = [-|(\boldsymbol{t}_k, \boldsymbol{s}_l)|^{\alpha}]_{k,l=1,\ldots,L},$$

$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_L)'$ and $D_1 = (\Re \log \Phi_{\boldsymbol{X}}(\boldsymbol{t}_1), \ldots, \Re \log \Phi_{\boldsymbol{X}}(\boldsymbol{t}_L))'$. Then, based on (10), we have $\Delta_1 \boldsymbol{\gamma} = D_1$. If Δ_1 is nonsingular, then the vector $\boldsymbol{\gamma}$ has a unique solution $\boldsymbol{\gamma} = \Delta_1^{-1} D_1$. Now, we can estimate the vector $\boldsymbol{\gamma}$ from an i.i.d. sequence $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. By estimating α we can estimate Δ_1. Using the two methods introduced in Sect. 2, we can estimate the parameter α. Note that the presented estimators of α are for strictly α-stable random vectors. Using the strong law of large numbers, we can estimate the vector D_1. The following theorem summarizes the obtained results.

Theorem 3.1 *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be an i.i.d. sequence of α-stable random vectors with spectral measure (9). Assume that $\hat{\alpha}_n$ is an estimator for α. Let*

$$\hat{\Delta}_1 = [-|(\boldsymbol{t}_k, \boldsymbol{s}_l)|^{\hat{\alpha}_n}]_{k,l=1,\ldots,L}$$

be nonsingular and

$$\hat{D}_1 = \left(\Re \log\left(\frac{1}{n}\sum_{j=1}^{n}(\cos((\boldsymbol{t}_1, \boldsymbol{X}_j)) + i\sin((\boldsymbol{t}_1, \boldsymbol{X}_j)))\right), \ldots, \right.$$
$$\left. \Re \log\left(\frac{1}{n}\sum_{j=1}^{n}(\cos((\boldsymbol{t}_L, \boldsymbol{X}_j)) + i\sin((\boldsymbol{t}_L, \boldsymbol{X}_j)))\right)\right)'.$$

Then we have the followings:

(a) If $\hat{\alpha}_n \to \alpha$ almost surely, then $\hat{\boldsymbol{\gamma}} = \hat{\Delta}_1^{-1}\hat{D}_1 \to \boldsymbol{\gamma}$, almost surely, as $n \to +\infty$.
(b) If $\hat{\alpha}_n \to \alpha$ in probability, then $\hat{\boldsymbol{\gamma}} = \hat{\Delta}_1^{-1}\hat{D}_1 \to \boldsymbol{\gamma}$ in probability, as $n \to +\infty$.

3.2 The second method

Consider L vectors $\boldsymbol{t}_k = (t_{k1}, \ldots, t_{kd}) \in \mathbb{R}^d$, $k = 1, \ldots, L$. Let

$$\Delta_2 = \left[-\left|(\boldsymbol{t}_k, \boldsymbol{s}_l)\right|^{\alpha}\left(1 - \text{sign}((\boldsymbol{t}_k, \boldsymbol{s}_l))G(\alpha, \boldsymbol{t}_k, \boldsymbol{s}_l)\right)\right]_{k,l=1,\ldots,L}$$

$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_L)'$ and

$$D_2 = \left(\Re \log \Phi_{\boldsymbol{X}}(\boldsymbol{t}_1) + \Im \log \Phi_{\boldsymbol{X}}(\boldsymbol{t}_1), \ldots, \Re \log \Phi_{\boldsymbol{X}}(\boldsymbol{t}_L) + \Im \log \Phi_{\boldsymbol{X}}(\boldsymbol{t}_L)\right)'.$$

Then, from (10) we have $\Delta_2\boldsymbol{\gamma} = D_2$. If Δ_2 is nonsingular, the vector $\boldsymbol{\gamma}$ has the unique solution $\boldsymbol{\gamma} = \Delta_2^{-1}D_2$. As in the case of the first method, by estimating α, we can estimate Δ_2, and the strong law of large numbers enables us to find an estimator for the vector D_2. We summarize the obtained results in the following theorem.

Theorem 3.2 *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be an i.i.d. sequence of α-stable random vectors with the spectral measure (9). Assume that $\hat{\alpha}_n$ is an estimator for α. Let*

$$\hat{\Delta}_2 = [-|(\boldsymbol{t}_k, \boldsymbol{s}_l)|^{\hat{\alpha}_n}(1 - sign((\boldsymbol{t}_k, \boldsymbol{s}_l))G(\hat{\alpha}_n, \boldsymbol{t}_k, \boldsymbol{s}_l))]_{l,k=1,\ldots,L}$$

be nonsingular and

$$\hat{D}_2 = \left(\Re \log \left(\frac{1}{n} \sum_{j=1}^{n} (\cos((\boldsymbol{t}_k, \boldsymbol{X}_j)) + i \sin((\boldsymbol{t}_k, \boldsymbol{X}_j))) \right) \right.$$
$$\left. + \Im \log \left(\frac{1}{n} \sum_{j=1}^{n} (\cos((\boldsymbol{t}_k, \boldsymbol{X}_j)) + i \sin((\boldsymbol{t}_k, \boldsymbol{X}_j))) \right) \right)'_{k=1,\ldots,L}.$$

Then we have the following:

(a) *If $\hat{\alpha}_n \to \alpha$ almost surely, then $\hat{\boldsymbol{\gamma}} = \hat{\Delta}_2^{-1} \hat{D}_2 \to \boldsymbol{\gamma}$, almost surely, as $n \to +\infty$.*

(b) *If $\hat{\alpha}_n \to \alpha$ in probability, then $\hat{\boldsymbol{\gamma}} = \hat{\Delta}_2^{-1} \hat{D}_2 \to \boldsymbol{\gamma}$ in probability, as $n \to +\infty$.*

Remark 3.3 In some cases, the matrices $\hat{\Delta}_1$ and $\hat{\Delta}_2$ may be singular. This difficulty can be circumvented by choosing appropriate linear combinations. In some cases, $\hat{\Delta}_1$ is singular for every linear combination. For example, if Γ concentrates on the symmetric points, then $\hat{\Delta}_1$ is singular for every linear combination. In these cases, the presented method does not work. If $d = 2$, by estimating the skewness intensity parameter (β) one can estimate Γ, see Example 2.3.5 of Samorodnitsky and Taqqu (1994).

Remark 3.4 One can obtain other estimators based on minus of the real part and imaginary part or only on the imaginary part. In these cases, it seems that the obtained estimators are similar to the second method and the first method, respectively.

Remark 3.5 The method introduced in Nolan et al. (2001) is also based on the characteristic function. They presented a vector I and a matrix Ψ such that $\boldsymbol{\gamma}$ satisfies the equation $\Psi\boldsymbol{\gamma} = I$. The vector I is approximated by two methods, characteristic function method and the projection method. Elements in I and Ψ are imaginary (they are not necessary real). Since elements in Δ_1, Δ_2, D_1, and D_2 are real, we conclude that Δ_1 and Δ_2 are different from Ψ; and D_1 and D_2 are different from I.

4 Simulation studies

Through a simulation study, the presented estimators are now compared to the following four estimators: three kinds of GEL method (i.e. empirical likelihood (EL), continuous updating estimator (CUE) and exponential tilting (ET) estimator) and the introduced method in Nolan et al. (2001), say, NPM. These three kinds of GEL methods have been produced by three different risk functions. These four estimators are the same as those considered in Ogata (2013). We consider the root mean square error (RMSE) criterion to investigate the accuracy of the estimators. We use the examples in Ogata (2013) and compare the presented estimators to the mentioned four estimators. In the examples, we consider multivariate α-stable distributions of the form

$$\boldsymbol{X} = \sum_{l=1}^{L} \gamma_l^{1/\alpha} Z_l \boldsymbol{s}_l, \tag{11}$$

Table 1 RMSE's of Examples 4.1 and 4.3

	Example 4.1				Example 4.3		
	EL	ET	CUE	PE1	EL	NPM	PE1
α	0.037	0.036	0.039	0.057	0.017	0.021	0.029
γ_1	0.028	0.021	0.025	0.028	0.041	0.041	0.013
γ_2	0.049	0.040	0.038	0.015	0.043	0.032	0.009
γ_3	0.043	0.035	0.030	0.022	0.032	0.005	0.005

Table 2 RMSE's of Examples 4.2 and 4.4

	Example 4.2				Example 4.4		
	EL	ET	CUE	PE2	EL	NPM	PE2
α	0.063	0.051	0.048	0.079	0.043	0.020	0.037
γ_1	0.065	0.045	0.047	0.038	0.037	0.000	0.017
γ_2	0.061	0.039	0.036	0.046	0.073	0.001	0.022
γ_3	0.059	0.053	0.044	0.041	0.035	0.002	0.020
γ_4	0.052	0.040	0.050	0.042	0.054	0.004	0.023

where $\gamma_l \geq 0$ and $Z_l, l = 1, \ldots, L$ is an i.i.d. sequence of α-stable random variables such that they are totally skewed to the right with location 0 and scale 1. Modarres and Nolan (1994) showed that for $\alpha \neq 1$ the vector $\boldsymbol{X}$ in (11) has an α-stable distribution with $\boldsymbol{\mu} = \mathbf{0}$ and the spectral measure $\Gamma(.) = \sum_{l=1}^{L} \gamma_l \eta_{\boldsymbol{s}_l}(.)$. In Examples 4.1 and 4.2, we set $\boldsymbol{s}_l = (\cos(\theta_l), \sin(\theta_l))$ and in Examples 4.3 and 4.4, we set $\boldsymbol{s}_l = \big(\sin(\psi_l)\cos(\theta_l), \sin(\psi_l)\sin(\theta_l), \cos(\psi_l)\big)$, $0 \leq \theta_l < 2\pi$, $0 \leq \psi_l \leq \pi$, $l = 1, \ldots, L$. To estimate the spectral measure, we need to estimate α, and we used Corollary 2.4 with $m = 2$ for that. In Tables 1 and 2, PE1 and PE2 denote the RMSE based on the first and second new estimators of the spectral measure, respectively. In simulations, the obtained negative values of γ_i, $i = 1, \ldots, L$, are set to 0. The RMSE's in tables for the mentioned four estimators (in the above) are given in Ogata (2013). From the simulation study, we found that the convergence rate of RMSE in Example 4.2 is small; therefore, for this example we used 500 iterations.

Example 4.1 Let $\alpha = 1.5$, $L = 3$, $\gamma_l = 1/3$, $\theta_1 = \pi/3$, $\theta_2 = \pi$ and $\theta_3 = 3\pi/2$. We take $\boldsymbol{t}_1 = (1, -1)$, $\boldsymbol{t}_2 = (1, 0)$, $\boldsymbol{t}_3 = (0, -1)$, $N = 2{,}000$ and 100 iterations.

Example 4.2 Let $\alpha = 1.75$, $L = 4$, $\gamma_i = 1/4$, $\theta_1 = 0$, $\theta_2 = \pi/2$, $\theta_3 = \pi$ and $\theta_4 = 3\pi/2$. We take $\boldsymbol{t}_i = \boldsymbol{s}_i$, $i = 1, \ldots, 4$, $N = 2{,}000$ and 500 iterations.

Example 4.3 Let $\alpha = 1.7$, $L = 3$, $(\gamma_l, \gamma_2, \gamma_3) = (0.5, 0.4, 0.1)$, $(\theta_1, \psi_1) = (\pi/4, \pi/3)$, $(\theta_2, \psi_2) = (0, \pi/2)$ and $(\theta_3, \psi_3) = (3\pi/2, 5\pi/6)$. We take $\boldsymbol{t}_1 = (1, 0, 0)$, $\boldsymbol{t}_2 = (0, 1, 0)$, $\boldsymbol{t}_3 = (0, 0, 1)$, $N = 10{,}000$ and 100 iterations.

Example 4.4 Let $\alpha = 1.7$, $L = 4$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (0.1, 0.2, 0.3, 0.4)$, $\boldsymbol{s}_1 = (1/2, 1/2, 1/\sqrt{2})$, $\boldsymbol{s}_2 = (-1/2, -1/2, 1/\sqrt{2})$, $\boldsymbol{s}_3 = -\boldsymbol{s}_1$, $\boldsymbol{s}_4 = -\boldsymbol{s}_2$ (Note that the expressions of $(\theta_3, \psi_3) = (-\pi/4, -\pi/4)$ and $(\theta_4, \psi_4) = (-5\pi/4, -\pi/4)$ in Ogata (2013) are wrong). We take $\boldsymbol{t}_i = \boldsymbol{s}_i$, $i = 1, \ldots, 4$, $N = 10{,}000$ and 100 iterations.

In Example 4.1, for α and γ_1, ET works better than other three estimators; and for γ_2 and γ_3 PE1 works best. In Example 4.3, only for α, EL and NPM work better than PE1, but for other parameters, PE1 works best. In Example 4.2, estimation of α using EL, ET and CUE are more accurate than PE2. In the estimation of $\boldsymbol{\gamma}$, PE2 works better than EL. In Example 4.4, PE2 is more accurate than EL and NPM works best.

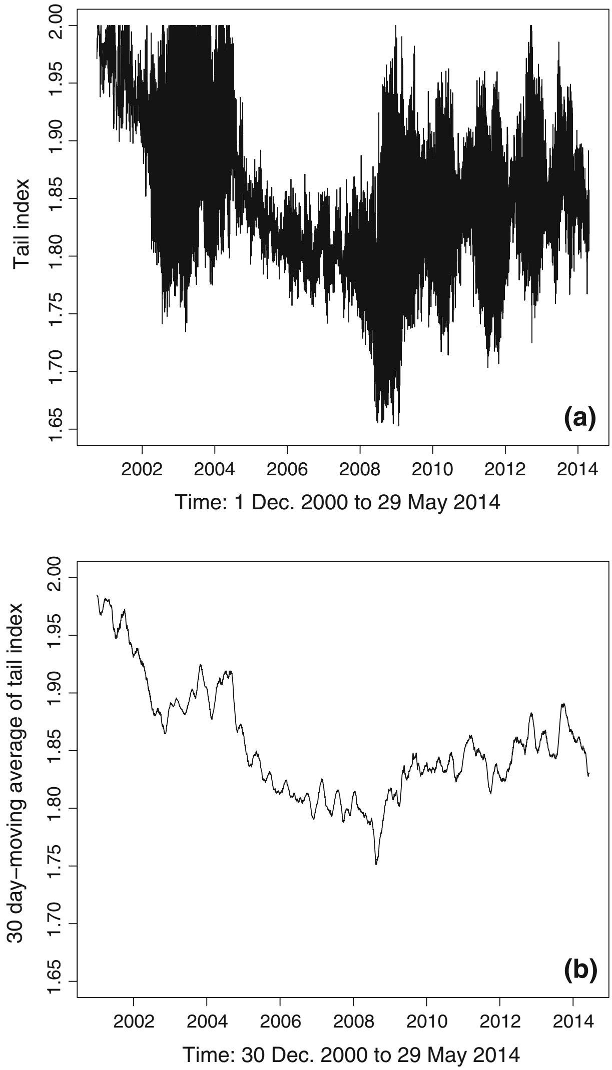

Fig. 1 **a** Estimated tail index for 10 assets in SP500 with the highest average volumes from 1 Dec. 2000 to 29 May 2014 and **b** 30 day-moving average of estimated tail index from 30 Dec. 2000 to 29 May 2014

5 Application in finance

As an illustration, we consider estimating the tail index of financial assets included in the SP500. Since the estimator in Corollary 2.4 converges to the true value almost surely, we use this estimator for estimating the tail index.

Rachev et al. (2005) considered 382 companies in the SP500 market. They showed that ARMA(1,1)-GARCH(1,1) with innovations under α-stable distribution is an appropriate model for daily returns. Similar to Rachev et al. (2010), we consider 10 assets in the SP500 with the highest large volumes during 1 Dec. 2000–29 May 2014. We estimated the tail index after the ARMA(1,1)-GARCH(1,1) effect is removed from the marginal distributions, and a symmetric multivariate α-stable distribution is fitted to the innovations of the 10 univariate ARMA(1,1)-GARCH(1,1) time series fitted in the individual returns of the assets in 10 stocks in the SP500. We plot the tail index of the innovations of an ARMA(1,1)-GARCH(1,1) model fitted on a 500-day rolling window and 30-day moving average of the tail index in the two graphs in Fig. 1. For numerical calculations, we used the *GEVStableGarch* and *STABLE* packages in R. They are available at http://www.cran.r-project.org/package=GEVStableGarch and http://www.RobustAnalysis.com, respectively. From the graph (a) in Fig. 1, we can find that the tail index is time varying and the values around 2008–2009 are small, which indicates that the market tail is volatile in that period. The graph (b) shows long term (monthly) behavior of daily tail index. Figure 1 shows that after 2013, the tail index is almost greater than 1.8. From this, we can conclude that the currently the market is "almost normal", which is good news for financial markets.

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